Title: On space of integrable quantum field theories

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Abstract:

On space of Integrable Field Theories

Joint work with F.Smirnov

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In 2D, many Integrable Quantum Field theories are known. In pragmatic terms, IQFT can be regarded as "reference points" on the map of the space of generic Quantum Field Theories: if some QFT is in some sense "close" to certain IQFT, it is likely that its physics is similar. Therefore it is important to understand in which manner the IQFT reside inside the space of generic QFT.

Integrable Quantum Field Theories fill a subspace $\Sigma^{(\mathrm{Int})}$ in the space Σ of generic QFT in 2D. As the tangent $T\Sigma$ is the space of scalar local operators (modulo derivatives), the tangent $T\Sigma^{(\mathrm{Int})}$ is the span of operators which, when taken as infinitesimal perturbations of the IQFT, preserve its integrability. We call such operators "integrable". They play the role analogous to "exactly marginal operators" in CFT:

$$\Sigma^{(\mathrm{CFT})} \subset \Sigma \qquad T\Sigma^{(\mathrm{CFT})} = \mathrm{Span}(\text{Exactly marginal operators})$$

$$\Sigma^{(\mathrm{Int})} \subset \Sigma \qquad T\Sigma^{(\mathrm{Int})} = \mathrm{Span}(\text{Integrable operators})$$

It turns out that $T\Sigma^{(\mathrm{Int})}$ is infinite-dimensional. Given an IQFT, an infinite set of "integrable" operators can be constructed explicitly, in terms of local currents of IQFT.

Form-factors of integrable operators have special properties. The form-factors are constructed explicitly in the case of sine-Gordon model.

I. Playground:

1. Flat two dimensional (Euclidean) space $\mathbb{R}^2$. Points are usually denoted as z, complex coordinates

$$z = (\mathsf{z}, \bar{\mathsf{z}}) : \qquad \mathsf{z} = \mathsf{x} + i\mathsf{y}\,, \qquad \bar{\mathsf{z}} = \mathsf{x} - i\mathsf{y}$$

are used.

2. QFT will be understood in abstract sense, as an (infinite-dimensional) space of local fields

$$\mathcal{F} = \mathrm{Span}\{\mathcal{O}_a\},$$

and a collection of the correlation functions

$$\langle\, \mathcal{O}_{a_1}(z_1)\mathcal{O}_{a_2}(z_2)\cdots\mathcal{O}_{a_n}(z_n)\,\rangle$$

which satisfy such-and-such properties (some specified below, as needed), most importantly the OPE algebra

$$\mathcal{O}_a(z_1)\mathcal{O}_b(z_2) = \sum_c C_{ab}^c(z_1 - z_2)\,\mathcal{O}_c(z_2)\,.$$

One may (or may not) think in terms Lagrangian Field Theories (paths integral)

$$\langle \cdots \rangle = Z^{-1} \int D[\phi] ... e^{-\mathcal{A}[\phi]}\,, \;\; \mathcal{A}[\phi] = \int \mathcal{L}\left(\phi(z), \partial_\mu \phi(z), \partial_\mu \partial_\nu \phi(z), ...\right) d^2 z$$

Some UV regularization (with the UV cutoff distance ϵ) is assumed.

Such actions form the (infinite-dimensional) space of quasi-local field theories Σ. The coordinates $\{g^i\}$ on Σ are "coupling parameters" in $\mathcal{L} = \mathcal{L}(\phi(x), ...|g^i)$.

There are other definitions of QFT, i.e. as a "perturbed CFT", via "formal action"

$$\mathcal{A} = \mathcal{A}_{\text{CFT}} + \sum_i g^i \int O_i(z)\, d^2 z\,, \quad O_i \in \mathcal{F}_0$$

($\mathcal{F}_0 \subset \mathcal{F}$ - spaces of scalar (spinless) fields of the CFT) which may be understood in terms of Conformal Perturbation Theory (with some UV regularization assumed). Again, the couplings $\{g^i\}$ provide (local) coordinates on Σ .

3. In any case, let $\{g^i\}$ be coordinates on $\mathbf{\Sigma}$, QFT$_g \in \mathbf{\Sigma}$, and let

$\mathcal{F}_0^{(g)} \subset \mathcal{F}^{(g)}$ – space of scalar fields of QFT$_g$, with the basis $\{O_i\}$

"Deformation formula"

$$\delta_g \langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle_g = - \sum_i \delta g^i \int d^2 z \langle O_i(z) \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle_g + \sum_{k=1}^{n} \langle \mathcal{O}(z_1) \cdots \delta_g \mathcal{O}_k(z_k) \cdots \mathcal{O}(z_n) \rangle_g$$

where $\delta_g \mathcal{O}_a = \sum_i \delta g^i (\hat{B}_i(g)\mathcal{O})_a$. (The integral over z can - and usually does - diverge as $z \to z_k$, and $\delta \mathcal{O}_k(z_k)$ includes the counterterms.) That is, $\mathcal{F}_0^{(g)} = T\mathbf{\Sigma}|_g$. More precisely

$$T\mathbf{\Sigma}|_g = \hat{\mathcal{F}}_0^{(g)} := \mathcal{F}_0^{(g)} / \partial \mathcal{F}^{(g)}, \qquad \partial \mathcal{F}^{(g)} = \{\partial_\mu \mathcal{O}_a\}$$

Subtlety with locality: Even if QFT_g is strictly local, adding generic combination of fields O_i typically spoils the UV properties and breaks locality at the scales $\sim \epsilon$. Therefore, the space of truly local QFT constitutes but a small subspace in $\boldsymbol{\Sigma}$. Isolating this subspace requires full understanding of the RG flow in $\boldsymbol{\Sigma}$ - exceedingly difficult problem, which I generally ignore in this discussion.

Integrable Field Theories is another subspace in $\boldsymbol{\Sigma}^{(\mathrm{Int})} \subset \boldsymbol{\Sigma}$. Correspondingly, given an IQFT

$$T\boldsymbol{\Sigma}^{(\mathrm{Int})}|_{\mathrm{IQFT}} \subset T\boldsymbol{\Sigma}|_{\mathrm{IQFT}}$$

i.e. there is a subspace in $\hat{\mathcal{F}}_0$ associated with "integrable perturbations".

Main result of this work: Characterization (constructive, and likely close to complete) of $T\boldsymbol{\Sigma}^{(\mathrm{Int})}|_{\mathrm{IQFT}}$ for generic IQFT.

As it turns, the question of integrability can be largely separated from the problem of true locality. Perhaps, it will be easier to solve the latter for IQFT?

4. Integrable Field Theories: Common property - an infinite set of Local Integrals of Motion.

$$P_s = \int_C \left[T_{s+1}(z) dz + \Theta_{s-1}(z) d\bar{z} \right]$$

where the local currents (T_{s+1}, Θ_{s-1}) satisfy the continuity equation

$$\partial_{\bar{z}} T_{s+1}(z) = \partial_z \Theta_{s-1}(z)$$

The label s takes some (infinite) set of values $\mathcal{S}$, typically associated with the spin(s) of the fields, $\mathcal{S} \subset \mathbb{Z}/2$.(More generally, fractional-spin IM with quasi-local currents are possible). The set $\mathcal{S}$ is an important characteristic of IQFT.

It is often convenient to make separate notations for the negative-spin currents, $(T_{-s+1}, \Theta_{-s-1}) = (\bar{\Theta}_{s-1}, \bar{T}_{s+1})$, so that

$$\partial_z \bar{T}_{s+1}(z) = \partial_{\bar{z}} \bar{\Theta}_{s-1}(z)$$

$$\bar{P}_s = \int_C \left[\bar{T}_{s+1}(z) d\bar{z} + \bar{\Theta}_{s-1}(z) dz \right] , \qquad s > 0 .$$

- Conservation: The integrals over C in P_s, $\bar{P}_s$ do not change under "trivial" deformations of the integration contours, e.g.

$$\oint_C \langle\, [T_{s+1}(z)d\mathsf{z} + \Theta_{s-1}(z)d\bar{\mathsf{z}}]\ \mathcal{O}_1(z_1)\cdots\mathcal{O}_n(z_n)\,\rangle = 0$$

as long as the contour C leaves outside the insertion points $z_1, z_2, ..., z_n$.

- The IM form a commuting set,

$$[P_s, P_\sigma] = [P_s, \bar{P}_\sigma] = [\bar{P}_s, \bar{P}_\sigma] = 0\,.$$

The commutativity implies

$$[P_\sigma, T_{s+1}(z)] = \partial_{\bar{\mathsf{z}}} A_{\sigma,s}(z)\,, \quad [P_\sigma, \Theta_{s-1}(z)] = \partial_{\mathsf{z}} A_{\sigma,s}(z)\,,$$

$$[P_\sigma, \bar{T}_{s+1}(z)] = \partial_{\mathsf{z}} B_{\sigma,s}(z)\,, \quad [P_\sigma, \bar{\Theta}_{s-1}(z)] = \partial_{\bar{\mathsf{z}}} B_{\sigma,s}(z)\,,$$

and similar equations for the commutators of $\bar{P}_\sigma$ with the currents. The commutators (as usual in Euclidean QFT) are expressed as the contour integrals, e.g.

$$[P_\sigma, \mathcal{O}(z)] = (1/2\pi i)\oint_{C_z} [T_{\sigma+1}(w)d\mathsf{w} + \Theta_{\sigma-1}(w)d\bar{\mathsf{w}}]\mathcal{O}(z)$$

over a small contour C_z encircling z.

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over a small contour C_z encircling z.

5. Operators X_s

Given the set of currents (T_{s+1}, Θ_{s-1}), it is possible to construct the set of special scalar fields $X_s \in \hat{\mathcal{F}}_0$, one for each current. The fields X_s are suitably defined limits

$$X_s(z') = \lim_{z \to z'} \left\{ T_{s+1}(z)\bar{T}_{s+1}(z') - \Theta_{s-1}(z)\Theta_{s-1}(z') \right\}.$$

Generally, the operator products like those in the r.h.s are singular at $z = z'$, and the naive limit does not exist. It turns out that for these special combinations the limit is straightforward, if one ignores terms from $\partial\mathcal{F}$.

Lemma: For any local fields $\mathcal{O}_a, \mathcal{O}_b \in \mathcal{F}$ the operator product

$$(\partial_{\mathsf{z}} + \partial_{\mathsf{z}'})\, \mathcal{O}_a(z)\mathcal{O}_b(z') \ \in \ \partial\mathcal{F}$$

where

$$\partial\mathcal{F} = \mathrm{Span}\,\{\partial_{\mathsf{z}}\mathcal{O}_a(z), \partial_{\bar{\mathsf{z}}}\mathcal{O}_a(z)\}\ ,$$

and similarly with $(\partial_{\mathsf{z}} + \partial_{\mathsf{z}'})$ replaced by $(\partial_{\mathsf{z}} + \partial_{\mathsf{z}'})$.

