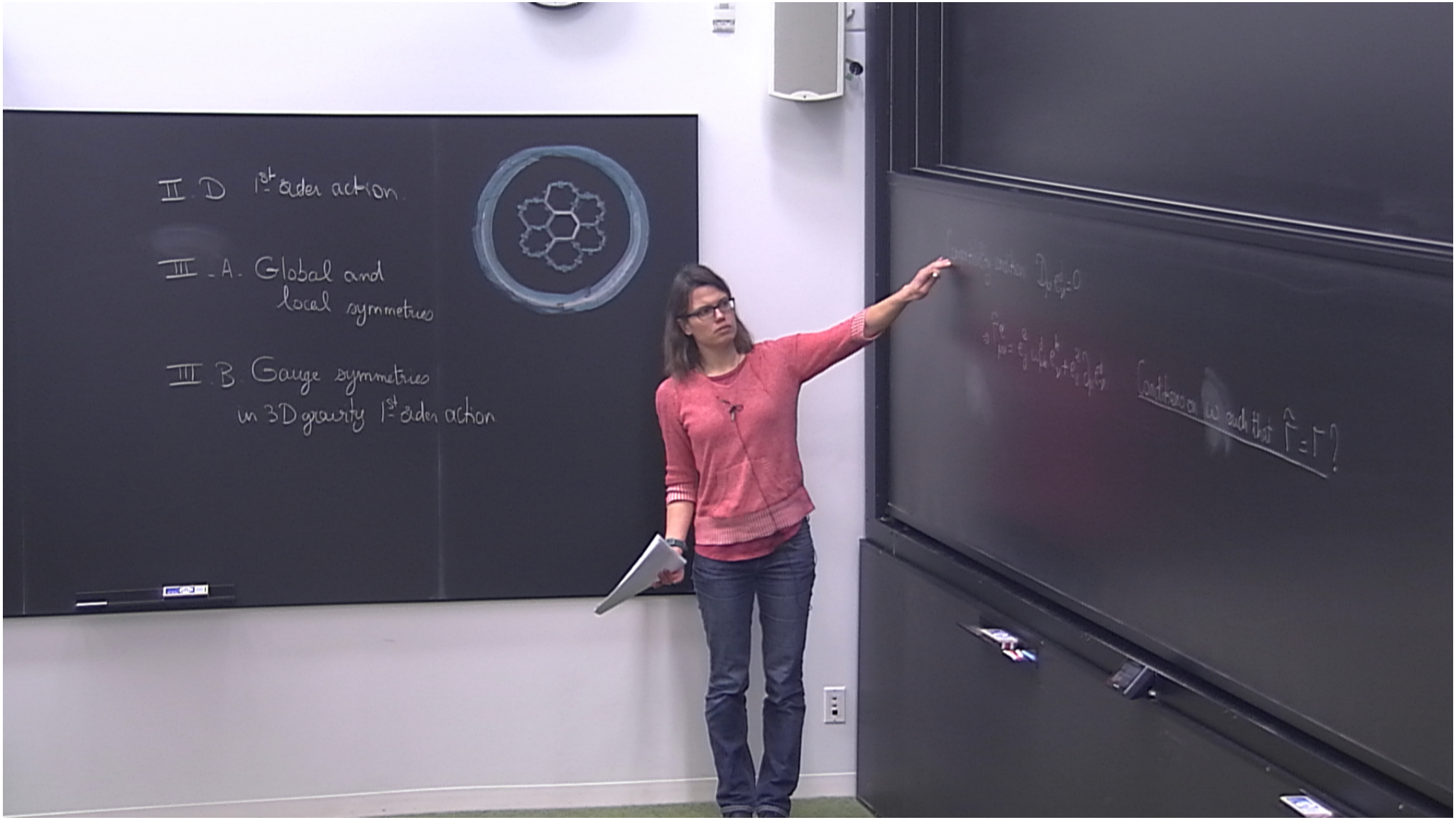
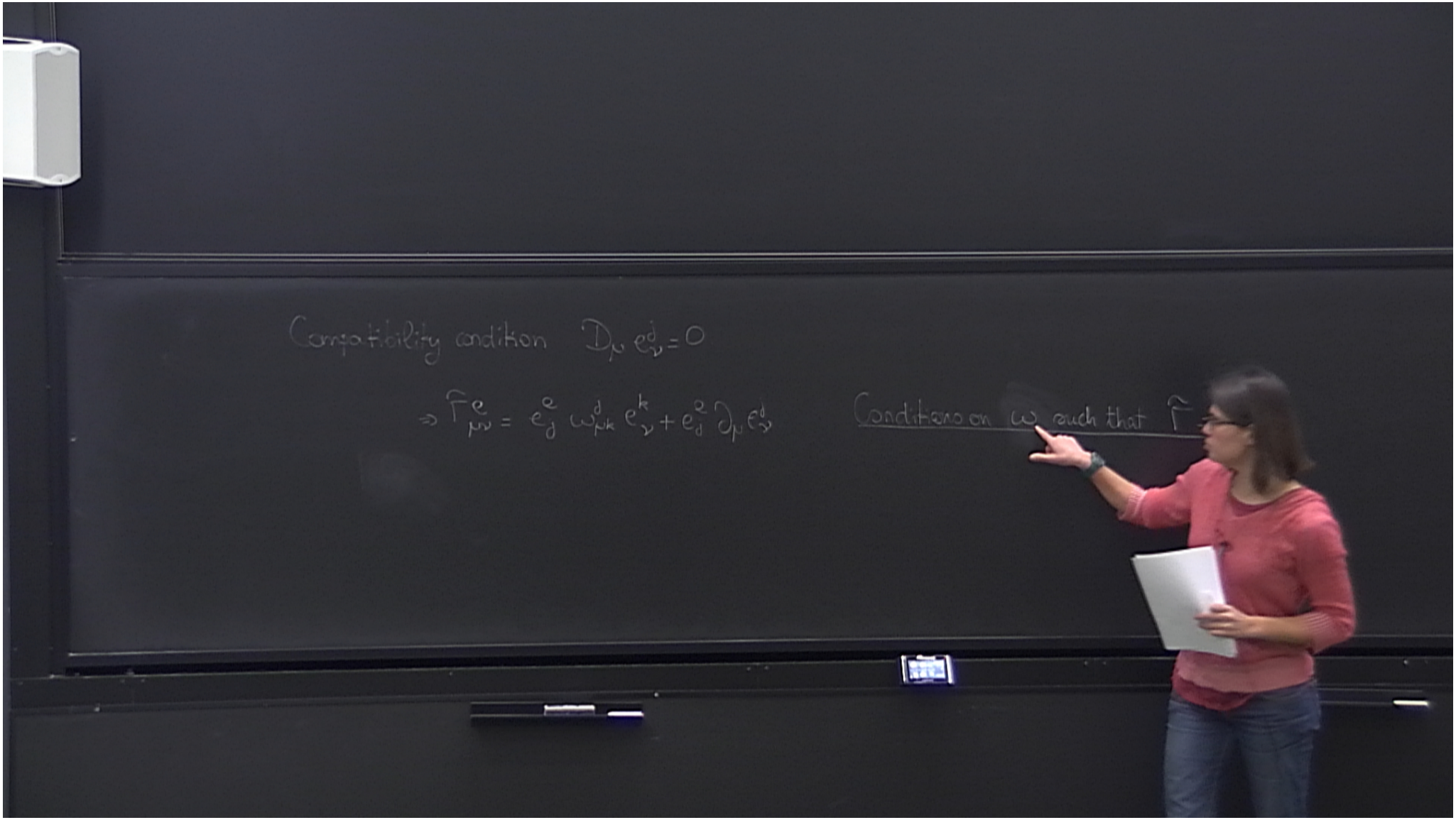
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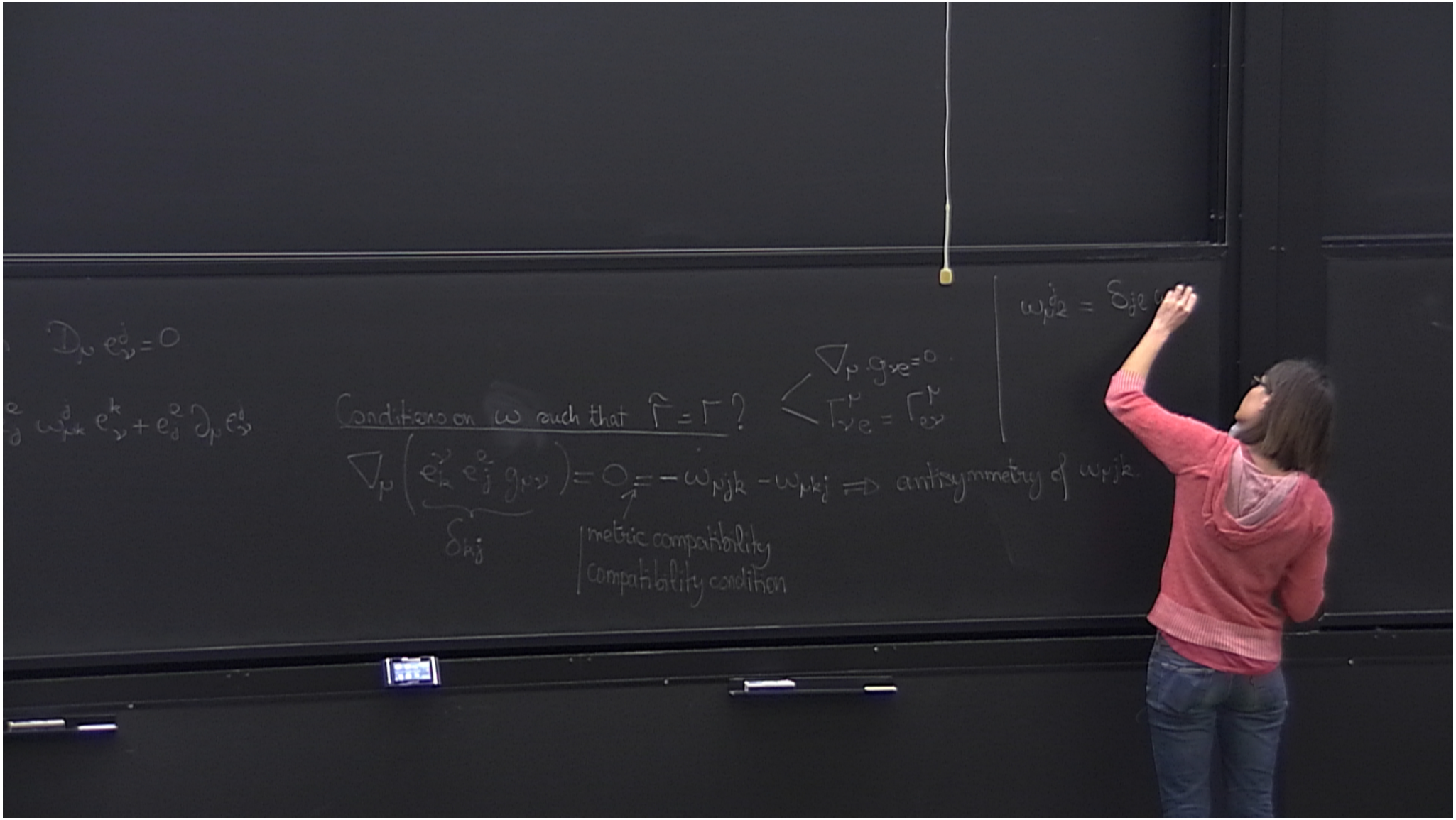
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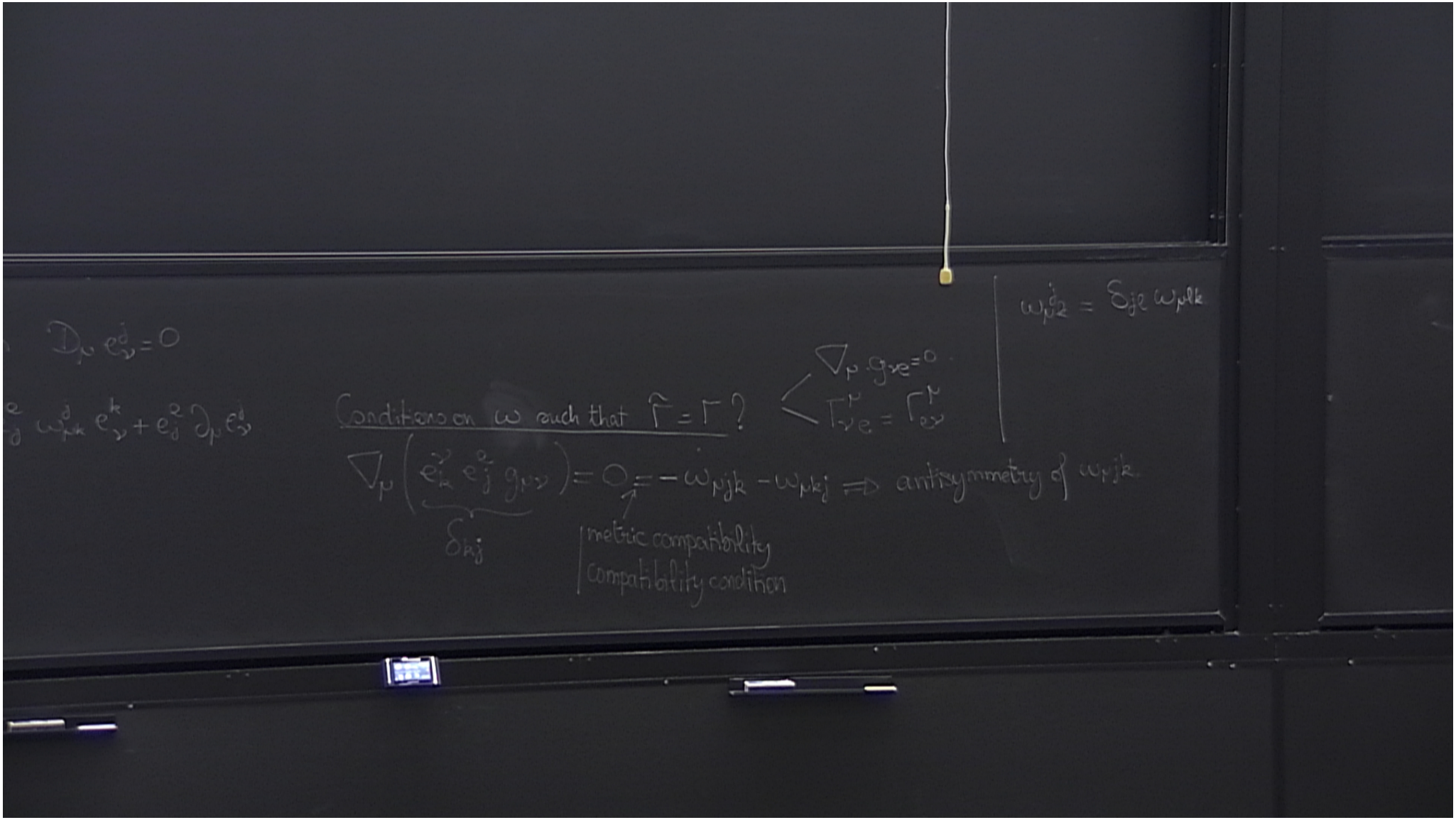
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Abstract:


II. D 1st order action
III. A. Global and local symmetries
III. B Gauge symmetries in 3D gravity 1st order action


Compatibility condition $D_\mu e^a_\nu = 0$
$\Rightarrow \Gamma^\rho_{\mu\nu} = e^\rho_j \omega^j_{\mu k} e^k_\nu + e^\rho_j \partial_\mu e^j_\nu$
Conditions on ω such that Γ


$D_\mu e^i_\nu = 0$
$\omega^i_{\mu k} e^k_\nu + e^i_\delta \partial_\mu e^\delta_\nu$
Conditions on ω such that $\tilde{\Gamma} = \Gamma$?
$\nabla_\mu g_{\nu\rho} = 0$
$\Gamma^\nu_{\mu\rho} = \Gamma^\nu_{\rho\mu}$
$\nabla_\mu \left(e^\nu_k e^\rho_j g_{\nu\rho} \right) = 0 = -\omega_{\mu jk} - \omega_{\mu kj} \Rightarrow$ antisymmetry of $\omega_{\mu jk}$
δ_{kj}
metric compatibility
compatibility condition
$\omega^i_{\mu 2} = \delta_{j\ell}$


$D_\mu e^a_\nu = 0$
$\omega^a_{\mu k} e^k_\nu + e^a_j \partial_\mu e^j_\nu$
Conditions on ω such that $\hat{\Gamma} = \Gamma$?
$\nabla_\mu g_{\nu\rho} = 0$
$\Gamma^\nu_{\nu\rho} = \Gamma^\nu_{\rho\nu}$
$\omega^i_{\mu k} = \delta_{jl} \omega_{\mu lk}$
$\nabla_\mu (e^\nu_i e^\rho_j g_{\nu\rho}) = 0 \Rightarrow -\omega_{\mu ji} - \omega_{\mu ij} \Rightarrow$ antisymmetry of $\omega_{\mu ji}$
δ_{ij}
metric compatibility
compatibility condition

$$\nabla_\mu(\delta_{ij}) = 0 \Leftrightarrow D_\mu(\delta_{ij}) = 0$$

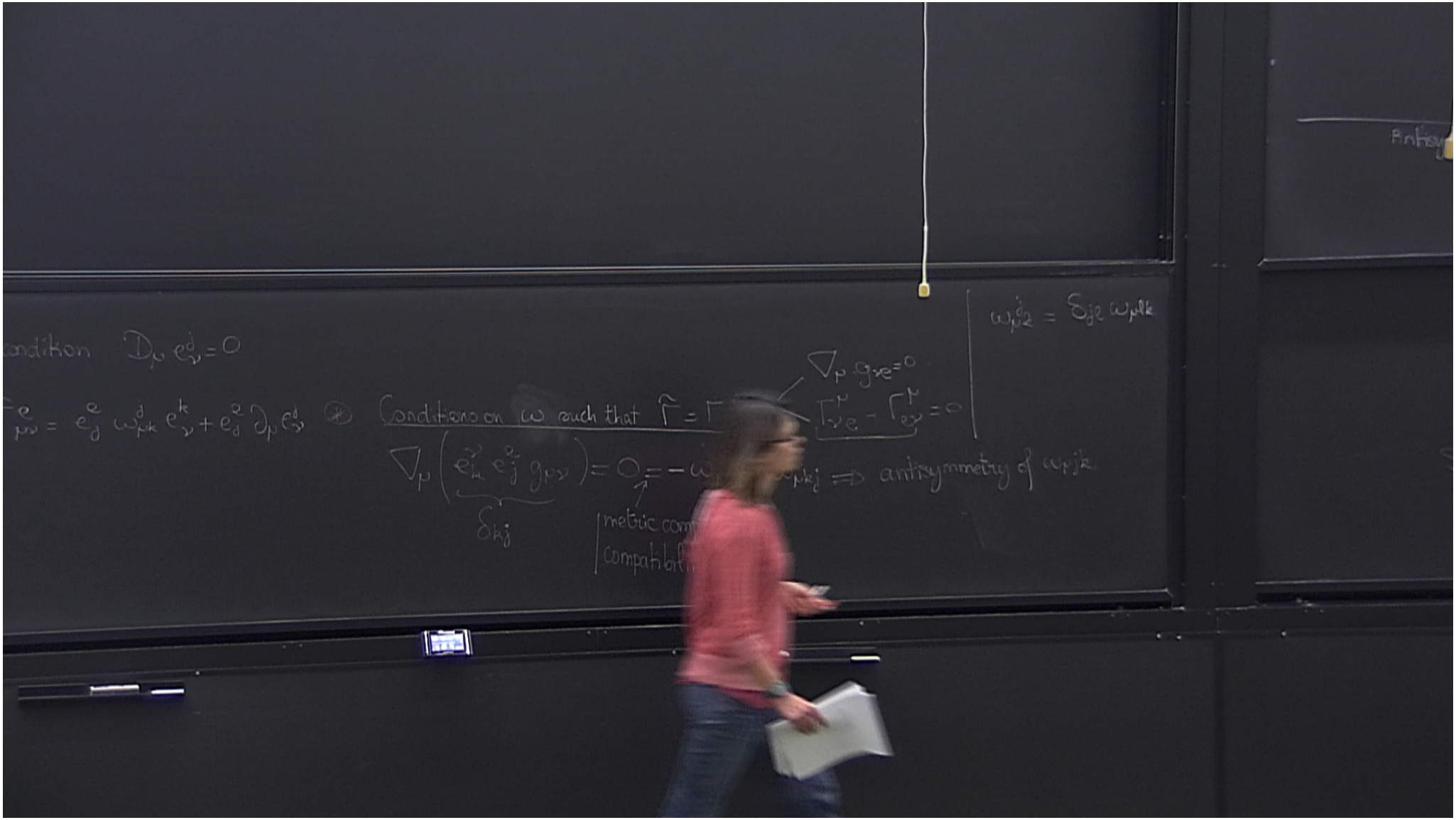
$$= \partial_\mu(\delta_{ij}) - \omega_{\mu i}^{\ \ k}\,\delta_{jk} - \omega_{\mu k}^{\ \ j}\,\delta_{ij}$$

