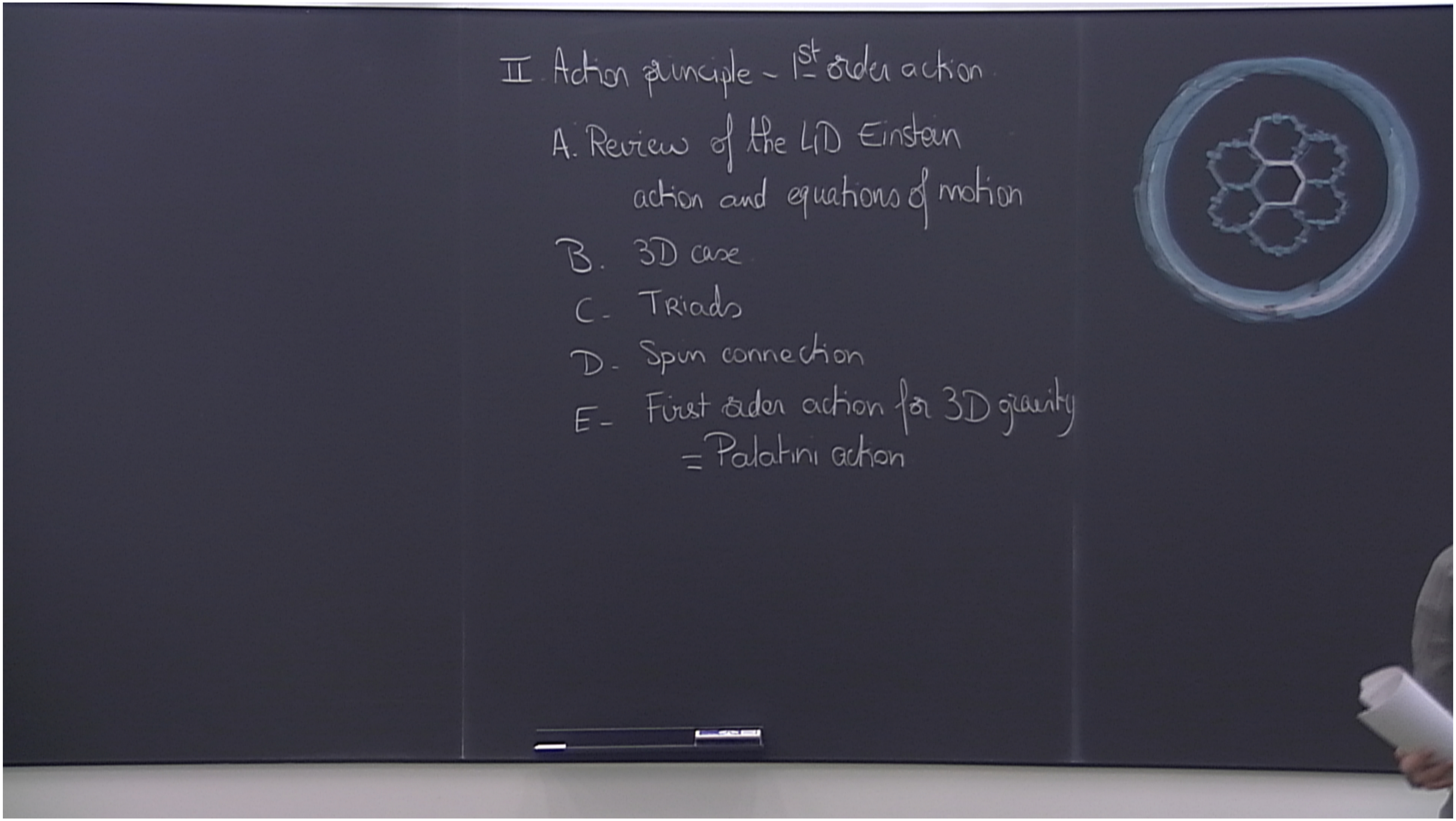
Title: PSI 2015/2016 Quantum Gravity - Lecture 2

Date: Feb 23, 2016 10:15 AM

URL: http://pirsa.org/16020073

Abstract:



II. Action principle - 1^{st} order action

A. Review of the 4D Einstein action and equations of motion

B. 3D case

C. Triads

D. Spin connection

E. First order action for 3D gravity = Palatini action

II. A.

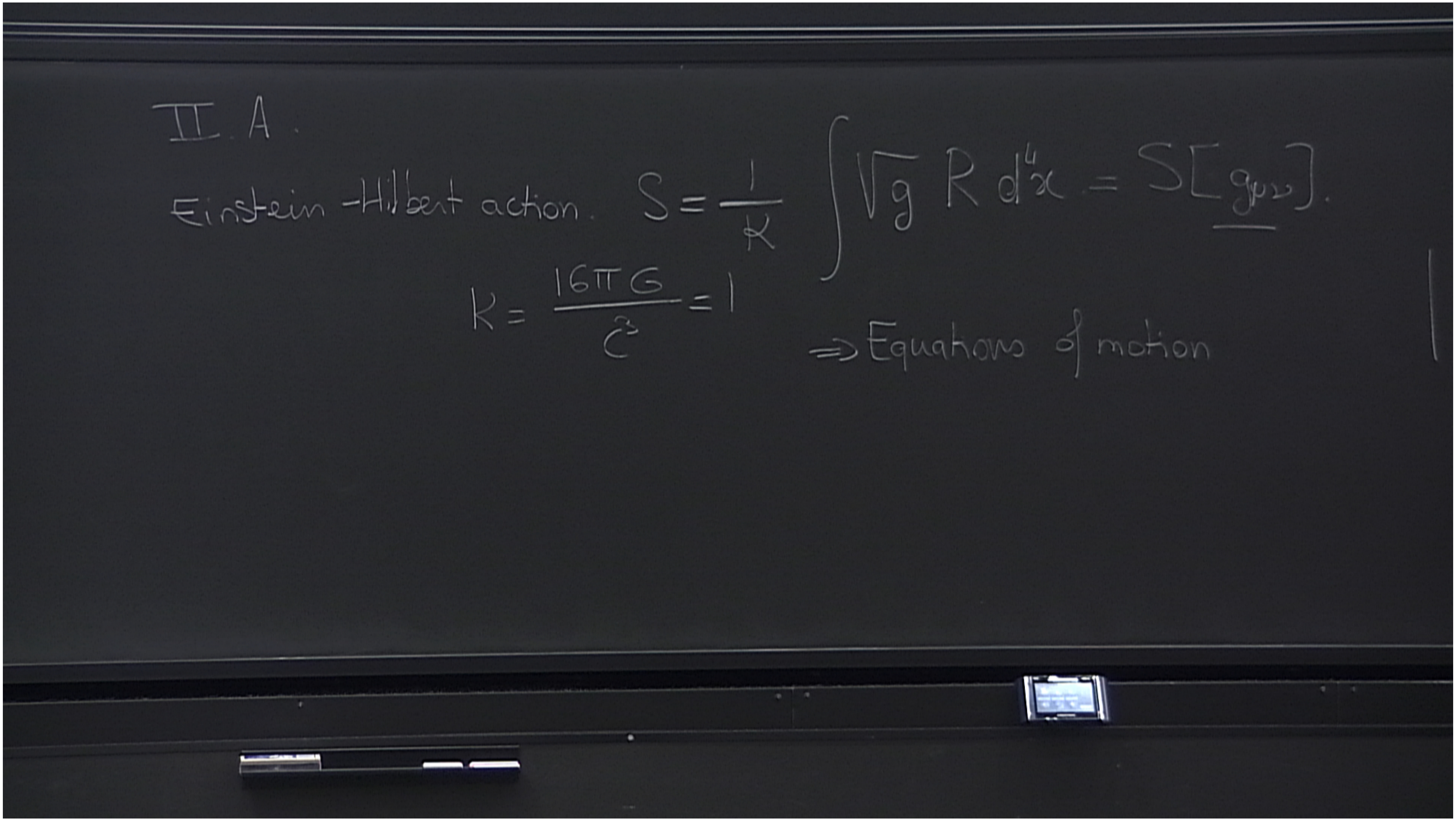
Einstein-Hilbert action. $S = \frac{1}{K} \int \sqrt{g}\, R\, d^4x$.

$$K = \frac{16\pi G}{c^3} = 1$$

II. A.

Einstein-Hilbert action. $S = \frac{1}{\kappa} \int \sqrt{g}\, R\, d^4x = S[\underline{g_{\mu\nu}}]$.

$$\kappa = \frac{16\pi G}{c^3} = 1$$

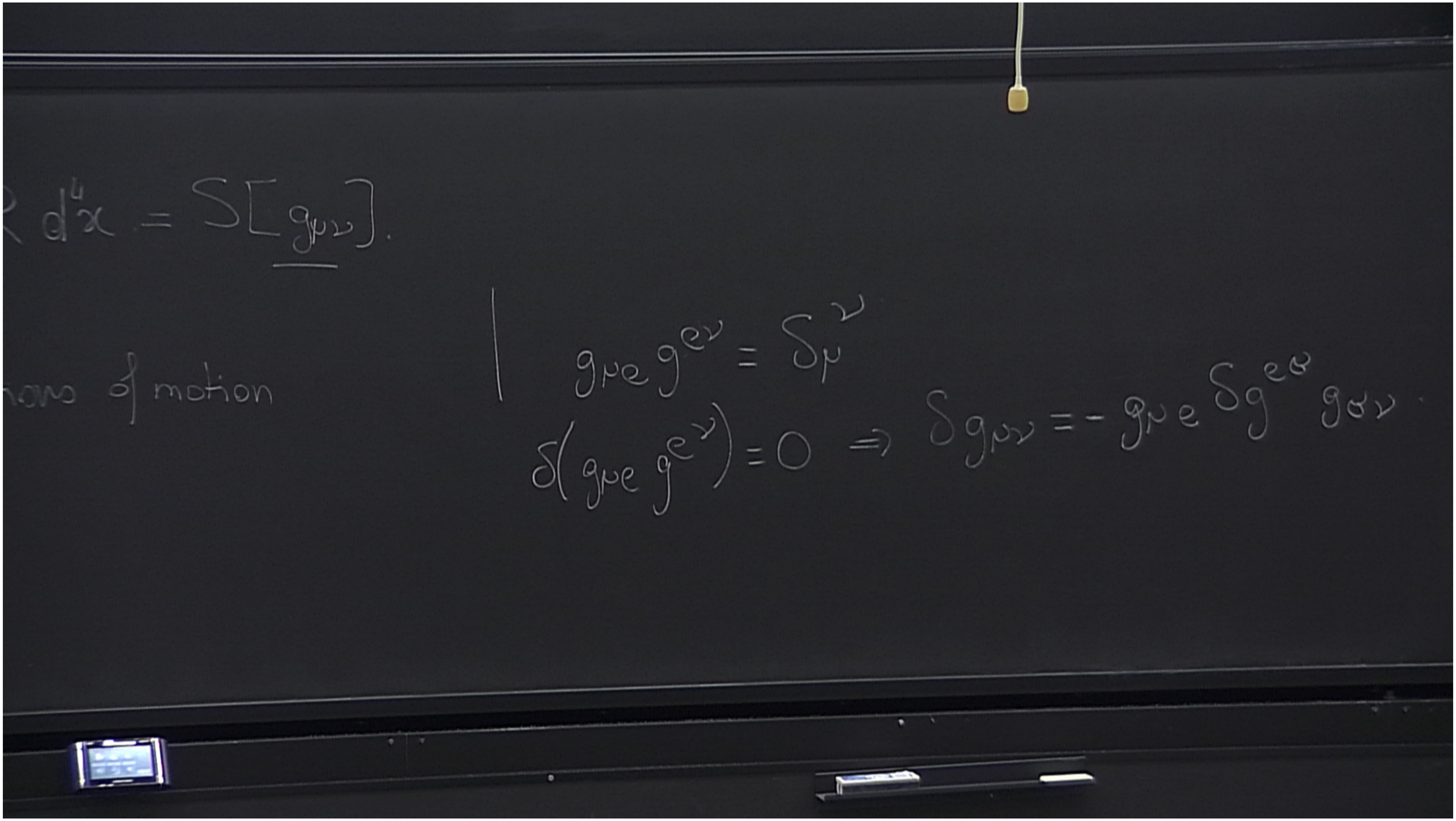


II. A.

Einstein-Hilbert action. $S = \frac{1}{K} \int \sqrt{g}\, R\, d^4x = S[\underline{g_{\mu\nu}}]$.

$K = \frac{16\pi G}{c^3} = 1$

$\Rightarrow$ Equations of motion



$R\, d^4x = S[\underline{g_{\mu\nu}}].$

ions of motion

$$g_{\mu\rho} g^{\rho\nu} = \delta_\mu^\nu$$

$$\delta(g_{\mu\rho} g^{\rho\nu}) = 0 \Rightarrow \delta g_{\mu\nu} = - g_{\mu\rho}\, \delta g^{\rho\sigma} g_{\sigma\nu}.$$

ction. $S = \frac{1}{\kappa} \int \sqrt{g}\, R\, d^4x = S[\underline{g_{\mu\nu}}]$.

