Title: Quantum Correlations: Dimension Bounds and Conic Formulations

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Abstract: <p>In this talk, I will discuss correlations that can be generated by performing local measurements on bipartite quantum systems. I'll present an algebraic characterization of the set of quantum correlations which allows us to identify an easy-to-compute lower bound on the smallest Hilbert space dimension needed to generate a quantum correlation. I will then discuss some examples showing the tightness of our lower bound. Also, the algebraic characterization can be used to express the set of quantum correlations as the projection of an affine section of the cone of completely positive semidefinite matrices. Using this, we identify a semidefinite programming outer approximation to the set of quantum correlations which is contained in the first level of the NavascuÃ©s, Pironio and AcÃ-n hierarchy, and a linear conic programming problem formulating exactly the quantum value of a nonlocal game. Time permitting, I will discuss other consequences of these conic formulations and some interesting special cases.

This talk is based on work with Antonios Varvitsiotis and Zhaohui Wei, arXiv:1507.00213 and arXiv:1506.07297.</p>

Quantum Correlations: Dimension Bounds and Conic Formulations

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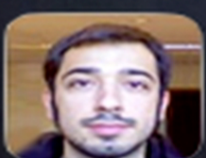


arXiv:1506.07297 with Antonios Varvitsiotis
arXiv:1507.00213 with Antonios Varvitsiotis and Zhaohui Wei

Outline

- Introduction
- Alternative Characterizations of the Sets of Quantum and Classical Correlations
- Physical Application
- Conic Formulations
  - Nonlocal Games
  - Special Strategies
- Open Problems

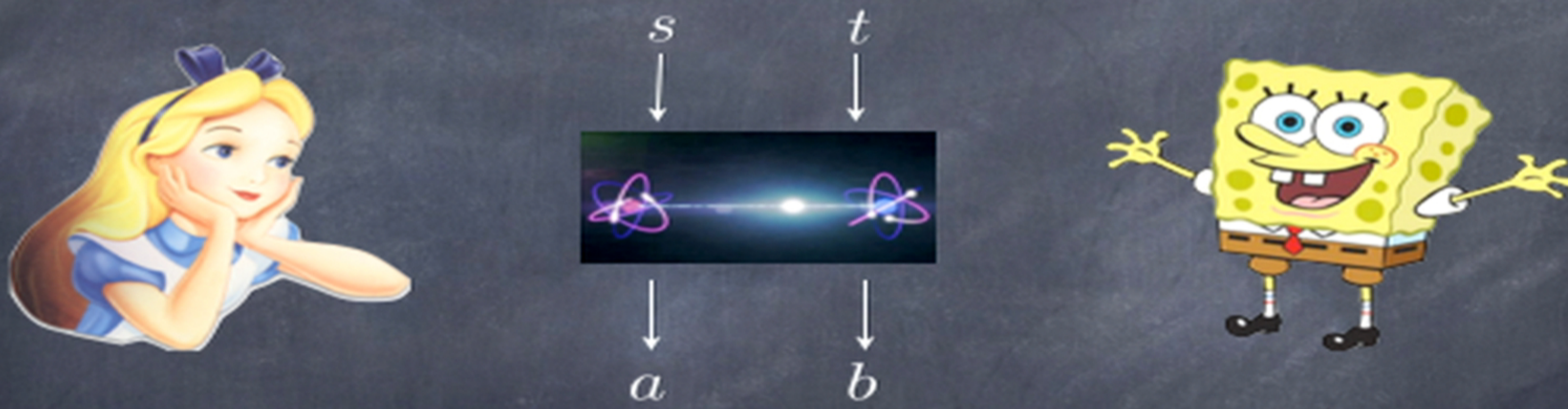
Bell scenario

Alice

Shared
Physical
System

Bob

Bell scenario



The scenario is characterized by the conditional probabilities

$$(p(a,b|s,t))_{a,b,s,t}$$

Correlation vectors

Every correlation $(p(a,b|s,t))$ satisfies:

- $p(a,b|s,t) \geq 0$ for all a,b,s,t

- $\sum_{a,b} p(a,b|s,t) = 1$ for all s,t

Each physical model adds additional constraints on the correlations that can be generated