Indeed, the diff. operators $(\partial + \partial')$ annihilates the coefficient functions $C^c_{ab}(z - z')$ in the OPE

$$\mathcal{O}_a(z)\mathcal{O}_b(z') = \sum_c C^c_{ab}(z - z')\, \mathcal{O}_c(z')\,.$$

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The identity

$$\partial_{\bar{z}}\Big\{T_{s+1}(z)\bar{T}_{s+1}(z') - \Theta_{s-1}(z)\bar{\Theta}_{s-1}(z')\Big\} =$$
$$(\partial_z + \partial_{z'})\ \Theta_{s-1}(z)\bar{T}_{s+1}(z') - (\partial_{\bar{z}} + \partial_{\bar{z}'})\ \Theta_{s-1}(z)\bar{\Theta}_{s-1}(z') \in \partial\mathcal{F}\,,$$

and similar identity with $\partial_{\bar{z}} \to \partial_z$, is an elementary consequence of the continuity equations

$$\partial_{\bar{z}}T_{s+1}(z) = \partial_z\Theta_{s-1}(z)\,, \quad \partial_z\bar{T}_{s+1}(z) = \partial_{\bar{z}}\bar{\Theta}_{s-1}(z)\,.$$

Hence

$$T_{s+1}(z)\bar{T}_{s+1}(z') - \Theta_{s-1}(z)\Theta_{s-1}(z') \;=\; X_s(z') \quad \text{mod} \quad \partial\mathcal{F}\,.$$

$(X_s(z') = X_s(z)\ \ \text{mod}\ \ \partial\mathcal{F}.)$

This gives unique (modulo $\partial\mathcal{F}$) definition of scalar (spinless) field X_s. I.e. X_s uniquely specifies a vector in $T\Sigma|_{\text{IQFT}}$. Next, we want to show that in fact

$$X_s \;\in\; T\Sigma^{(\text{Int})}|_{\text{IQFT}}$$

First, we prove that for any $\sigma, s \in \mathcal{S}$

$$[P_\sigma, X_s(z)] = 0 \quad \text{mod} \quad \partial\mathcal{F}\,,$$

where, by definition,

$$[P_\sigma, X_s(z)] = (1/2\pi i) \oint_{C_z} [T_{\sigma+1}(w)d\mathrm{w} + \Theta_{\sigma-1}(w)d\bar{\mathrm{w}}]\, X_s(z)$$

Indeed, consider

$$\Big[P_\sigma, \Big\{T_{s+1}(z)\bar{T}_{s+1}(z') - \Theta_{s-1}(z)\bar{\Theta}_{s-1}(z')\Big\}\Big] = \text{"}z\text{-term"} + \text{"}z'\text{-term"}$$

Recall that, as the condition for $[P_\sigma, P_s] = 0$,

$$[P_\sigma, T_{s+1}(z)] = \partial_{\mathrm{z}} A_{\sigma,s}(z)\,, \quad [P_\sigma, \Theta_{s-1}(z)] = \partial_{\bar{\mathrm{z}}} A_{\sigma,s}(z)\,,$$

Then

$$\text{"}z\text{-term"} = \partial_{\mathrm{z}} A_{\sigma,s}(z)\,\bar{T}_{s+1}(z') - \partial_{\bar{\mathrm{z}}} A_{\sigma,s}(z)\bar{\Theta}_{s-1}(z') =$$
$$= (\partial_{\mathrm{z}} + \partial_{\mathrm{z}'})\,A_{\sigma,s}(z)\,\bar{T}_{s+1}(z') - (\partial_{\bar{\mathrm{z}}} + \partial_{\bar{\mathrm{z}}'})\,A_{\sigma,s}(z)\bar{\Theta}_{s-1}(z') \;\in\; \partial\mathcal{F}\,,$$

Likewise

$$\text{"}z'\text{-term"} = 0 \quad \text{mod} \quad \partial\mathcal{F}\,.$$

The desired statement follows in the limit $z' \to z$.

6. Integrability of X_s perturbations

Consider IQFT_0 with the set of local IM $\{P_\sigma, \sigma \in \mathcal{S}\}$, and let $\mathcal{A}_0$ denote its formal action. Let us show that the infinitesimal perturbation

$$\mathcal{A}_0 + \delta\mathcal{A}_s\,, \qquad \delta\mathcal{A}_s = \delta g_s \int X_s(w)\, d^2w$$

with any $s \in \mathcal{S}$ preserves the same set of IM. Conservation of P_σ in the unperturbed theory implies

$$\langle\, \mathcal{O}_1(z_1)\cdots\mathcal{O}_n(z_n)\ \oint_C \left[T_{\sigma+1}(z)dz + \Theta_{\sigma-1}(z)d\bar{z}\right] \rangle_0 = 0\,.$$

for any "trivial" contour C (simple contour which leaves outside all the insertion points). Here and below $\langle\cdots\rangle_0$ denote the correlation functions of the original unperturbed theory. We want to prove that the same relations (perhaps with somewhat modified currents $(T_{\sigma+1}, \Theta_{\sigma-1})$) hold in the theory $\mathcal{A}_0 + \delta\mathcal{A}_s$.

By the "deformation formula" (ignoring the terms with $\delta\mathcal{O}_k(z_k)$)

$$\langle\, \mathcal{O}_1(z_1)\cdots\mathcal{O}_n(z_n)\ \oint_C \left[T_{\sigma+1}(z)dz + \Theta_{\sigma-1}(z)d\bar{z}\right] \rangle_{0+\delta\mathcal{A}_s} =$$

$$\langle\, \mathcal{O}_1(z_1)\cdots\mathcal{O}_n(z_n)\ \oint_C \left[T_{\sigma+1}(z)dz + \Theta_{\sigma-1}(z)d\bar{z}\right] \rangle_0 \quad -$$

$$\delta g_s \int d^2w\,\langle\, \mathcal{O}_1(z_1)\cdots\mathcal{O}_n(z_n)\ \oint_C \left[T_{\sigma+1}(z)dz + \Theta_{\sigma-1}(z)d\bar{z}\right]\ X_s(w)\rangle_0$$

The first term is zero (by definition). In the second term, split the integral over w as

$$-\int d^2w = -\int_{D(C)} d^2w - \int_{D'(C)} d^2w$$

where $D(C)$ is the part of $\mathbb{R}^2$ bounded by C, and $D'(C)$ is its complement. The second term vanishes (since the contour C there remains "trivial"). In the first term C encloses the point w; closing it on that point yields the commutator $[P_\sigma, X_s(w)]$, yet to be integrated in w over the domain $D(C)$.

As we have seen, $[P_\sigma, X_s(w)] \in \partial\mathcal{F}$, i.e.

$$-2i\,[P_\sigma, X_s(w)] = \partial_{\mathrm{w}}\hat{\Theta}_{s,\sigma-1}(w) - \partial_{\bar{\mathrm{w}}}\hat{T}_{s,\sigma+1}(w)\,,$$

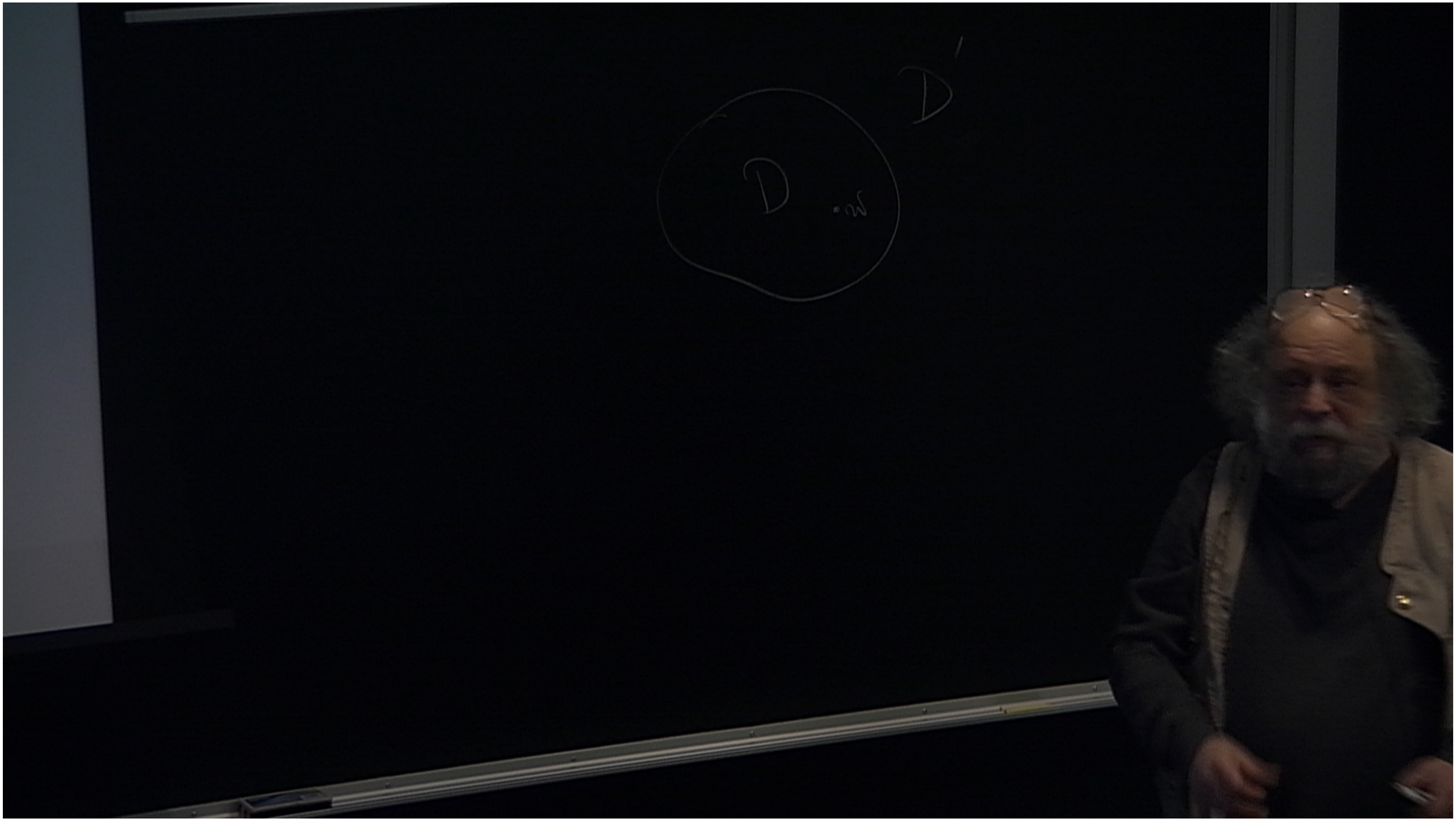
with some local fields $\hat{T}, \hat{\Theta}$. Therefore, by the Stokes, the integral over w evaluates to

$$-\langle \mathcal{O}_1(z_1)\cdots\mathcal{O}_n(z_n)\ \oint_C \left[\hat{T}_{s,\sigma+1}(z)d\mathrm{z} + \hat{\Theta}_{s,\sigma-1}(z)d\bar{\mathrm{z}}\right]\rangle_0$$