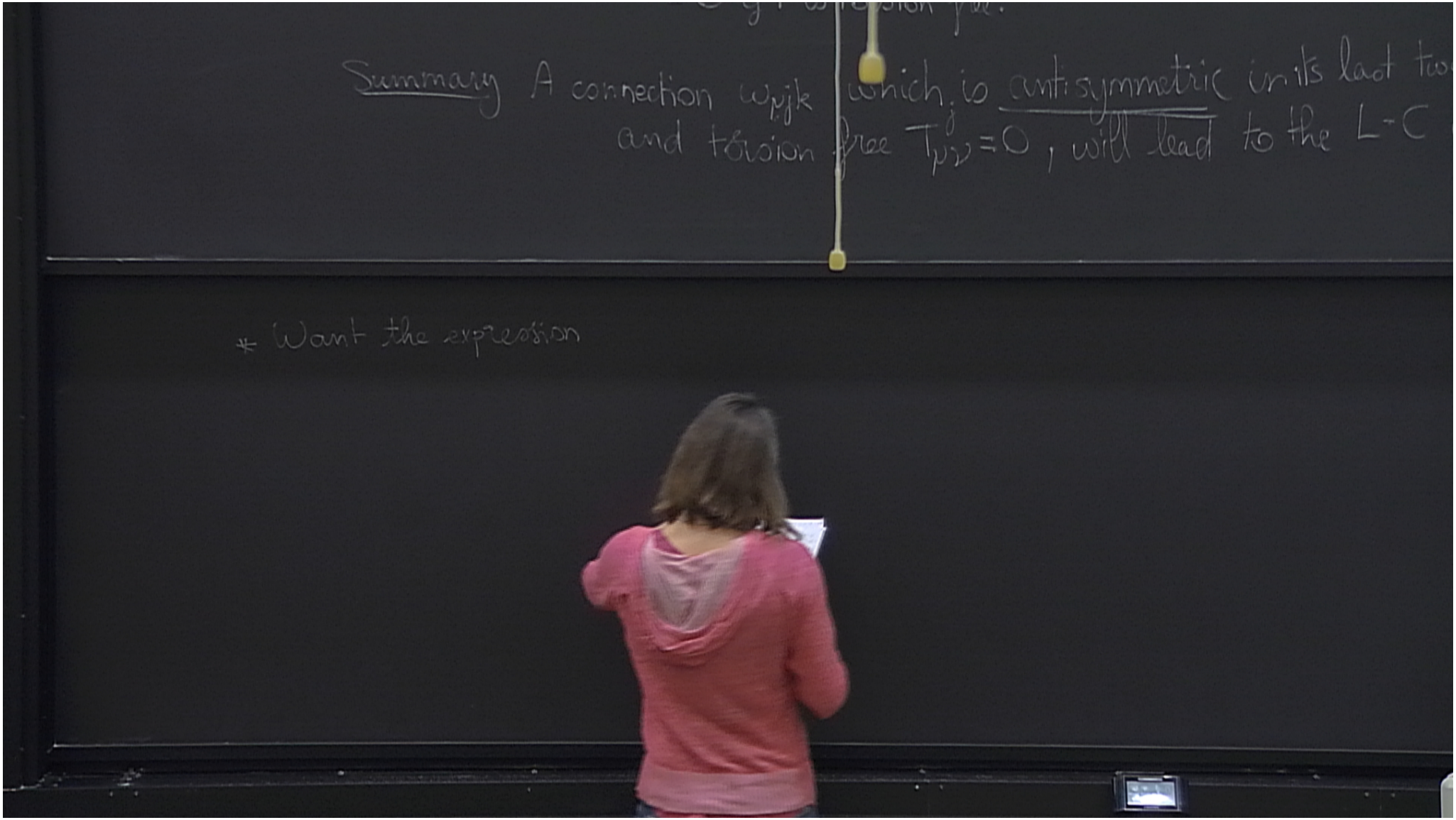
$$= -\omega_{\mu ki} - \omega_{\mu ik}$$

Antisymmetry of the spin connection implies that the covariant is metric.

Antisymmetrized derivative has to vanish

$$D_\mu e^j_\nu - D_\nu e^j_\mu = 0 = \partial_\mu e^j_\nu - \partial_\nu e^j_\mu - \Gamma^{\ \ e}_{\mu\nu}\, e^j_e + \Gamma^{\ \ e}_{\nu\mu}\, e^j_e$$

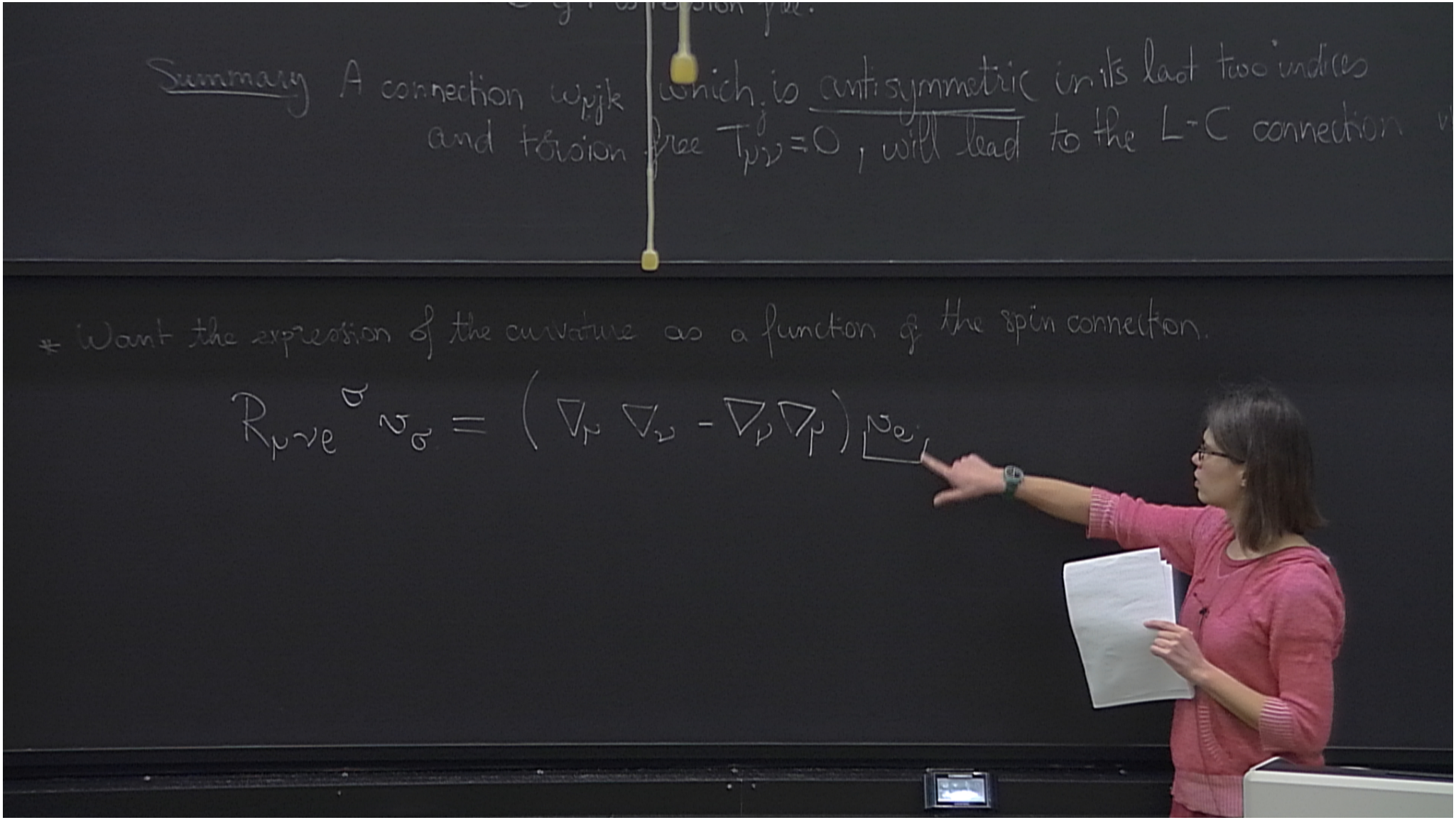

$\omega_{\mu jk} = \delta_{jl}\,\omega_{\mu lk}$
$\nabla_\mu g_{\nu\rho} = 0$
$\Gamma^\nu_{\rho\mu} - \Gamma^\nu_{\mu\rho} = 0$
Conditions on ω such that $\hat{\Gamma} = \Gamma$
$\nabla_\mu \left(e^i_\nu e^j_\rho g_{\nu\rho} \right) = 0$
δ_{ij}
metric
$\Rightarrow$ antisymmetry of $\omega_{\mu jk}$


Summary A connection $\omega_{\mu jk}$ which is antisymmetric in its last two
and torsion free $T_{\mu\nu}{}^{\lambda} = 0$, will lead to the L-C
* Want the expression

Summary A connection $\omega_{\mu jk}$ which is antisymmetric in its last two indices and torsion free $T^{\lambda}_{\mu\nu} = 0$, will lead to the L-C connection

* Want the expression of the curvature as a function of the spin connection.

$$R_{\mu\nu e}{}^{\sigma} v_{\sigma} = \left(\nabla_{\mu} \nabla_{\nu} - \nabla_{\nu} \nabla_{\mu} \right) v_{e}$$

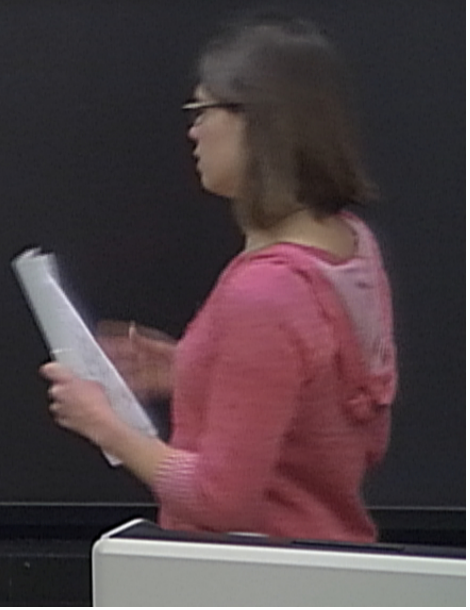


Summary A connection $\omega_{\mu ij}$ which is anti-symmetric in its last two indices and torsion free $T^{\lambda}_{\mu\nu} = 0$, will lead to the L-C connection via

Want the expression of the curvature as a function of the spin connection.

$$R_{\mu\nu\rho}{}^{\sigma} v_{\sigma} = \left(\nabla_{\mu} \nabla_{\nu} - \nabla_{\nu} \nabla_{\mu} \right) \underbrace{v_{\rho}}_{e^{j}_{\rho} v_{j}}$$

$$= e^{j}_{\rho} \left(D_{\mu} D_{\nu} - D_{\nu} D_{\mu} \right) v_{j} =$$



summary A connection $\omega_{\mu jk}$ which is antisymmetric in its last two indices and torsion free $T^{\lambda}_{\mu\nu} = 0$, will lead to the L-C connection via (*)

The expression of the curvature as a function of the spin connection.

$$R_{\mu\nu}{}^{\sigma}{}_{\sigma} = \left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu\right) \underbrace{v_a}_{e^j_a v_j}$$

$$= e^j_a \left(D_\mu D_\nu - D_\nu D_\mu\right) v_j = e^j_a \left[\partial_\mu \omega_{\nu jk} - \partial_\nu \omega_{\mu jk} + \omega\right.$$