Classical correlations

A correlation $(p(a,b|s,t))$ is **classical** if

$$p(a,b|s,t) = \sum_{i=1}^{n} k_i x_a^{s,i} y_b^{t,i}$$

where:

- $\sum_{i \in [n]} k_i = 1$
- $\sum_{a \in A} x_a^{s,i} = \sum_{b \in B} y_b^{t,i} = 1$

(variables are all nonnegative)

Quantum correlations

A correlation $(p(a,b|s,t))$ is quantum if

$$p(a,b|s,t) = \mathrm{Tr}((X_a^s \otimes Y_b^t)\rho)$$

where:

- X_a^s, Y_b^t, ρ are positive semidefinite
- $\sum_{a \in A} X_a^s = \sum_{b \in B} Y_b^t = I$ for all s, t
- $\mathrm{Tr}(\rho) = 1$

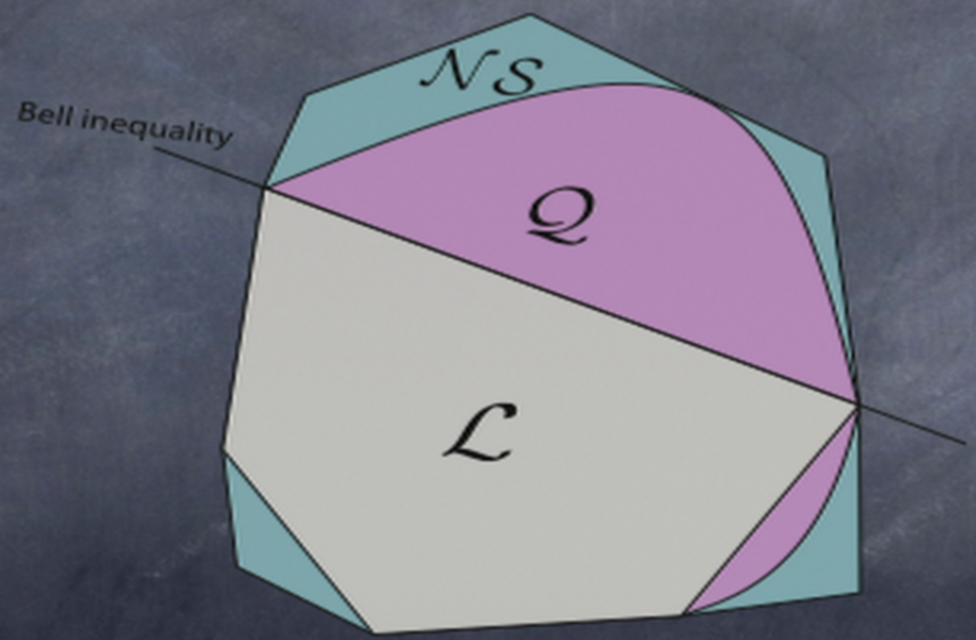
No-signaling correlations 

A correlation $p = (p(a,b|s,t))$ is **no-signaling** if

- $\sum_{b \in B} p(a,b|s,t) = \sum_{b \in B} p(a,b|s,t')$ for all $s, t \neq t'$

- $\sum_{a \in A} p(a,b|s,t) = \sum_{a \in A} p(a,b|s',t)$ for all $t, s \neq s'$

How do they relate?



Credit: arXiv:1303.2849v3

Why do we care about quantum correlations?

- To better understand quantum physics
- Leads to cool optimization problems
- Cryptography
  - Key distribution
  - Device-independent implementations
  - Relativistic bit-commitment

Alternative Characterizations of the Sets of Quantum and Classical Correlations

Algebraic Characterization

Thm: A correlation $p = (p(a,b|s,t))$ is classical if and only if there exist nonnegative vectors $\{e_{s,a}\}, \{f_{t,b}\}$ satisfying:

- $p(a,b|s,t) = \langle e_{s,a}, f_{t,b} \rangle$ for all a,b,s,t,
- $\sum_a e_{s,a} = \sum_b f_{t,b}$ for all s,t

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  - Nonlocal Games
  - Special Strategies
- Open Problems

Physical Application: Dimension Bounds

Quantum resources

Question: What is the smallest size of a quantum system needed to generate a quantum correlation?