Finally, we have

$$\langle \mathcal{O}_1(z_1)\cdots\mathcal{O}_n(z_n)\ \oint_C \left[T_{\sigma+1}(z)d\mathrm{z} + \Theta_{\sigma-1}(z)d\bar{\mathrm{z}}\right]\rangle_{0+\delta\mathcal{A}_s} =$$

$$-\delta g_s \int \langle \mathcal{O}_1(z_1)\cdots\mathcal{O}_n(z_n)\ \oint_C \left[\hat{T}_{s,\sigma+1}(z)d\mathrm{z} + \Theta_{s,\sigma-1}(z)d\bar{\mathrm{z}}\right]\rangle_0$$



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That is, the IM P_σ, with

$$T_{\sigma+1} \to T_{\sigma+1} + \delta g_s \hat{T}_{s,\sigma+1}\,, \qquad \Theta_{\sigma-1} \to \Theta_{\sigma-1} + \delta g_s \hat{\Theta}_{s,\sigma-1}$$

still conserves in the perturbed theory $\mathcal{A}_0 + \delta\mathcal{A}_s$.

7. Conclusion:

• With any Integrable Field Theory, with the set $\{P_s, s \in \mathcal{S}\}$ of local IM, one can associate an infinite-dimensional space Σ_{Int} of Integrable quasi-local theories, so that

$$T\Sigma_{\text{Int}} = \text{Span}(X_s, s \in \mathcal{S})\,.$$

8. Remarks:

- UV regularization does not seem to spoil the analysis.

- Although I explicitly discussed strictly local IM (with (T_{s+1}, Θ_{s-1}) strictly local fields), the analysis extends to "semi-local" IM, of fractional spins s, with "parafermionic" currents.

- Although only Relativistic (Rotationally invariant) Field Theories were considered, the analysis straightforwardly generalizes to non-invariant theories by including perturbations of non-zero spins $s-s'$

$$X_{s,s'}(z) = \lim_{z' \to z} \left\{ T_{s+1}(z)\bar{T}_{s'+1}(z') - \Theta_{s-1}(z)\bar{\Theta}_{s'-1}(z') \right\}$$

- Integrable Theories are amendable to analysis by various exact methods (formfactor bootstrap, TBA, QISM), which may help in addressing the problem of true locality.

9. Factorizable S-matrix

Massive IQFT: Presence of $\{P_s\} \to$ Factorizable S-matrix.

- The set of particle momenta conserves in the scattering, i.e.

$$^{(out)}\langle \theta'_1, \cdots \theta'_M \mid \theta_1, \cdots, \theta_N \rangle^{(in)} = 0 \quad \text{unless} \quad \{\theta'_1, \cdots, \theta'_M\} = \{\theta_1, \cdots, \theta_N\}$$

In particular, $M = N$. Here θ's denote the rapidities of the particles.

- $N \to N$ S-matrix factorizes in terms of the $2 \to 2$ S-matrix $\widehat{S}(\theta) \equiv \widehat{S}_{2\to 2}(\theta)$,

$$\mid \theta_1, \theta_2 \rangle^{(in)} = \widehat{S}(\theta_1 - \theta_2) \mid \theta_1, \theta_2 \rangle^{(out)}$$

(generally, $\widehat{S}$ is an operator in the spaces of "colors" of the particles).

The 2-particle S-matrix satisfies a number of general conditions (Unitarity, Analyticity, Crossing, Yang-Baxter Relation). The conditions generally admit the so-called CDD ambiguity:

If $\hat{S}(\theta)$ satisfies the conditions, then so do $\hat{S}(\theta)\,U(\theta)$, where U arbitrary meromorphic function (analytic& bounded in the domain $0 < \Im m\theta < \pi$), such that

$$U(\theta)U(-\theta) = 1\,, \quad U(\theta) = U(i\pi - \theta)$$

In particular, $U(\theta + 2\pi i) = U(\theta)$. Locally (i.e. ignoring global analyticity condition) these conditions are solved by

$$U(\theta) = \exp\{i\Delta(\theta)\}\ , \qquad \Delta(\theta) = \sum_{s=1,3,5,7,\dots} G_s \sinh(s\theta)$$

10. Formfactor perturbation theory

Let again $\mathcal{A}_0$ denote IQFT, and consider an infinitesimal perturbation

$$\mathcal{A}_0 + \delta\mathcal{A}, \qquad \delta\mathcal{A} = \delta g \int \mathcal{O}(w)\, d^2 w$$

The S-matrix version of the "deformation formula" reads

$$\delta\ {}_{(out)}\langle \theta'_1, \cdots \theta'_M \mid \theta_1, \cdots, \theta_N \rangle_{(in)} =$$
$$= \delta g \int d^2 w\ {}_{(out)}\langle \theta'_1, \cdots \theta'_M \mid \mathcal{O}(w) \mid \theta_1, \cdots, \theta_N \rangle_{(in)} =$$
$$\delta g\ (2\pi)^2 \delta^{(2)} \left(P^\mu_{in} - P^\mu_{out} \right)\ {}_{(out)}\langle \theta'_1, \cdots \theta'_M \mid \mathcal{O}(0) \mid \theta_1, \cdots, \theta_N \rangle_{(in)}$$

The connected part of the matrix element here - the "form-factor" - is analytic function of the rapidities $\rightarrow$ if not identically zero, can not vanish at $\{\theta'_j\} \neq \{\theta_i\}$ $\rightarrow$ inelastic scattering (breakdown of integrability).

However, for special $\mathcal{O}$ the multi-particle ($N \geq 4$) formfactors could all vanish on the "Energy-Momentum Surface" $P^\mu_{in} = P^\mu_{out}$.

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To see that this is a consistent proposition, recall that the formfactor generally satisfy the "formfactor bootstrap equations" (FBE) - the system of linear relations (which guarantee locality of $\mathcal{O}$):

i) Scattering relation

$$F_N(\theta_1,\ldots,\theta_i,\theta_{i+1},\ldots\theta_N) = S(\theta_i-\theta_{i+1})\, F_N(\theta_1,\ldots,\theta_{i+1},\theta_i,\ldots,\theta_N)\,,$$

ii) Crossing relation

$$F_N(\theta_1+2\pi i,\theta_2,\ldots,\theta_N) = F_N(\theta_2,\ldots,\theta_N,\theta_1)\,,$$

iii) "Annihilation poles"

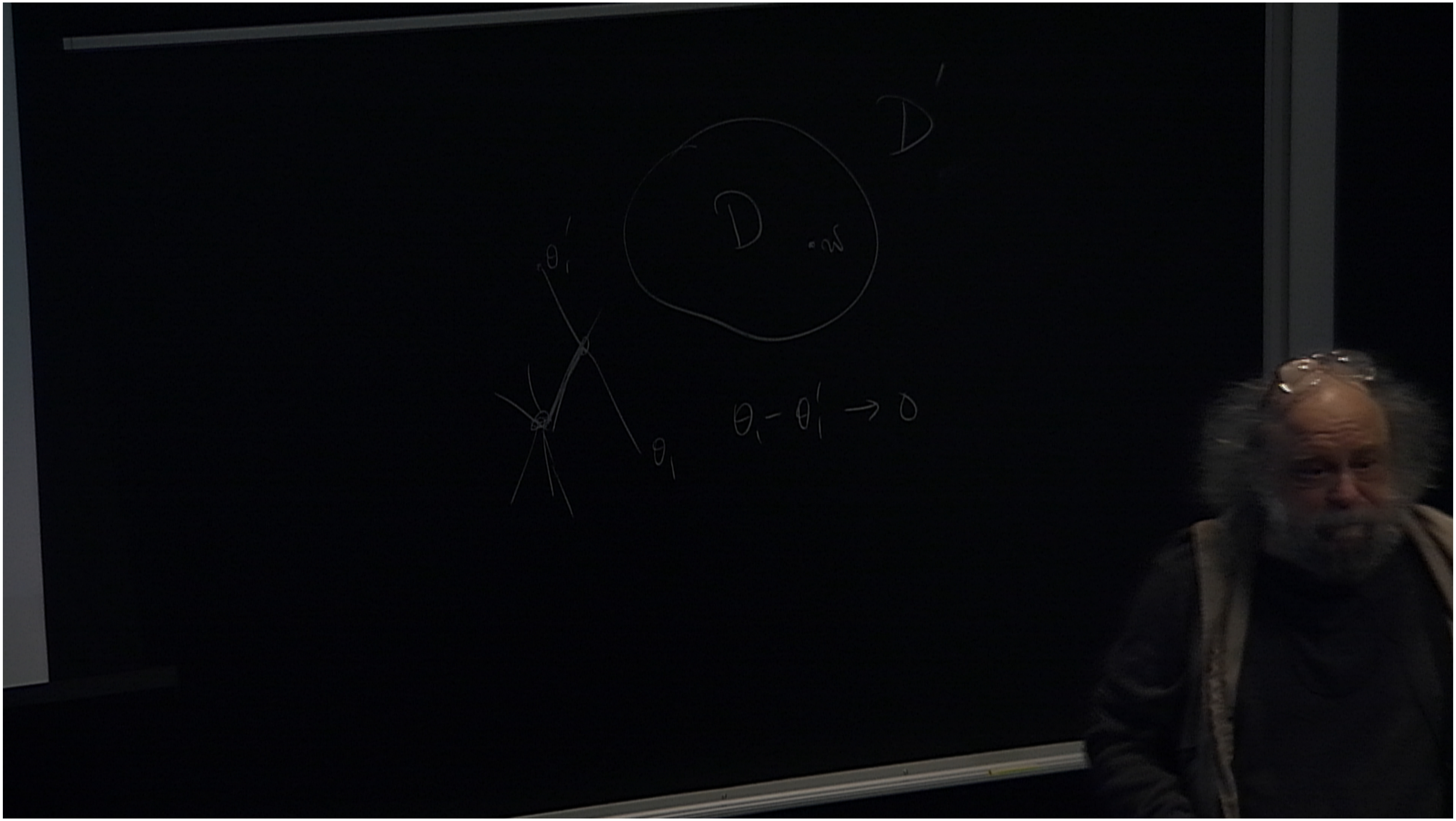
$$F_{N+2}(\theta+i\pi+\epsilon,\theta,\theta_1,\ldots,\theta_N) \to \frac{i}{\epsilon}\left[1-\prod_{j=1}^{n} S(\theta-\theta_j)\right] F_N(\theta_1,\ldots,\theta_N)$$

as $\epsilon \to 0$. Here

$$F_N(\theta_1,\ldots,\theta_N) = \langle 0 \mid \mathcal{O}(0) \mid \theta_1,\cdots,\theta_N\,\rangle_{(in)}$$