$= \frac{16\pi G}{c^3} = 1$

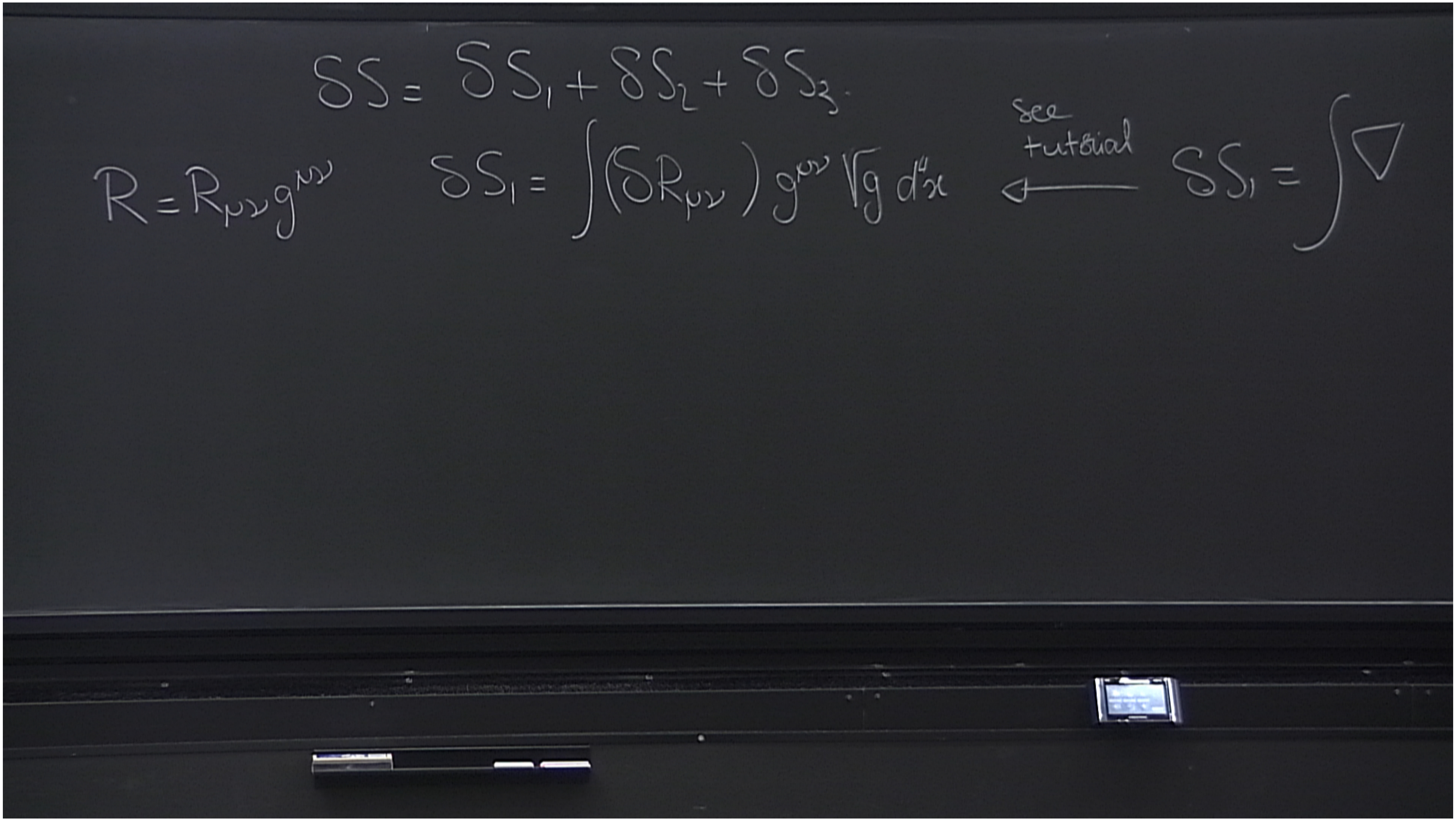
$\Rightarrow$ Equations of motion

variations of the action

w.r.t metric & inverse metric

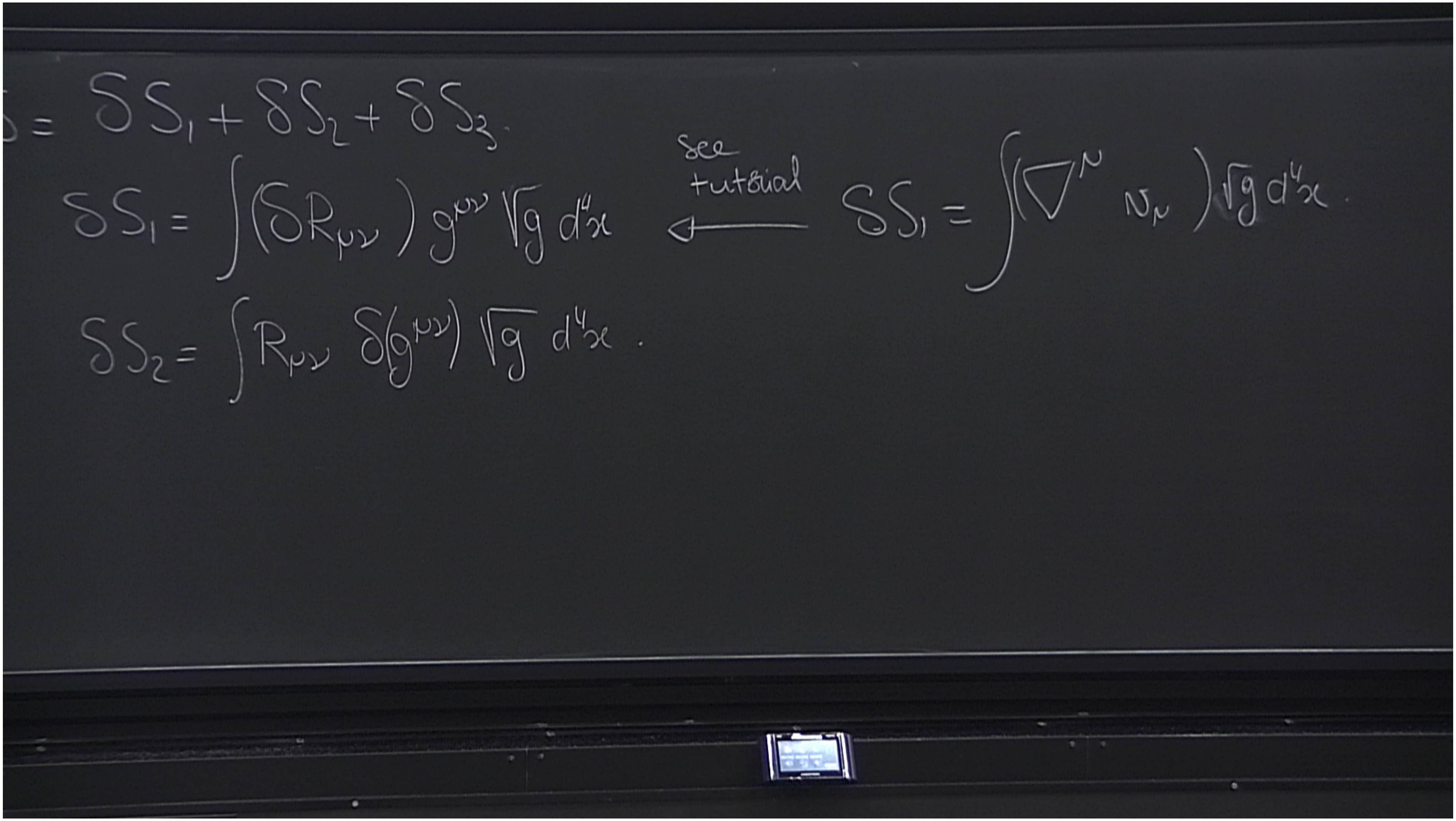
$$g_{\mu\rho} g^{\rho\nu} = \delta_\mu{}^\nu$$

$$\delta(g_{\mu\rho} g^{\rho\nu}) = 0 \Rightarrow \delta$$



$$\delta S = \delta S_1 + \delta S_2 + \delta S_3 .$$

$$R = R_{\mu\nu} g^{\mu\nu} \qquad \delta S_1 = \int (\delta R_{\mu\nu}) g^{\mu\nu} \sqrt{g}\, d^4x \xleftarrow{\text{see tutorial}} \delta S_1 = \int \nabla$$



$$= \delta S_1 + \delta S_2 + \delta S_3 .$$

$$\delta S_1 = \int (\delta R_{\mu\nu}) g^{\mu\nu} \sqrt{g}\, d^4x \xleftarrow{\text{see tutorial}} \delta S_1 = \int (\nabla^{\mu} \, v_{\mu}) \sqrt{g}\, d^4x .$$

$$\delta S_2 = \int R_{\mu\nu}\, \delta(g^{\mu\nu}) \sqrt{g}\, d^4x .$$

$S = \delta S_1 + \delta S_2 + \delta S_3.$

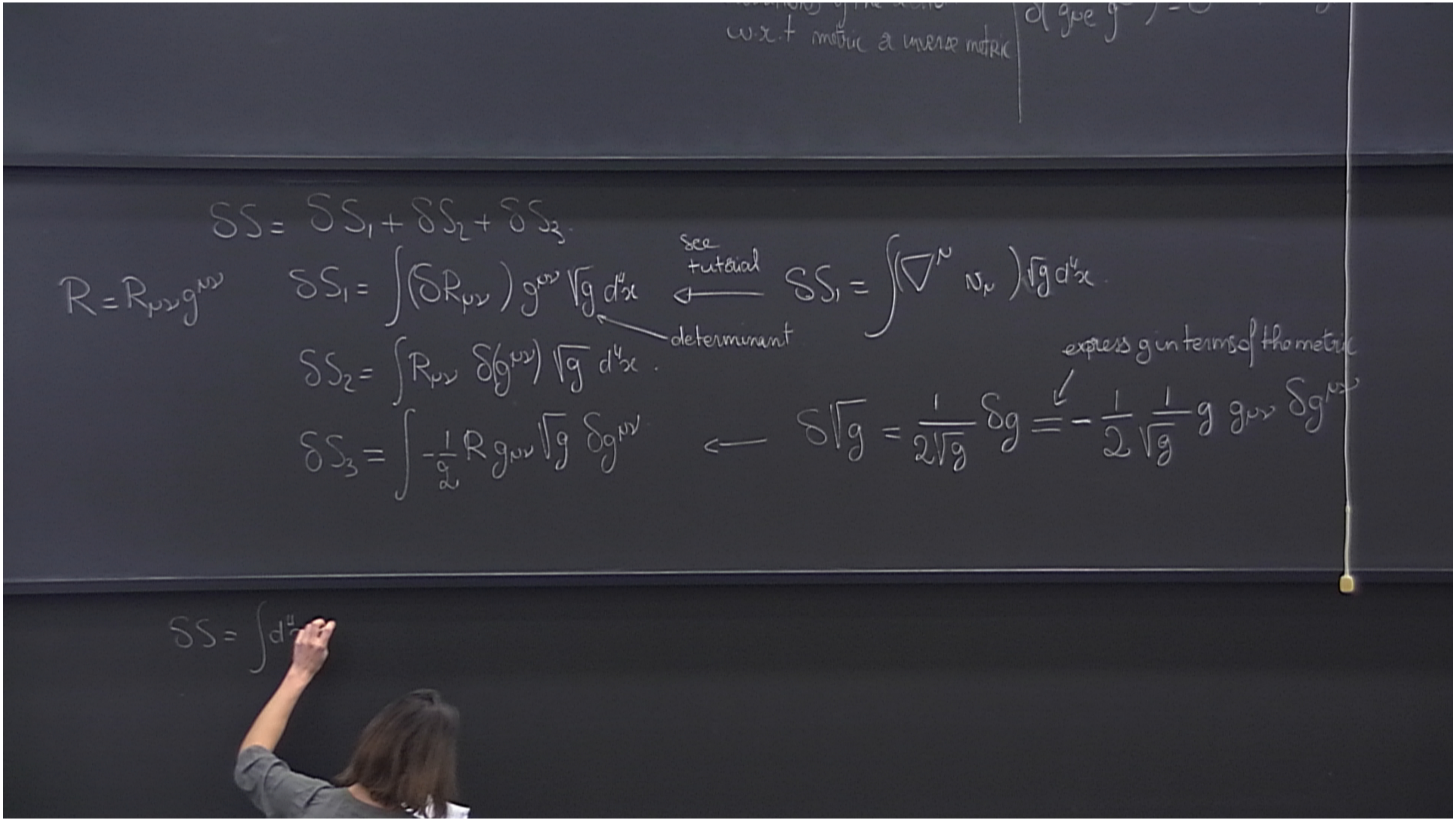
$$\delta S_1 = \int (\delta R_{\mu\nu})\, g^{\mu\nu} \sqrt{g}\, d^4x \xleftarrow{\text{see tutorial}} \delta S_1 = \int (\nabla^{\nu} v_{\nu}) \sqrt{g}\, d^4x.$$