Recall that $(p(a,b|s,t))$ is quantum iff

$$p(a,b|s,t) = \mathrm{Tr}((X_a^s \otimes Y_b^t)\rho)$$

POVMs State

Quantum resources

Thm: For any correlation $p = (p(a,b|s,t))$

we need a state of (local) dimension at least

$$\max_{t,t' \in T} \left[\sum_{b,b' \in B} \min_{s \in S} \left(\sum_{a \in A} \sqrt{p(a,b|s,t)} \sqrt{p(a,b'|s,t')} \right)^2 \right]^{-1}$$

Example: PR-box

$$p(a,b|s,t) = \begin{cases} 1/2 & \text{if } a \oplus b = s \cdot t \\ 0 & \text{otherwise} \end{cases}$$

- No-signaling
- NOT quantum [Popescu, Rohrlich '94]

Example: PR-box

Recall:

$$\max_{t,t' \in T} \left[\sum_{b,b' \in B} \min_{s \in S} \left(\sum_{a \in A} \sqrt{p(a,b|s,t)} \sqrt{p(a,b'|s,t')} \right)^2 \right]^{-1}$$

For $t = 0, t' = 1$ we show that $\forall b, b' \in \{0,1\}$

$$\min_{s \in S} \left(\sum_{a \in A} \sqrt{p(a,b|s,0)} \sqrt{p(a,b'|s,1)} \right) = 0$$

Example: PR-box

$$\min_{s\in S}\left(\sum_{a\in A}\sqrt{p(a,b|s,0)}\sqrt{p(a,b'|s,1)}\right)=0$$

Correlation matrix:

$$\left[\begin{array}{cc|cc} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ \hline 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \end{array}\right]$$

rows indexed by (s,a),
columns indexed by (t,b)

Example: PR-box

$$\min_{s\in S}\left(\sum_{a\in A}\sqrt{p(a,b|s,0)}\sqrt{p(a,b'|s,1)}\right)=0$$

Correlation matrix:

$t=0 \qquad t=1$

$$\left[\begin{array}{cc|cc} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ \hline 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \end{array}\right]$$

$s=0$

$b=0, b'=0$

Example: PR-box

$$\min_{s\in S}\left(\sum_{a\in A}\sqrt{p(a,b|s,0)}\sqrt{p(a,b'|s,1)}\right)=0$$

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$s=0$

$b=0, b'=0 \quad \rightarrow 0$

$b=0, b'=1$

Example: PR-box

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Correlation matrix:

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$b = 0, b' = 0 \quad \rightarrow 0$

$b = 0, b' = 1 \quad \rightarrow 0$

$s = 0$ minimum!

Example: PR-box

$$\min_{s\in S}\left(\sum_{a\in A}\sqrt{p(a,b|s,0)}\sqrt{p(a,b'|s,1)}\right)=0$$

Correlation matrix:

$t=0 \qquad t=1$

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$b=0, b'=0 \quad \rightarrow 0$

$b=0, b'=1 \quad \rightarrow 0$

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$b=1, b'=1 \quad \rightarrow 0$

Other Examples

- Can show (popular) optimal strategies for nonlocal games (such as CHSH and Magic Square) use a state of minimum dimension
- Can show that other games (such as the Fortnow-Feige-Lovász game) have no perfect strategies