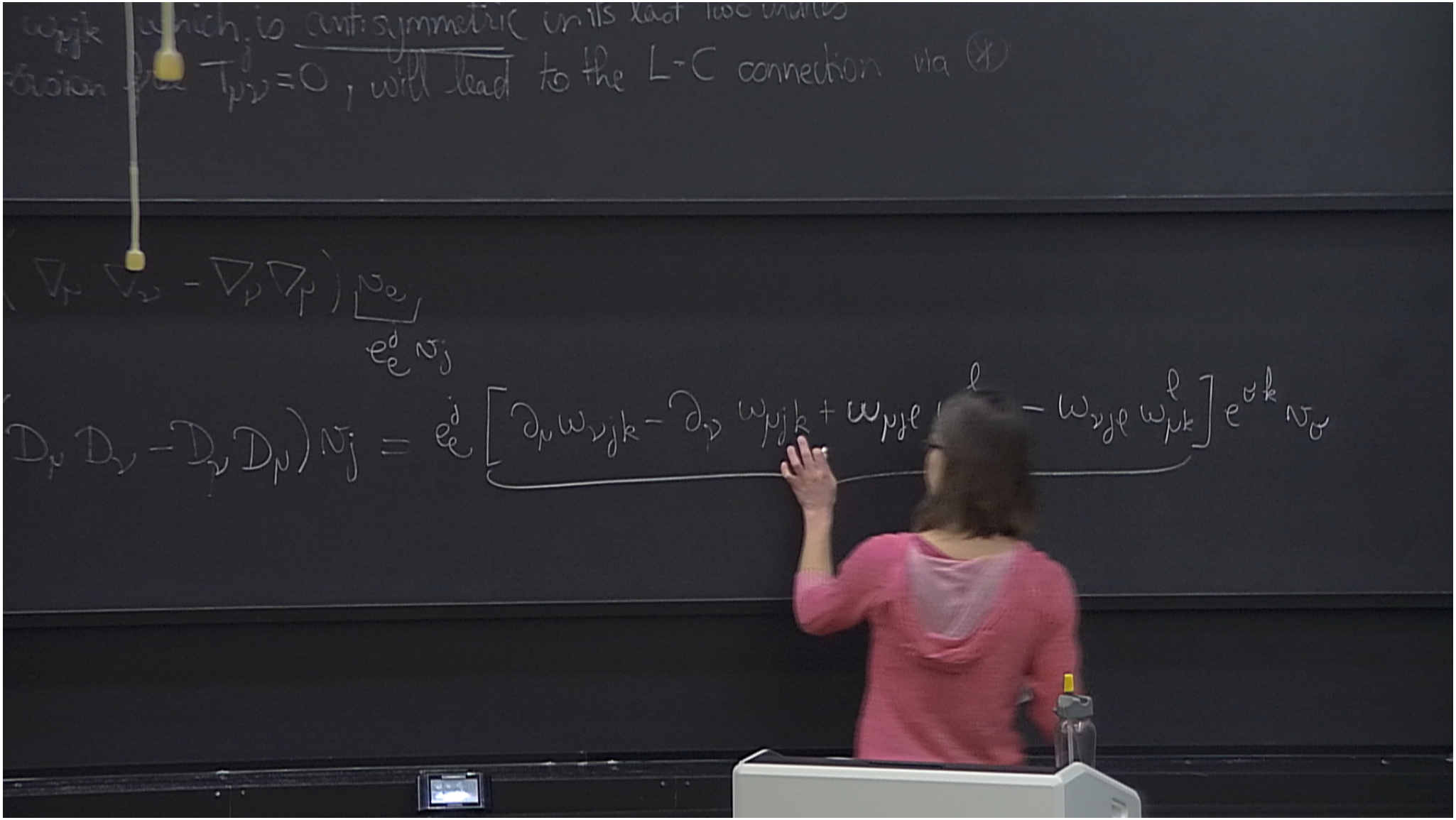
$\omega_{\mu jk}$ which is anti-symmetric in its last two indices

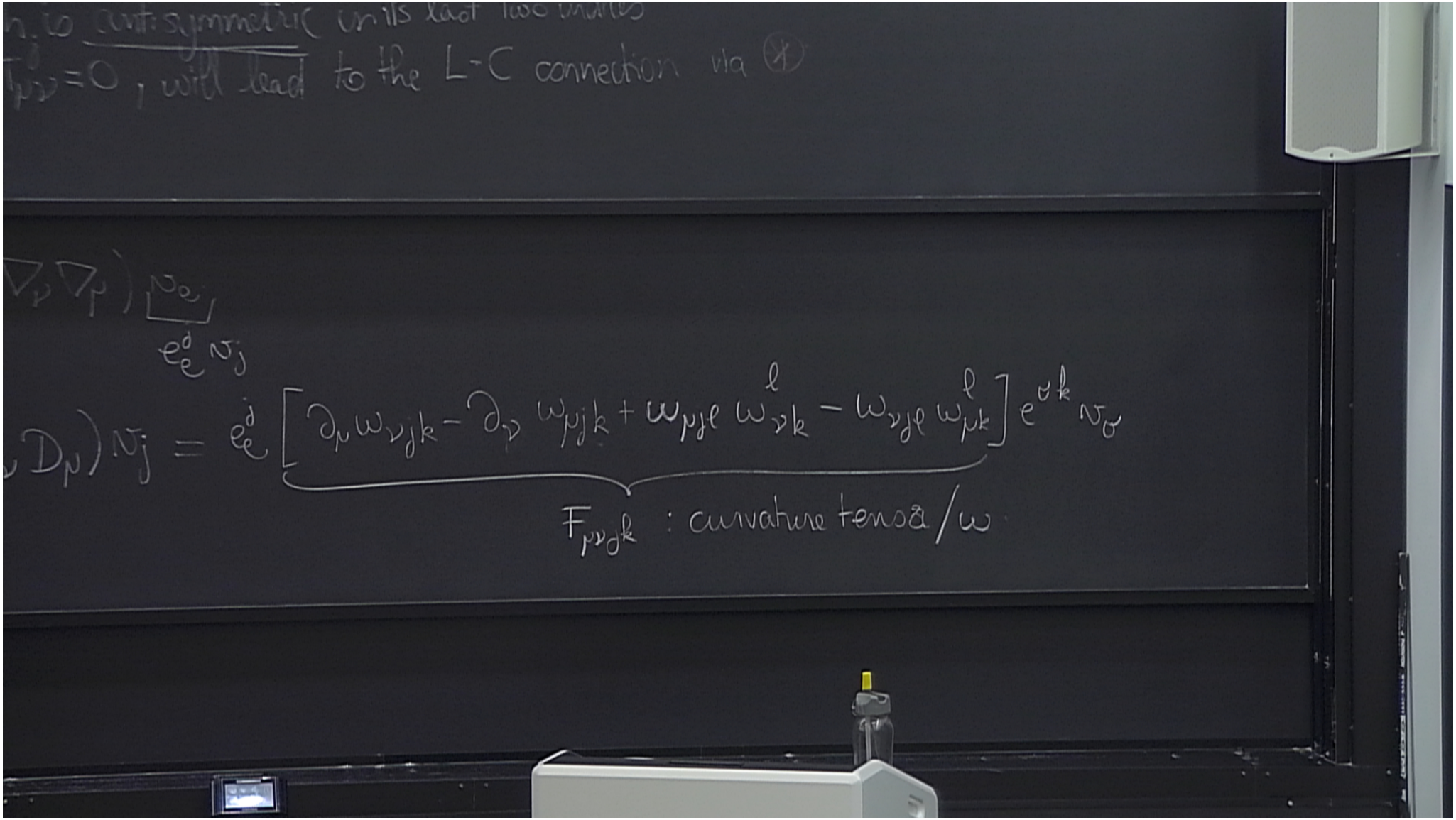
torsion free $T^{\lambda}_{\mu\nu} = 0$, will lead to the L-C connection via (*)

curvature as a function of the spin connection.

$$\left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu\right) \underbrace{\nu_e}_{e^d_e \nu_j}$$

$$\left(D_\mu D_\nu - D_\nu D_\mu\right) \nu_j = e^d_e \left[\partial_\mu \omega_{\nu jk} - \partial_\nu \omega_{\mu jk} + \omega_{\mu j\ell}\, \omega^{\ell}_{\nu k} - \omega_{\nu j\ell}\, \omega^{\ell}_{\mu k}\right] e^{\nu k} \nu$$


which is antisymmetric
$T_{\mu\nu}=0$, will lead to the L-C connection via (*)
$\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu$
$e^d_e V_j$
$\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V_j = e^d_e [\partial_\mu \omega_{\nu jk} - \partial_\nu \omega_{\mu jk} + \omega_{\mu je}$
$- \omega_{\nu je} \omega^e_{\mu k}] e^k_\sigma V_\sigma$


$T_{\mu\nu}=0$, will lead to the L-C connection via (*)
$\nabla_\nu \nabla_\mu)\, V_e$
$e^d_e V_j$
$\nu D_\mu) V_j = e^j_e \left[\partial_\mu \omega_{\nu jk} - \partial_\nu \omega_{\mu jk} + \omega_{\mu j\ell}\, \omega^{\ell}_{\nu k} - \omega_{\nu j\ell}\, \omega^{\ell}_{\mu k} \right] e^{vk} V_v$
$F_{\mu\nu jk}$: curvature tensor / ω

Summary: A connection $\omega_{\mu jk}$ which is anti-symmetric (in its last two indices) and torsion free $T^{j}_{\mu\nu}=0$, will lead to the L-C connection via (*)

$$R_{\mu\nu e}{}^{\sigma} v_{\sigma} = \left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu\right) \underbrace{v_e}_{e^d_e v_j}$$

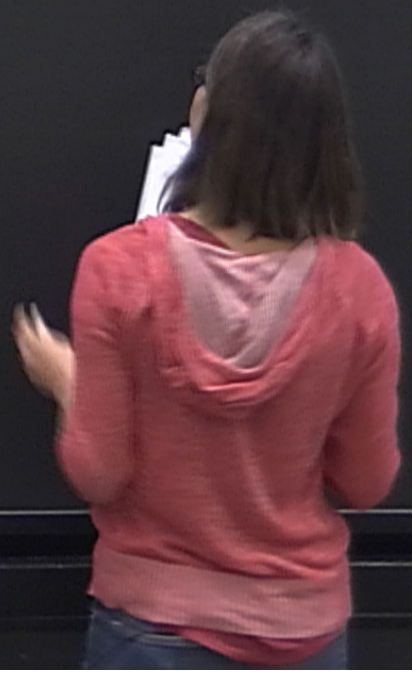
$$= e^j_e \left(D_\mu D_\nu - D_\nu D_\mu\right) v_j = e^d_e \underbrace{\left[\partial_\mu \omega_{\nu jk} - \partial_\nu \omega_{\mu jk} + \omega_{\mu je}\, \omega^{\ell}_{\nu k} - \omega\right.}_{F_{\mu\nu jk}\ :\ \text{curvature ten}}$$