It is generally assumed that the space of solutions of FBE is isomorphic to the space of local fields

$$\mathcal{F}^{\mathsf{FF}} \simeq \mathcal{F}$$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as $\epsilon \to 0$. Here

$$F_N(\theta_1,\ldots,\theta_N) = \langle\, 0 \mid \mathcal{O}(0) \mid \theta_1,\cdots,\theta_N\,\rangle_{(in)}$$

It is generally assumed that the space of solutions of FBE is isomorphic to the space of local fields

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- It is easy to check that the FBE admit the ansatz

$$F_N(\theta_1,\cdots,\theta_N) = \left[\sum_{i=1}^{N} P^\mu(\theta_i)\right] G_{\mu,N}(\theta_1,\cdots,\theta_N)$$

where $P^\pm(\theta) = m\,e^{\pm\theta}$. Let $\mathcal{F}^{\mathsf{FF}}_{\mathsf{special}}$ be the subspace of solutions of FBE of this form.

Formfactors of the derivatives $\partial_\mu\mathcal{O}$ constitute trivial part of $\mathcal{F}^{\mathsf{FF}}_{\mathsf{special}}$. Existence of "integrable perturbations" suggests

$$\mathcal{F}^{\mathsf{FF}}_{\mathsf{special}}/\partial\mathcal{F} \neq \emptyset$$

How it is possible? For $N = 4$ the Energy-Momentum surface $\sum_{i=1}^{4} P_\mu(\theta_i) = 0$ lays entirely within the locus of the "annihilation poles" at $\theta_i = \theta_j + i\pi$, and the zeroes of the factor $\sum_{i=1}^{N} P_\mu(\theta_i)$ can be canceled by the annihilation poles. As the result, solutions are possible such that

$$F_N(\theta_1,\cdots,\theta_N)|_{\mathsf{Energy\text{-}Momentum\ surface}} = \begin{cases} 0 \text{ for } N > 4 \\ \# \text{ for } N = 4 \end{cases}$$

Space of such solutions = space of "integrable deformations".

As usual, $F_4 \to$ correction $\sim \delta g$ to the $2 \to 2$ S-matrix $\hat{S}(\theta)$:

$$\hat{S}(\theta) = \hat{S}_0(\theta) \left\{ I + \frac{i\delta g}{\sinh\theta} \left[f^{\text{reg}}_{2\to 2}(\theta) + 2\cosh\theta \; \Delta_0'(\theta) \right] \right\}$$

By "annihilation poles" condition

$$_{(out)}\langle \theta_1 + \varepsilon_1, \theta_2 + \varepsilon_2 \mid \mathcal{O} \mid \theta_1, \theta_2 \rangle_{(in)} = - \left(\frac{\varepsilon_1}{\varepsilon_2} + \frac{\varepsilon_1}{\varepsilon_2} \right) \Delta_0'(\theta) \, \langle \theta \mid \mathcal{O} \mid \theta \rangle + f^{\text{reg}}_{2\to 2}(\theta)$$

which is the definition of f^{reg}.

11. sine-Gordon model

$$\mathcal{L}_{\text{SG}} = \frac{1}{16\pi}(\partial_\mu\phi)^2 - 2\mu\cos(\beta\phi)$$

Has infinite set of conserved currents (T_{s+1}, Θ_{s-1}), $(\bar{T}_{s+1}, \bar{\Theta}_{s-1})$, with

$$s \in \mathcal{S} = 1, 3, 5, 7, \ldots$$

Particles: Soliton/Antisolton, Bound States. The space of solutions of FBE was completely sorted out by Miwa, Jimbo, Smirnov (2007-2012):

$\mathcal{F}^{\text{FF}}$ consists of module generated from "primary fields" (exponentials) by operators $P_s, \bar{P}_s$

$$P_s\mathcal{O}(z) \equiv [P_s, \mathcal{O}(z)]$$

and a set of fermionic operators β_s^*, γ_s^* and $\bar{\beta}_s^*, \bar{\gamma}_s^*$ (again, $s \in \mathcal{S} = \{1, 3, 5, \ldots\}$), with certain explicit action on the formfactors. In particular, it is possible to show that

$$\bar{P}_1\beta_s^*\gamma_1^* \cdot I = P_1\beta_s^*\bar{\gamma}_1^* \cdot I\,,$$

i.e.

$$T_{s+1} = \beta_s^*\gamma_1^* \cdot I\,, \quad \Theta_{s-1} = \beta_s^*\bar{\gamma}_1^* \cdot I$$

Moreover, it is possible to show that (this is a new result)

$$X_s = \beta_1^* \gamma_s^* \bar{\beta}_1^* \bar{\gamma}_s^* \cdot I \in \mathcal{F}^{\mathsf{FF}}_{\mathsf{special}}$$

• Substantiates (for the sine-Gordon) the general conclusion about "integrable deformations" $\{X_s\}$.

• Yields explicit formfactors of X_s. In particular, for $\mathcal{O} = X_s$ we find

$$f^{\mathsf{reg}}_{2\to 2}(\theta) = c_s \sinh(\theta) \sinh(s\theta)$$

That is, for the SG model, the "integrable deformations" X_s are in one-to-one correspondence with the infinitesimal changes by the CDD factors: For

$$\mathcal{L} = \mathcal{L}_{\mathsf{SG}} + \sum_{s=1,3,5,\ldots} \delta g_s \int X_s$$

$$\hat{S}(\theta) = \hat{S}_0 \left\{ I + \sum_{s=1,3,5,\ldots} c_s \, \delta g_s \sinh(s\theta) + \ldots \right\}$$

11. More remarks

Take expectation values of both sides of the identity

$$T_{s+1}(z)\bar{T}_{s+1}(z') - \Theta_{s-1}(z)\bar{\Theta}_{s-1}(z') \;=\; X_s(z') \;+\; \text{derivatives}.$$

Since in space-time homogeneous system $\langle\,\text{derivatives}\,\rangle = 0$, we have

$$\langle\,\Big(T_{s+1}(z)\bar{T}_{s+1}(z') - \Theta_{s-1}(z)\bar{\Theta}_{s-1}(z')\Big)\,\rangle = \langle\,X_s(z')\,\rangle = \langle\,X_s\,\rangle\,.$$

If also the space-time geometry is such that z and z' can be brought arbitrarily far apart, then

$$\text{l.h.s.} = \langle T_{s+1}\rangle\langle\bar{T}_{s+1}\rangle - \langle\Theta_{s-1}\rangle\langle\bar{\Theta}_{s-1}\rangle$$

Important example of such geometry is Euclidean cylinder, with "spatial" coordinate compactified on a circle on a circle of finite circumference R. In this case the spectrum of Hamiltonian is discrete, and expectation values in above relations can be understood as

$$\langle \dots \rangle = \langle n \mid \dots \mid n \rangle \,,$$

where $\mid n\rangle$ is any non-degenerate Energy-Momentum eigenstate.

Consider coordinate system in $\Sigma^{(\mathrm{Int})}$ in which X_s is tangent to coordinate line of g_s, i.e. in terms formal action

$$\frac{\partial}{\partial g_s}\mathcal{A} = \int X_s(z)\, d^2 z \,.$$

Then

$$\frac{\partial}{\partial g_s} E_n(R,g) = \int_0^R dx \,\langle X_s \rangle = R\left[\langle T_{s+1}\rangle\langle \bar{T}_{s+1}\rangle - \langle \Theta_{s-1}\rangle\langle \bar{\Theta}_{s-1}\rangle\right]$$

The r.h.s. can be evaluated explicitly in special case $s=1$.

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The r.h.s. can be evaluated explicitly in special case $s = 1$.

$s = 1$: corresponding currents are components of the energy-momentum tensor

$$T_2 = T_{zz} \equiv T\,, \quad \bar{T}_2 = T_{\bar{z}\bar{z}} \equiv \bar{T}\,, \quad \Theta_0 = \bar{\Theta}_0 = -T_{z\bar{z}} \equiv \Theta$$

The associated field X_1 is not trivial

$$X_1 = \lim_{z' \to z} \left[T(z)\bar{T}(z') - \Theta(z)\Theta(z') \right]$$

We have

$$\langle X_1 \rangle = \langle T \rangle \langle \bar{T} \rangle - \langle \Theta \rangle^2 = \frac{1}{16}\Big(-4\,\langle T_{xx} \rangle \langle T_{yy} \rangle + 4\,\langle T_{xy} \rangle \langle T_{xy} \rangle \Big)$$