determinant

$$\delta S_2 = \int R_{\mu\nu}\, \delta(g^{\mu\nu}) \sqrt{g}\, d^4x.$$

express g in terms of the metric

$$\delta S_3 = \qquad \longleftarrow \quad \delta\sqrt{g} = \frac{1}{2\sqrt{g}}\,\delta g = -\frac{1}{2}\frac{1}{\sqrt{g}}\, g\, g_{\mu\nu}\, \delta g^{\mu\nu}$$



w.r.t metric & inverse metric | $\delta(g_{\mu e}\, g$

$$\delta S = \delta S_1 + \delta S_2 + \delta S_3 .$$

$$R = R_{\mu\nu} g^{\mu\nu}$$

$$\delta S_1 = \int (\delta R_{\mu\nu}) g^{\mu\nu} \sqrt{g}\, d^4x \xleftarrow[\text{}]{\text{see tutorial}} \delta S_1 = \int (\nabla^\nu v_\nu) \sqrt{g}\, d^4x .$$

determinant

$$\delta S_2 = \int R_{\mu\nu}\, \delta(g^{\mu\nu}) \sqrt{g}\, d^4x .$$

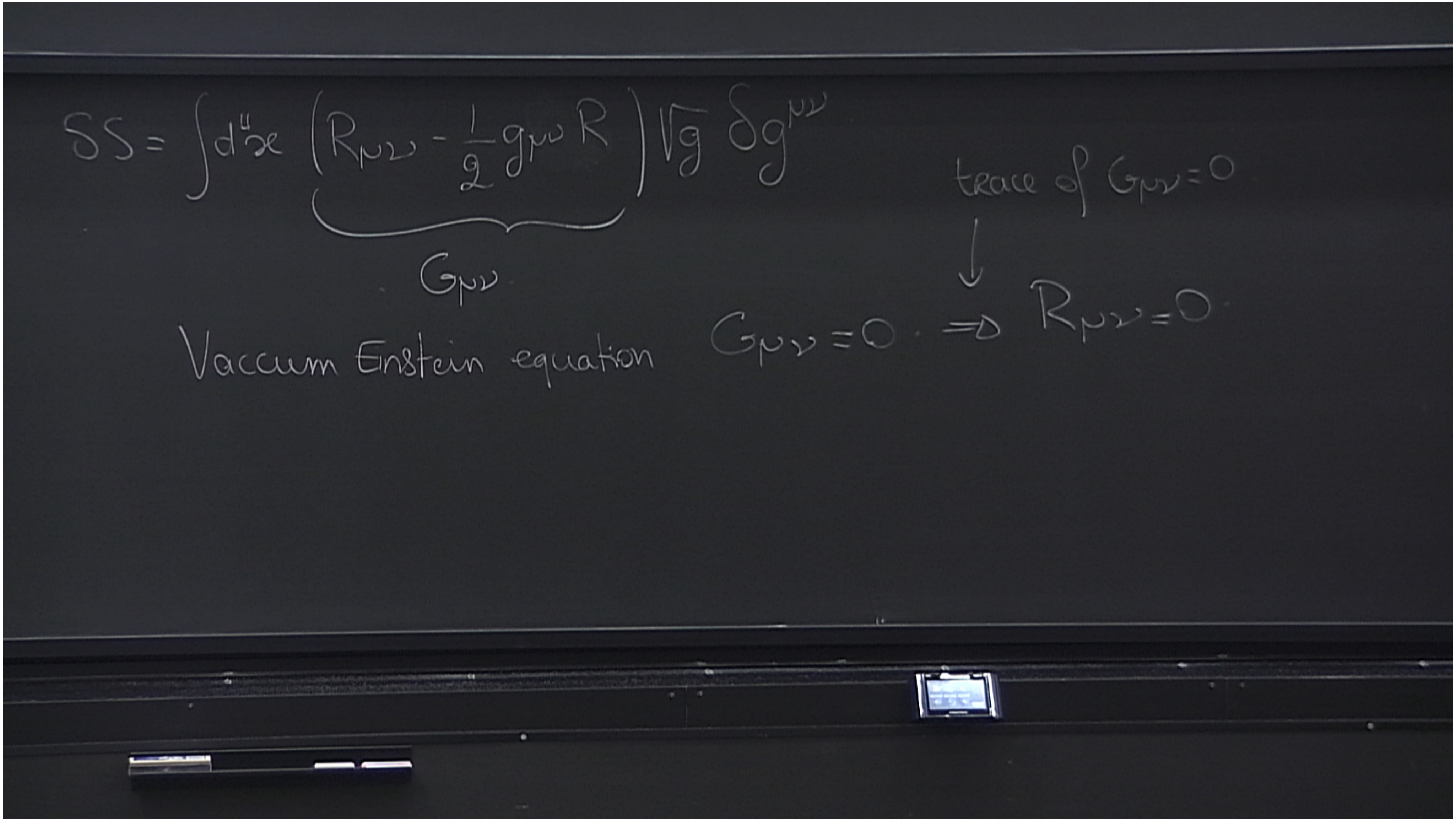
express g in terms of the metric

$$\delta S_3 = \int -\frac{1}{2} R\, g_{\mu\nu} \sqrt{g}\, \delta g^{\mu\nu} \longleftarrow \delta\sqrt{g} = \frac{1}{2\sqrt{g}} \delta g = -\frac{1}{2} \frac{1}{\sqrt{g}} g\, g_{\mu\nu}\, \delta g^{\mu\nu}$$

$$\delta S = \int d^4x$$

$$\delta S = \int d^4x \underbrace{\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right)}_{G_{\mu\nu}} \sqrt{g}\, \delta g^{\mu\nu}$$

Vaccum Einstein

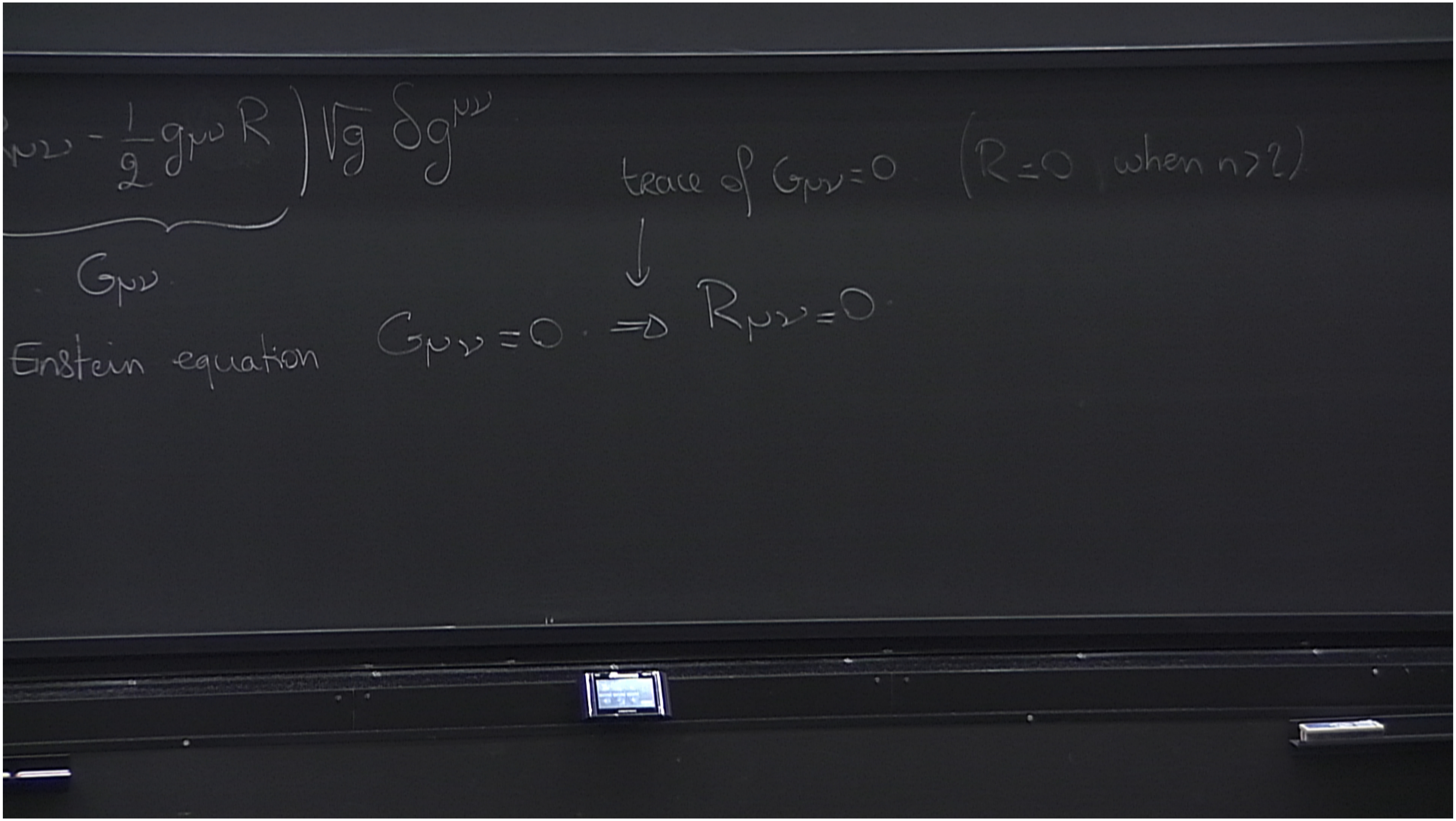


$$\delta S = \int d^4x \, \underbrace{\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right)}_{G_{\mu\nu}} \sqrt{g} \, \delta g^{\mu\nu}$$

trace of $G_{\mu\nu} = 0$

$\downarrow$

Vaccum Einstein equation $G_{\mu\nu} = 0 \Rightarrow R_{\mu\nu} = 0$

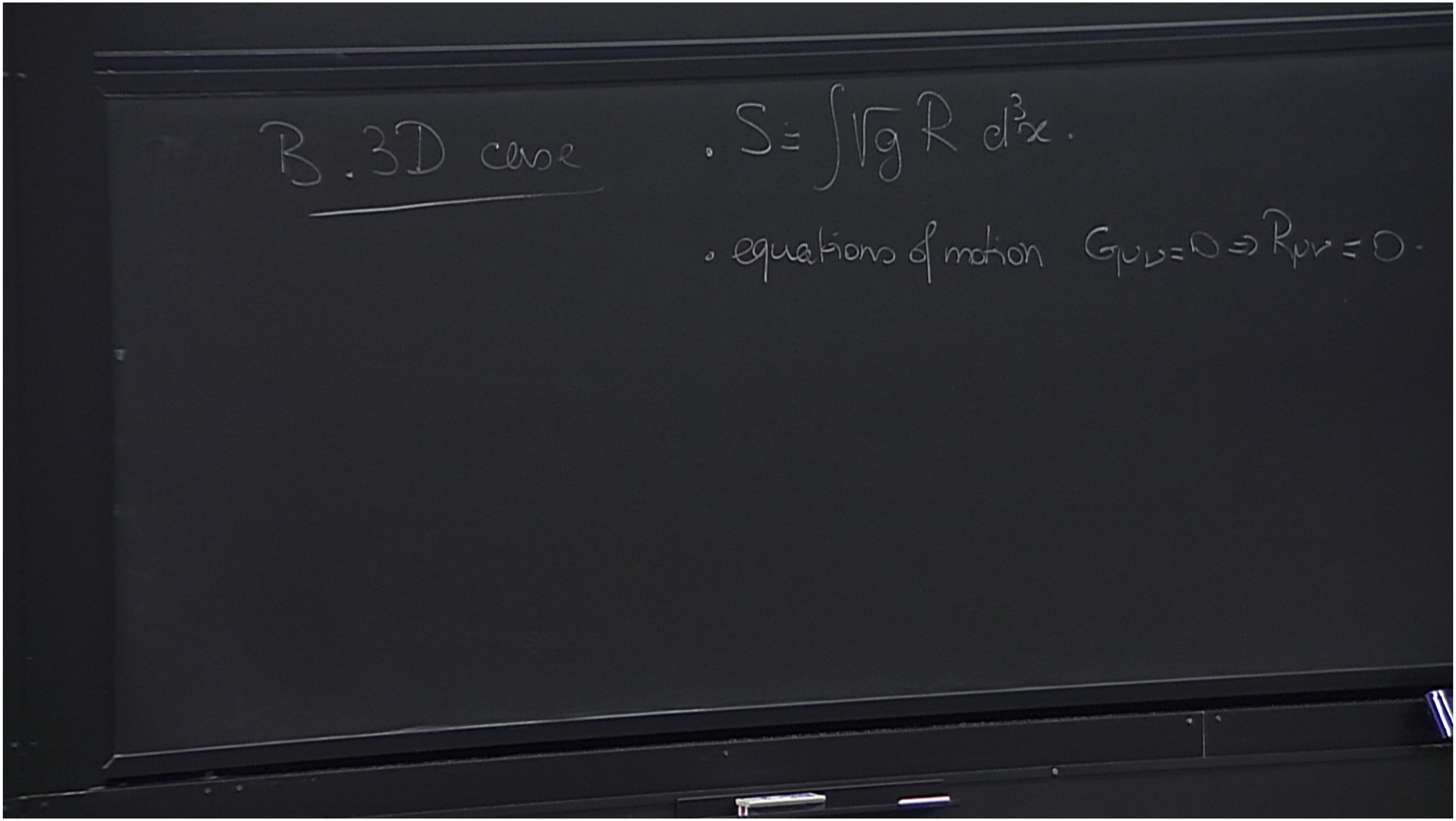


$$\underbrace{{}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}_{G_{\mu\nu}} \Big) \sqrt{g}\, \delta g^{\mu\nu}$$

trace of $G_{\mu\nu} = 0$ ($R = 0$, when $n > 2$)

