Title: PSI 2015/2016 Quantum Gravity - Lecture 2

Date: Feb 23, 2016 10:15 AM

URL: http://pirsa.org/16020073

Abstract:

II Action principle - 1st order action

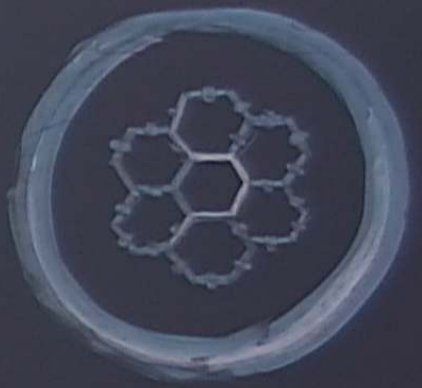
A. Review of the 4D Einstein action and equations of motion

B. 3D case

C. Triads

D. Spin connection

E- First order action for 3D gravity = Palatini action



II A.

Einstein-Hilbert action $S = \frac{1}{K} \int \sqrt{g}\, R\, d^4x$

$K = \frac{16\pi G}{c^3} = 1$

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Einstein-Hilbert action. $S = \frac{1}{\kappa} \int \sqrt{g}\, R\, d^4x = S[g_{\mu\nu}]$.

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$\kappa = \frac{16\pi G}{c^4} = 1$

$\Rightarrow$ Equations of motion

$\int d^4x = S[g_{\mu\nu}]$.

...ions of motion

$g_{\mu\rho} g^{\rho\nu} = \delta_\mu^{\ \nu}$

$\delta(g_{\mu\rho} g^{\rho\nu}) = 0 \Rightarrow \delta g_{\mu\nu} = - g_{\mu\rho}\, \delta g^{\rho\sigma}\, g_{\sigma\nu}$.

ction. $S = \frac{1}{\kappa} \int \sqrt{g}\, R\, d^4x = S[g_{\mu\nu}]$.

$= \frac{16\pi G}{c^4} = 1$

$\Rightarrow$ Equations of motion

variations of the action

w.r.t metric & inverse metric

$g_{\mu\rho} g^{\rho\nu} = \delta_\mu^{\ \nu}$

$\delta(g_{\mu\rho} g^{\rho\nu}) = 0 \Rightarrow \delta$

$$\delta S = \delta S_1 + \delta S_2 + \delta S_3$$

$$R = R_{\mu\nu} g^{\mu\nu} \qquad \delta S_1 = \int (\delta R_{\mu\nu})\, g^{\mu\nu} \sqrt{g}\, d^n x$$

see tutorial ←

$$\delta S_1 = \int \nabla$$

$$\delta = \delta S_1 + \delta S_2 + \delta S_3 .$$

$$\delta S_1 = \int (\delta R_{\mu\nu})\, g^{\mu\nu} \sqrt{g}\, d^4x$$

see tutorial

$$\delta S_1 = \int (\nabla^\nu N_\nu) \sqrt{g}\, d^4x .$$

$$\delta S_2 = \int R_{\mu\nu}\, \delta(g^{\mu\nu}) \sqrt{g}\, d^4x .$$

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$$\delta S_1 = \int (\nabla^\nu N_\nu) \sqrt{g}\, d^4x$$

determinant

$$\delta S_2 = \int R_{\mu\nu}\, \delta(g^{\mu\nu}) \sqrt{g}\, d^4x$$

express g in terms of the metric

$$\delta\sqrt{g} = \frac{1}{2\sqrt{g}}\delta g = -\frac{1}{2}\frac{1}{\sqrt{g}}\, g\, g_{\mu\nu}\, \delta g^{\mu\nu}$$

$$\delta S_3 =$$

w.r.t metric & inverse metric

$$\delta S = \delta S_1 + \delta S_2 + \delta S_3$$

$R = R_{\mu\nu} g^{\mu\nu}$

$$\delta S_1 = \int (\delta R_{\mu\nu})\, g^{\mu\nu} \sqrt{g}\, d^4x \longleftarrow \text{see tutorial} \qquad \delta S_1 = \int (\nabla^\nu N_\nu) \sqrt{g}\, d^4x$$

determinant

$$\delta S_2 = \int R_{\mu\nu}\, \delta(g^{\mu\nu})\, \sqrt{g}\, d^4x$$

express g in terms of the metric

$$\delta S_3 = \int -\frac{1}{2} R\, g_{\mu\nu} \sqrt{g}\, \delta g^{\mu\nu} \longleftarrow \delta\sqrt{g} = \frac{1}{2\sqrt{g}} \delta g = -\frac{1}{2\sqrt{g}} \frac{1}{g}\, g\, g_{\mu\nu}\, \delta g^{\mu\nu}$$

$\delta S = \int d^4$

$$\delta S = \int d^4x \underbrace{\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right)}_{G_{\mu\nu}} \sqrt{g}\, \delta g^{\mu\nu}$$

Vacuum Einstein

$$\delta S = \int d^{11}x \underbrace{\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right)}_{G_{\mu\nu}} \sqrt{g}\, \delta g^{\mu\nu}$$

trace of $G_{\mu\nu} = 0$

Vacuum Einstein equation $G_{\mu\nu} = 0 \Rightarrow R_{\mu\nu} = 0$

$\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \sqrt{g}\, \delta g^{\mu\nu}$

$G_{\mu\nu}$

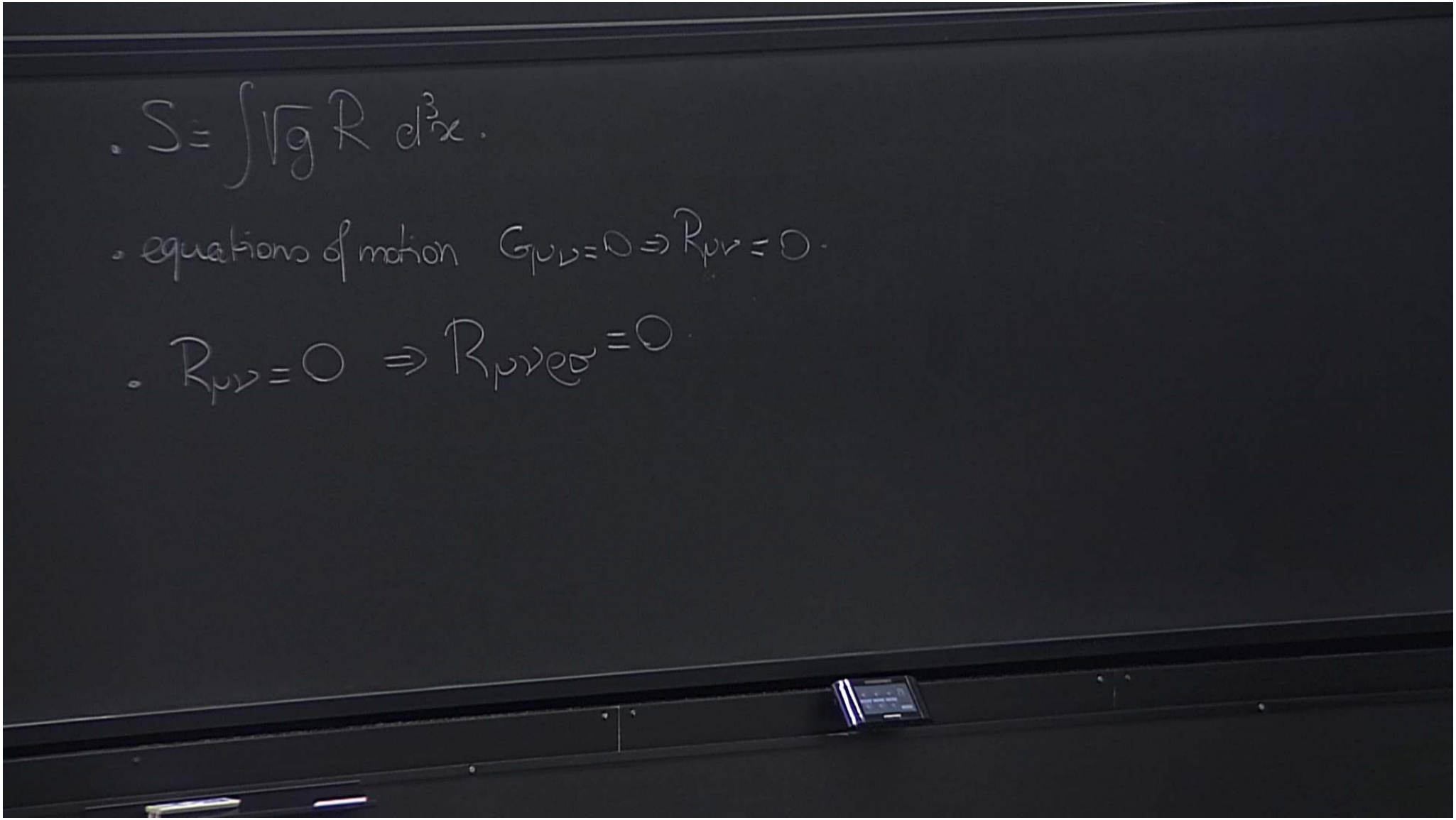
trace of $G_{\mu\nu} = 0$ $\quad (R = 0$ when $n > 2)$

$\downarrow$

Enstein equation $\quad G_{\mu\nu} = 0 \Rightarrow R_{\mu\nu} = 0$

B. 3D case

- $S = \int \sqrt{g} R \, d^3x$.
- equations of motion $G_{\mu\nu} = 0 \Rightarrow R_{\mu\nu} = 0$.