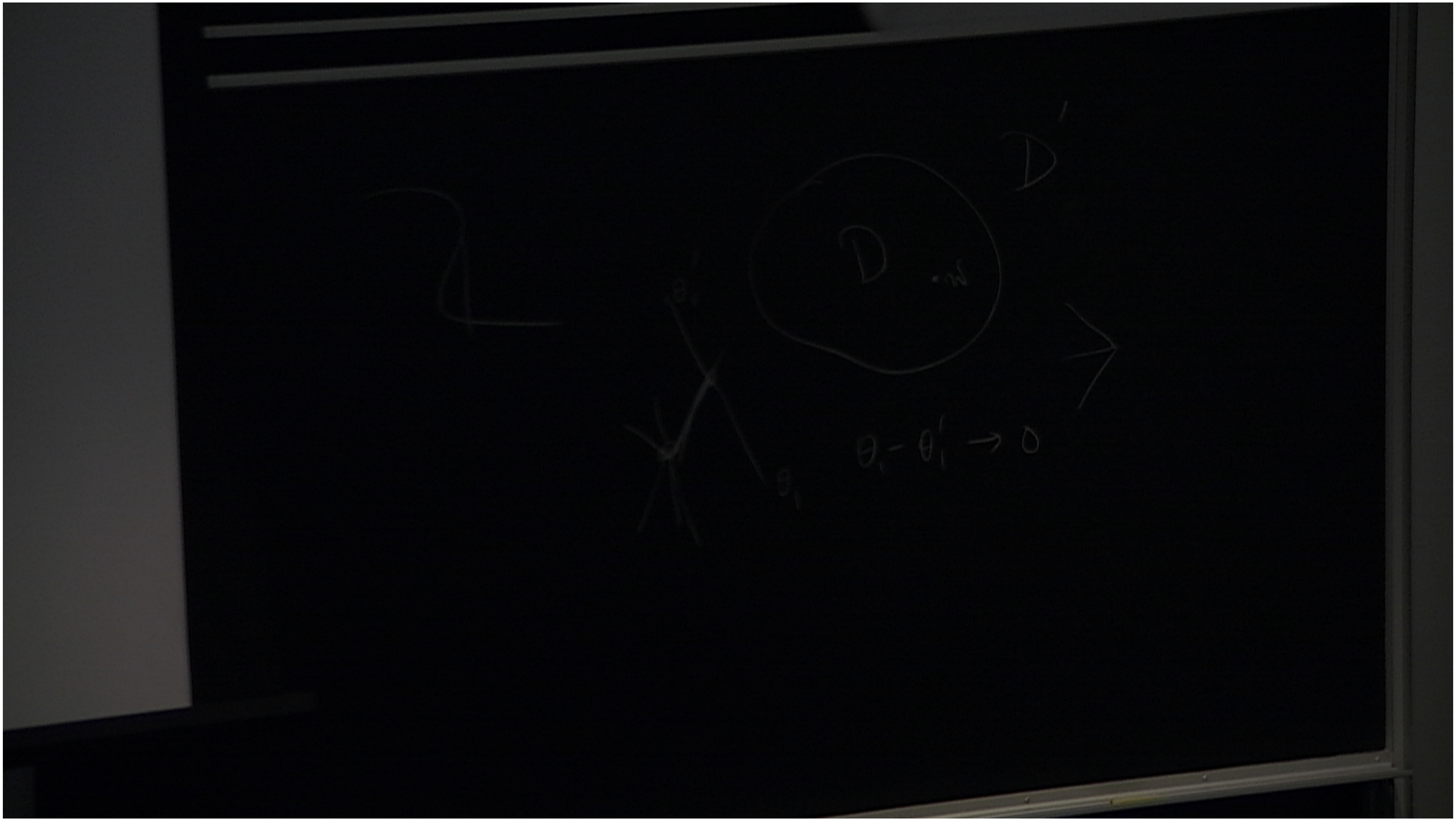
In the geometry of a cylinder, for a stationary state with energy and momentum $E(R), P(R)$

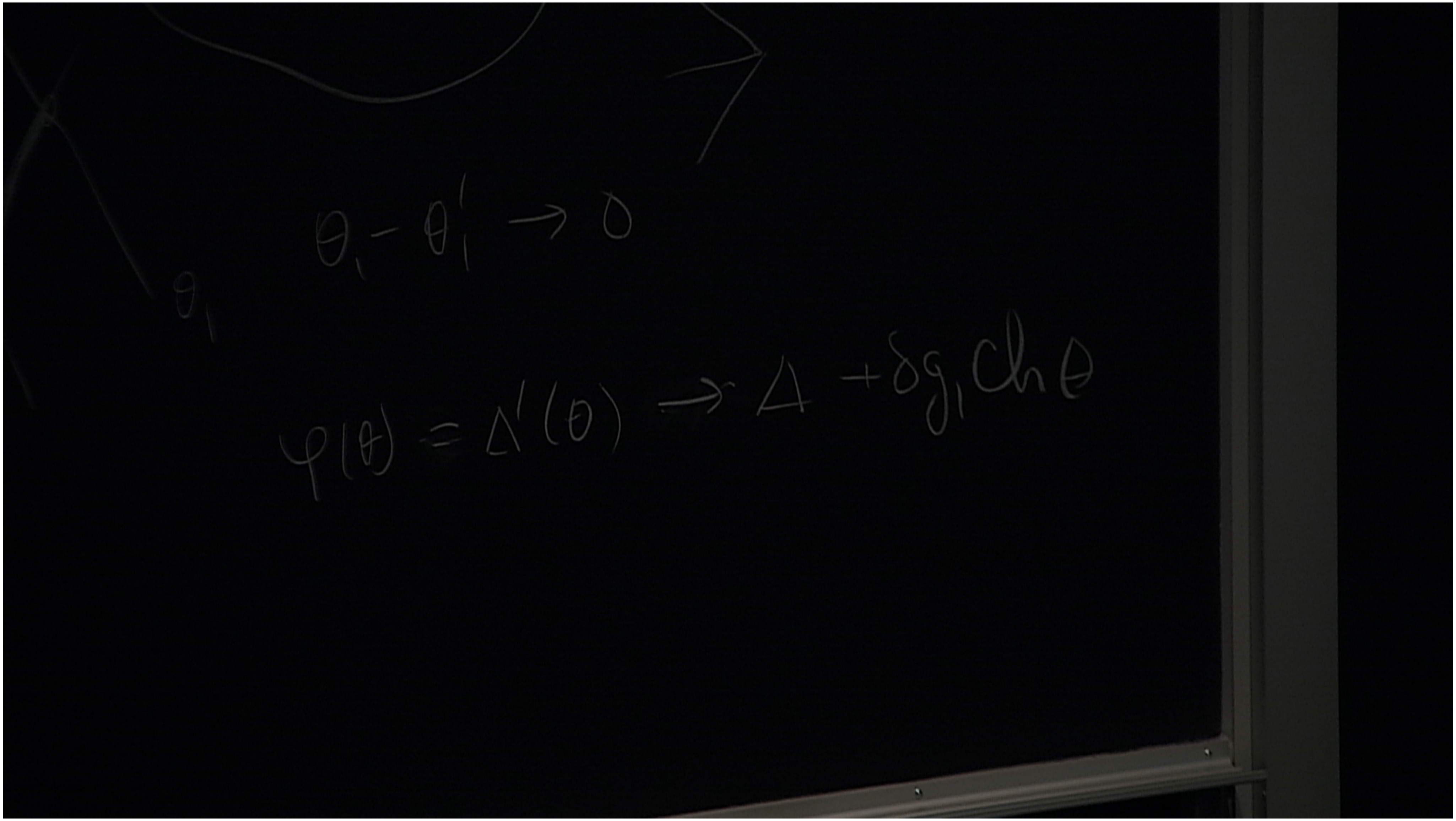
$$\langle X_1 \rangle = -\frac{(2\pi)^2}{4R} \left(E(R)\frac{d}{dR}E(R) + \frac{1}{R}P^2(R) \right).$$

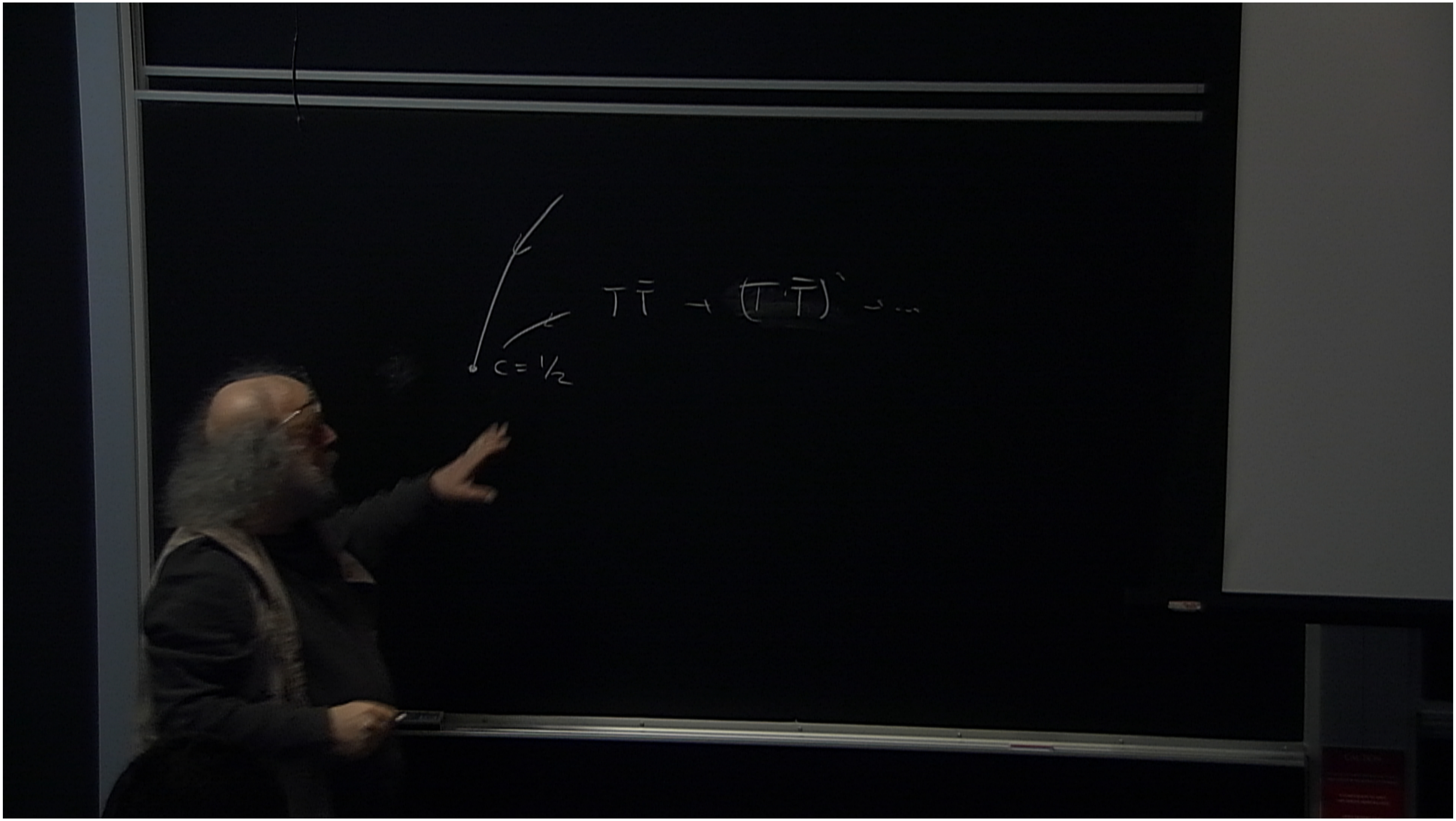
$(P(R) = \frac{2\pi\, k}{R})$.