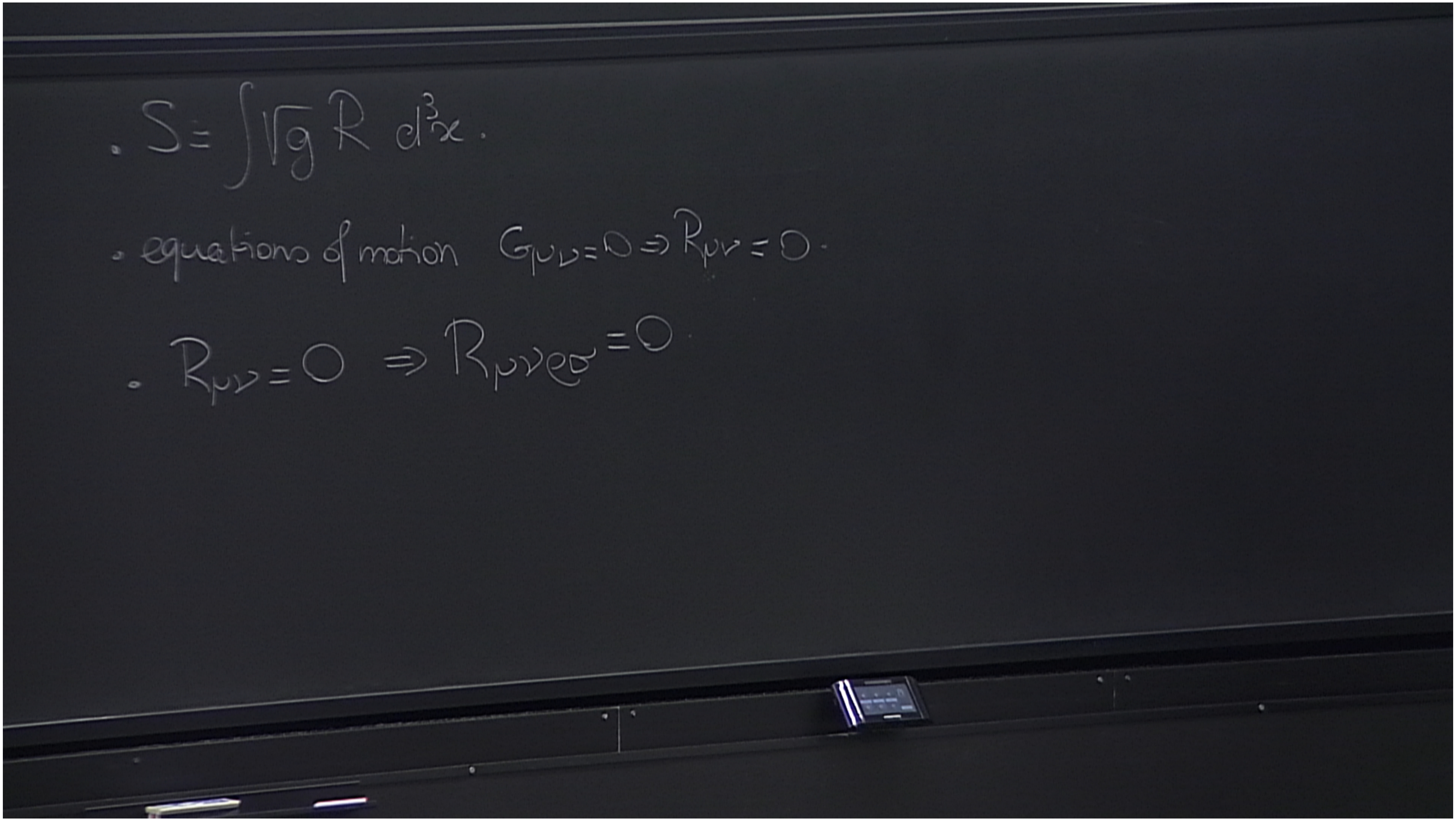
$\downarrow$

Einstein equation $G_{\mu\nu} = 0 \Rightarrow R_{\mu\nu} = 0$

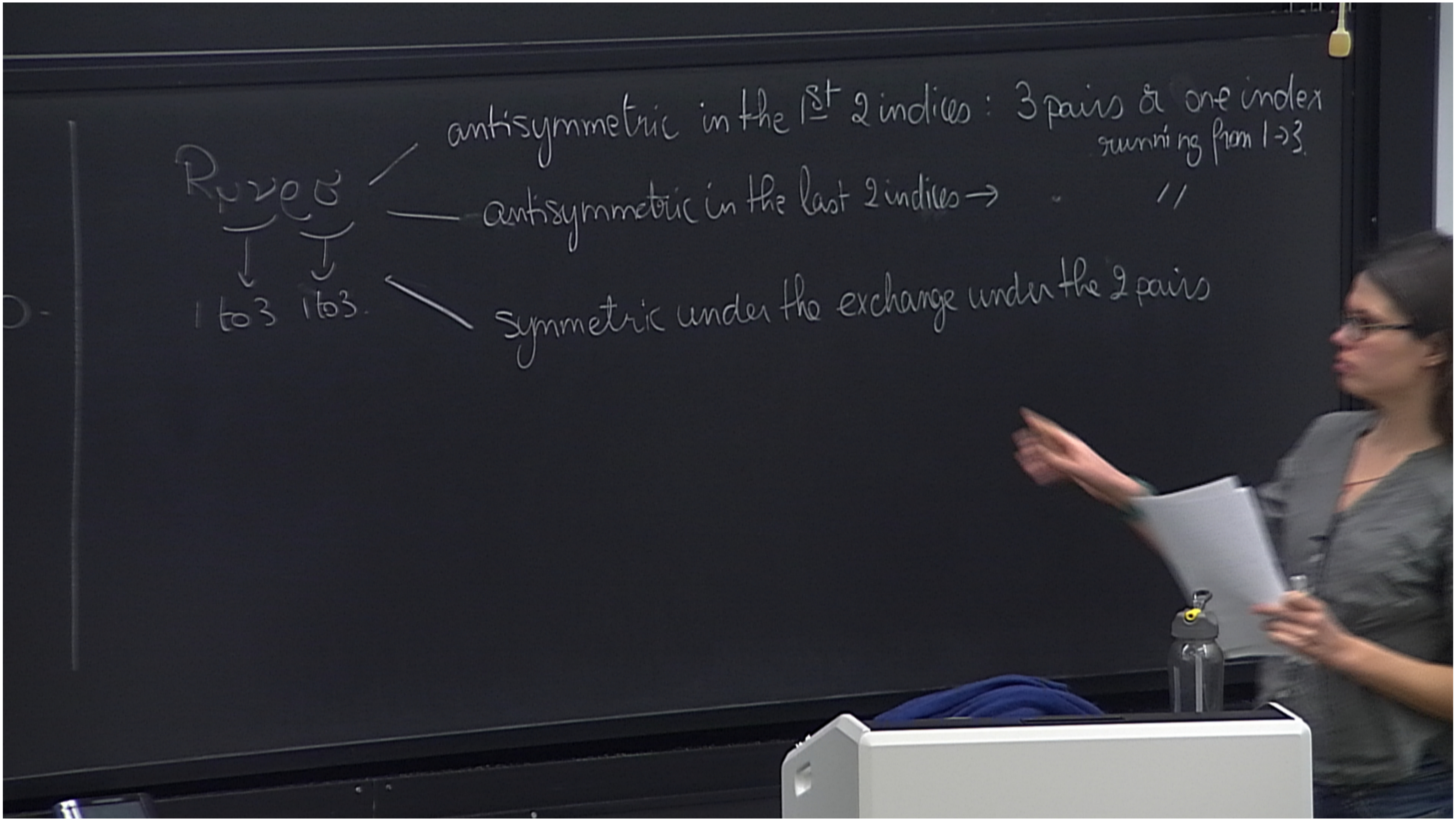


B. 3D case

- $S = \int \sqrt{g}\, R \, d^3x$.
- equations of motion $G_{\mu\nu} = 0 \Rightarrow R_{\mu\nu} = 0$.

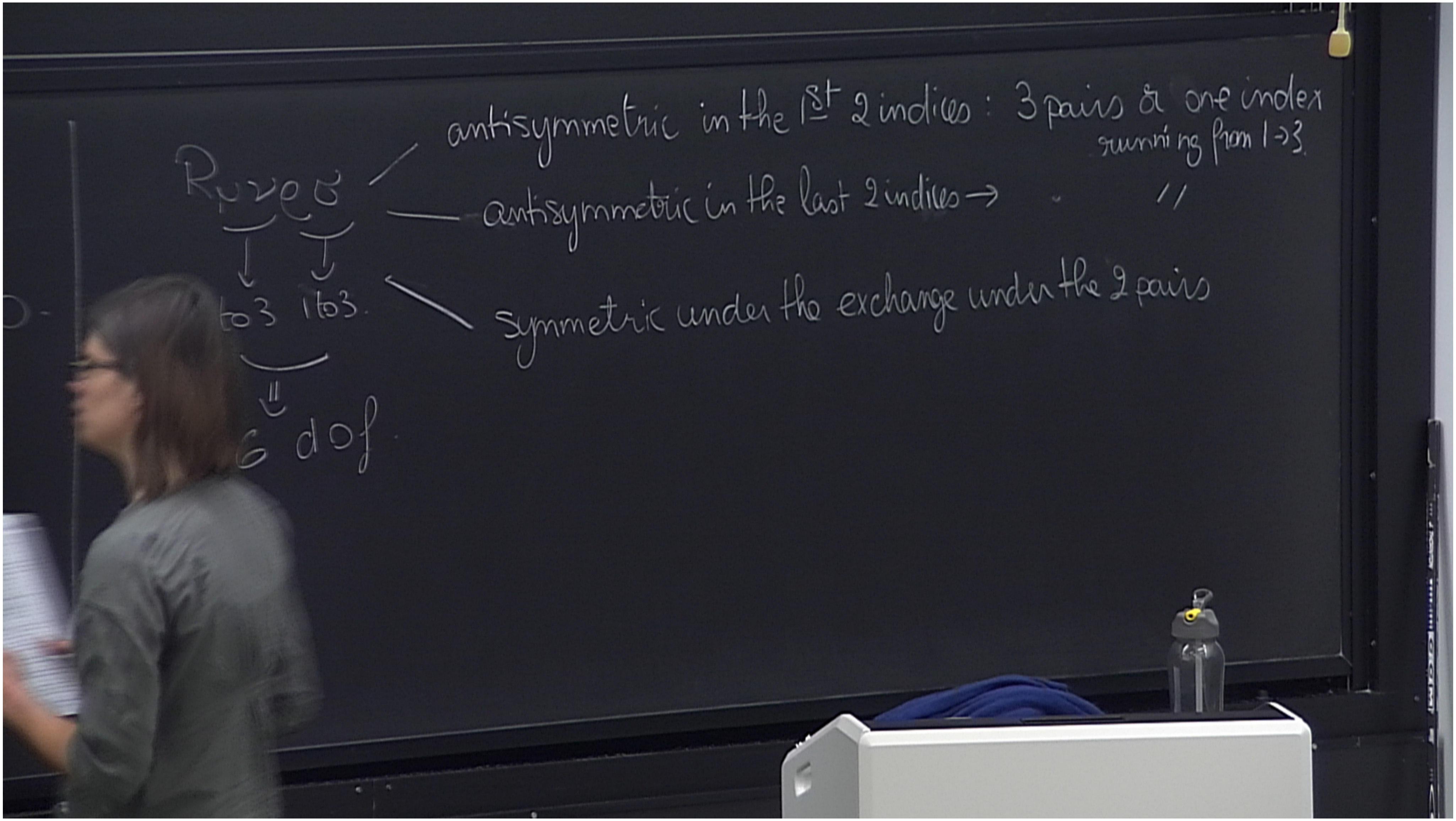


- $S = \int \sqrt{g}\, R \, d^3x$.
- equations of motion $G_{\mu\nu} = 0 \Rightarrow R_{\mu\nu} = 0$.
- $R_{\mu\nu} = 0 \Rightarrow R_{\mu\nu\rho\sigma} = 0$.