$S = \int \sqrt{g} R \, d^3x$.
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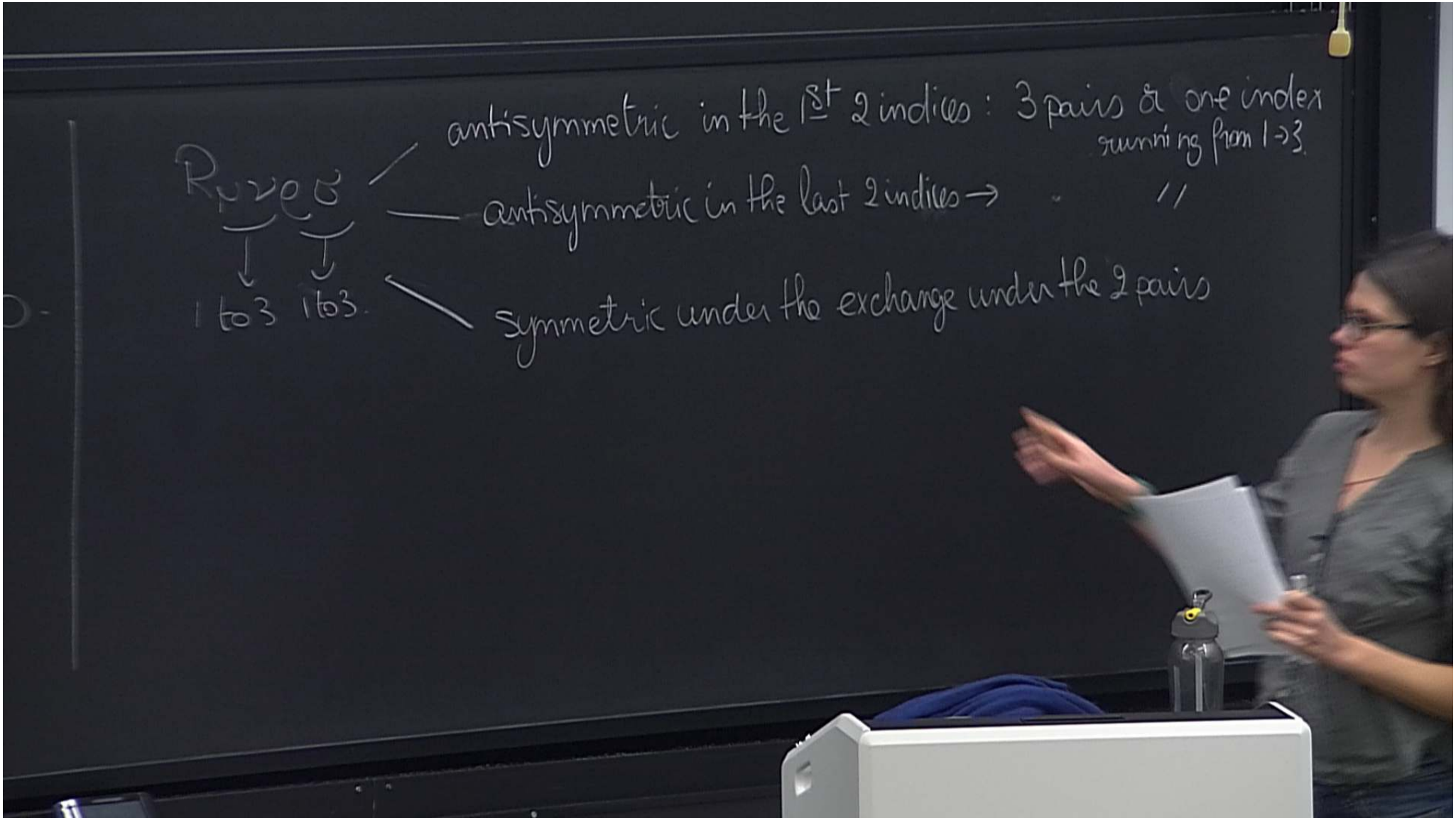
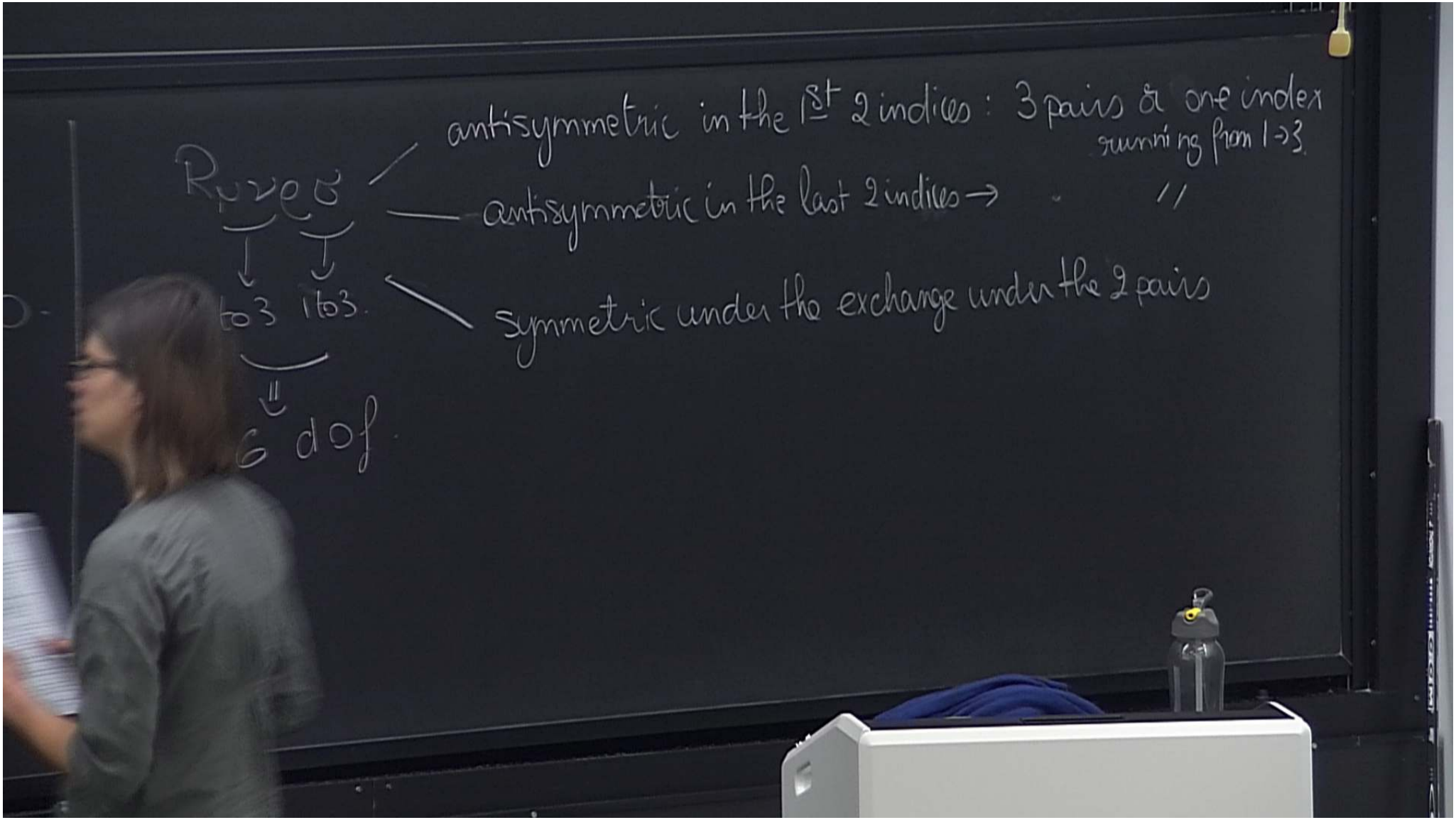
$R_{\mu\nu} = 0 \Rightarrow R_{\mu\nu\rho\sigma} = 0$


antisymmetric in the 1st 2 indices : 3 pairs & one index running from 1→3
antisymmetric in the last 2 indices →
1 to 3 1 to 3
symmetric under the exchange under the 2 pairs


$R_{\mu\nu\rho\sigma}$
antisymmetric in the 1st 2 indices : 3 pairs & one index running from 1→3
antisymmetric in the last 2 indices →
1 to 3
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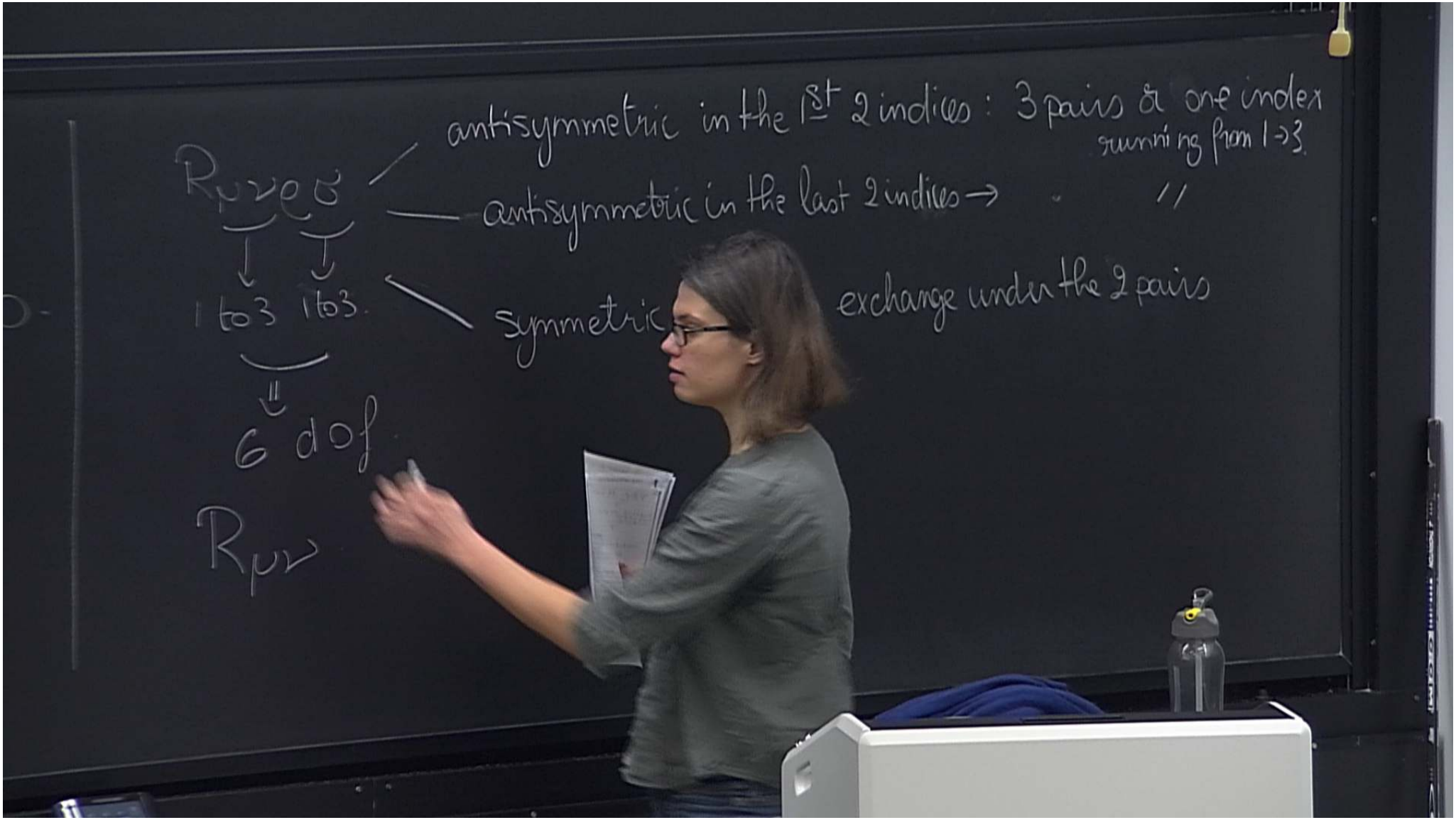
symmetric under the exchange under the 2 pairs
6 dof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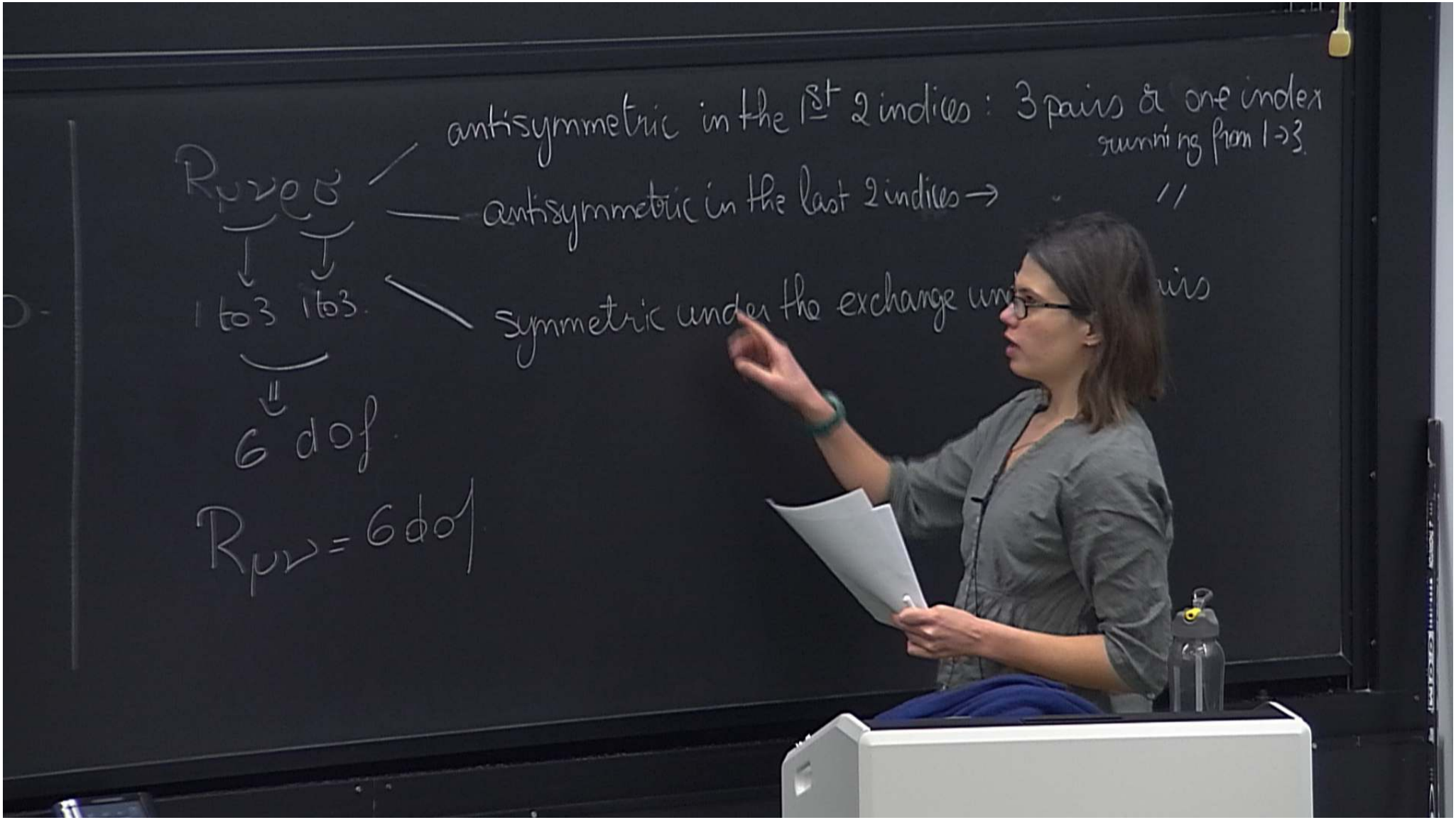
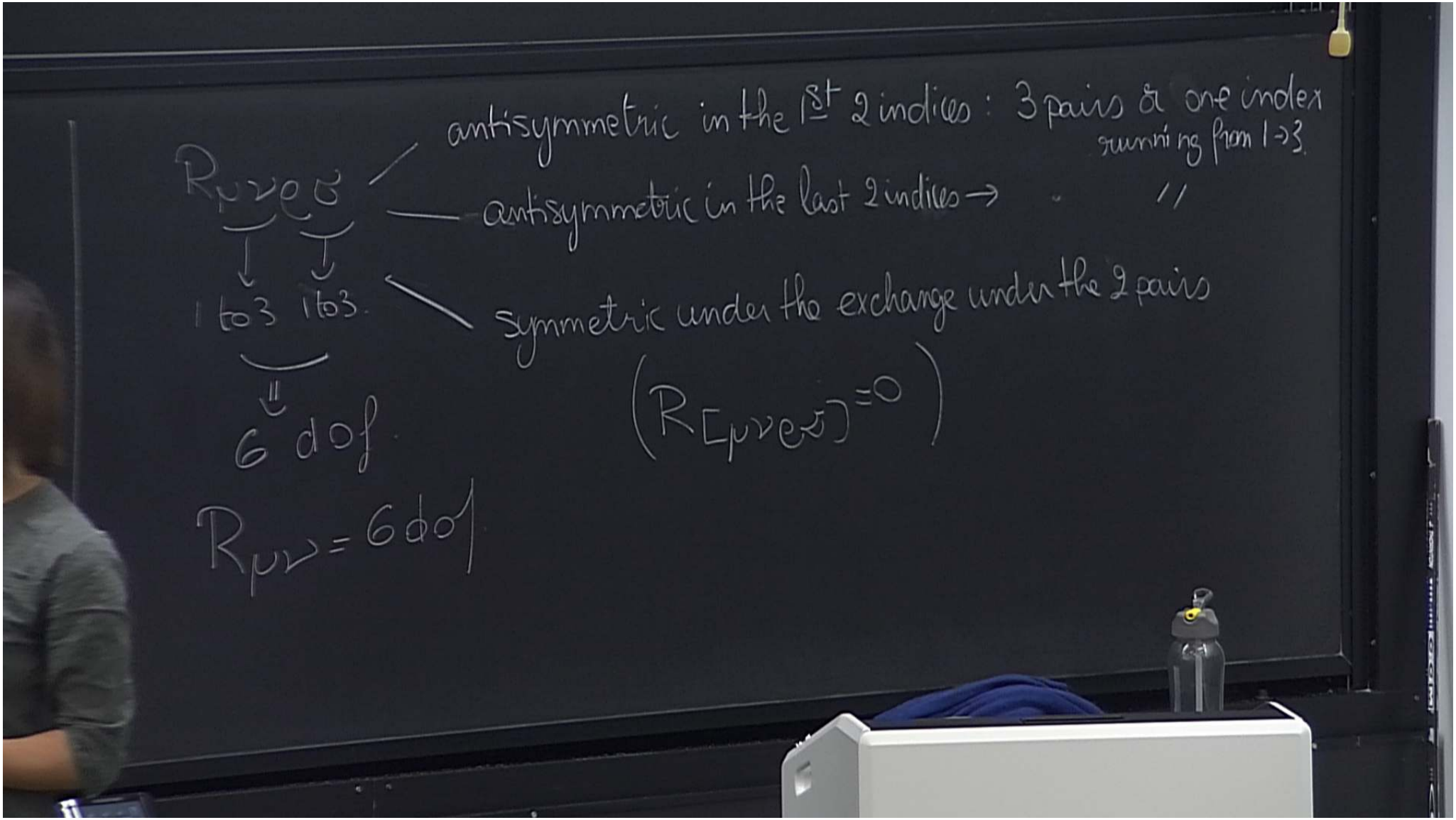