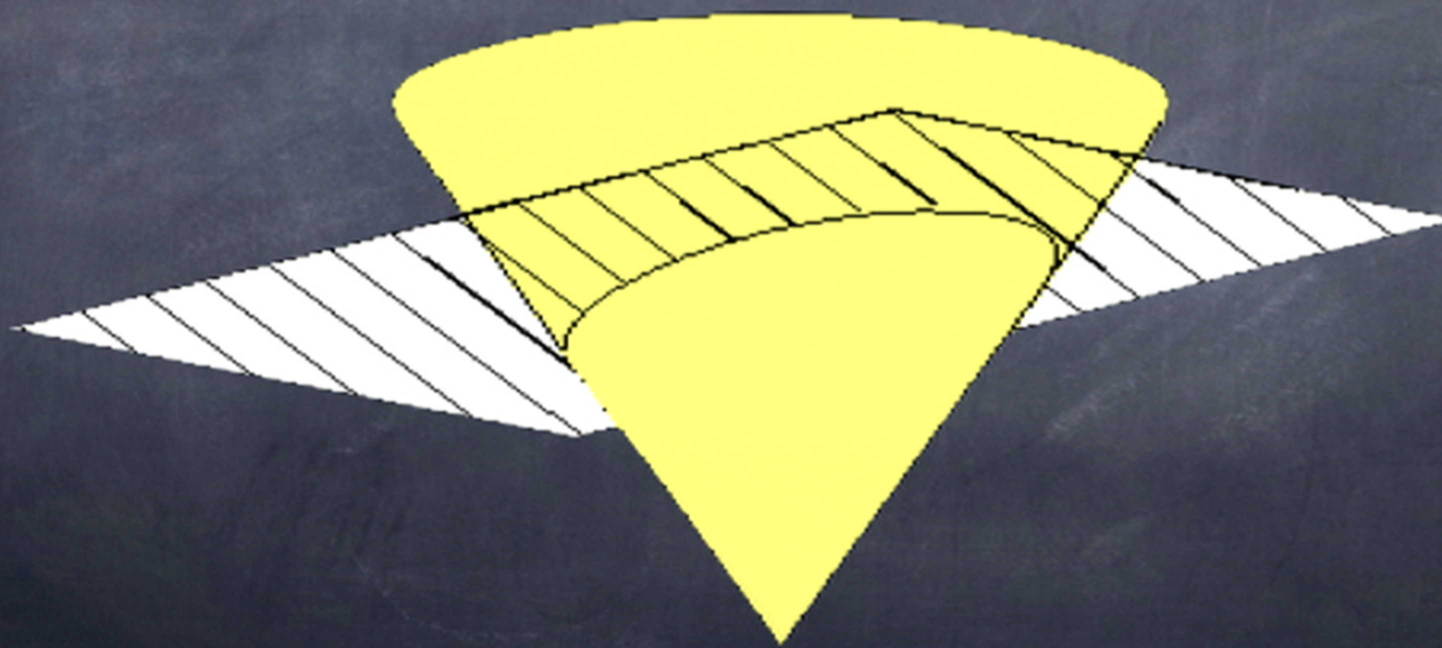
Experimentalist: Nobody is perfect!

Thm: For any correlation $p = (p(a,b|s,t))$ we need a state of (local) dimension at least

$$\max_{t,t'\in T}\left[\sum_{b,b'\in B}\min_{s\in S}\left(\sum_{a\in A}\sqrt{p(a,b|s,t)}\sqrt{p(a,b'|s,t')}\right)^2\right]^{-1}$$

Conic Formulations

We will look at affine slices of convex cones



Cones of interest

$\mathcal{H}_+^n$: Hermitian positive semidefinite

$\mathcal{CP}^n$: completely positive

$\mathcal{DNN}^n$: doubly nonnegative

$X \in \mathcal{DNN}^n \iff X \in \mathcal{H}_+^n$ with nonnegative entries

Cones of interest

$\mathcal{H}_+^n$: Hermitian positive semidefinite

$\mathcal{CP}^n$: completely positive

$\mathcal{DNN}^n$: doubly nonnegative

$\mathcal{CS}_+^n$: completely positive semidefinite

$X \in \mathcal{CS}_+^n \Longleftrightarrow \exists A_1, \ldots, A_n \in \mathcal{H}_+^d$ such that

$X_{ij} = \mathrm{Tr}(A_i A_j)$ for all i, j

Cones of interest

$\mathcal{H}_+^n$: Hermitian positive semidefinite

$\mathcal{CP}^n$: completely positive

$\mathcal{DNN}^n$: doubly nonnegative

$\mathcal{CS}_+^n$: completely positive semidefinite

$\mathcal{NN}^n$: nonnegative and symmetric matrices

$$\mathcal{CP} \subseteq \mathcal{CS}_+ \subseteq \mathcal{DNN} \subseteq \mathcal{NN}$$