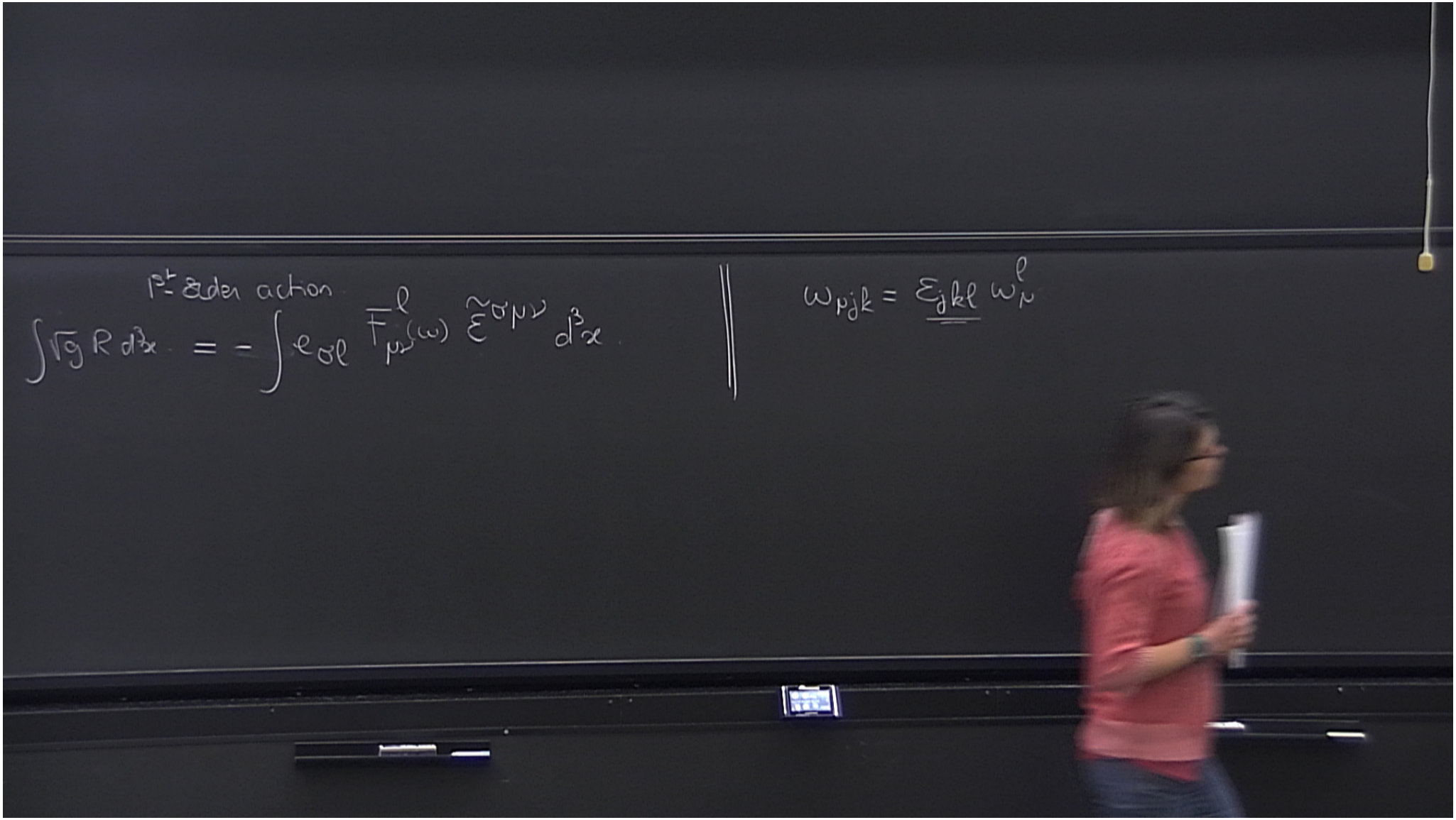
$$\Rightarrow \boxed{R_{\mu\nu e}{}^{\sigma} = e^d_e e^{\sigma k} F_{\mu\nu jk}}$$

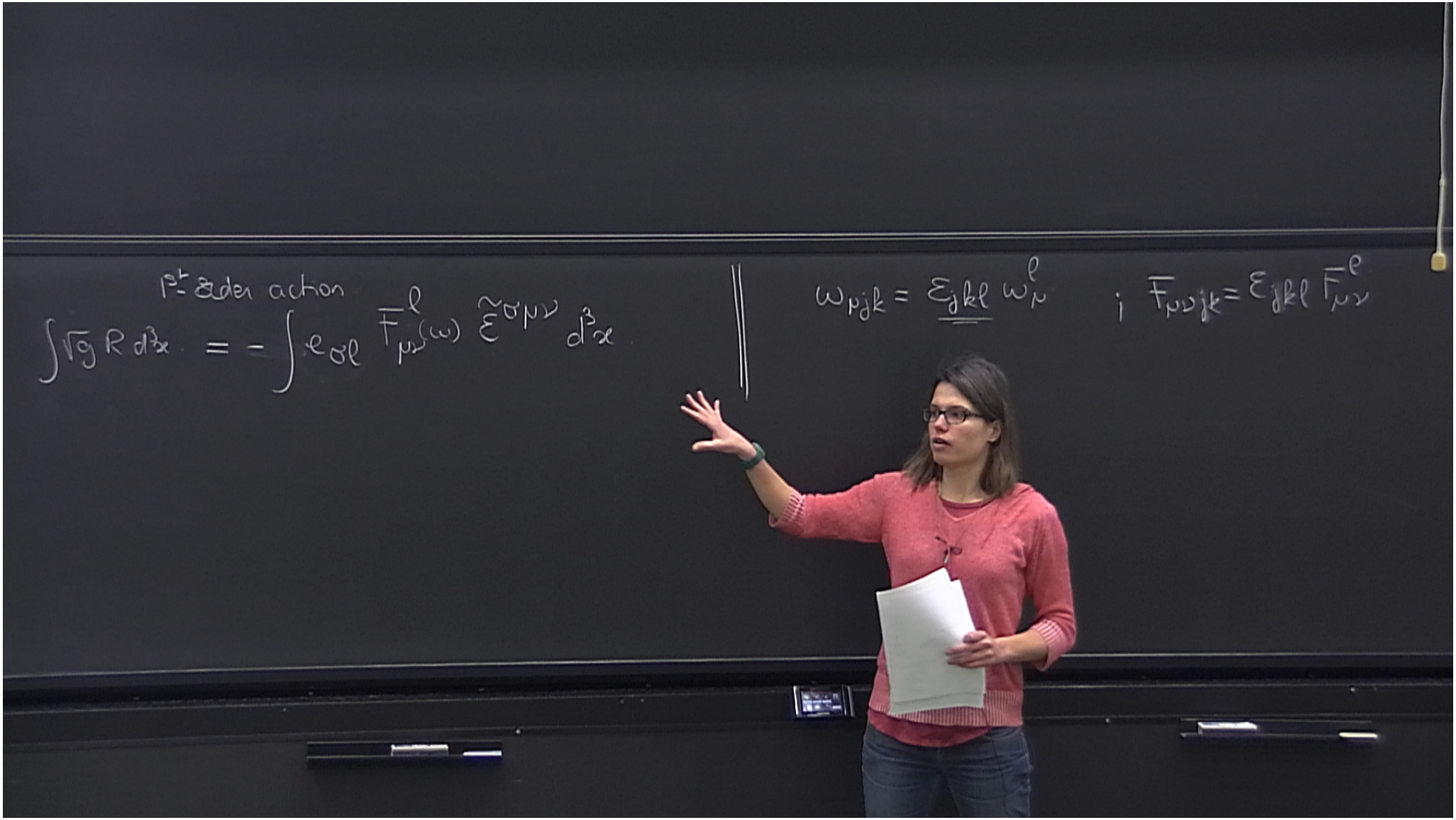
1st order action

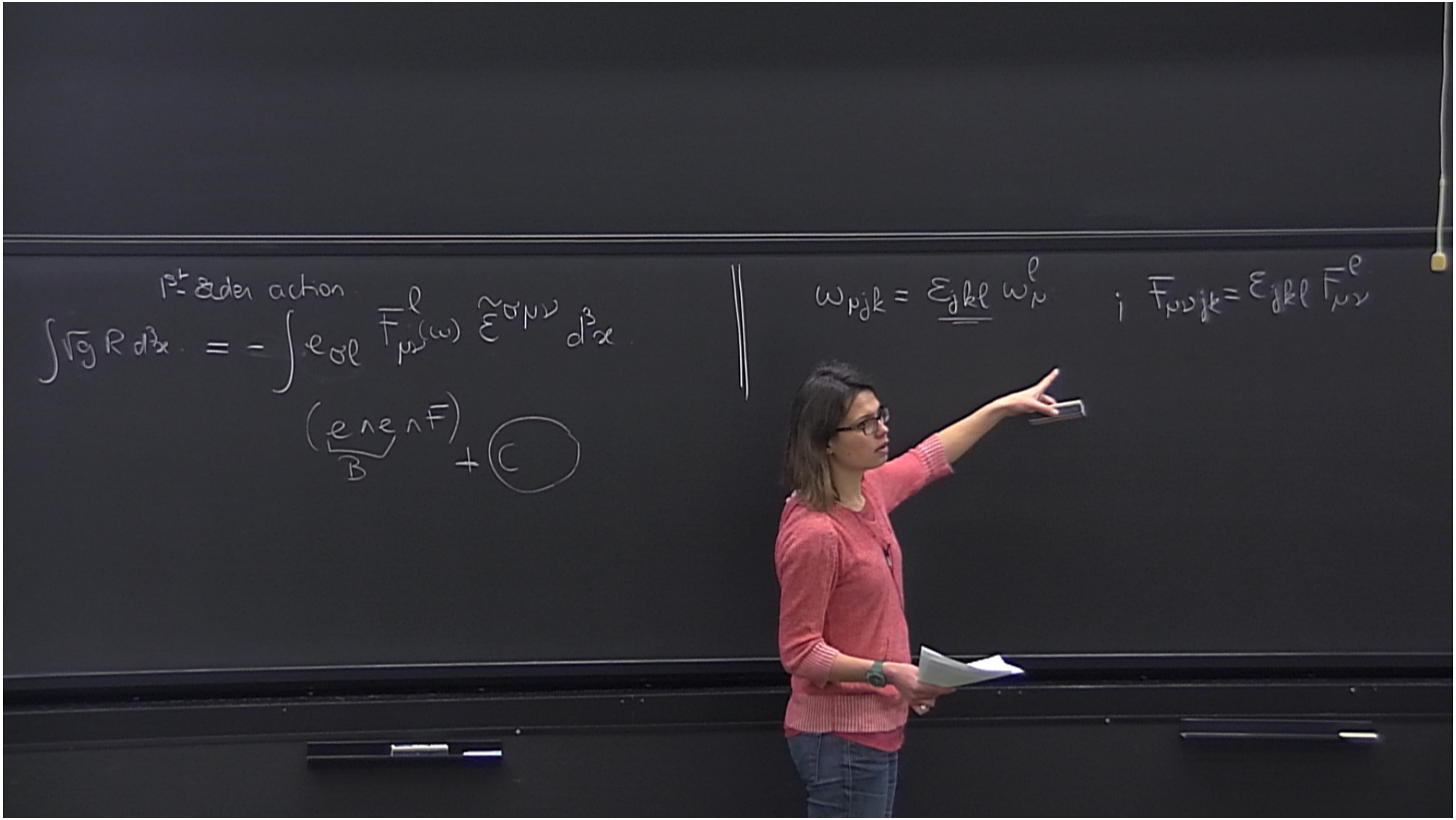
$$\int \sqrt{g}\, R\, d^3x = -\int e_{\sigma\ell}\, F^{\ell}_{\mu\nu}(\omega)\, \tilde{\varepsilon}^{\sigma\mu\nu}\, d^3x$$

$$\omega_{\mu jk} =$$




1st order action
$\int \sqrt{g} R \, d^3x = - \int e_{\sigma\ell} \, \tilde{F}^{\ell}_{\mu\nu}(\omega) \, \tilde{\varepsilon}^{\sigma\mu\nu} \, d^3x$
$\omega_{\mu jk} = \varepsilon_{jk\ell} \, \omega^{\ell}_{\mu}$


1st order action
$\int \sqrt{g} R \, d^3x = - \int e_{\sigma l} F^{l}_{\mu\nu}(\omega) \tilde{\varepsilon}^{\sigma\mu\nu} d^3x$
$\omega_{\mu jk} = \varepsilon_{jkl} \omega^{l}_{\mu}$; $F_{\mu\nu jk} = \varepsilon_{jkl} F^{l}_{\mu\nu}$