antisymmetric in the 1st 2 indices : 3 pairs of one index running from 1→3
antisymmetric in the last 2 indices →
1 to 3 1 to 3
6 dof
$R_{\mu\nu}$ = 6 dof


$R_{\mu\nu\rho\sigma}$
antisymmetric in the 1st 2 indices : 3 pairs & one index running from 1→3
antisymmetric in the last 2 indices →
1 to 3 1 to 3
symmetric under the exchange under the 2 pairs
6 dof
$(R_{[\mu\nu\rho\sigma]}=0)$
$R_{\mu\nu}$ = 6 dof

⇒ 3D gravity : no local degree of freedom

Global dof depend on the topology of the manifold.

↳ fundamental group.

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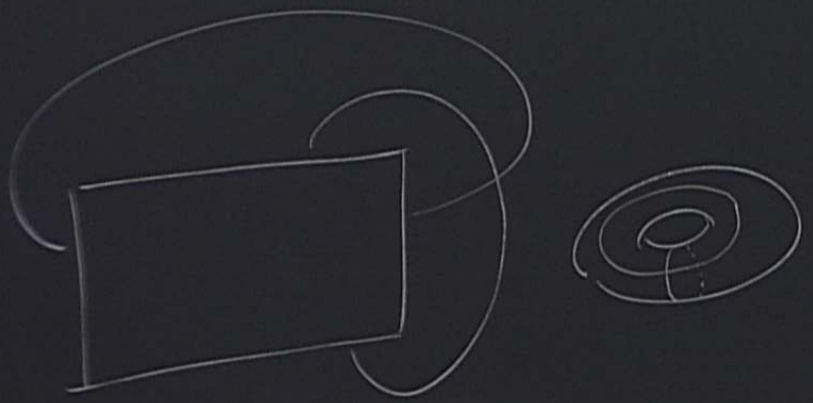
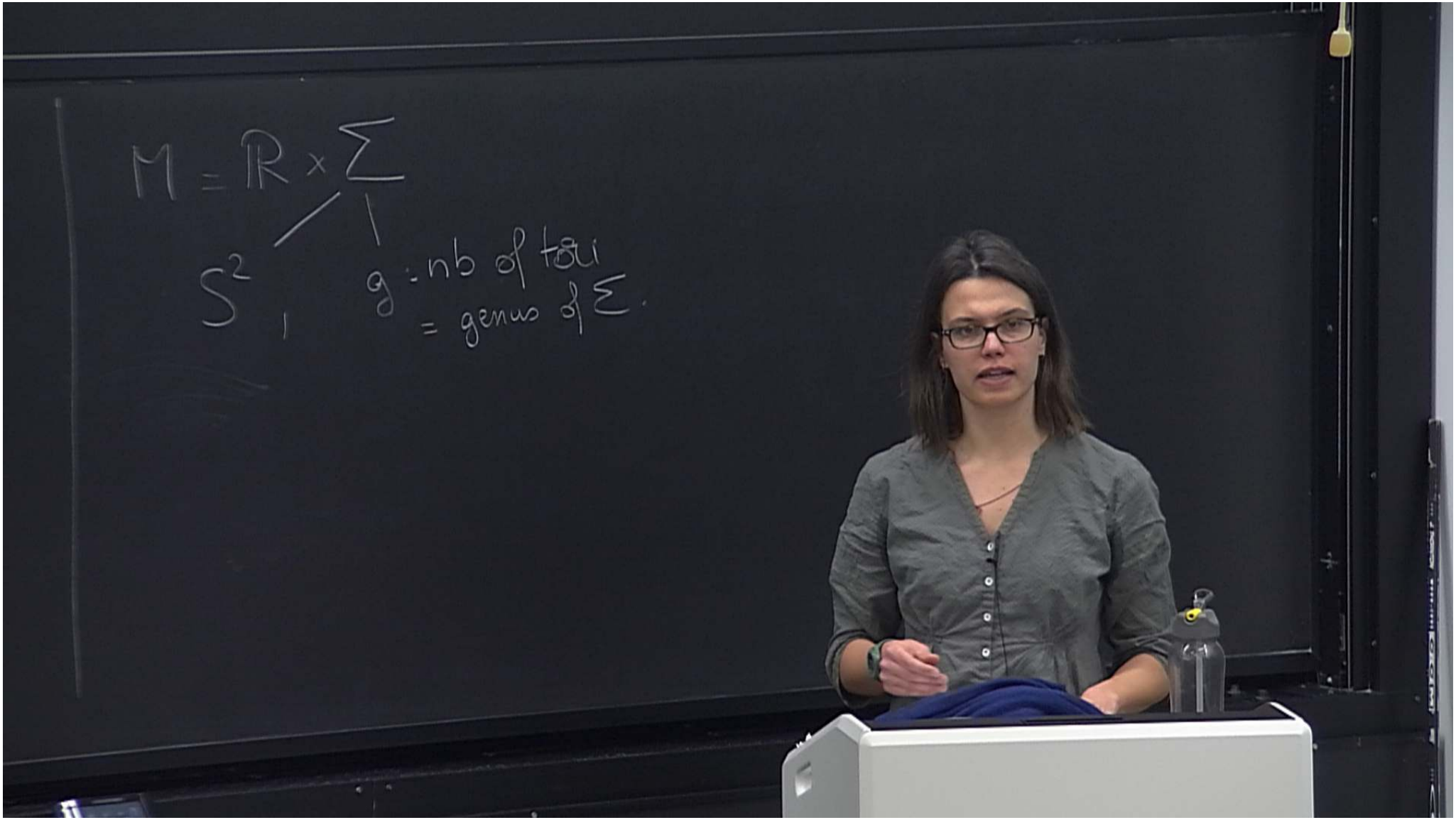
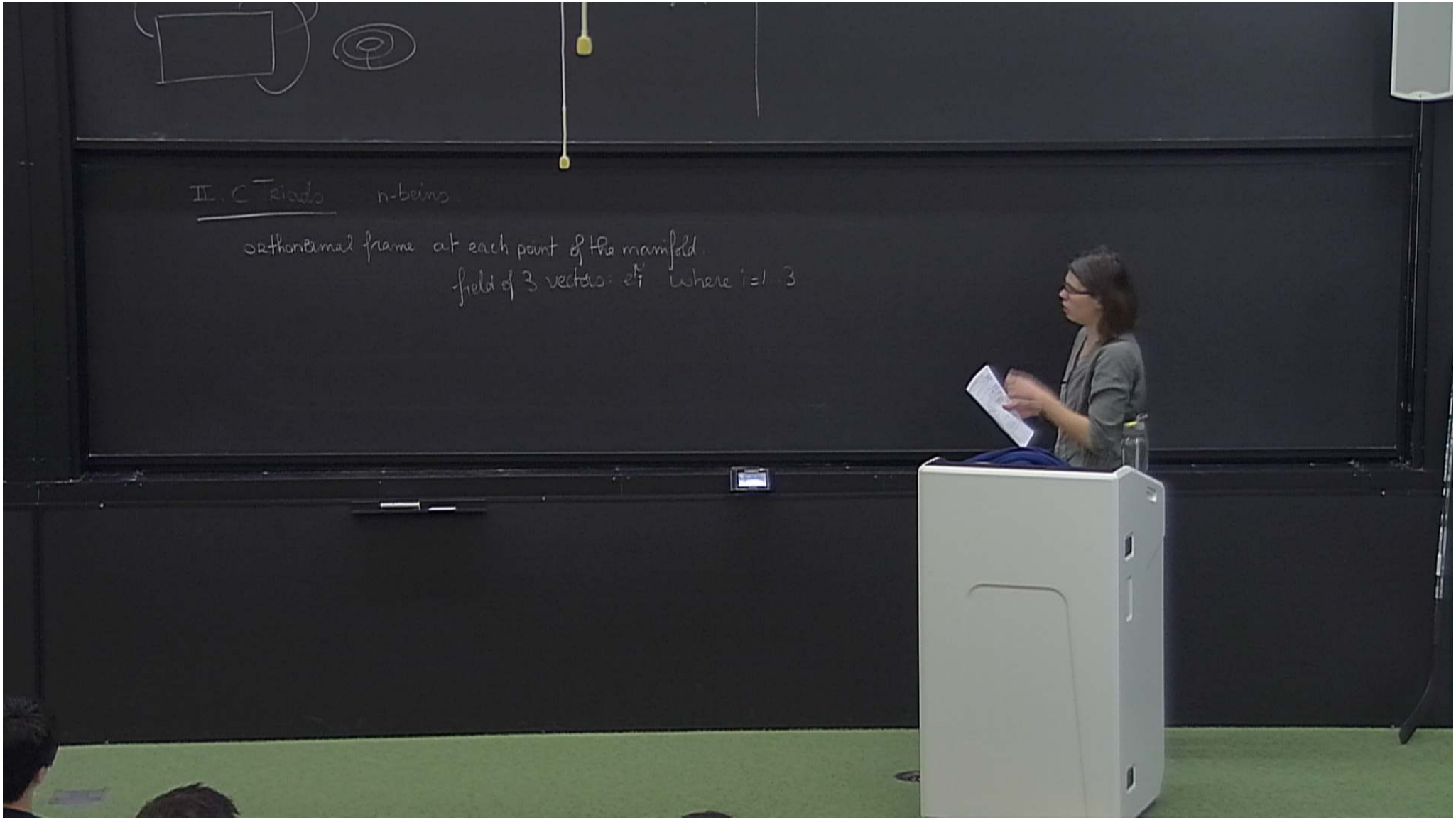



M = R × Σ
S²
g : nb of tori
= genus of Σ.