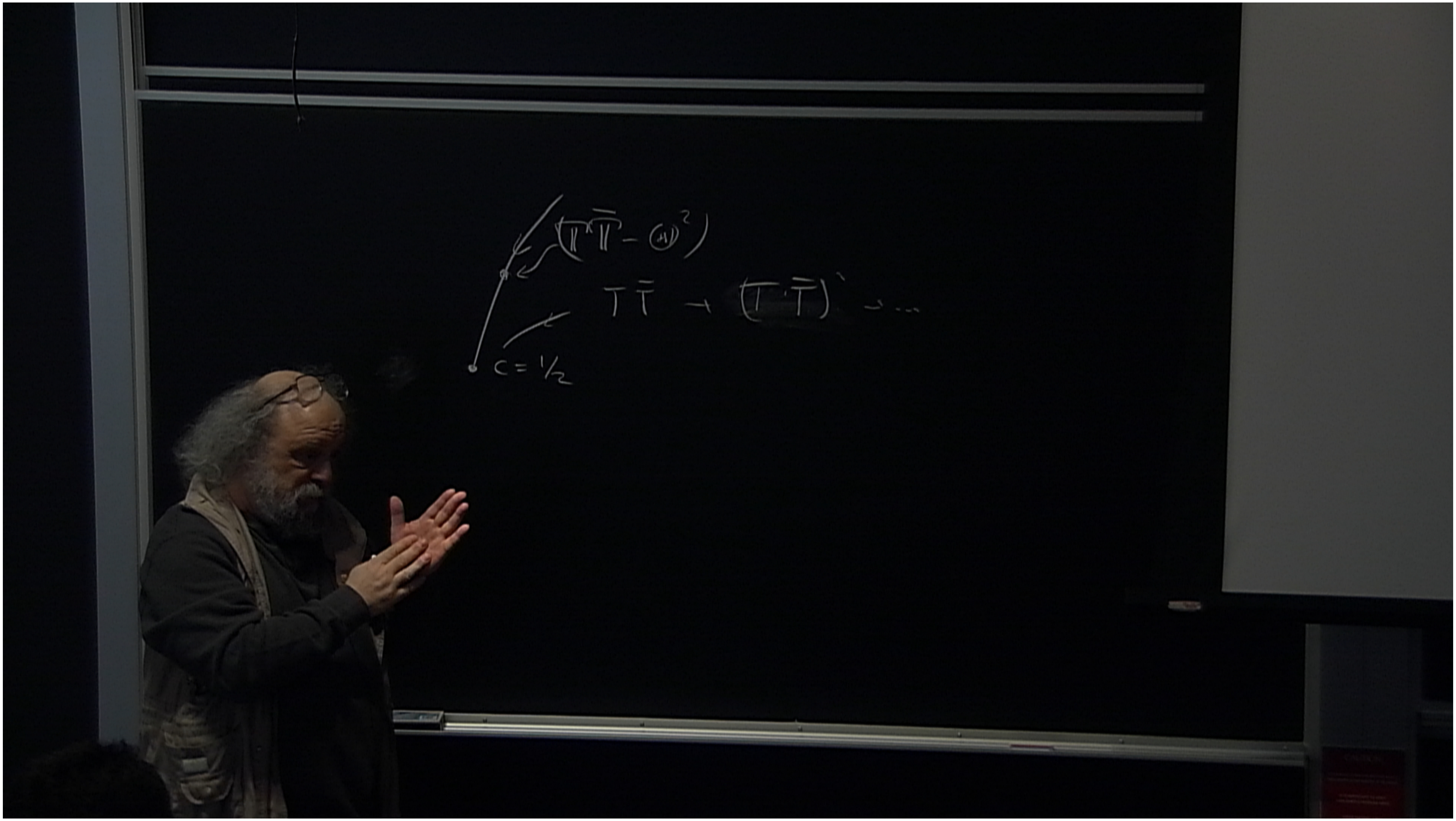
Then (with $g_1 = t$)

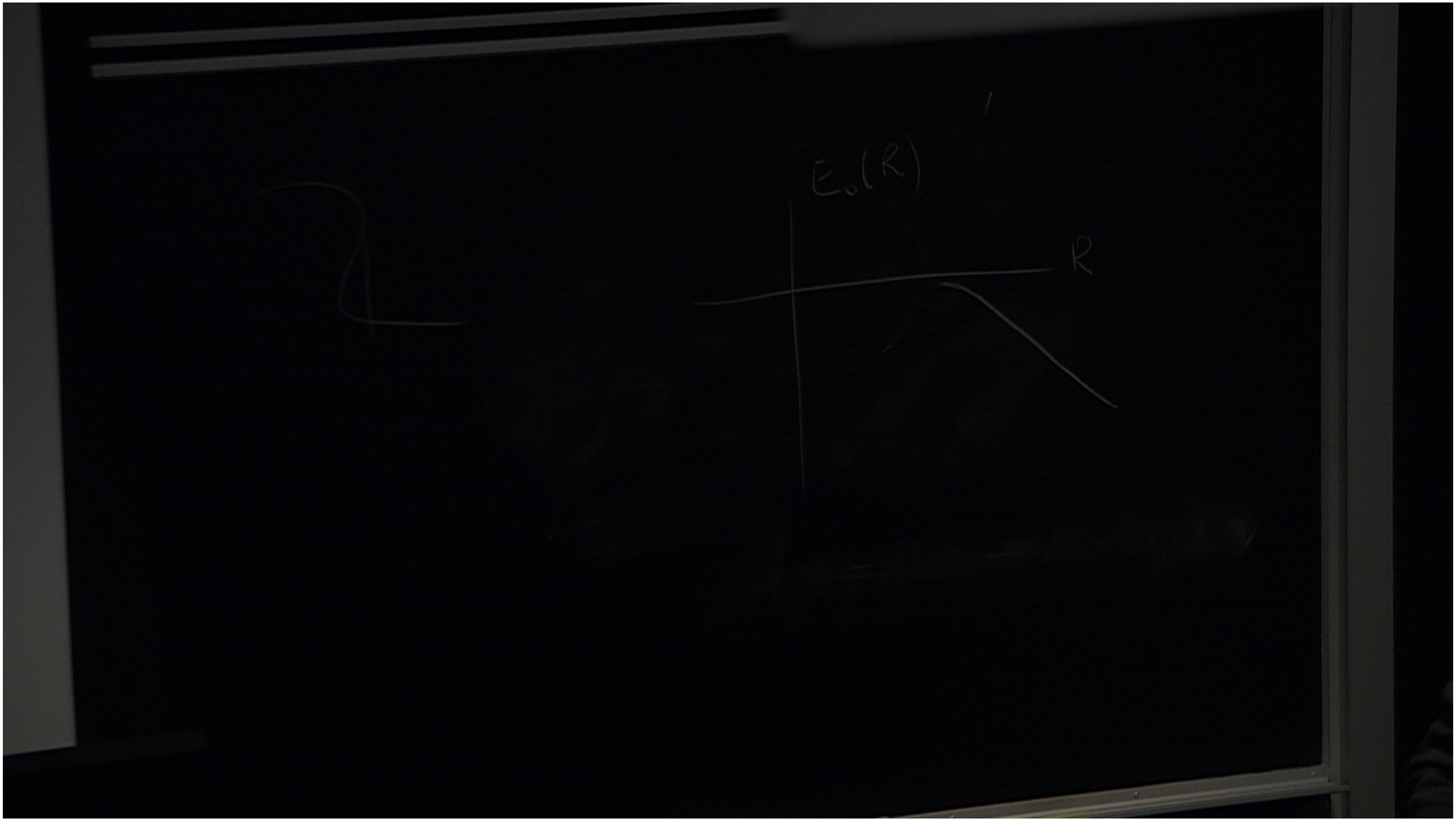
$$\frac{\partial}{\partial t}E(R,t) = E(R,t)\frac{\partial}{\partial R}E(R,t) + \frac{P^2(R)}{R}$$