1st order action
$\int \sqrt{g} R \, d^3x = -\int e_a e_b F^{ab}_{\mu\nu}(\omega) \tilde{\varepsilon}^{\mu\nu} d^3x$
$(\underbrace{e \wedge e}_{B} \wedge F)$
$+ C$
$\omega_{\mu jk} = \varepsilon_{jkl} \omega^l_\mu$; $F_{\mu\nu jk} = \varepsilon_{jkl} F^l_{\mu\nu}$

1st order action

$$\int \sqrt{g}\, R\, d^3x = -\int e_{\sigma\ell}\, F^{\ell}_{\mu\nu}(\omega)\, \tilde{\varepsilon}^{\sigma\mu\nu}\, d^3x$$

$$\omega_{\mu jk} = \varepsilon_{jk\ell}\, \omega^{\ell}_{\mu} \quad ; \quad F_{\mu\nu jk} = \varepsilon_{jk\ell}\, F^{\ell}_{\mu\nu}$$

1st order action

$$\int \sqrt{g}\, R\, d^3x = -\int e_{\sigma\ell}\, F^{\ell}_{\mu\nu}(\omega)\, \tilde{\varepsilon}^{\sigma\mu\nu}\, d^3x$$

equations of motion

$$\begin{cases} F = 0 & (\delta e_\sigma) \\ T^{\sigma}_{\mu\nu} = 0 & (\delta\omega) \end{cases}$$

$\|$

$$\omega_{\mu jk} = \varepsilon_{jk\ell}\, \omega^{\ell}_{\mu} \quad ; \quad F_{\mu\nu jk} = \varepsilon_{jk\ell}\, F^{\ell}_{\mu\nu}$$

1st order action

$$\int \sqrt{g}\, R\, d^3x = -\int e_{\sigma\ell}\, F^{\ell}_{\mu\nu}(\omega)\, \tilde{\varepsilon}^{\sigma\mu\nu}\, d^3x$$

equations of motion

$$\begin{cases} F = 0 & (\delta e_\sigma) \\ T^{\sigma}_{\mu\nu} = 0 & (\delta\omega) \end{cases}$$

$\|$

$$\omega_{\mu jk} = \varepsilon_{jk\ell}\, \omega^{\ell}_{\mu} \quad ; \quad F_{\mu\nu jk} = \varepsilon_{jk\ell}\, F^{\ell}_{\mu\nu}$$

equations of motion $\Leftrightarrow T^{I}_{\mu\nu} = 0 \quad (\delta\omega)$

III. Global and local symmetries

Gauge transformation: local and non-trivial transfo of the fields

- local → can be restricted to some local region
- non-trivial → act non-trivially on solutions (extrema actions)

$\Rightarrow$ NON uniqueness of the solutions even if boundary & initial data are fixed.

$T^{\mu}_{\mu\nu} = 0 \quad (\delta\omega)$

local symmetries

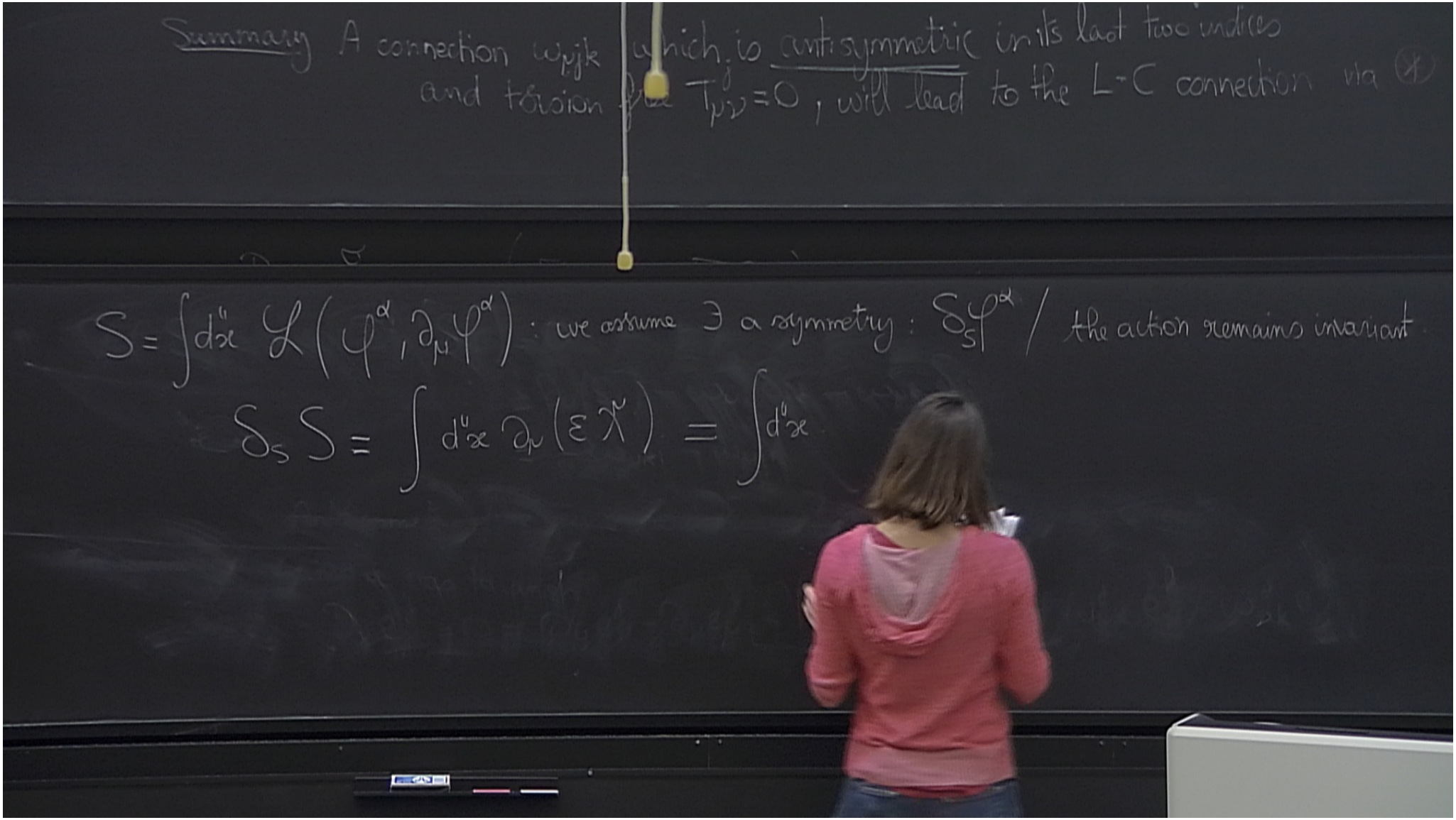
≠ Global transformation
(Noether theorem)

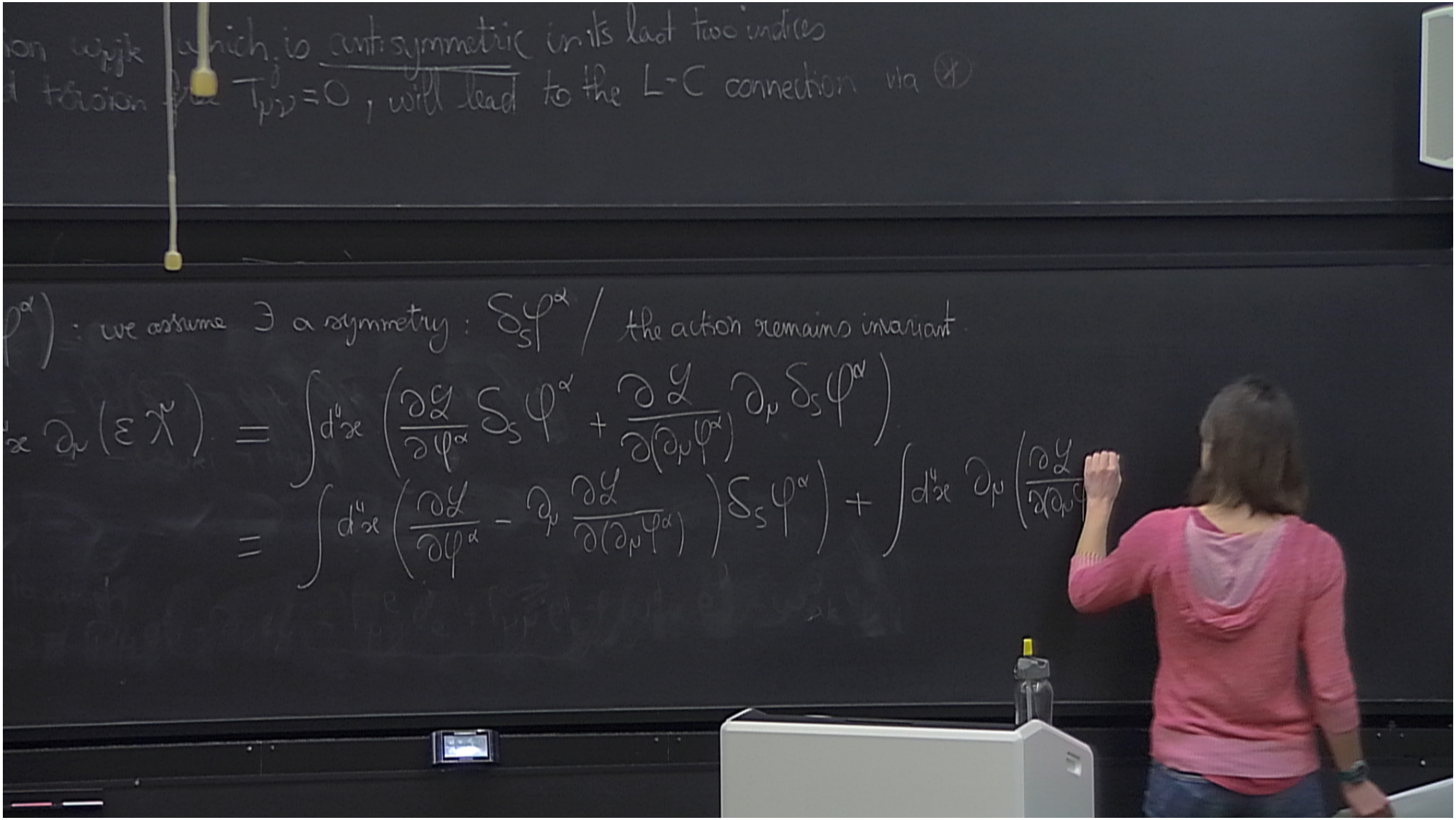
nation: local and non-trivial transfo of the fields