II. C Triads
n-beins
orthonormal frame at each point of the manifold
field of 3 vectors: e_i^a where i=1 3

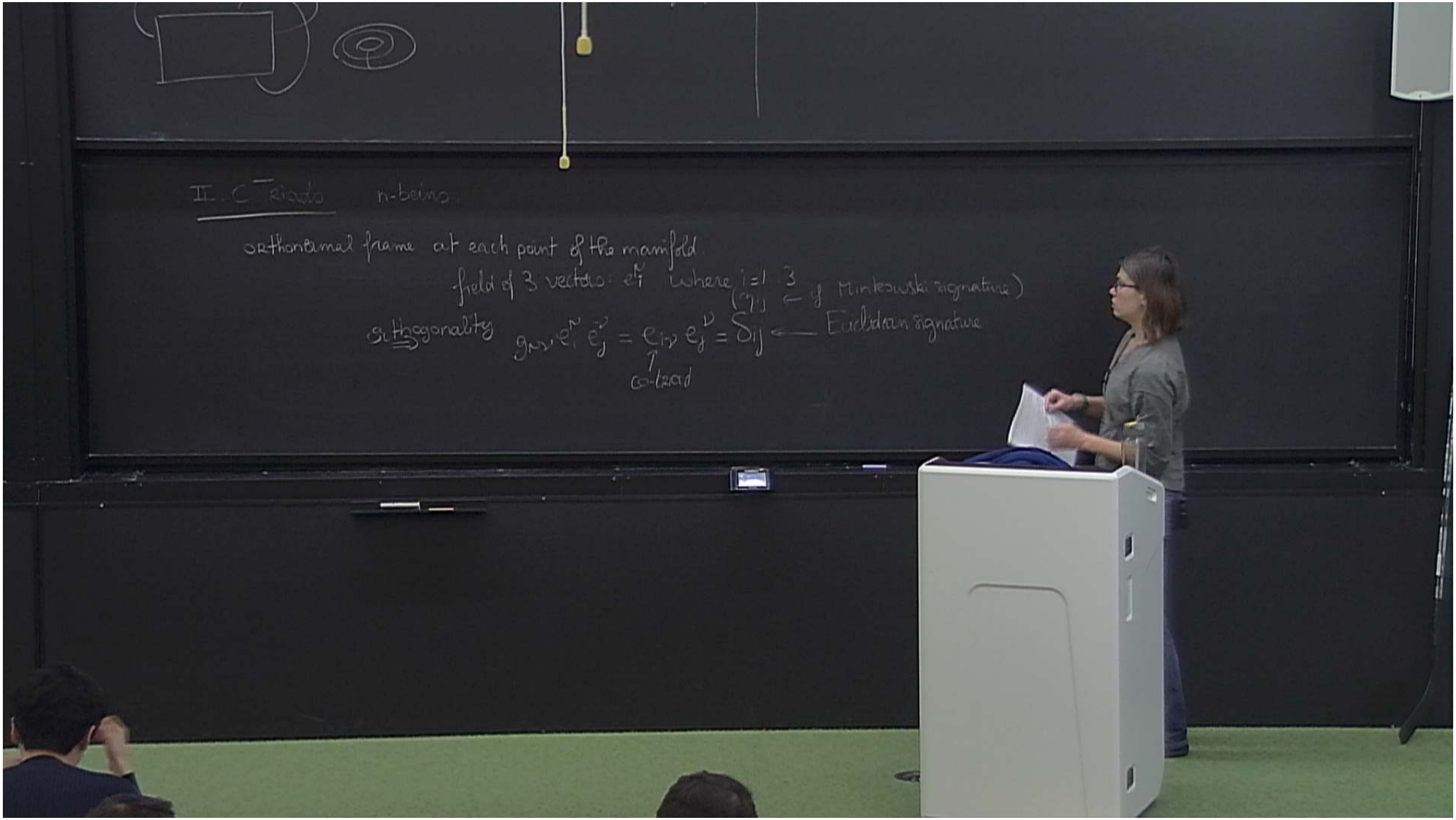
II. C Triads n-beins.

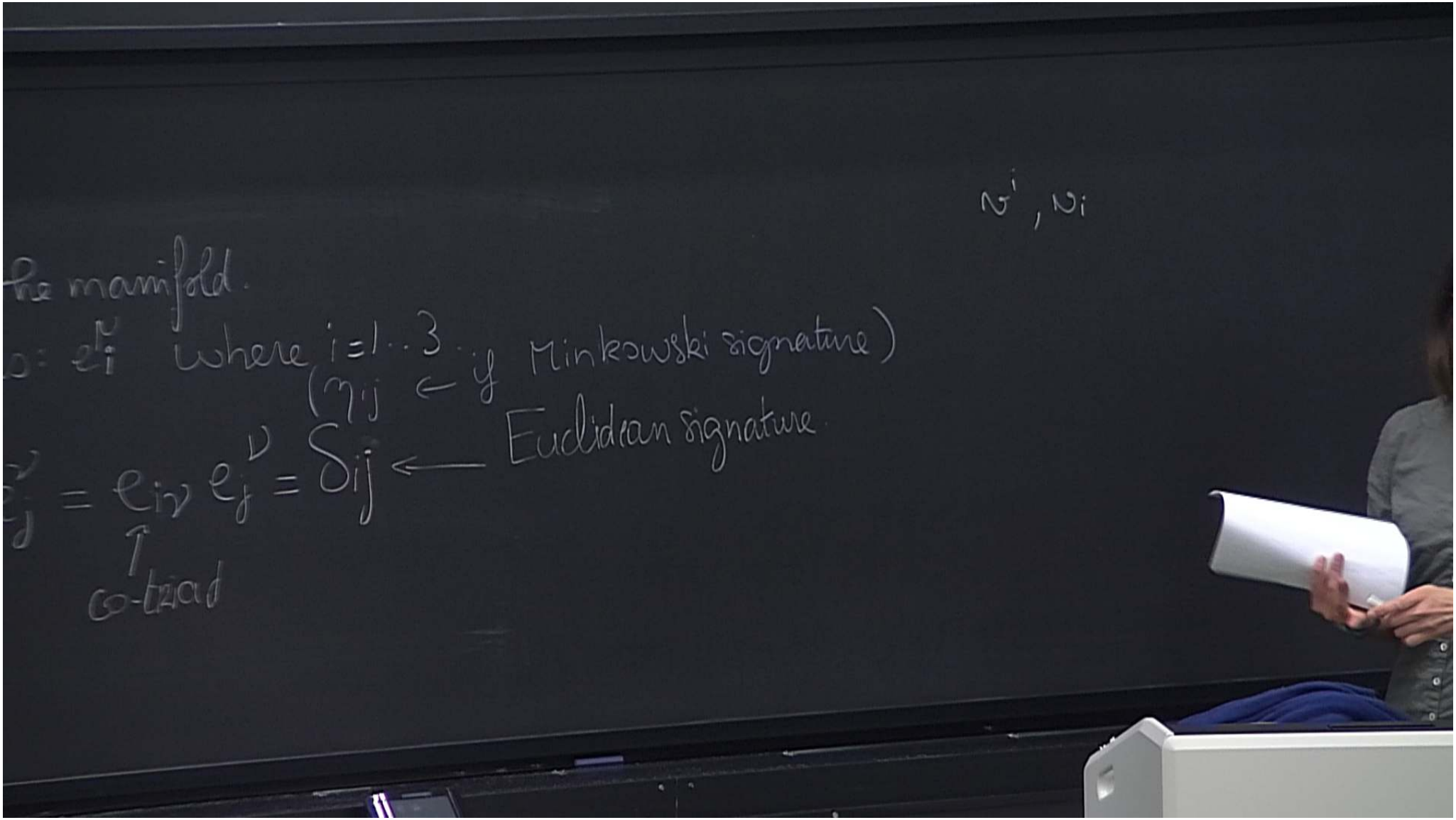
orthonormal frame at each point of the manifold.

field of 3 vectors: e_i^ν where $i = 1..3$.

orthogonality $g_{\mu\nu} e_i^\mu e_j^\nu = e_{i\nu} e_j^\nu = \delta_{ij}$

$e_{i\nu}$: co-triad


II. C Triads n-beins
orthonormal frame at each point of the manifold
field of 3 vectors: e_i^μ where $i=1..3$
(η_{ij} ← if Minkowski signature)
orthogonality $g_{\mu\nu} e_i^\mu e_j^\nu = e_{i\nu} e_j^\nu = \delta_{ij}$ ← Euclidean signature
co-triad

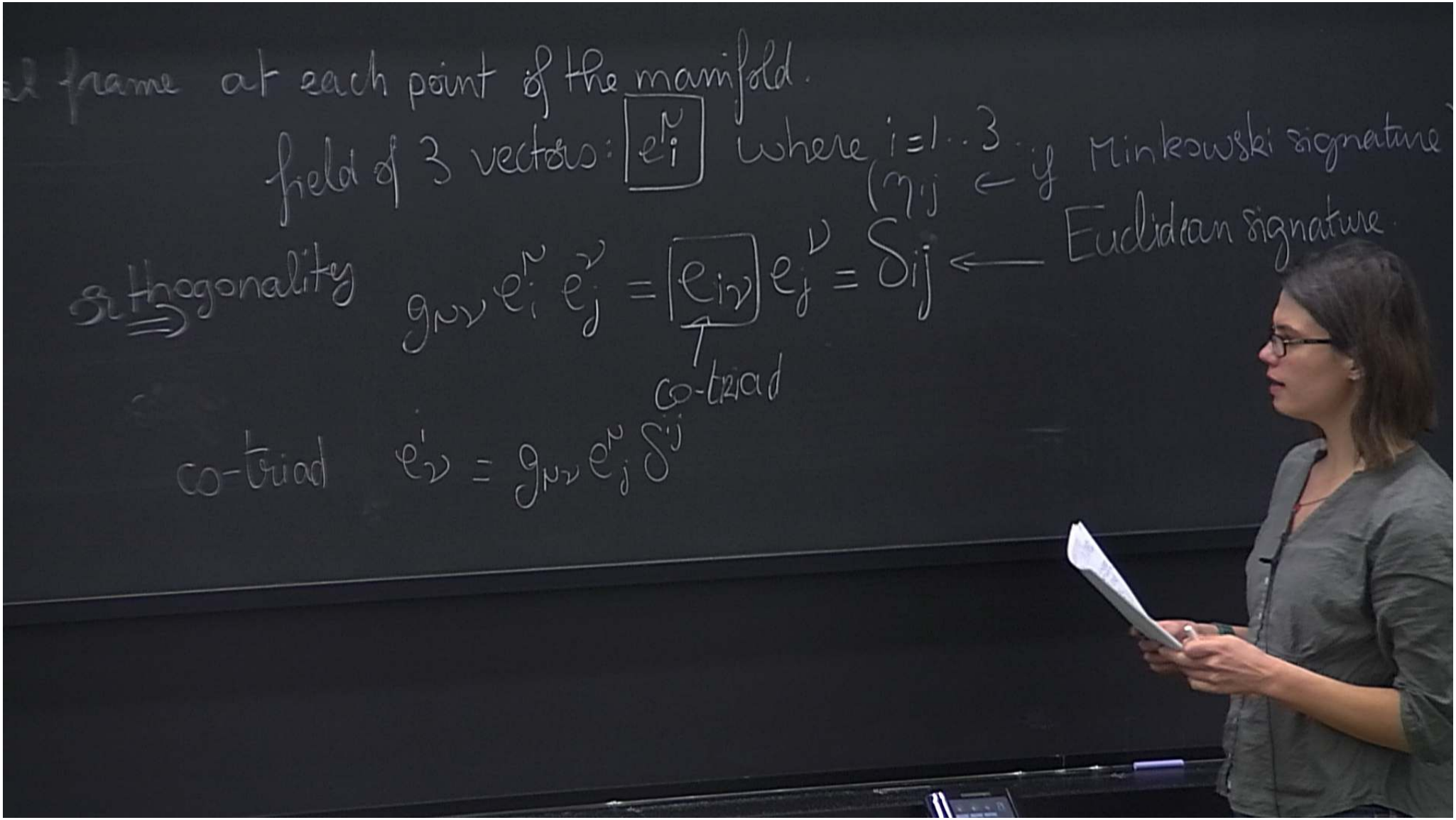

N^i, N_i
the manifold.
e^ν_i where $i=1..3$
(η_{ij} ← if Minkowski signature)
$= e_{i\nu} e^\nu_j = \delta_{ij}$ ← Euclidean signature
co-triad

at each point of the manifold.

field of 3 vectors: $\boxed{e_i^\nu}$ where $i = 1..3$ (η_{ij} ← if Minkowski signature)

gonality $g_{\mu\nu} e_i^\mu e_j^\nu = \boxed{e_{i\nu}} e_j^\nu = \delta_{ij}$ ← Euclidean signature