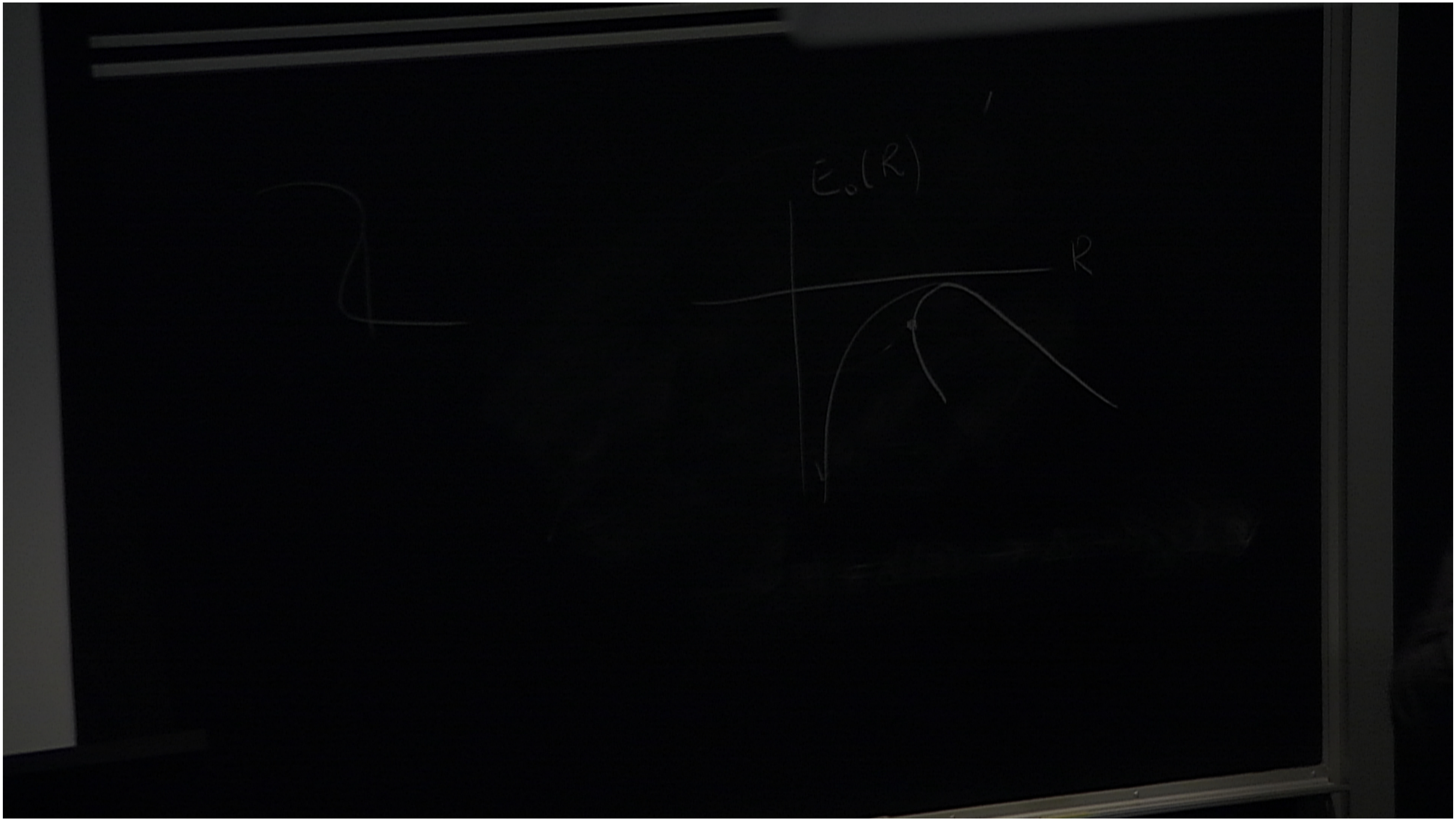

D'
D
θ1 − θ1' → 0


θ_1
$\theta_1 - \theta_1' \to 0$
$\varphi(\theta) = \Delta'(\theta) \to \Delta + \delta g_1 \mathrm{ch}\,\theta$