$$R_{\underbrace{\mu\nu}_{\downarrow\,1\text{ to }3}\underbrace{\rho\sigma}_{\downarrow\,1\text{ to }3}}$$

- antisymmetric in the 1st 2 indices : 3 pairs or one index running from 1→3.
- antisymmetric in the last 2 indices → //
- symmetric under the exchange under the 2 pairs

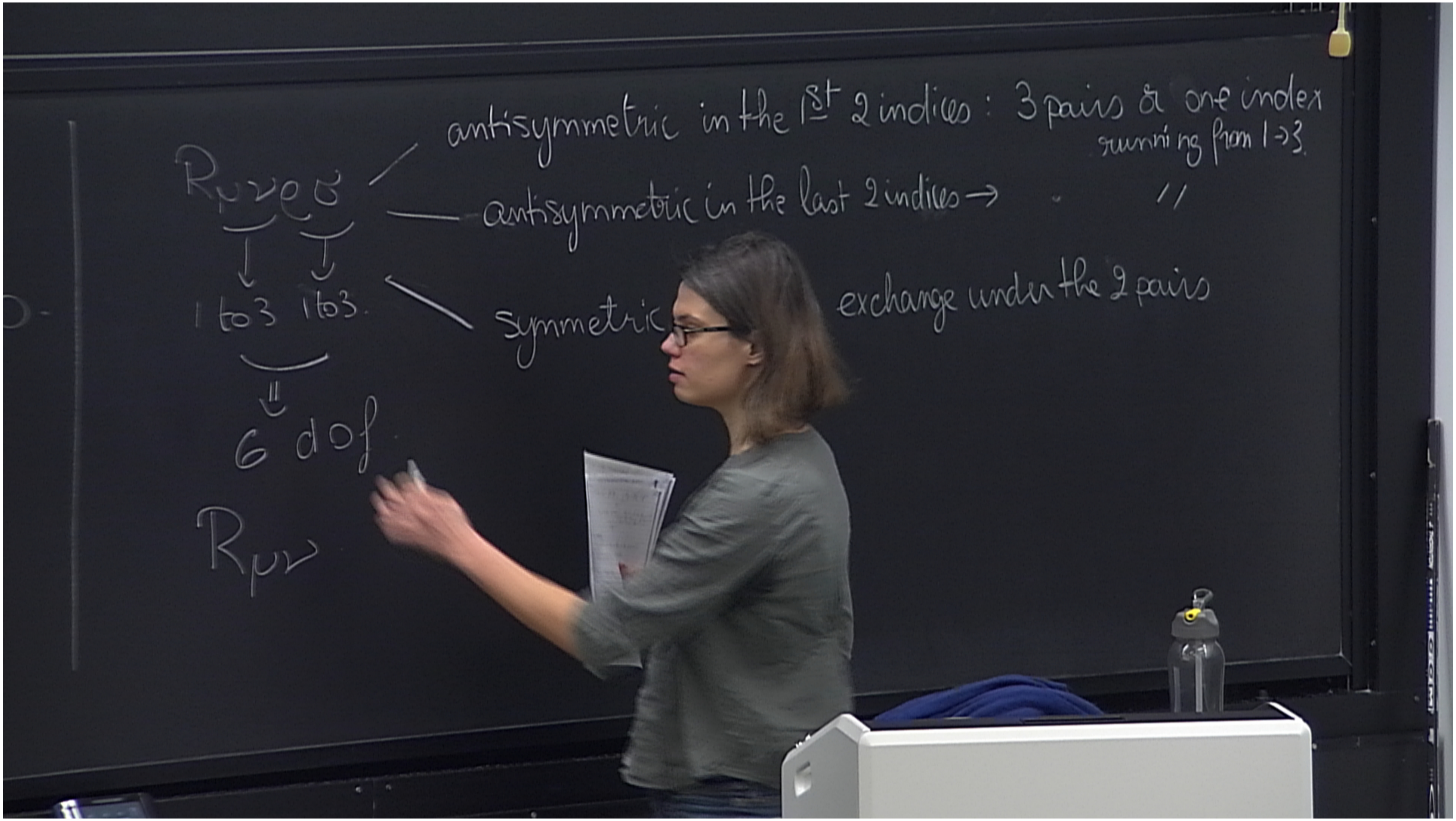


$$\underbrace{\underbrace{R_{\mu\nu}}_{\substack{\downarrow \\ 1\text{to}3}}\underbrace{{}_{\rho\sigma}}_{\substack{\downarrow \\ 1\text{to}3}}}_{\substack{\Downarrow \\ 6\ \text{dof}}}$$

antisymmetric in the 1st 2 indices : 3 pairs or one index running from 1→3

antisymmetric in the last 2 indices → //

symmetric under the exchange under the 2 pairs



$R_{\mu\nu\rho\sigma}$

antisymmetric in the 1st 2 indices : 3 pairs or one index running from 1→3.

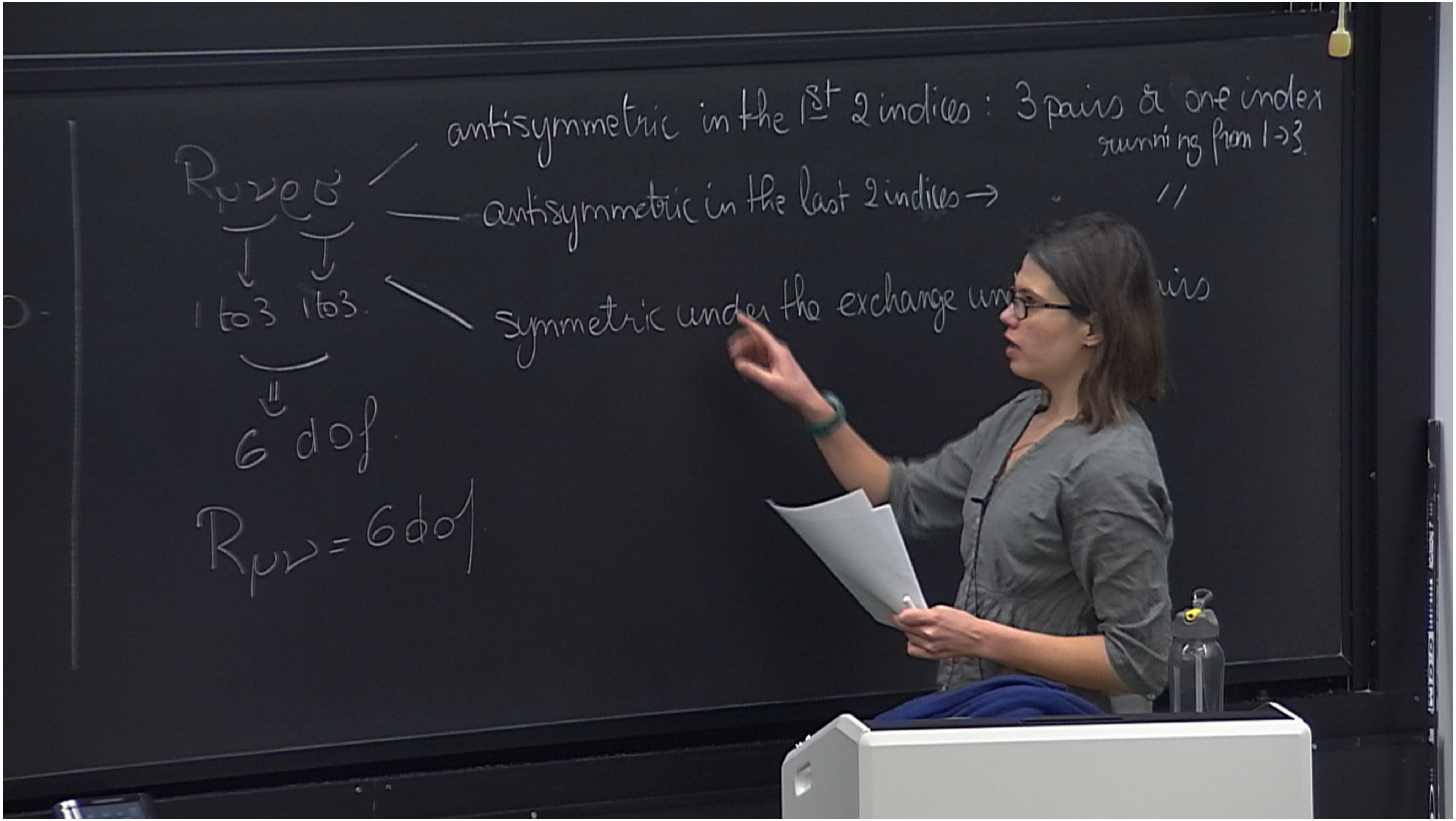
antisymmetric in the last 2 indices → //

symmetric exchange under the 2 pairs

$$\underbrace{\underbrace{\mu\nu}_{1 \text{ to } 3}\ \underbrace{\rho\sigma}_{1 \text{ to } 3}}_{\Downarrow}$$

6 dof

$R_{\mu\nu}$



$R_{\mu\nu\rho\sigma}$ — antisymmetric in the 1st 2 indices : 3 pairs or one index running from 1→3.

— antisymmetric in the last 2 indices → //

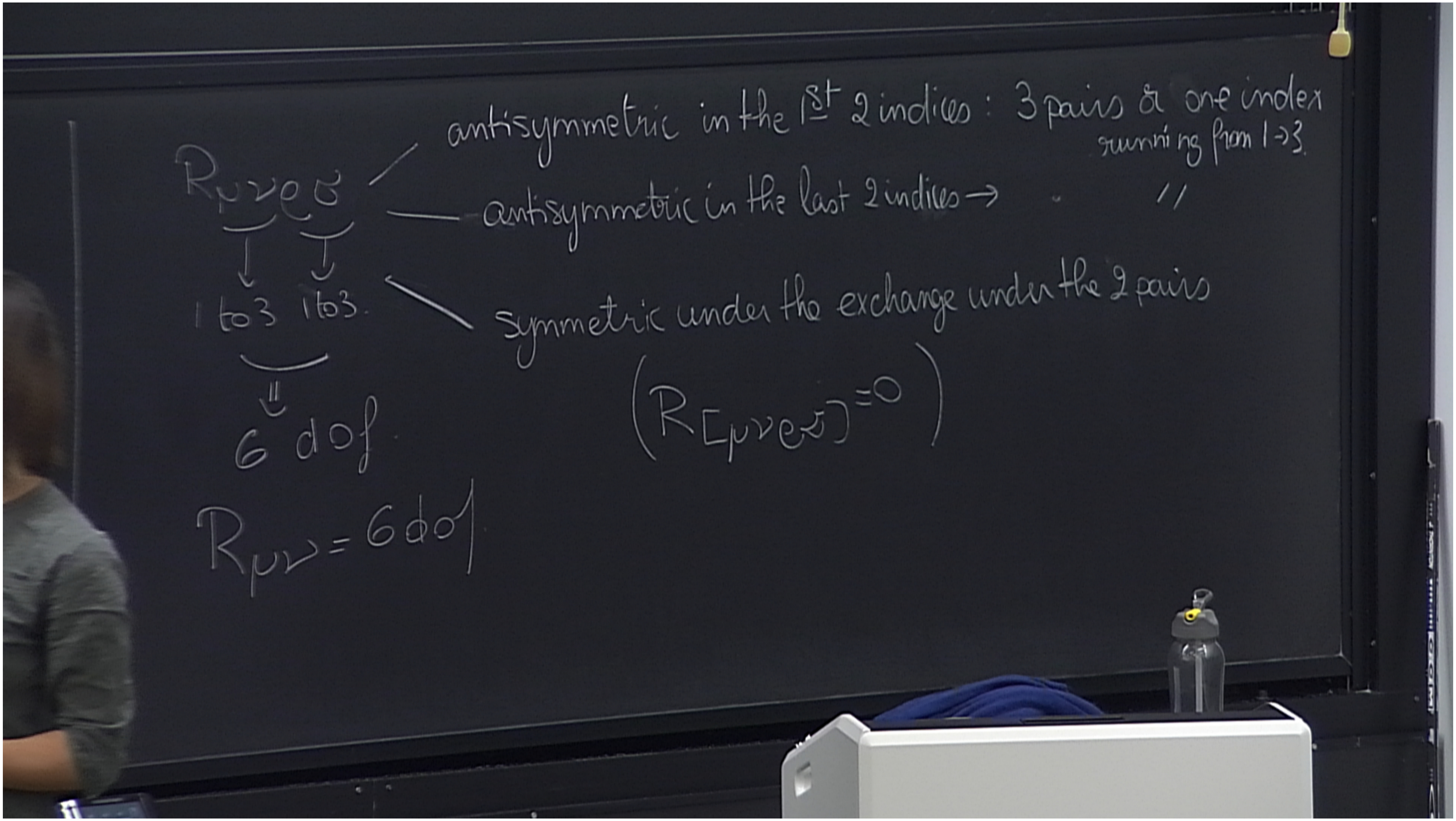
— symmetric under the exchange un pairs

1 to 3   1 to 3

$\Downarrow$

6 dof

$R_{\mu\nu} = 6$ dof



$$\underbrace{\underbrace{R_{\mu\nu}}_{1\text{ to }3}\underbrace{{}_{\rho\sigma}}_{1\text{ to }3}}_{6\text{ dof}}$$