↑ co-triad


frame at each point of the manifold.
field of 3 vectors: e^ν_i where $i = 1..3$
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co-triad
co-triad $e^i_\nu = g_{\mu\nu} e^\mu_j \delta^{ij}$

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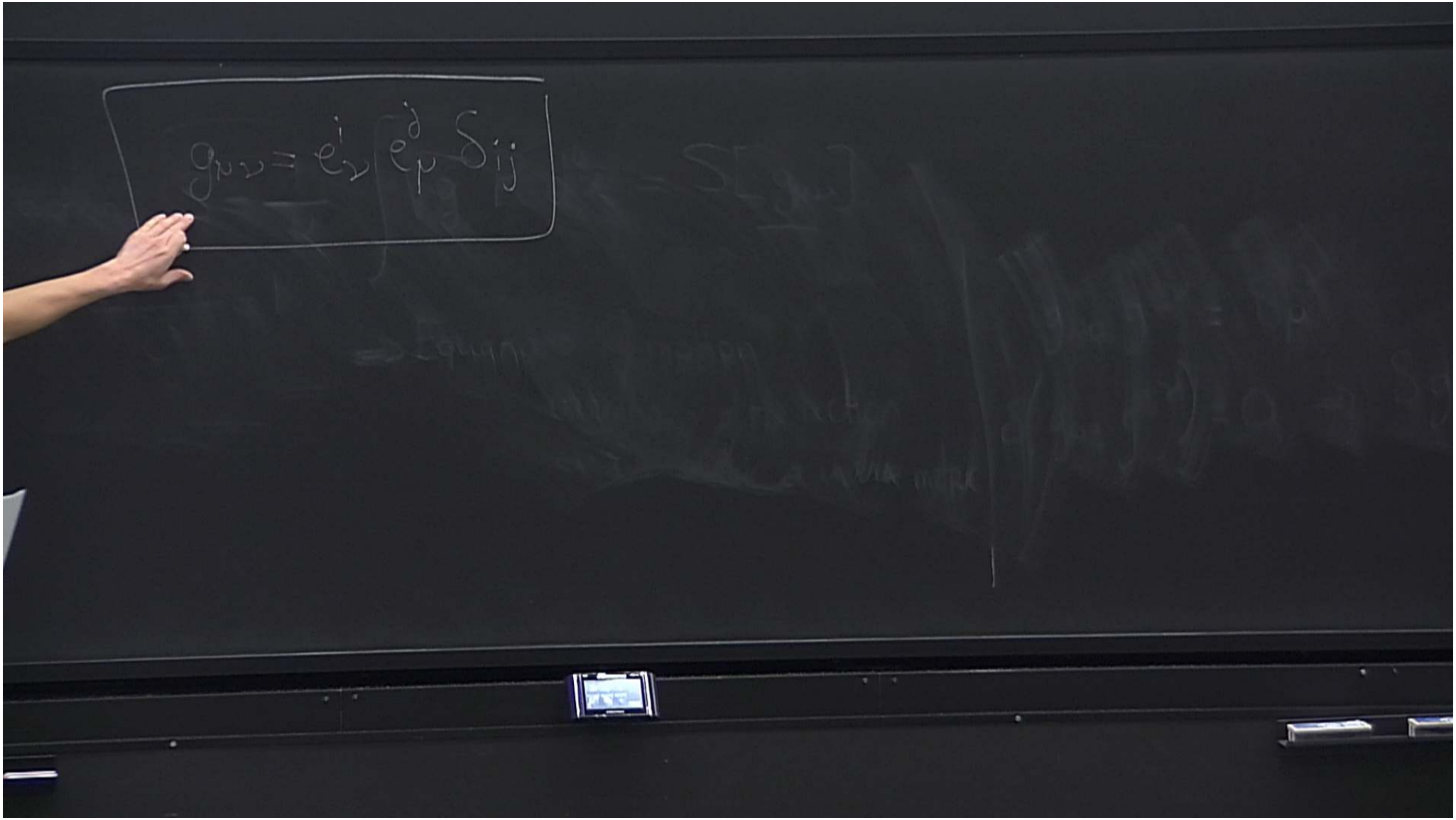
$(\eta_{ij} \leftarrow$ if Minkow

orthogonality $\Rightarrow$ $g_{\mu\nu} e^\mu_i e^\nu_j = \boxed{e_{i\nu}} e^\nu_j = \delta_{ij} \longleftarrow$ Euclidean

co-triad

co-triad $e^i_\nu = g_{\mu\nu} e^\mu_j \delta^{ij}$

contracting by e^ν_k $\Rightarrow$ $g_{\mu\nu} = e^i_\mu e^j_\nu \delta_{ij}$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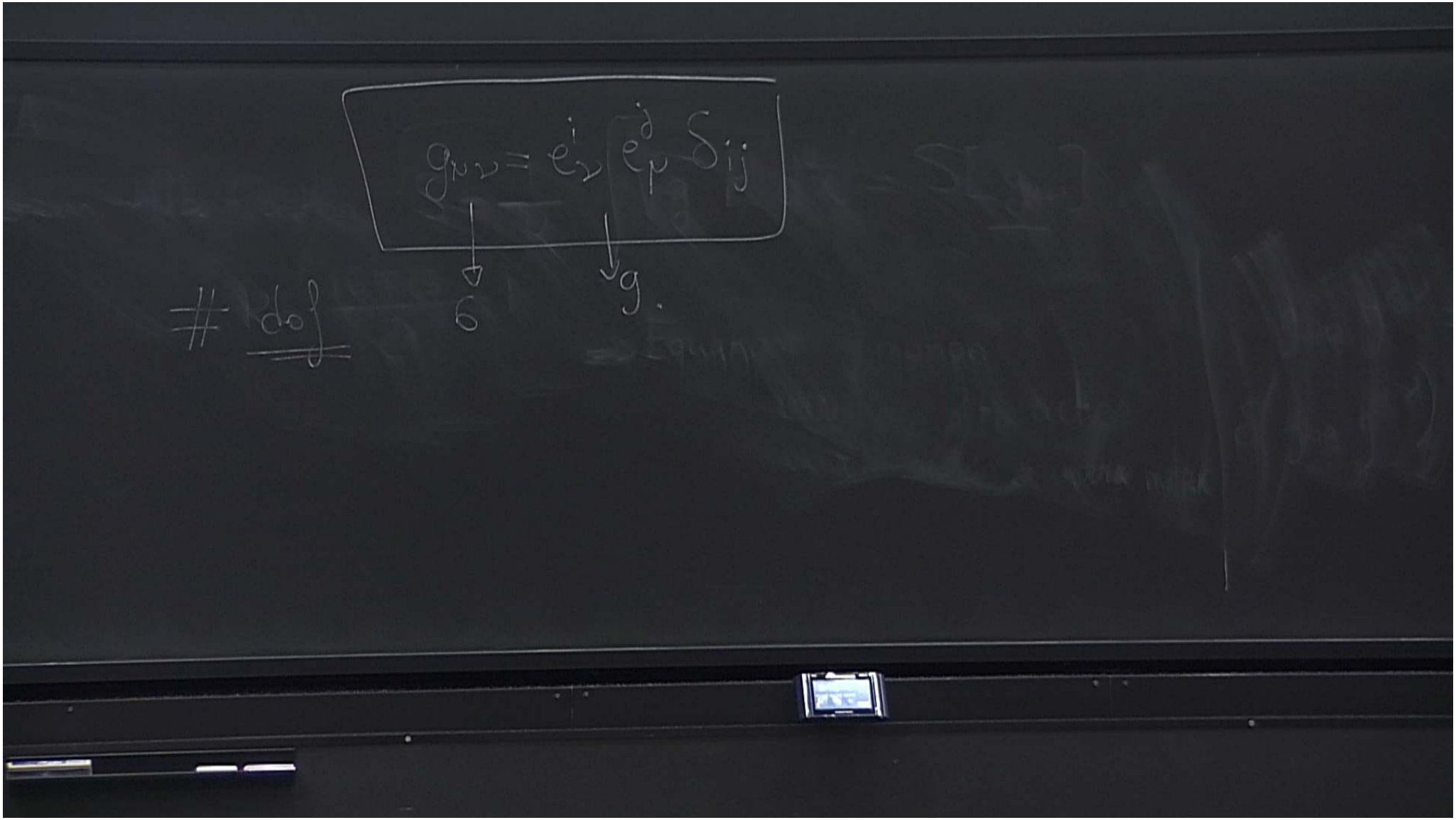
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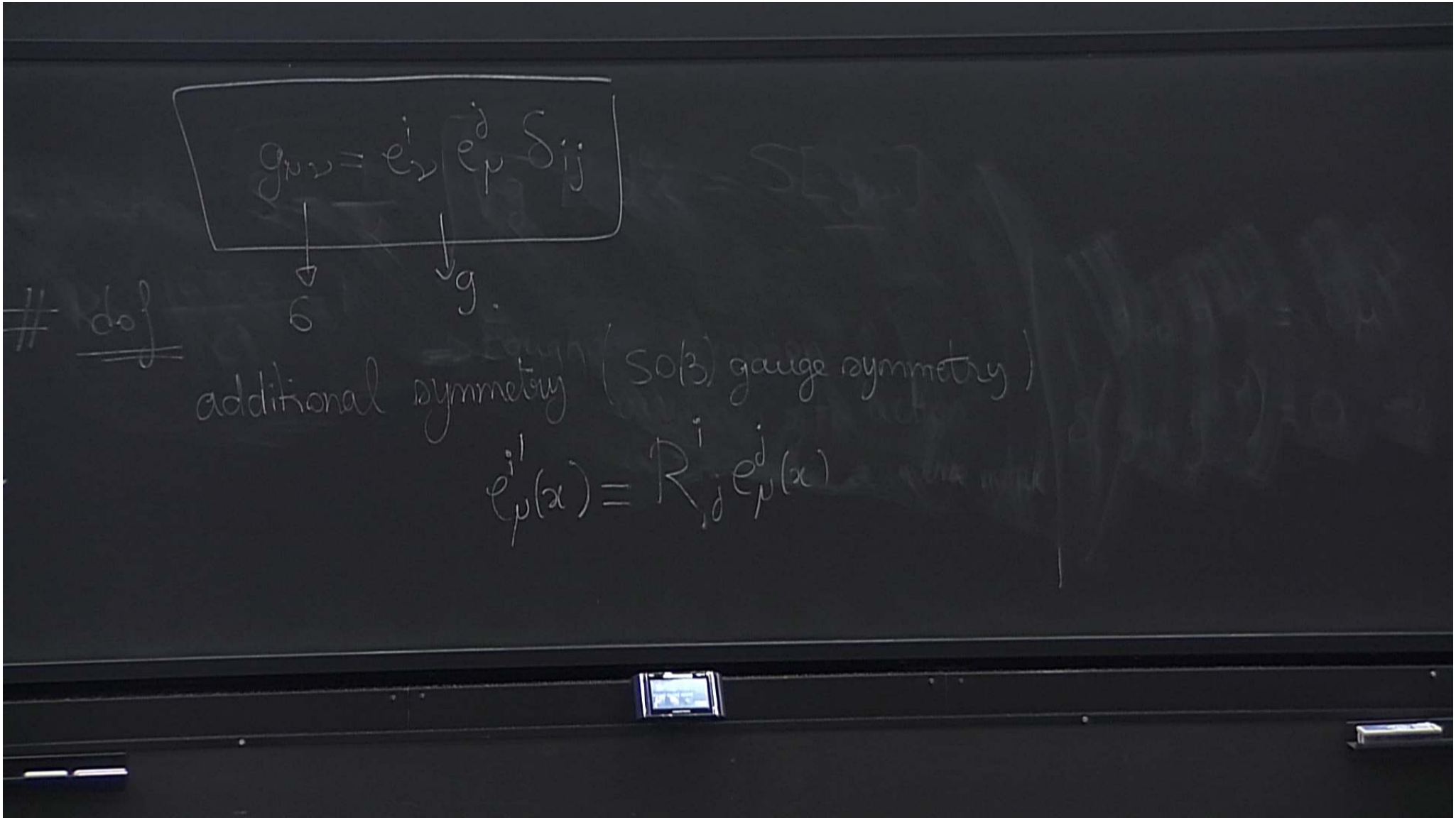
6
g.
dof