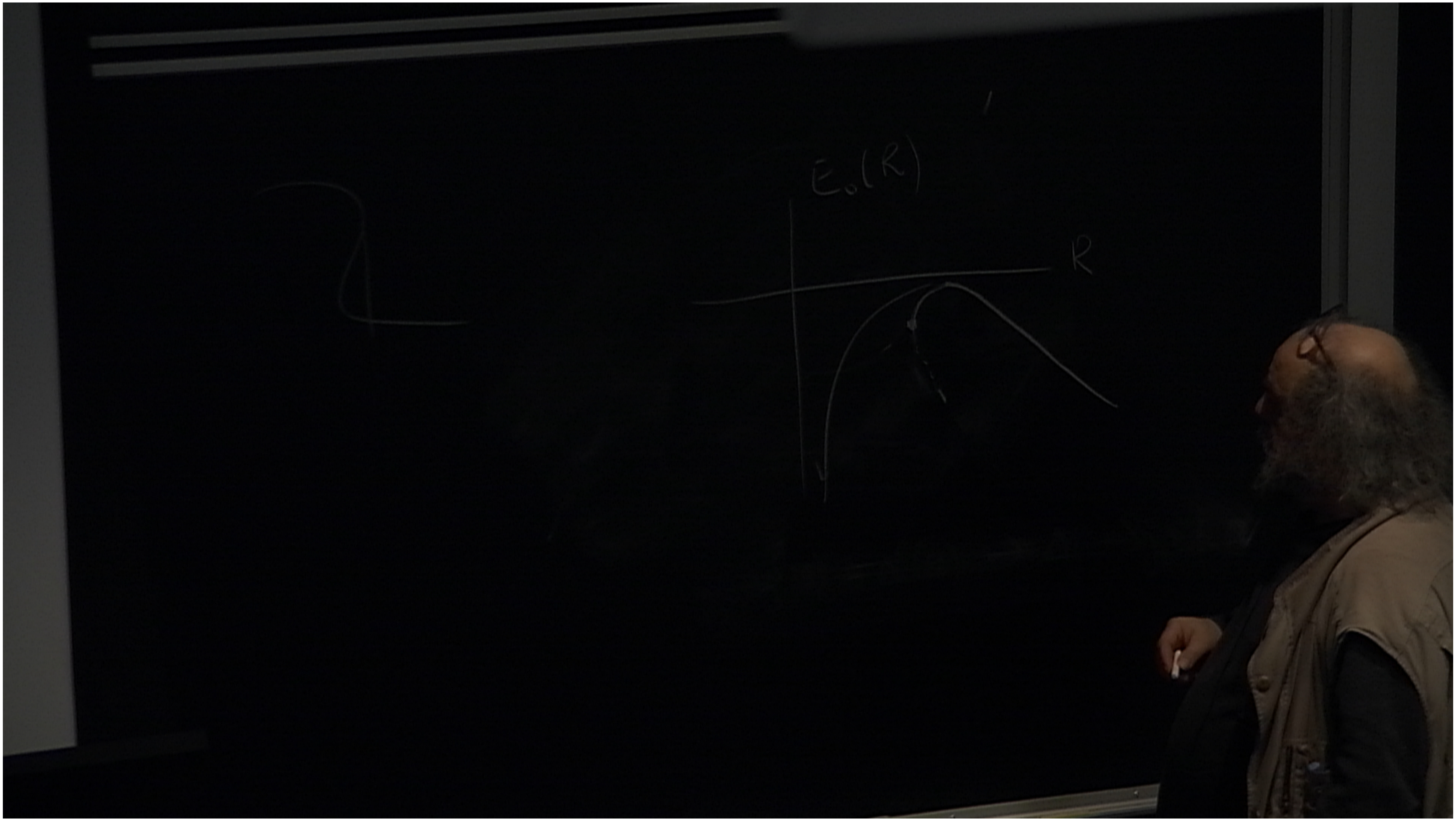

$c = 1/2$
$T\bar{T} \to (T\bar{T})' \to \dots$


c = 1/2

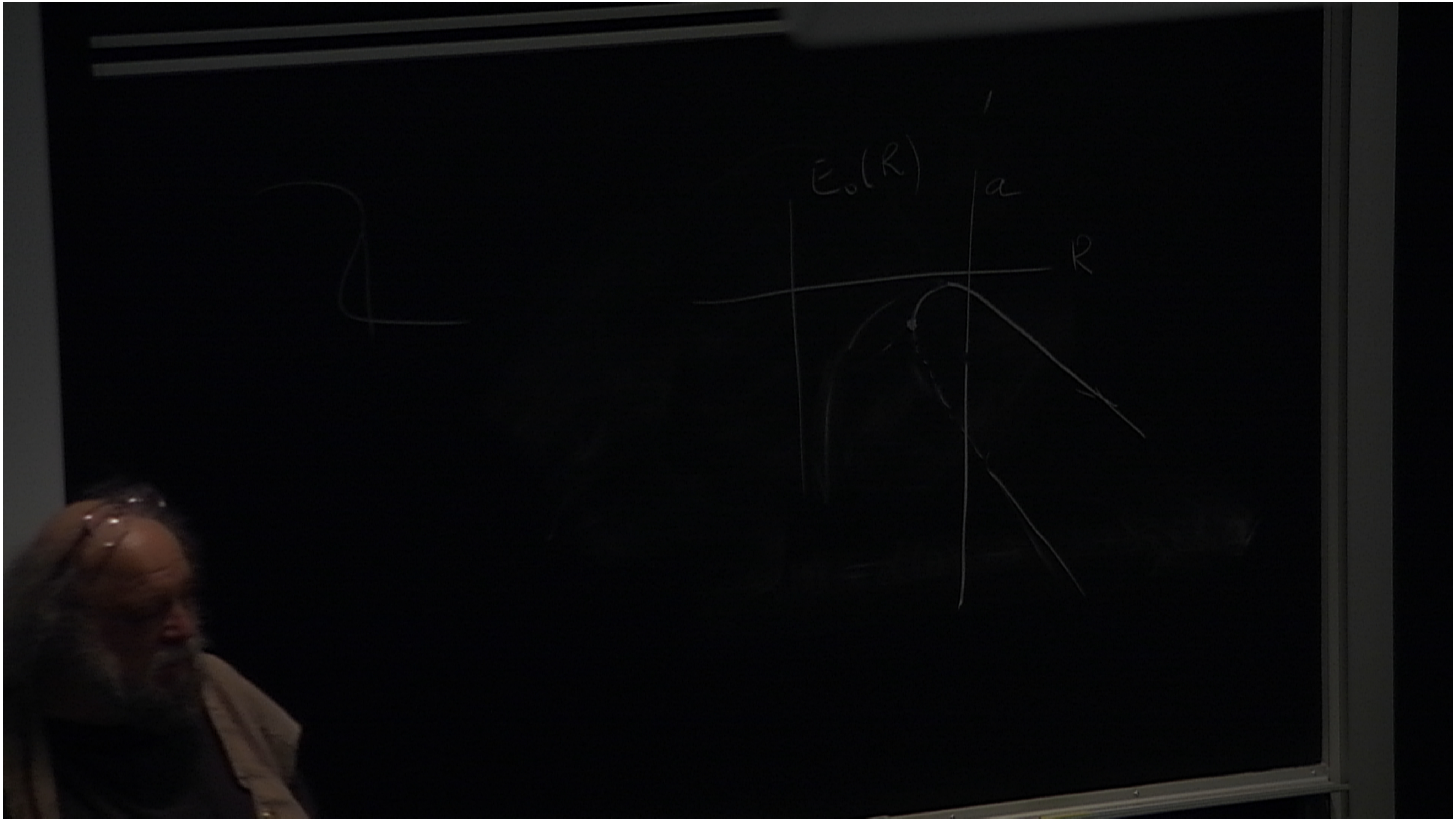

E₀(R)
R



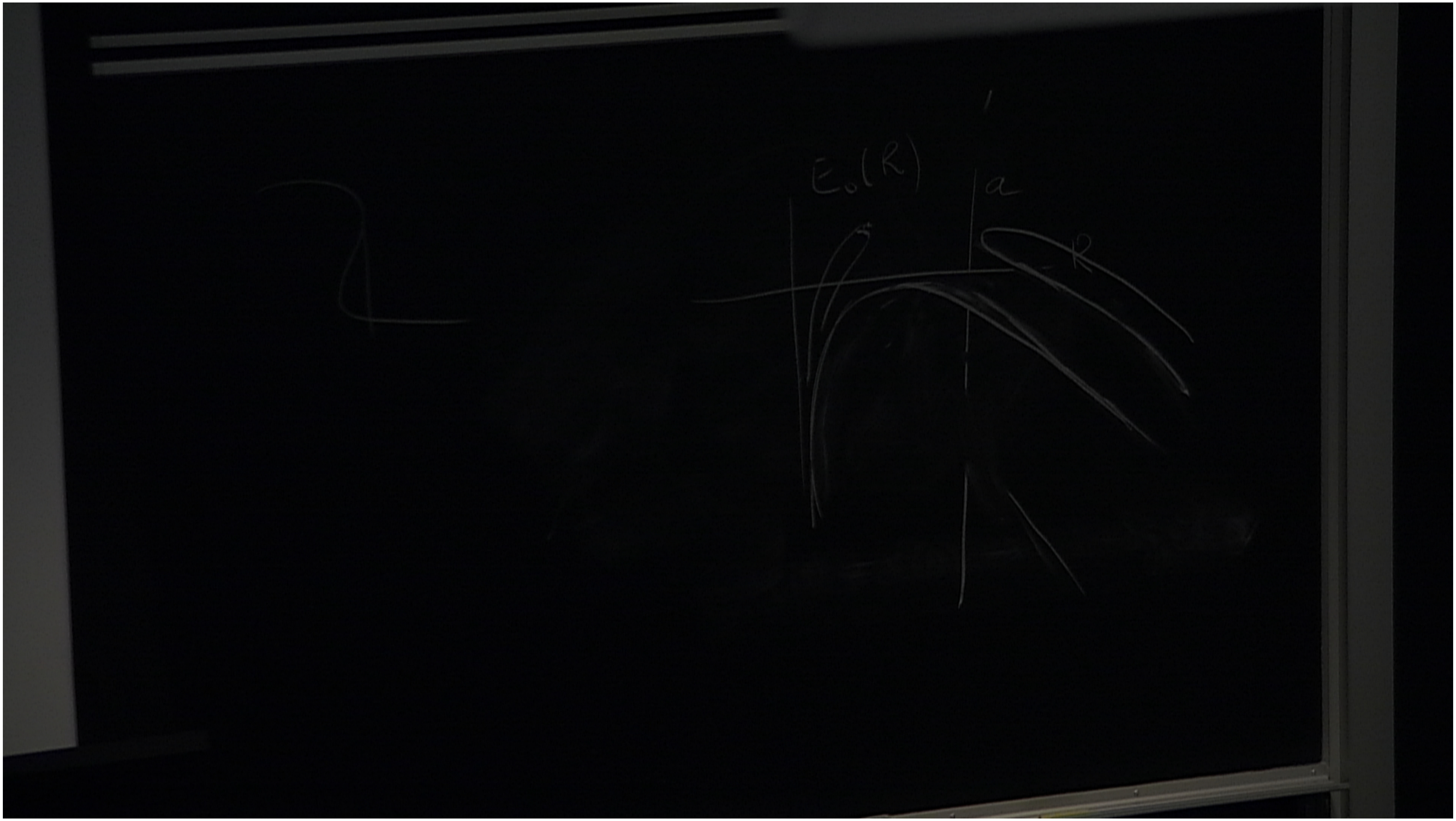
E_0(R)
R

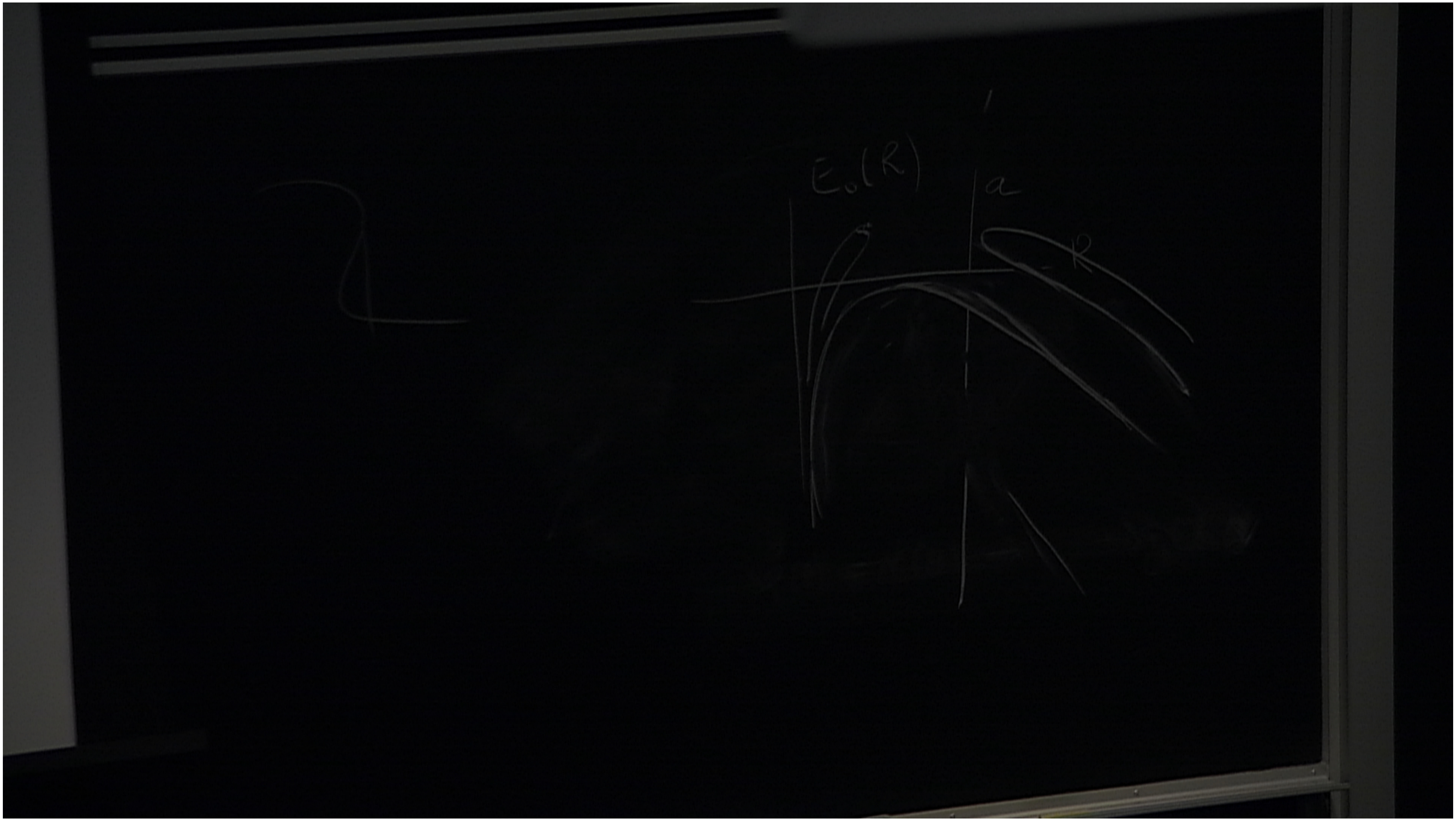


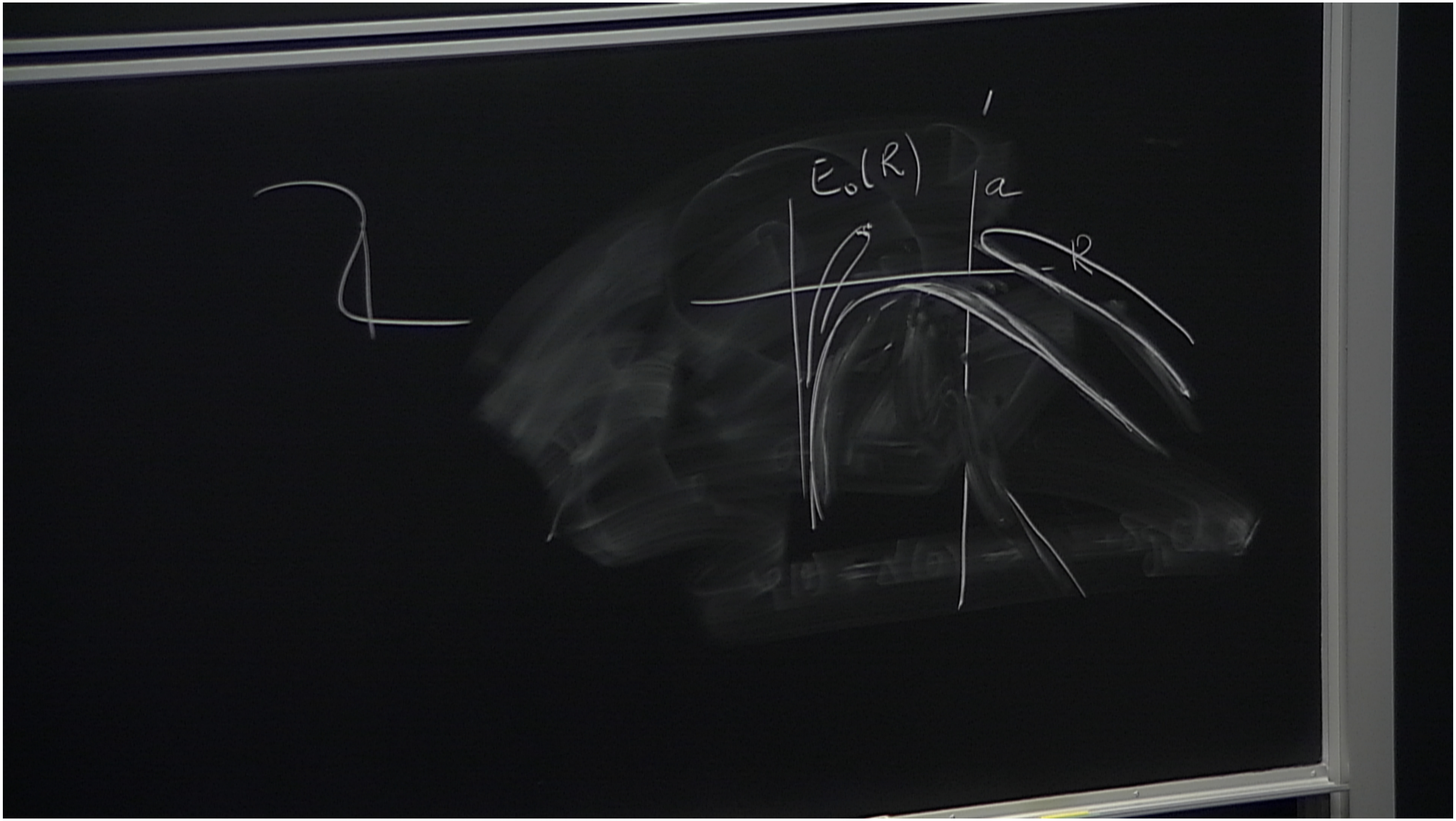
E₀(R)
R



E₀(R)
a
R


E₀(R)
a
R


E₀(R)
a
R



E₀(R)
a
R