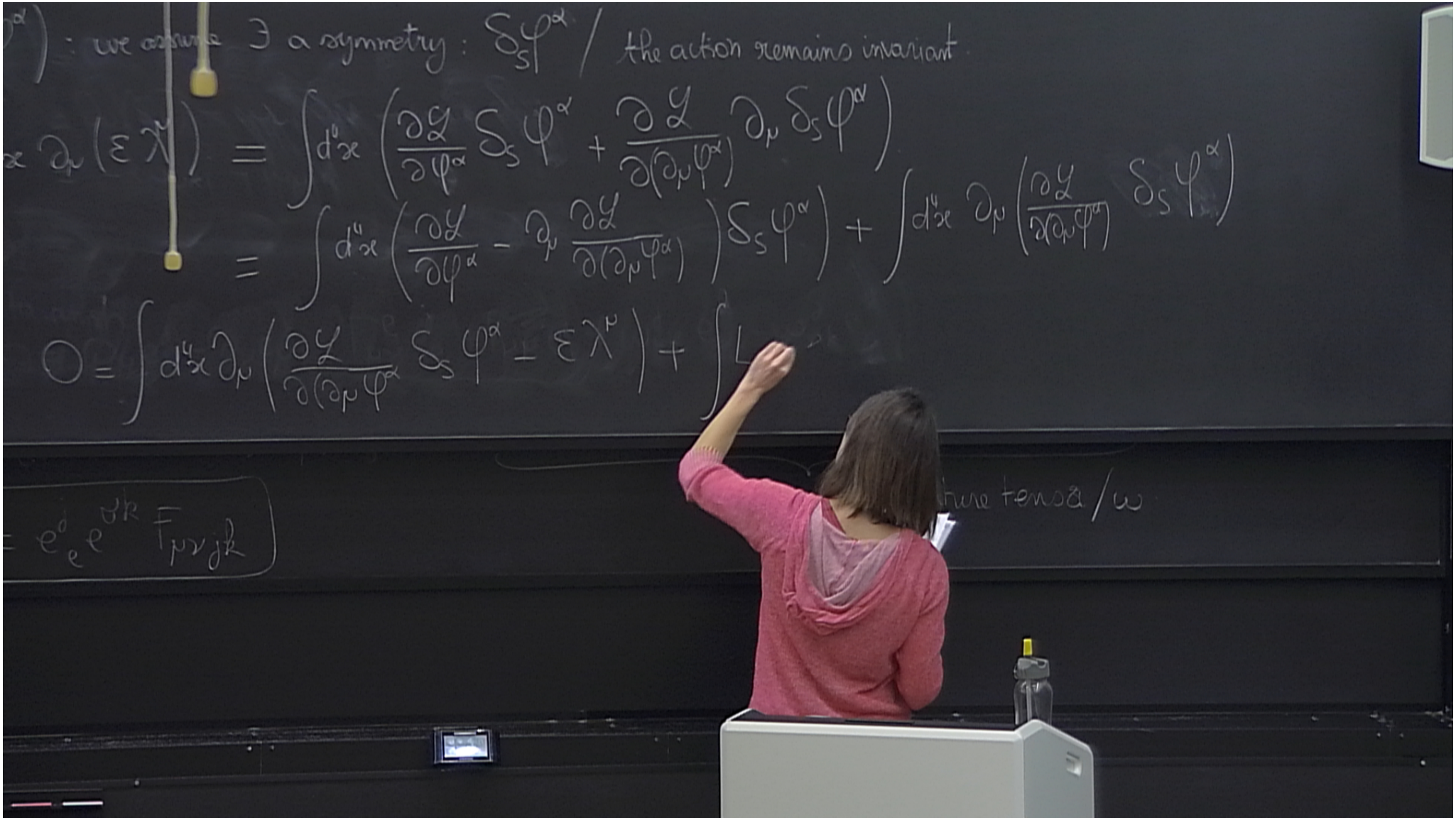
↓ can be restricted to some local region

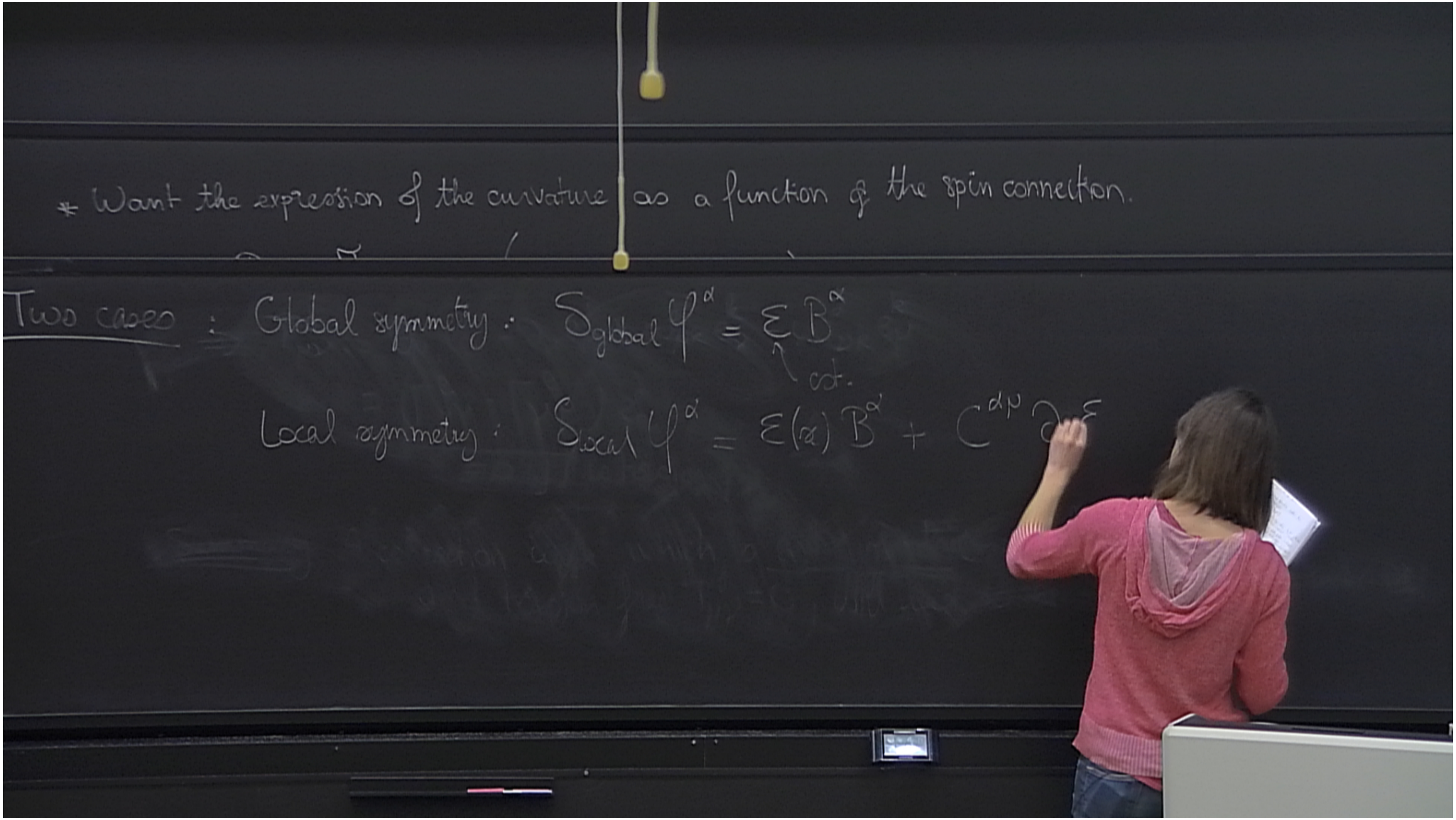
↘ act non-trivially on solutions (extrema actions)

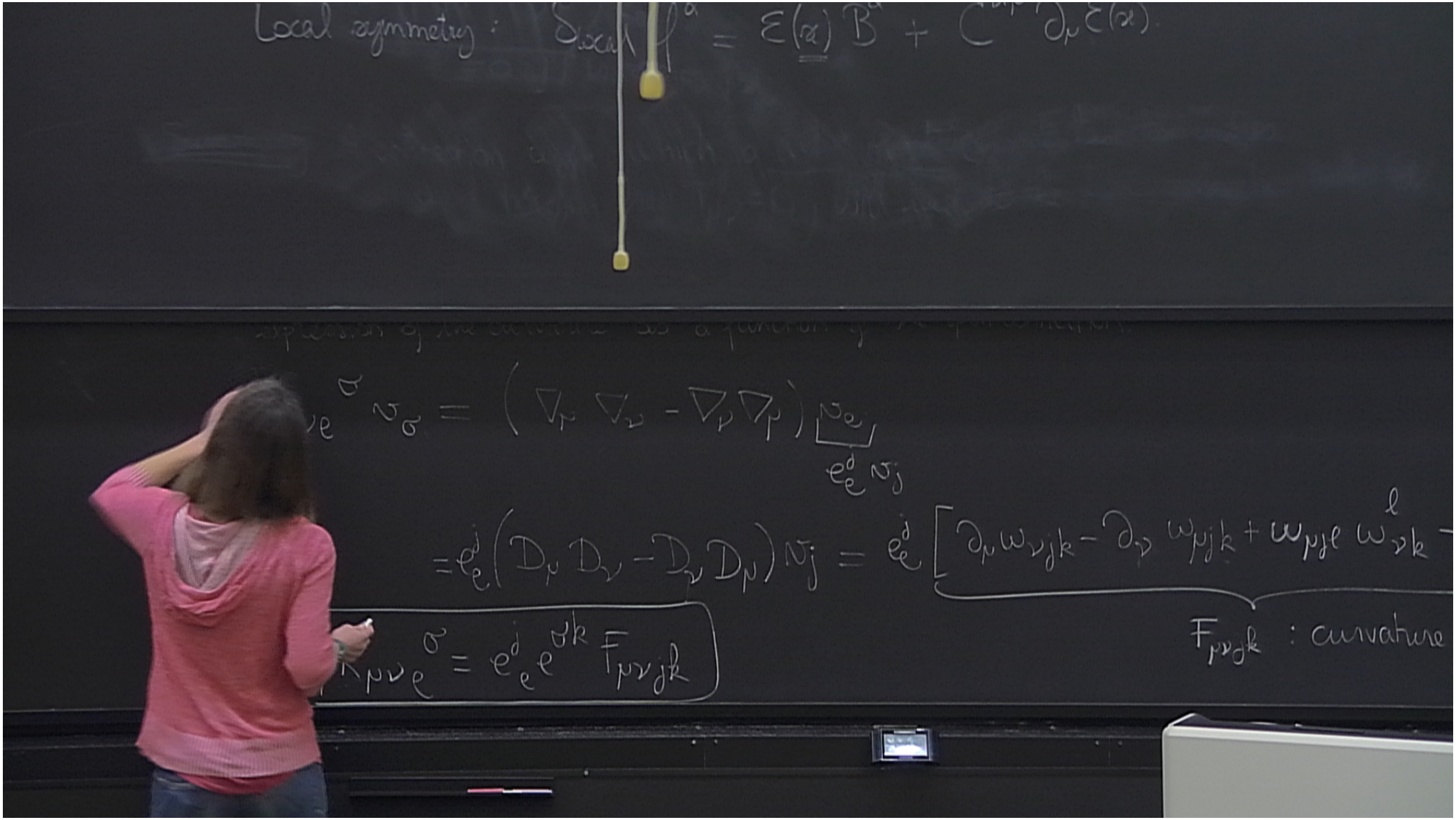
⇒ NON uniqueness of the solutions even if boundary & initial data are fixed