$g_{\mu\nu} = e^i_\mu e^j_\nu \delta_{ij}$
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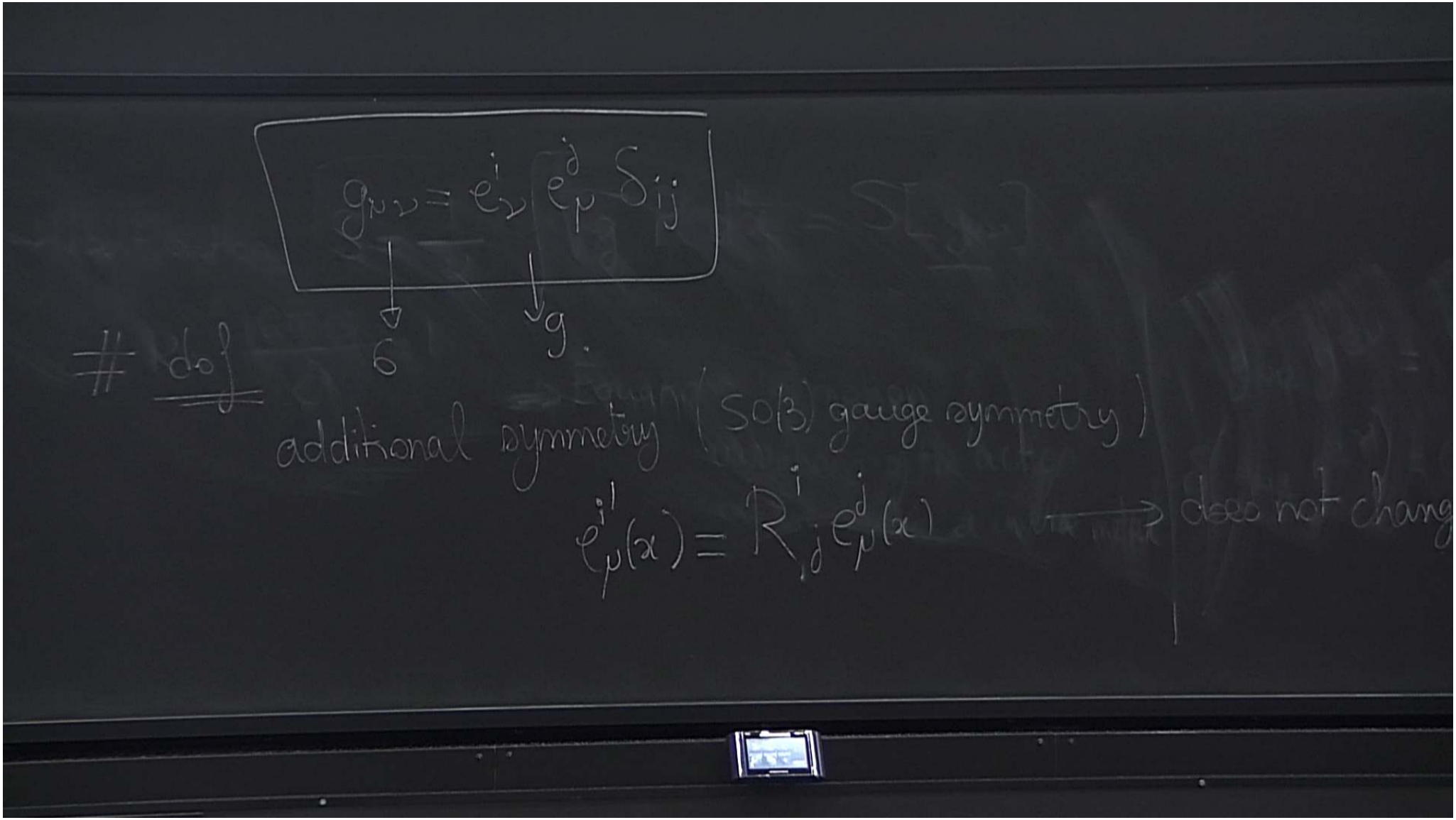
dof
additional symmetry (SO(3) gauge symmetry)
$e'^i_\mu(x) = R^i_j e^j_\mu(x)$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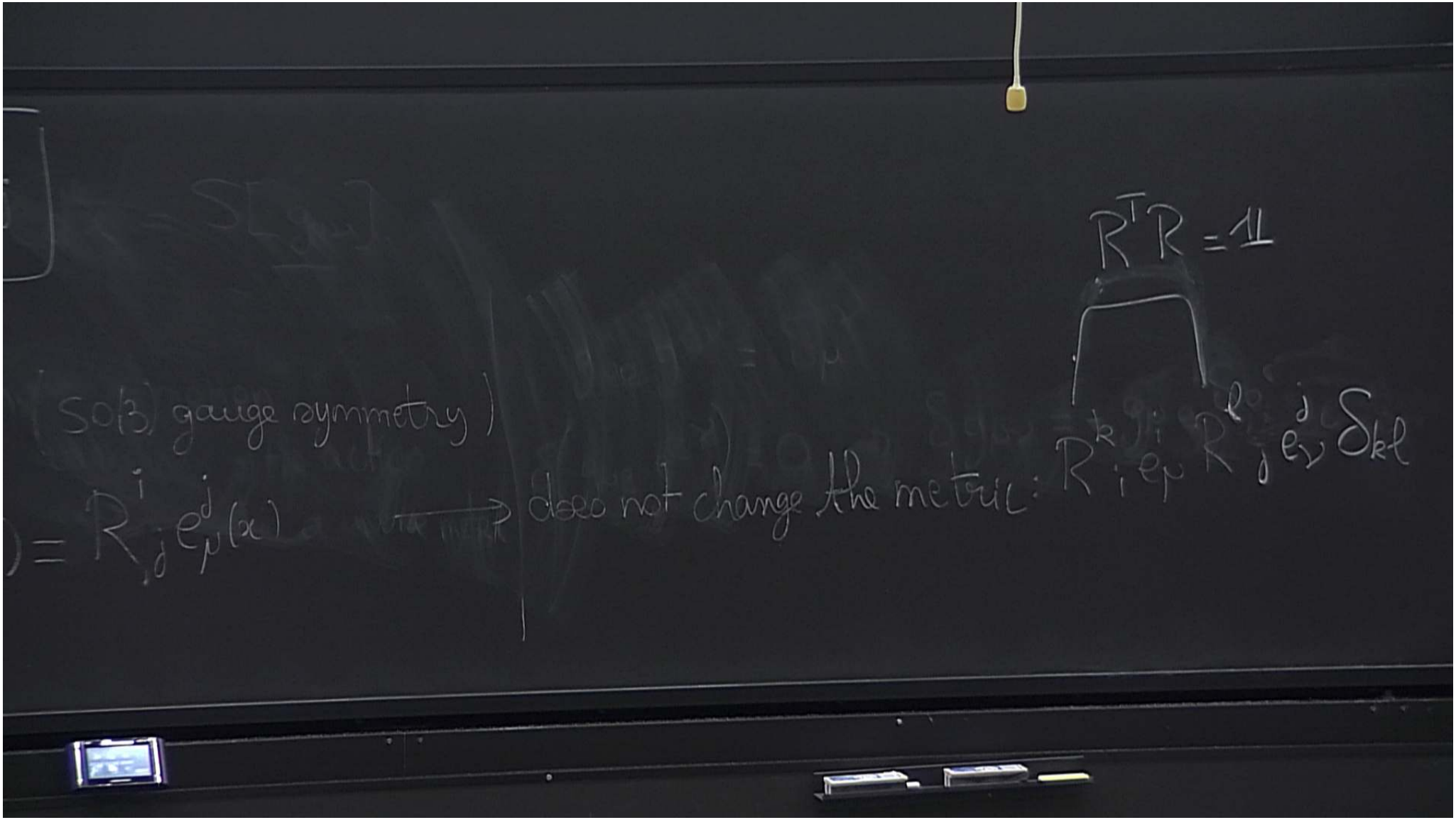
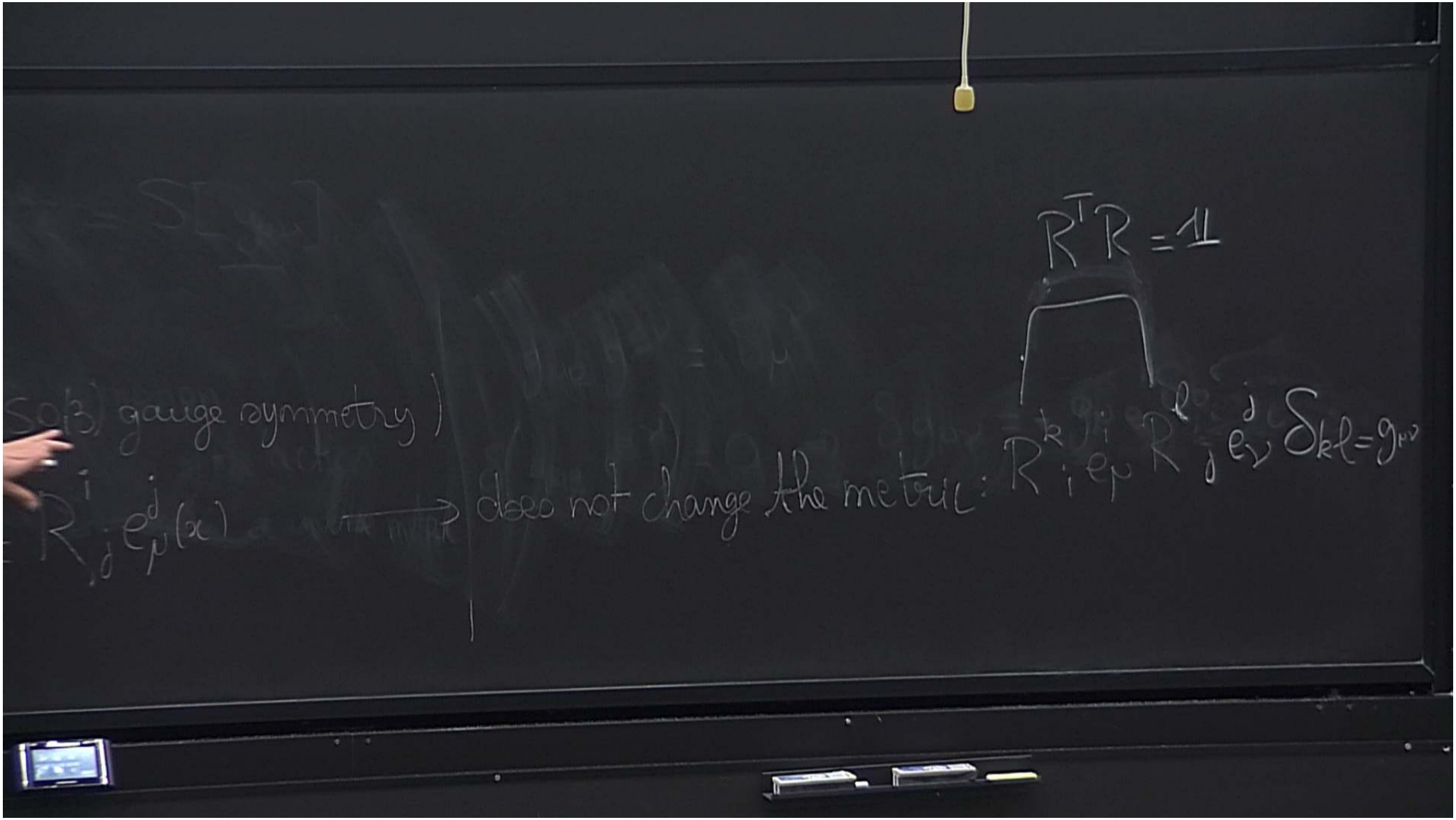
does not chang