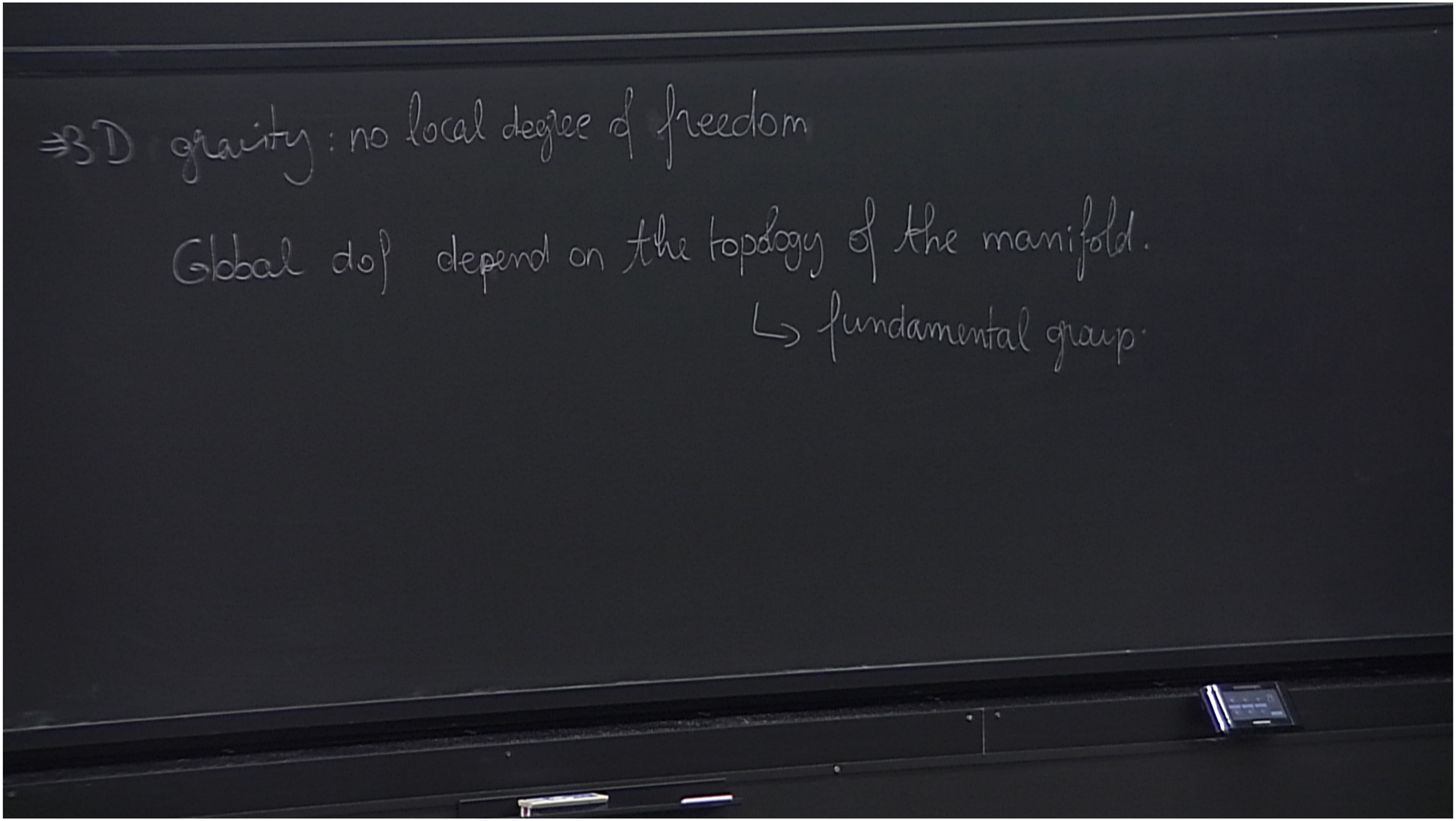
antisymmetric in the 1st 2 indices : 3 pairs or one index running from 1→3.

antisymmetric in the last 2 indices → //

symmetric under the exchange under the 2 pairs

$$\left(R_{[\mu\nu\rho\sigma]} = 0\right)$$

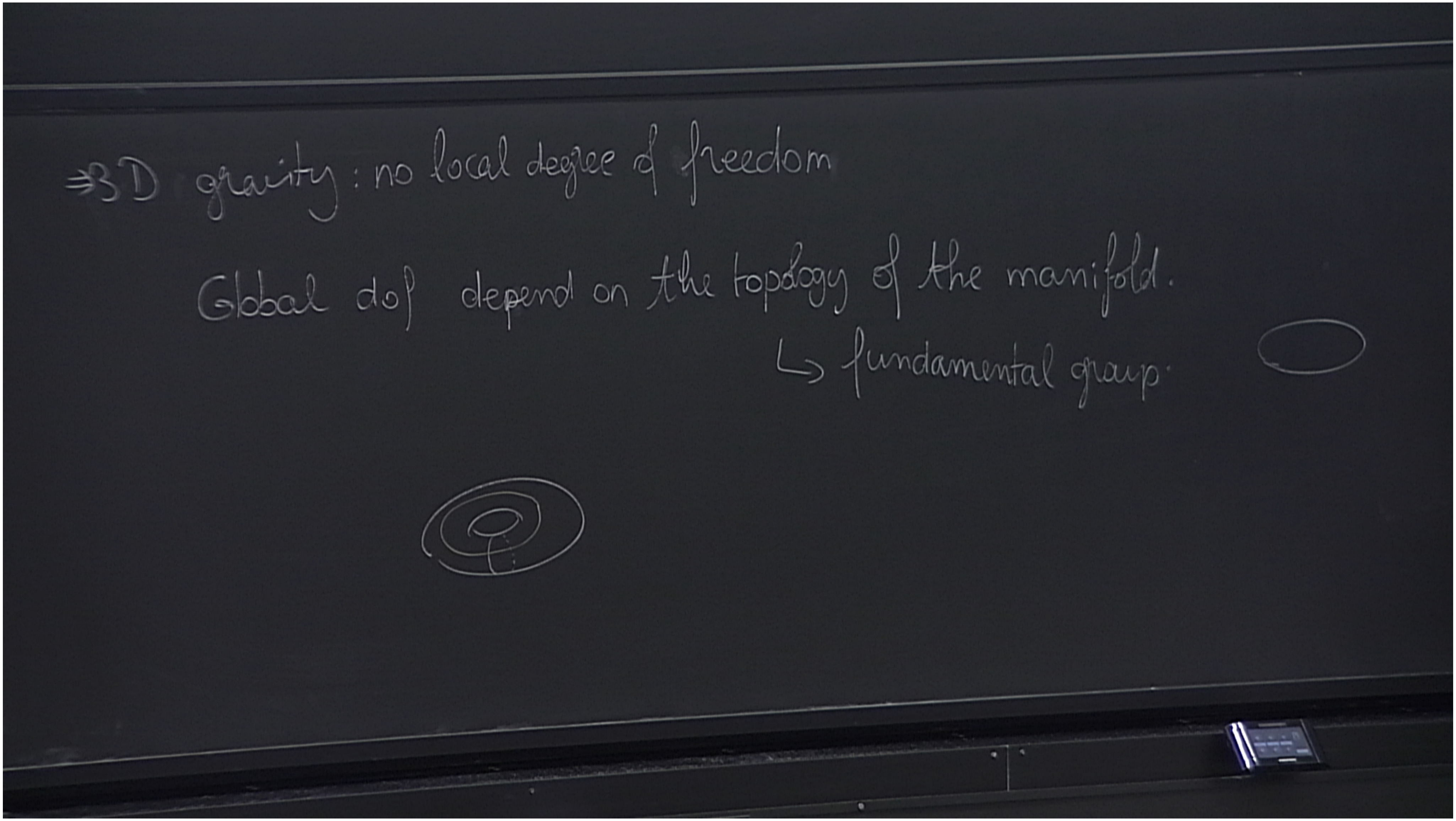
$R_{\mu\nu} = 6$ dof



⇒ 3D gravity : no local degree of freedom

Global dof depend on the topology of the manifold.

↳ fundamental group.



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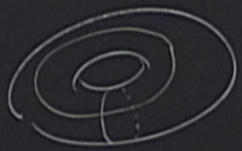
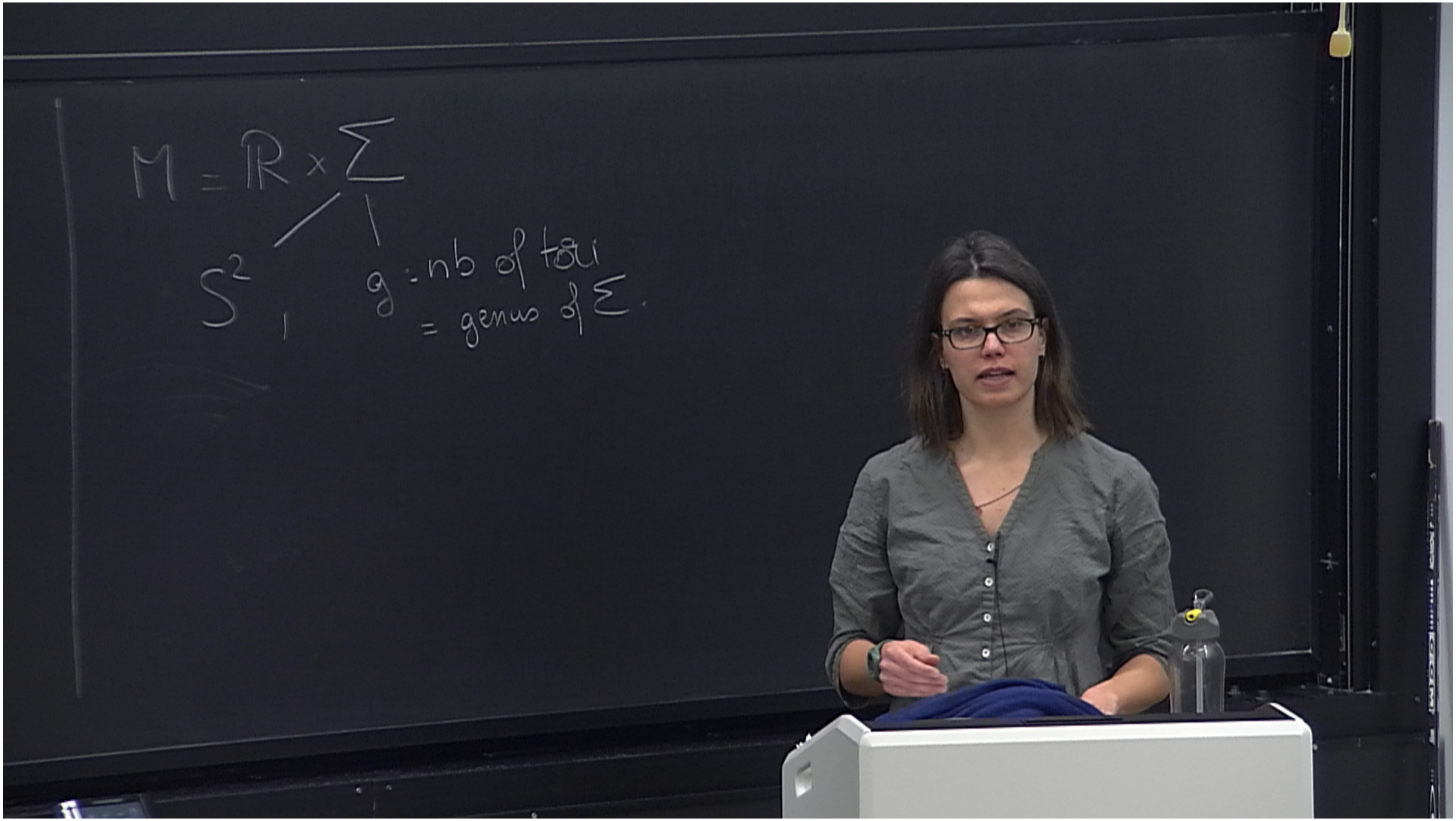
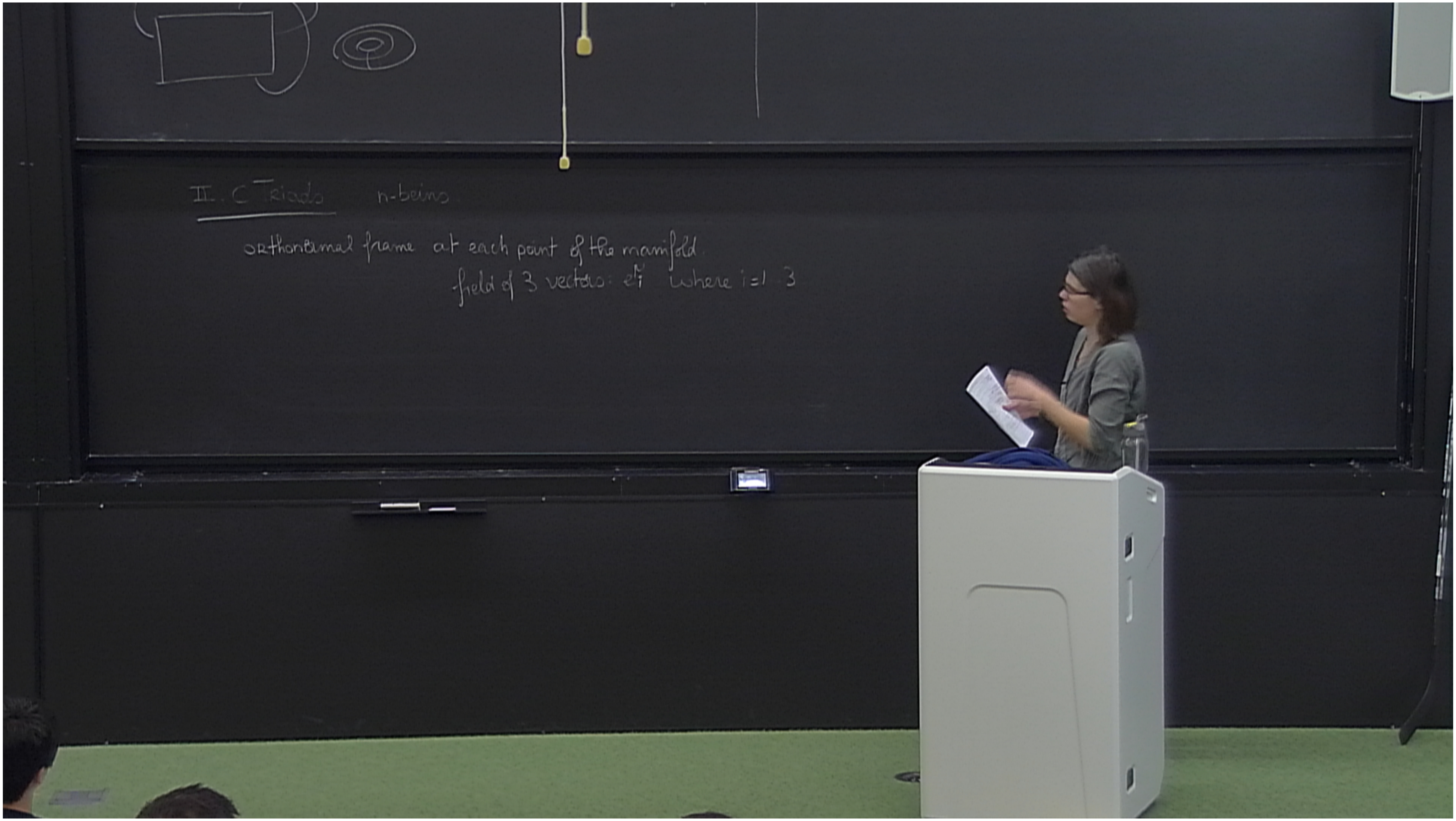