Cones of interest

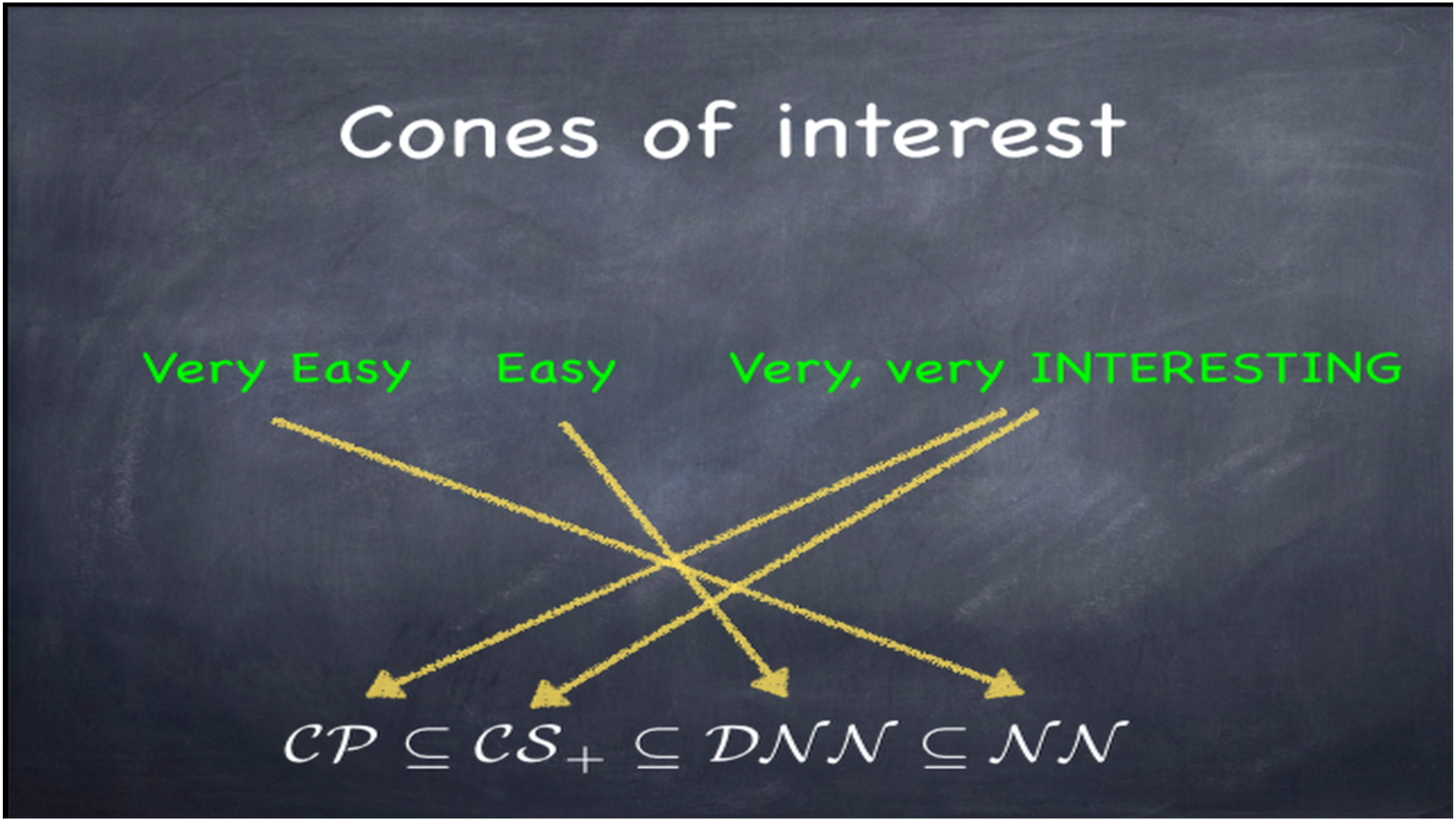
Easy to optimize/test

$$\mathcal{CP} \subseteq \mathcal{CS}_+ \subseteq \mathcal{DNN} \subseteq \mathcal{NN}$$

Cones of interest

Very Easy Easy

$$\mathcal{CP} \subseteq \mathcal{CS}_+ \subseteq \mathcal{DNN} \subseteq \mathcal{NN}$$


Cones of interest
Very Easy
Easy
Very, very INTERESTING
$\mathcal{CP} \subseteq \mathcal{CS}_+ \subseteq \mathcal{DNN} \subseteq \mathcal{NN}$

$\mathcal{CS}_+^n$ cone

- Introduced to give conic formulations for quantum graph parameters [Laurent, Piovesan' 14]
- Not known if it is closed
- Hard to test [Ji' 14 + Laurent, Piovesan' 14]
- No upper bound known on the dimension of the psd factors

We will consider matrices of this shape

This block can hold all the joint probabilities!