$R^T R = 1\!\!1$
(SO(3) gauge symmetry)
$= R^i_j e^d_\mu(x)$
does not change the metric: $R^k_i e_\mu R^l_j e_\nu \delta_{kl}$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SO(3) gauge symmetry)
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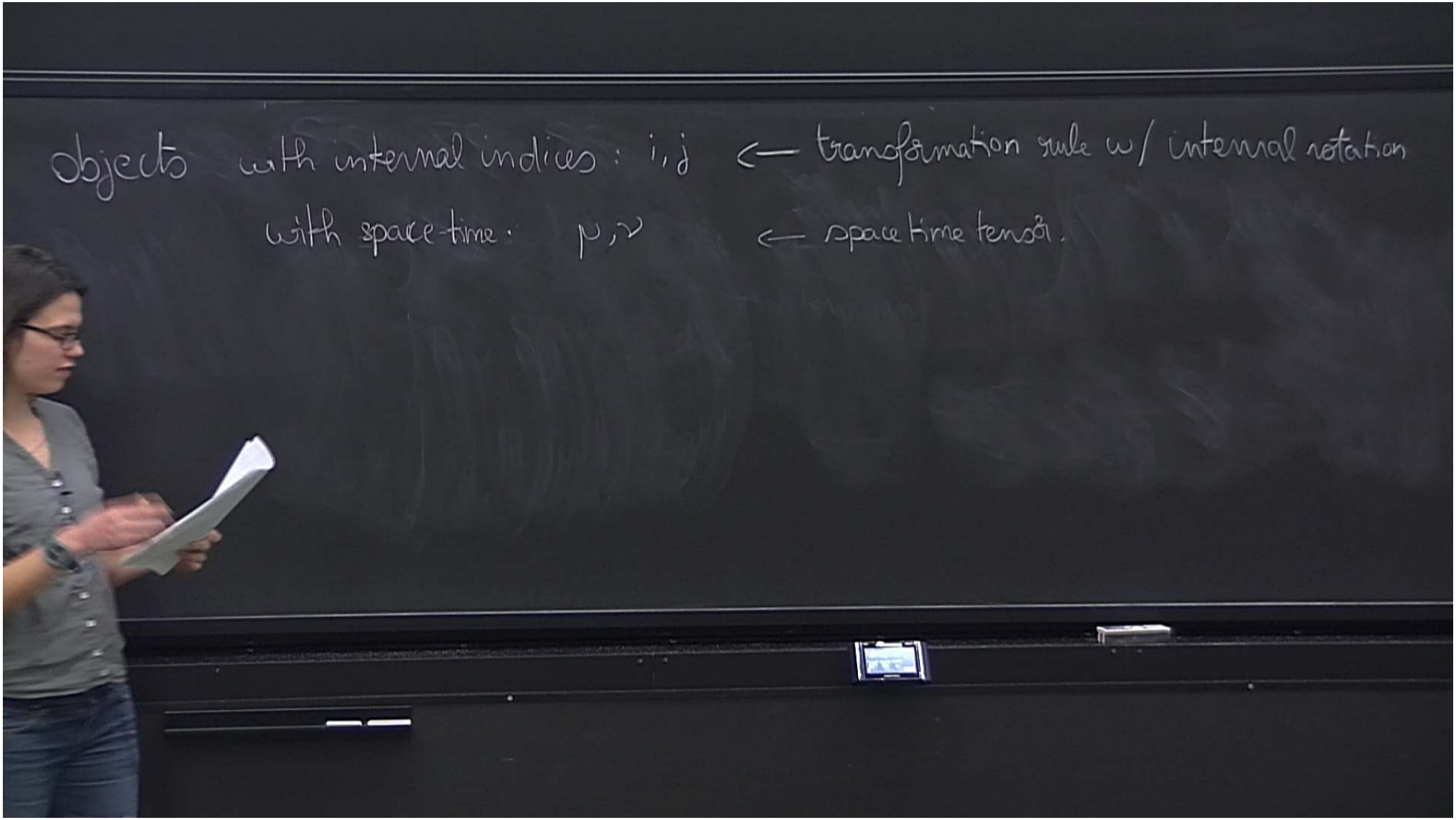
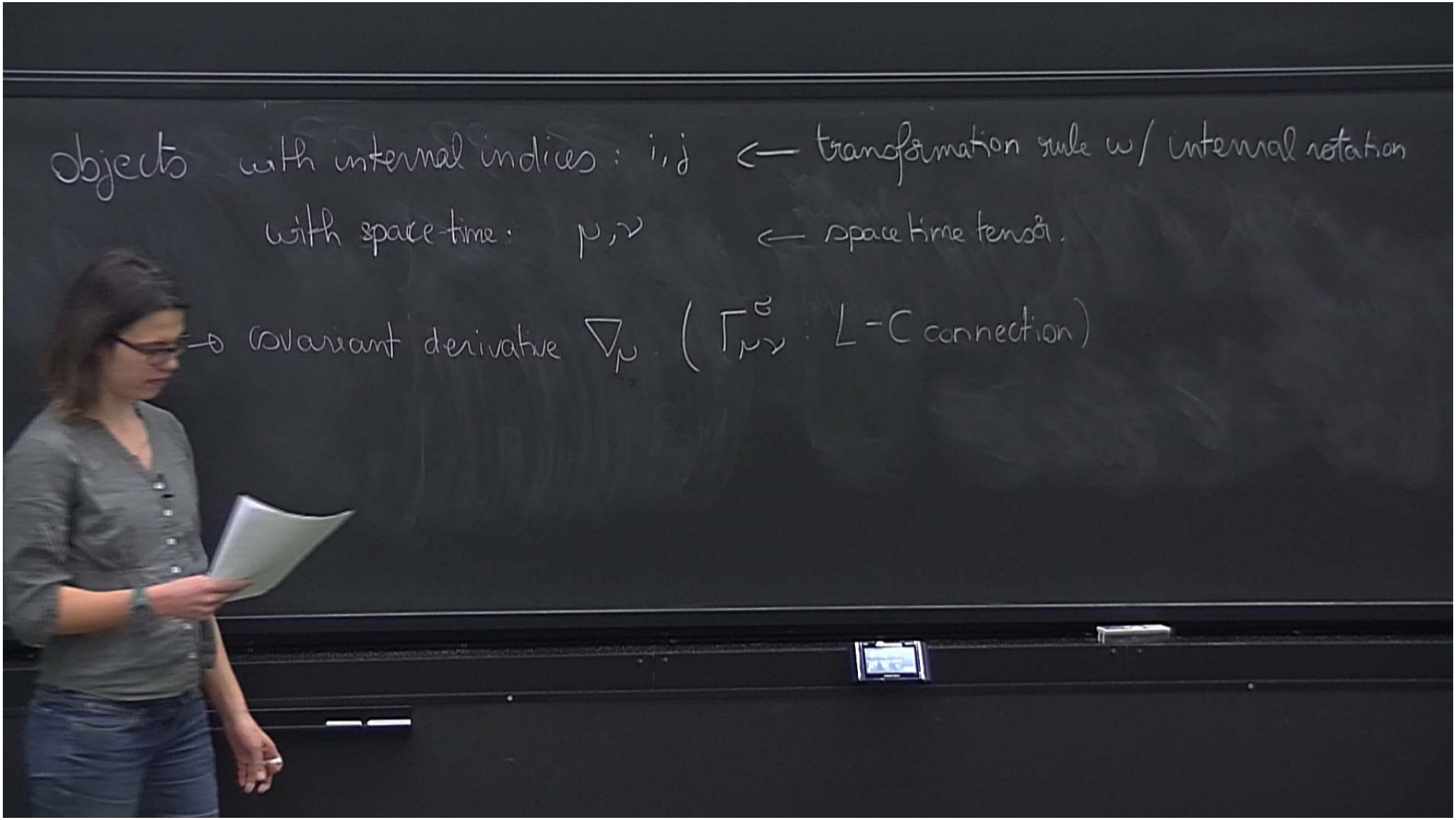

objects with internal indices: i, j ← transformation rule w/ internal rotation
with space-time: μ, ν ← space time tensor.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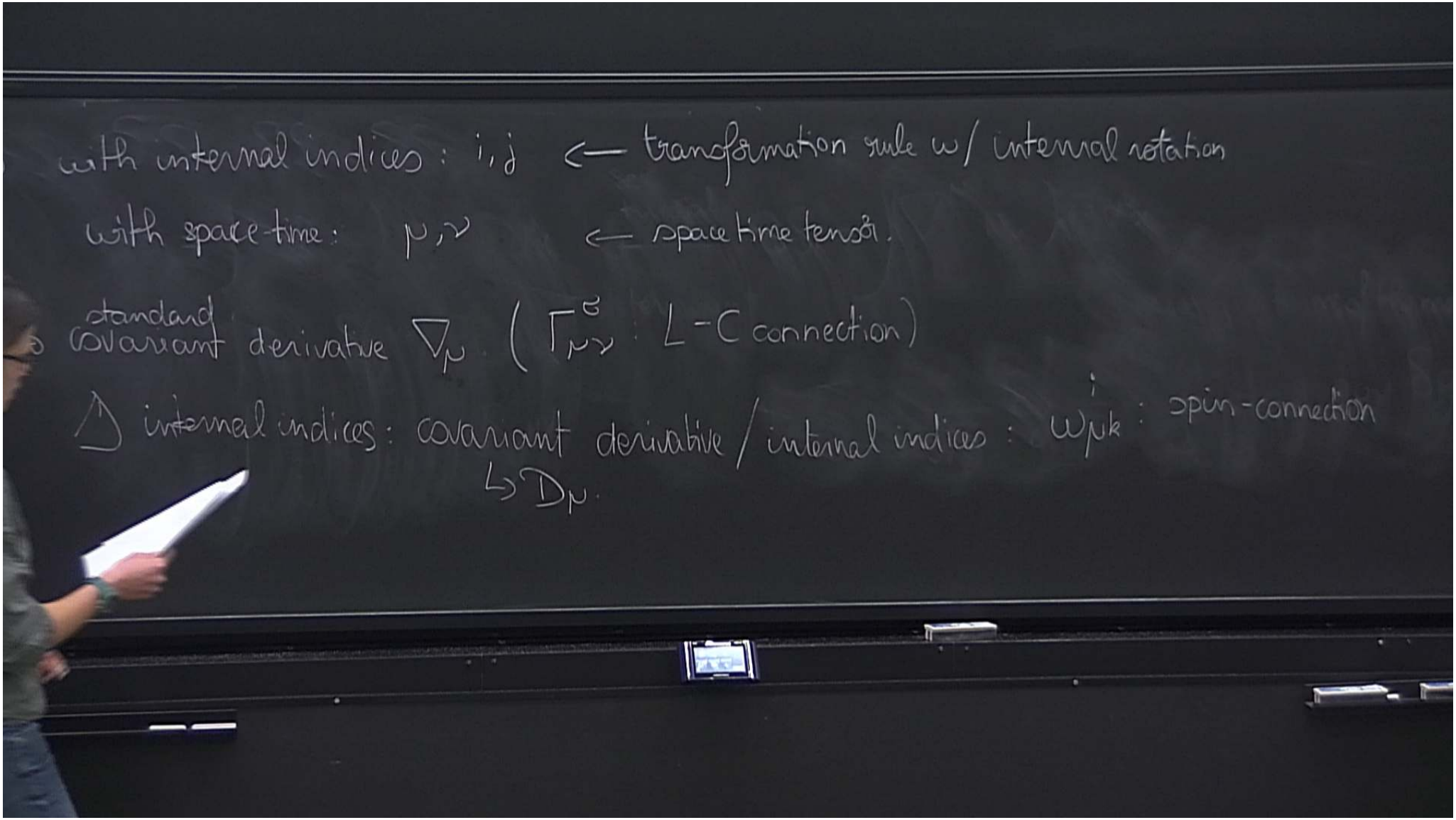
→ covariant derivative ∇_μ ($\Gamma^{\sigma}_{\mu\nu}$ L-C connection)

with internal indices: i, j ← transformation rule w/ internal rotation

with space-time: μ, ν ← space time tensor.

→ standard covariant derivative ∇_μ ($\overset{\circ}{\Gamma}{}^{\lambda}_{\mu\nu}$: L-C connection)

△ internal indices: covariant derivative / internal indices: $\omega^{i}_{\mu k}$: spin-connection