$M = \mathbb{R} \times \Sigma$
S^2
1
g := nb of tori
= genus of Σ.



II. C Triads n-beins.

orthonormal frame at each point of the manifold.

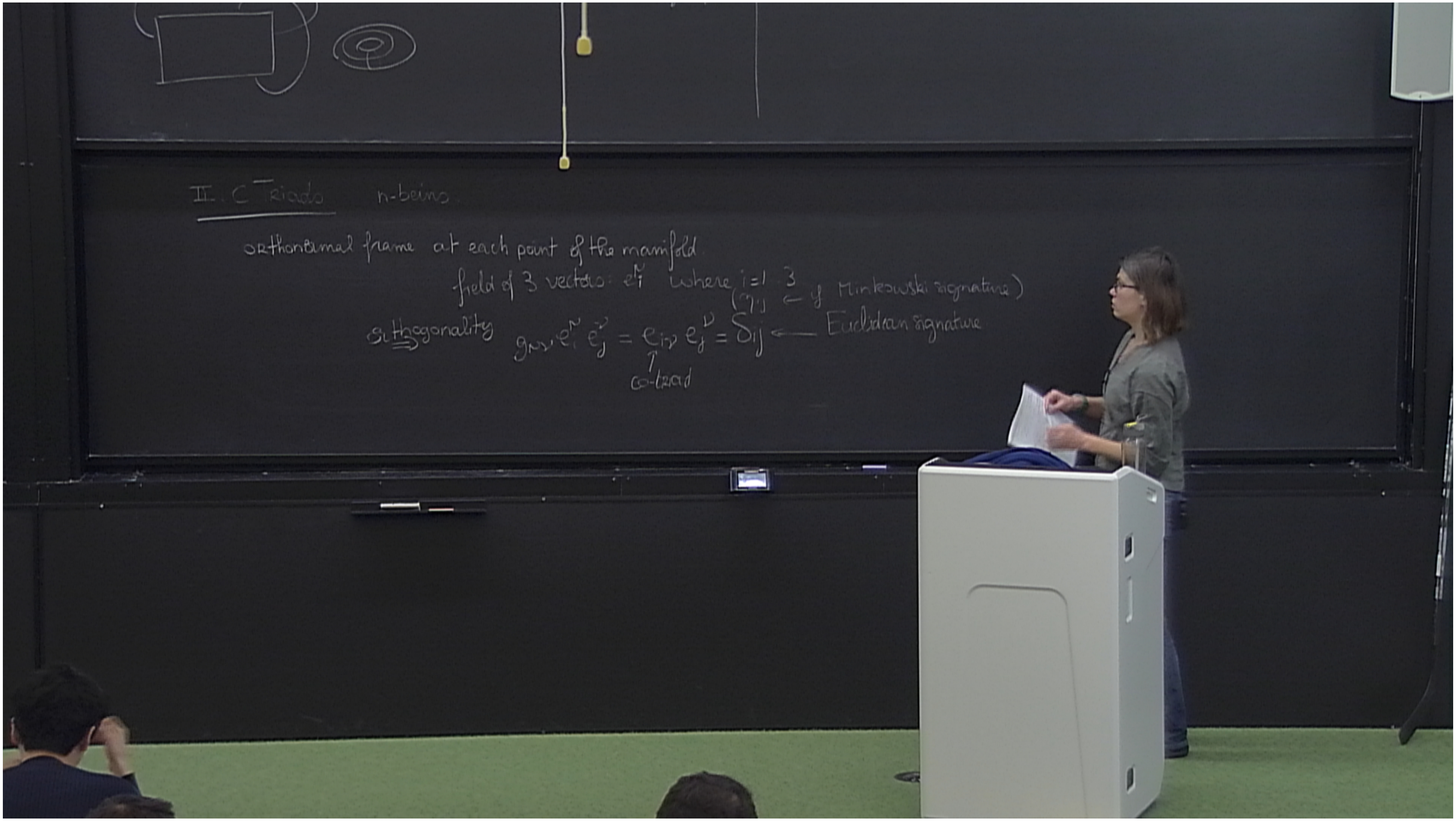
field of 3 vectors: e^a_i where $i=1..3$

II. C Triads   n-beins.

orthonormal frame at each point of the manifold.

field of 3 vectors: e_i^{μ} where $i = 1..3$.

orthogonality $$g_{\mu\nu}\, e_i^{\mu}\, e_j^{\nu} = \underset{\substack{\uparrow \\ \text{co-triad}}}{e_{i\nu}}\, e_j^{\nu} = \delta_{ij}$$



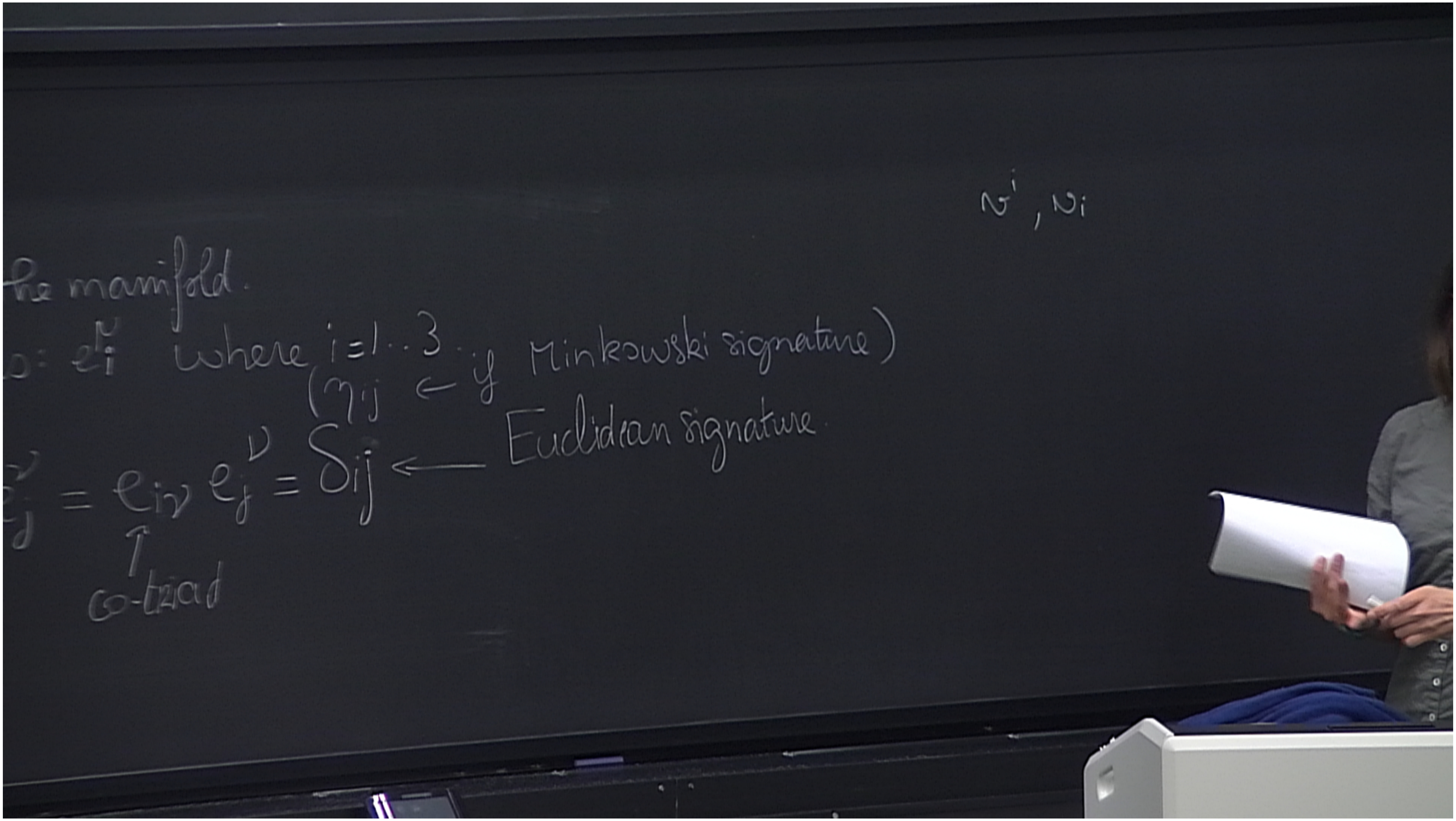
II. C Triads n-beins

orthonormal frame at each point of the manifold

field of 3 vectors: e^{μ}_{i} where $i=1 \ldots 3$

(η_{ij} ← if Minkowski signature)

orthogonality $g_{\mu\nu} e^{\mu}_{i} e^{\nu}_{j} = \underset{\substack{\uparrow \\ \text{co-triad}}}{e_{i\nu}} e^{\nu}_{j} = \delta_{ij}$ ← Euclidean signature