Conic correlations

Def: $(p(a,b|s,t))$ is a $\mathcal{K}$-correlation if $\exists X \in \mathcal{K}$:

- $\langle J_{i,j}, X \rangle = 1$ for all $i,j \in S \cup T$
- $X[(s,a),(t,b)] = p(a,b|s,t)$ for all a,b,s,t

Denoted: $\mathrm{Corr}(\mathcal{K})$

Geometry of correlations

Theorem: For any Bell scenario

- $p = (p(a,b|s,t))$ is classical iff $p \in \mathrm{Corr}(\mathcal{CP})$
- $p = (p(a,b|s,t))$ is quantum iff $p \in \mathrm{Corr}(\mathcal{CS}_+)$
- $p = (p(a,b|s,t))$ is no-signaling iff $p \in \mathrm{Corr}(\mathcal{NSO})$
- $p = (p(a,b|s,t))$ is unrestricted iff $p \in \mathrm{Corr}(\mathcal{NN})$

Consequence: The sets are closed when the cones are closed!

Applications of Conic Formulations

Efficient approximations

Question: Identify inner/outer Semidefinite Programming (SDP) approximations for quantum correlations

In [Navascués, Pironio, Acín '08] they give:

- A hierarchy of SDP outer approximations to $\mathcal{Q}$
- Not known if the NPA hierarchy converges onto $\mathcal{Q}$

SDP (efficient) approximations

Thm: For any $\mathcal{K} \subseteq \mathcal{DNN}$ we have $\mathrm{Corr}(\mathcal{K}) \subseteq \mathcal{NS}$

Since $\mathcal{CS}_+ \subseteq \mathcal{DNN}$ we get:

$$\mathcal{Q} \subseteq \mathrm{Corr}(\mathcal{DNN}) \subseteq \mathcal{NS}$$

Thm: We have $\mathrm{Corr}(\mathcal{DNN}) \subseteq \mathrm{NPA}^{(1)}$

Nonlocal Games

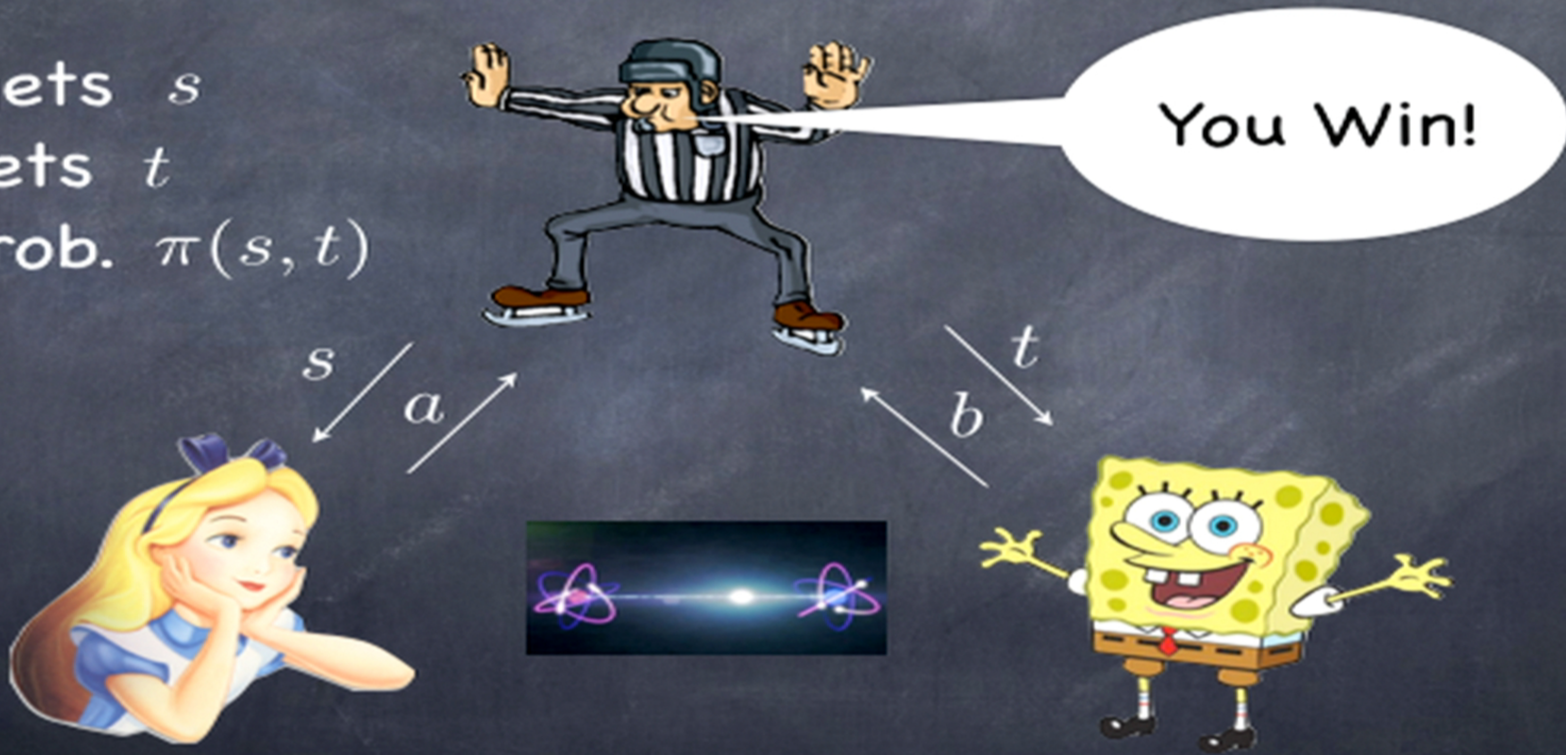
Nonlocal games

referee

He asks questions to Alice and Bob, and then accepts/rejects their answers

Nonlocal games

Alice gets s
Bob gets t
with prob. $\pi(s, t)$



Nonlocal games

The maximum probability of winning $\mathcal{G}(\pi, V)$ using $\mathcal{K}$-correlations is:

$$\begin{aligned}&\text{supremum} \quad \sum_{s\in S}\sum_{t\in T}\pi(s,t)\sum_{a\in A}\sum_{b\in B}V(a,b|s,t)p(a,b|s,t)\\&\text{subject to} \quad p=(p(a,b|s,t))\in \mathrm{Corr}(\mathcal{K}).\end{aligned}$$

One consequence...

Thm: For any game $\mathcal{G}(\pi, V)$ we have

$$\omega(\mathcal{CS}_+, \mathcal{G}) \leq \omega(\mathcal{DNN}, \mathcal{G}) \leq \mathrm{SDP}(\mathrm{NPA}^{(1)})$$

Quantum Value — Feige-Lovász — Optimizing over $\mathrm{NPA}^{(1)}$

Perfect strategies

Def: $\mathcal{G}(\pi, V)$ has a perfect $\mathcal{K}$-strategy if $\exists\ \mathcal{K}$-correlation that wins with probability 1

This happens if and only if there exists

$$p = (p(a,b|s,t)) \in \mathrm{Corr}(\mathcal{K})$$

such that

$$p(a,b|s,t) = 0 \text{ when } \pi(s,t) > 0 \text{ and } V(a,b|s,t) = 0$$

Perfect and synchronous strategies

Using this we recover:

- Conic formulations for quantum chromatic & stability number [Laurent, Piovesan '13]
- Conic formulations for quantum graph homomorphisms [Roberson '14]

Open problems

- Can we upper bound the Hilbert space dimension needed to generate a given quantum correlation?
- Can we prove the set of quantum correlations is closed?
- Can we prove anything interesting about the completely positive semidefinite cone? For example, can we bound the size of the matrices in the decomposition?

Future work

- "Completely Positive Semidefinite Rank" to soon appear on arXiv
- To apply dimension bounding techniques to the "Prepare and Measure" setting
- To find other models to which we can apply the study of completely positive semidefinite matrices

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Thank you!