with internal indices: i, j ← transformation rule w/ internal rotation
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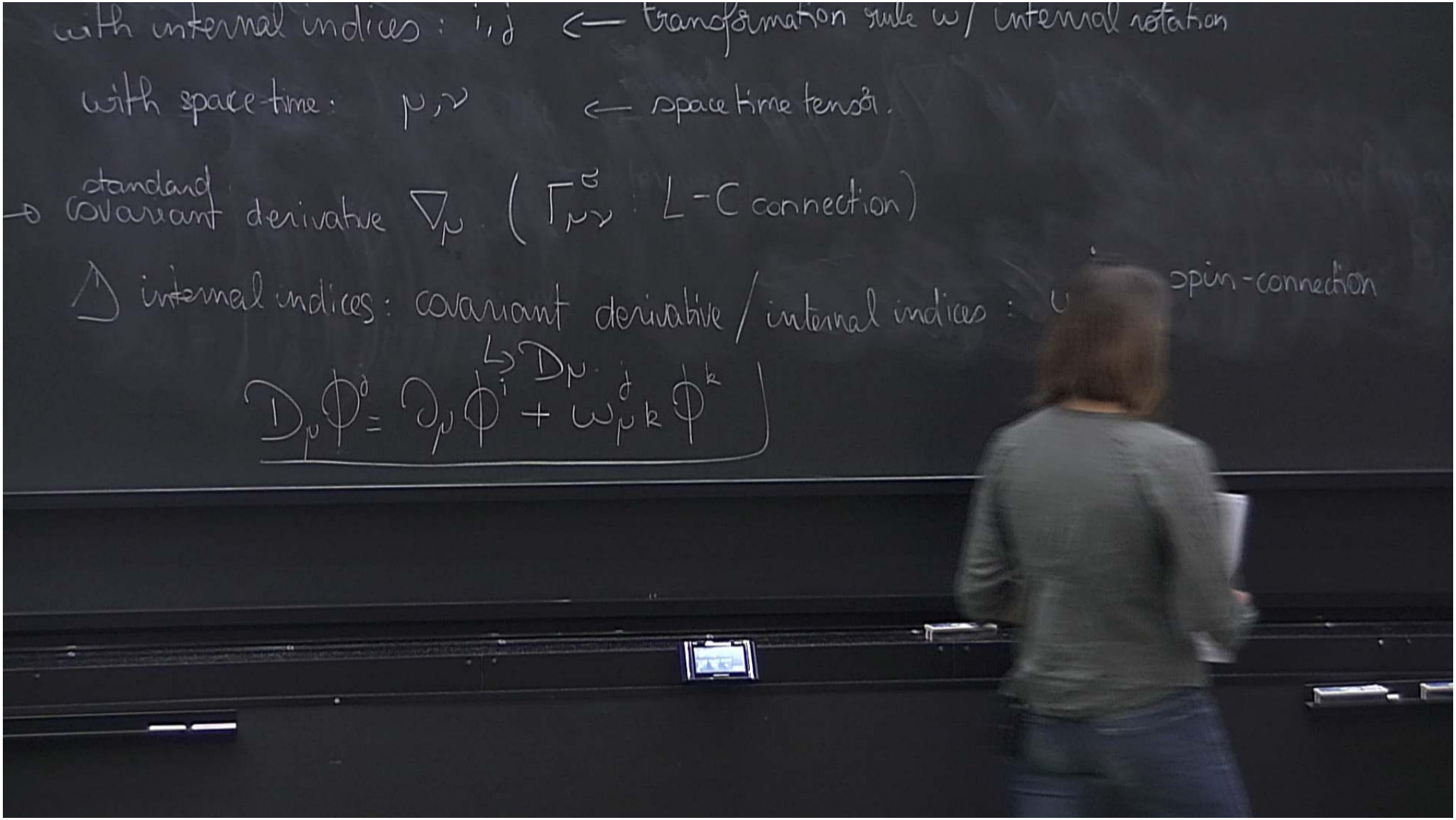
↳ D_μ.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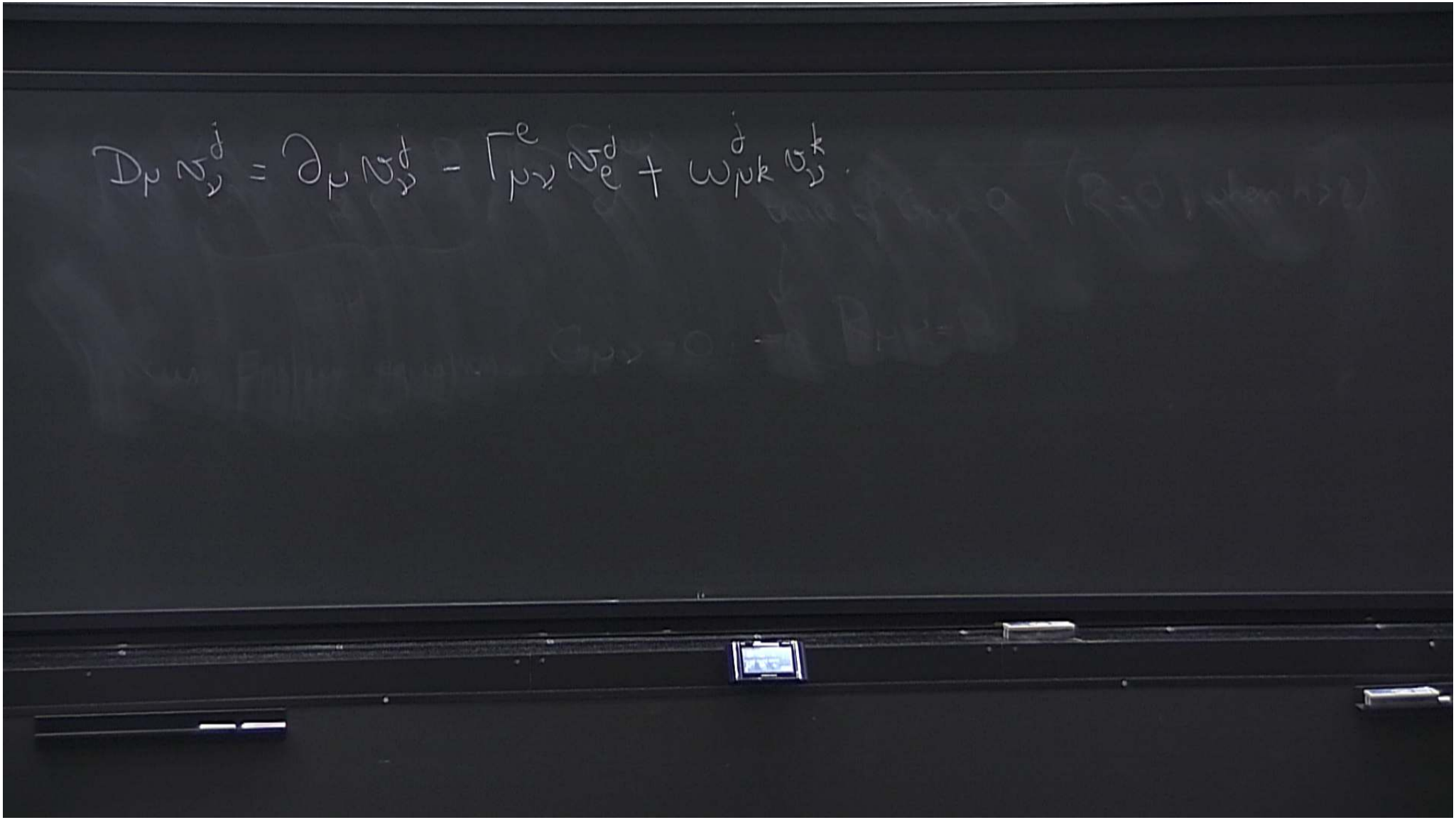
$D_\mu \phi^i = \partial_\mu \phi^i + \omega^i_{\mu k} \phi^k$
D_μ


$D_\mu V^a_\nu = \partial_\mu V^a_\nu - \Gamma^\rho_{\mu\nu} V^a_\rho + \omega^a_{\mu k} V^k_\nu$

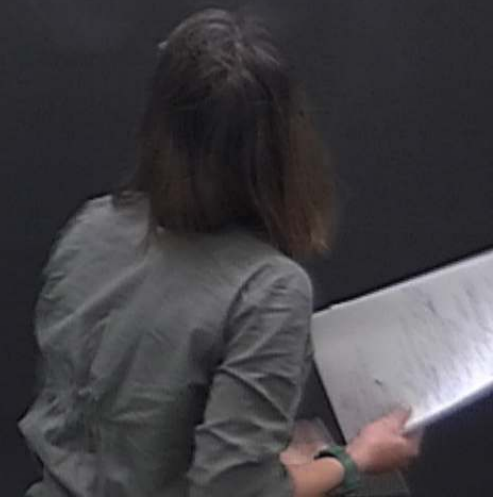
△ internal indices: covariant derivative / internal indices: $\omega^{i}_{\mu k}$: spin-connection

$$D_\mu \phi^i = \partial_\mu \phi^i + \omega^i_{\mu k} \phi^k$$

↳ D_μ

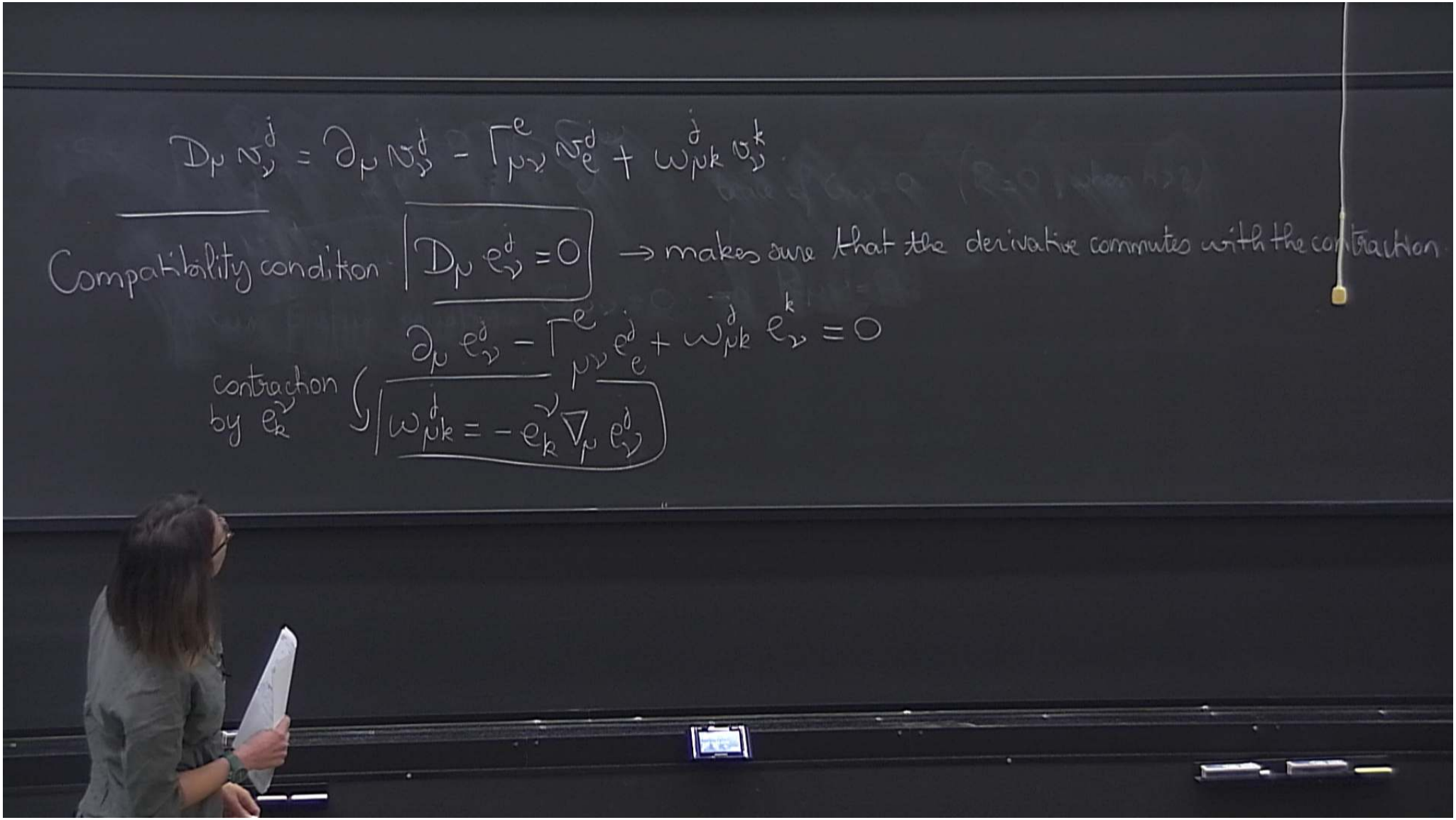
$\rightarrow \nabla_\mu \phi^i = \partial_\mu \phi$

$$D_\mu V^i_\nu = \partial_\mu V^i_\nu - \Gamma^e_{\mu\nu} V^i_e + \omega^i_{\mu k} V^k_\nu$$



$$D_\mu V^a_\nu = \partial_\mu V^a_\nu - \Gamma^\rho_{\mu\nu} V^a_\rho + \omega^a_{\mu k} V^k_\nu$$

Compatibility condition $D_\mu e^a_\nu = 0$


$D_\mu V^a_\nu = \partial_\mu V^a_\nu - \Gamma^e_{\mu\nu} V^d_e + \omega^a_{\mu k} V^k_\nu$
Compatibility condition $D_\mu e^a_\nu = 0$ → makes sure that the derivative commutes with the contraction
$\partial_\mu e^a_\nu - \Gamma^e_{\mu\nu} e^d_e + \omega^a_{\mu k} e^k_\nu = 0$
contraction by e^ν_k
$\omega^a_{\mu k} = -e^\nu_k \nabla_\mu e^a_\nu$

$$D_\mu V^i_\nu = \partial_\mu V^i_\nu - \Gamma^e_{\mu\nu} V^i_e + \omega^i_{\mu k} V^k_\nu$$

Compatibility condition $\boxed{D_\mu e^i_\nu = 0}$ → makes sure that the derivative commutes with the contraction

$$\partial_\mu e^i_\nu - \Gamma^e_{\mu\nu} e^i_e + \omega^i_{\mu k} e^k_\nu = 0$$

contraction by e^ν_k ↓

$$\boxed{\omega^i_{\mu k} = -e^\nu_k \nabla_\mu e^i_\nu} \quad (\Gamma)$$

$$\nabla_\mu V^\alpha_\nu = \partial_\mu V^\alpha_\nu - \Gamma^\rho_{\mu\nu} V^\alpha_\rho + \omega^\alpha_{\mu k} V^k_\nu$$

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$$\partial_\mu e^\alpha_\nu - \Gamma^\rho_{\mu\nu} e^\alpha_\rho + \omega^\alpha_{\mu k} e^k_\nu = 0$$

contraction by e^ν_k (Γ

$$\boxed{\omega^\alpha_{\mu k} = -e^\nu_k \nabla_\mu e^\alpha_\nu} \quad (\Gamma)$$