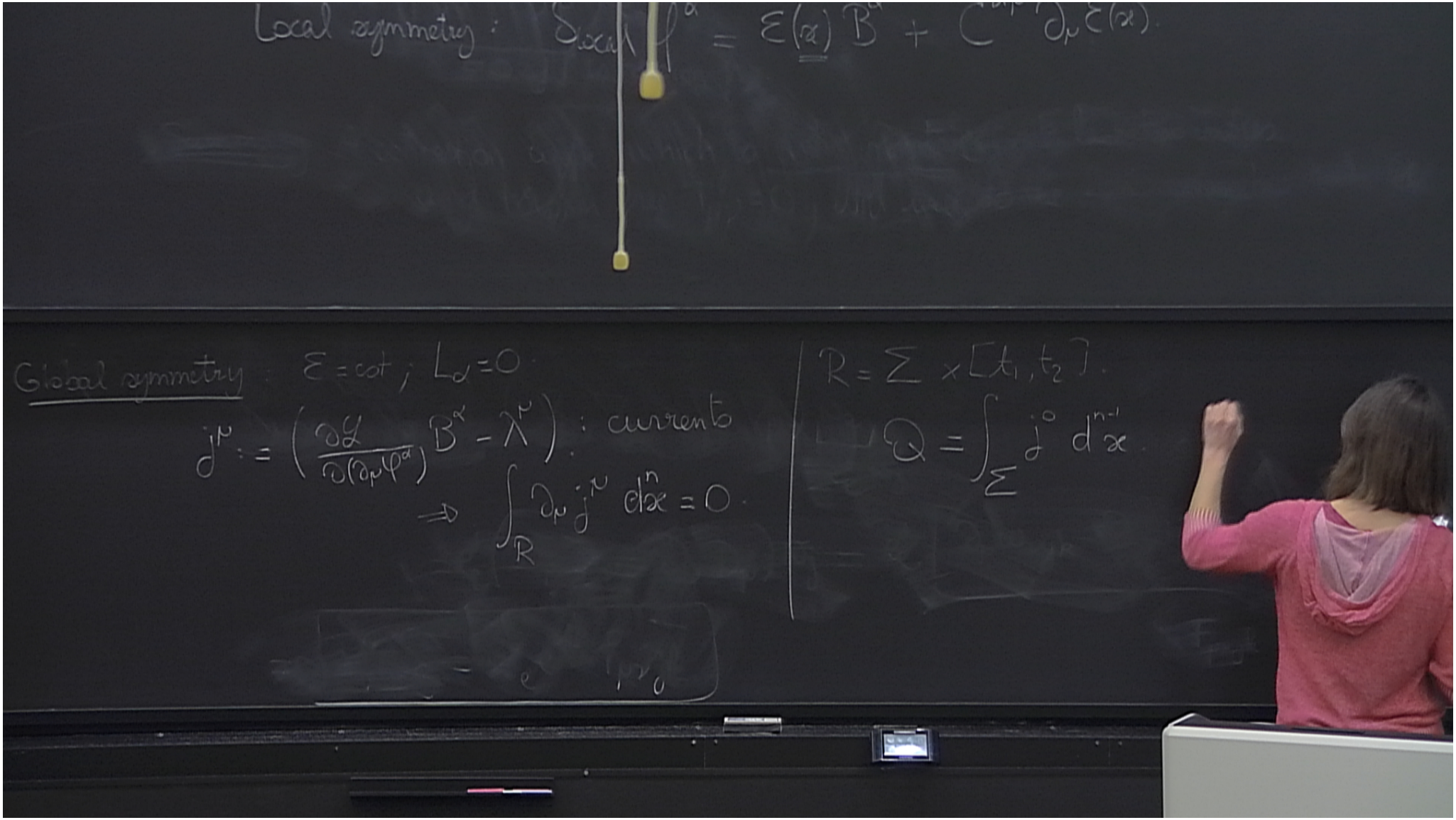

Summary A connection $\omega_{\mu jk}$ which is anti-symmetric in its last two indices and torsion $T^{j}_{\mu\nu}=0$, will lead to the L-C connection via (*)
$S=\int d^{n}x\ \mathcal{L}(\varphi^{\alpha}, \partial_{\mu}\varphi^{\alpha})$: we assume ∃ a symmetry : $\delta_{s}\varphi^{\alpha}$ / the action remains invariant
$\delta_{s} S = \int d^{n}x\ \partial_{\mu}(\varepsilon X^{\mu}) = \int d^{n}x$

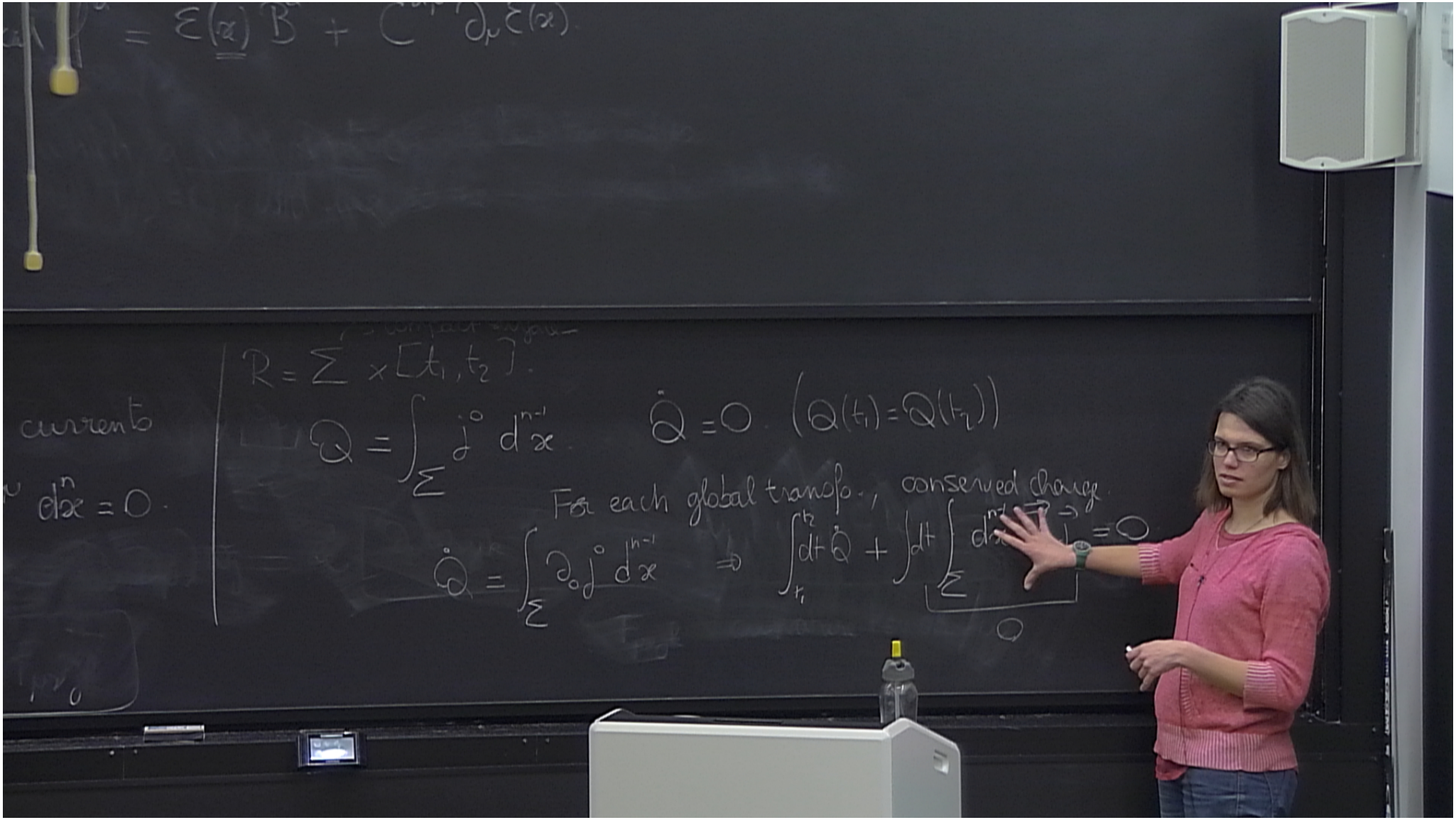

which is anti-symmetric in its last two indices
$T_{\mu\nu}=0$, will lead to the L-C connection via
: we assume ∃ a symmetry : $\delta_s\varphi^\alpha$ / the action remains invariant
$\int d^4x\, \partial_\nu(\varepsilon\lambda^\nu) = \int d^4x \left(\frac{\partial\mathcal{L}}{\partial\varphi^\alpha}\delta_s\varphi^\alpha + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^\alpha)}\partial_\mu\delta_s\varphi^\alpha\right)$
$= \int d^4x \left(\frac{\partial\mathcal{L}}{\partial\varphi^\alpha} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^\alpha)}\right)\delta_s\varphi^\alpha + \int d^4x\, \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu$


: we assume ∃ a symmetry : $\delta_s \varphi^\alpha$ / the action remains invariant
$\partial_\mu(\varepsilon \lambda^\mu) = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \varphi^\alpha} \delta_s \varphi^\alpha + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^\alpha)} \partial_\mu \delta_s \varphi^\alpha \right)$
$= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \varphi^\alpha} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^\alpha)} \right) \delta_s \varphi^\alpha + \int d^4x \, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^\alpha)} \delta_s \varphi^\alpha \right)$
$0 = \int d^4x \, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^\alpha)} \delta_s \varphi^\alpha - \varepsilon \lambda^\mu \right) + \int \mathcal{L}$
$= e^a_\varrho e^{\varrho k} F_{\mu\nu jk}$
ture tensor / ω


* Want the expression of the curvature as a function of the spin connection.
Two cases : Global symmetry : $\delta_{global}\varphi^\alpha = \varepsilon B^\alpha$
ε cst.
Local symmetry : $\delta_{local}\varphi^\alpha = \varepsilon(x) B^\alpha + C^{\alpha\mu}\partial_\mu\varepsilon$


Local symmetry :
$\delta_{local} A^a = \varepsilon(x) B^a + C^{ab} \partial_\mu \varepsilon(x)$
$\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu$
$e^d_e v_j$
$= e^j_e (D_\mu D_\nu - D_\nu D_\mu) v_j = e^d_e [\partial_\mu \omega_{\nu jk} - \partial_\nu \omega_{\mu jk} + \omega_{\mu je} \omega^{l}_{\nu k} -$
$R_{\mu\nu} e^\sigma = e^d_e e^{vk} F_{\mu\nu jk}$
$F_{\mu\nu jk}$: curvature


Local symmetry : $\delta_{local}\varphi^a = \varepsilon(x) B^a + C^{\dots} \partial_\mu \varepsilon(x)$
Global symmetry : $\varepsilon = cst$; $L_a = 0$
$j^\mu := \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^a)} B^a - \lambda^\mu\right)$: current
$\Rightarrow \int_R \partial_\mu j^\mu \, d^n x = 0$
$R = \Sigma \times [t_1, t_2]$
$Q = \int_\Sigma j^0 \, d^{n-1}x$


$= \varepsilon(x)\, B + C \partial_\mu \varepsilon(x)$
$R = \Sigma \times [t_1, t_2]$
$Q = \int_\Sigma j^0 \, d^{n-1}x$
$\dot{Q} = 0 \quad (Q(t_1) = Q(t_2))$
For each global transfo., conserved charge
$\dot{Q} = \int_\Sigma \partial_0 j^0 \, d^{n-1}x \Rightarrow \int_{t_1}^{t_2} dt\, \dot{Q} + \int dt \int_\Sigma d^{n-1}x \, \partial_\mu j^\mu = 0$
0
current
$d^n x = 0$