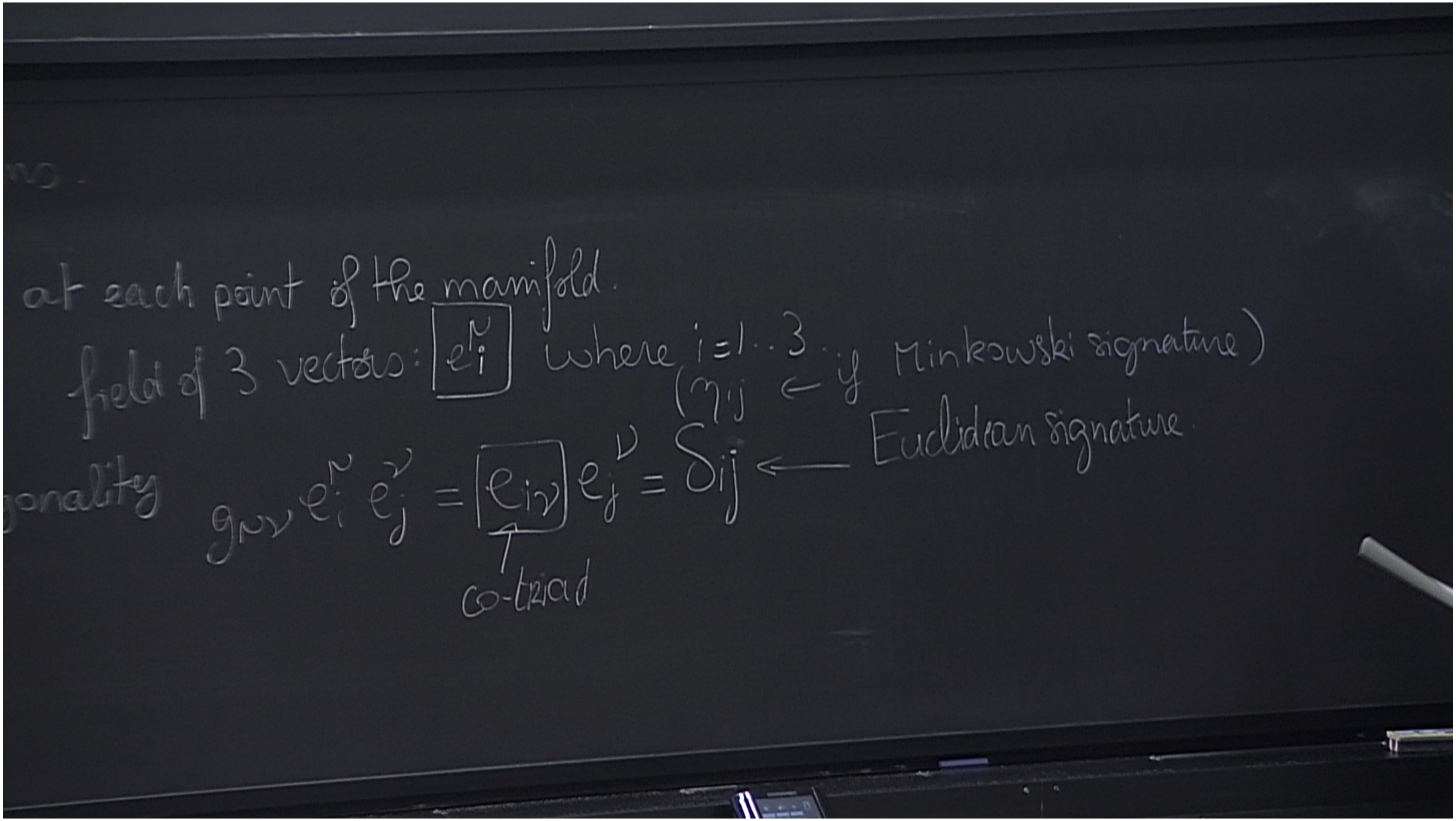
v^i, v_i

he manifold.

o: e^ν_i where $i=1..3$ ..

(η_{ij} ← if Minkowski signature)

$e^\nu_j = \underset{\substack{\uparrow \\ \text{co-triad}}}{e_{i\nu}} e^\nu_j = \delta_{ij} \longleftarrow$ Euclidean signature.



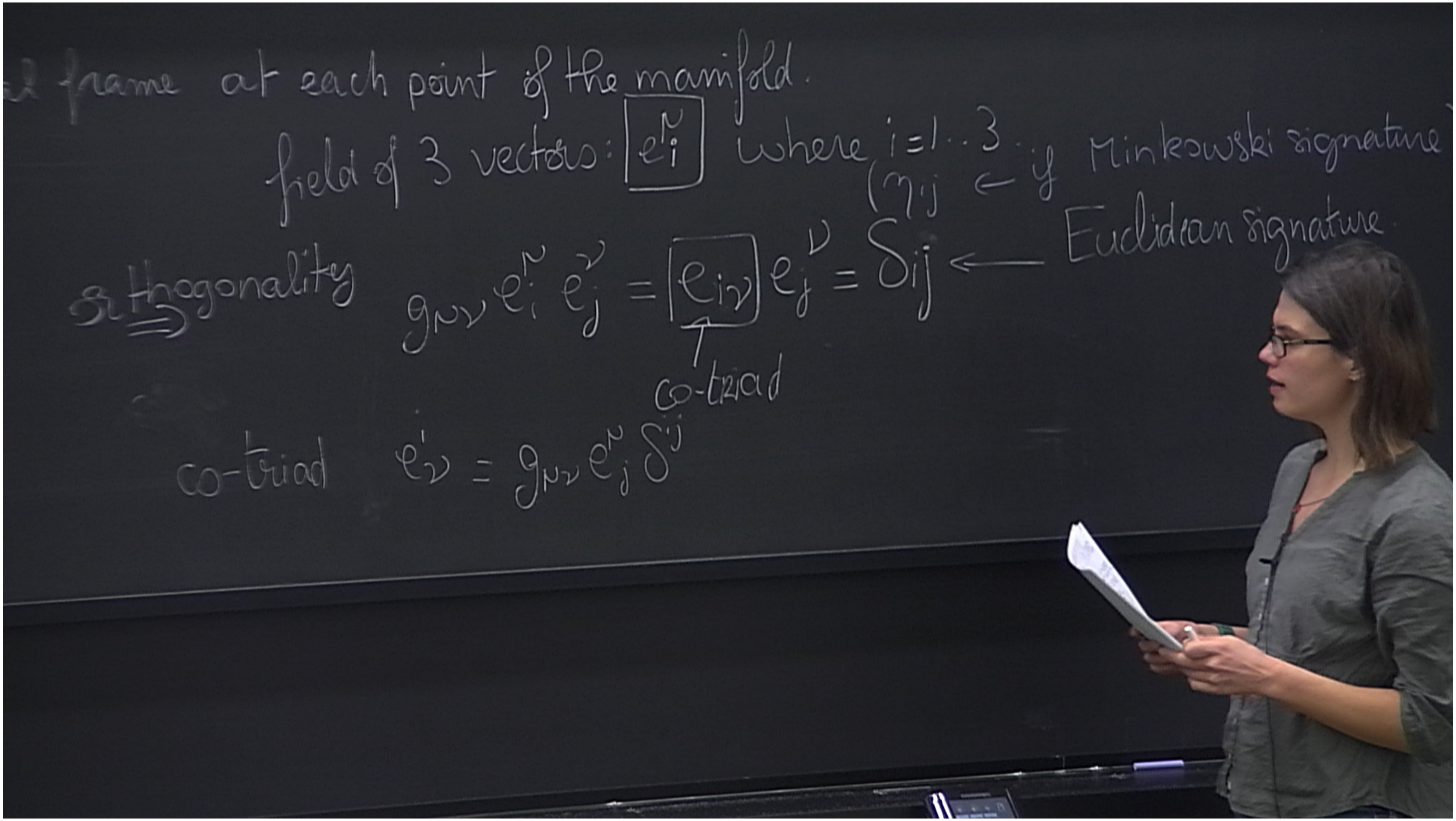
ms .

at each point of the manifold.

field of 3 vectors: $\boxed{e^{\mu}_{i}}$ where $i = 1 .. 3$ ..

(η_{ij} ← if Minkowski signature)

gonality $g_{\mu\nu} e^{\mu}_{i} e^{\nu}_{j} = \underset{\text{co-triad}}{\boxed{e_{i\nu}}} e^{\nu}_{j} = \delta_{ij}$ ← Euclidean signature.



l frame at each point of the manifold.

field of 3 vectors: $\boxed{e^{\mu}_{i}}$ where $i=1..3$

$(\eta_{ij} \leftarrow$ if Minkowski signature$)$

orthogonality $\Rightarrow$ $g_{\mu\nu} e^{\mu}_{i} e^{\nu}_{j} = \boxed{e_{i\nu}} e^{\nu}_{j} = \delta_{ij}$ $\longleftarrow$ Euclidean signature.

co-triad

co-triad $e^{i}_{\nu} = g_{\mu\nu} e^{\mu}_{j} \delta^{ij}$

orthonormal frame at each point of the manifold.

field of 3 vectors: $\boxed{e^{\mu}_{i}}$ where $i = 1 .. 3$ .. if Minkow

$(\eta_{ij}$ ← if Minkow

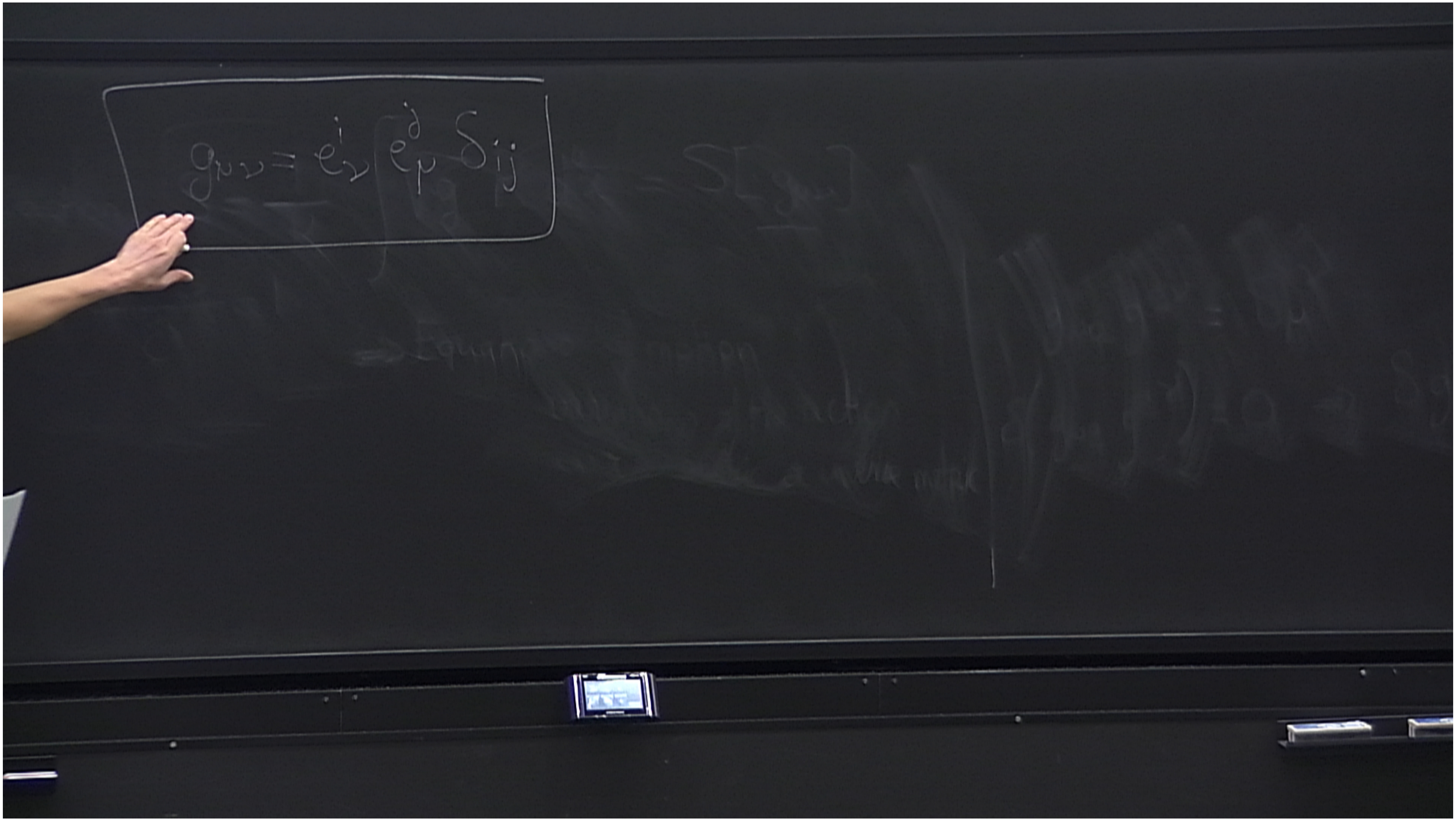
orthogonality $\Rightarrow$

$$g_{\mu\nu} e^{\mu}_{i} e^{\nu}_{j} = \boxed{e_{i\nu}} e^{\nu}_{j} = \delta_{ij} \leftarrow \text{Euclidea}$$

↑ co-triad

co-triad $\quad e^{i}_{\nu} = g_{\mu\nu} e^{\mu}_{j} \delta^{ij}$

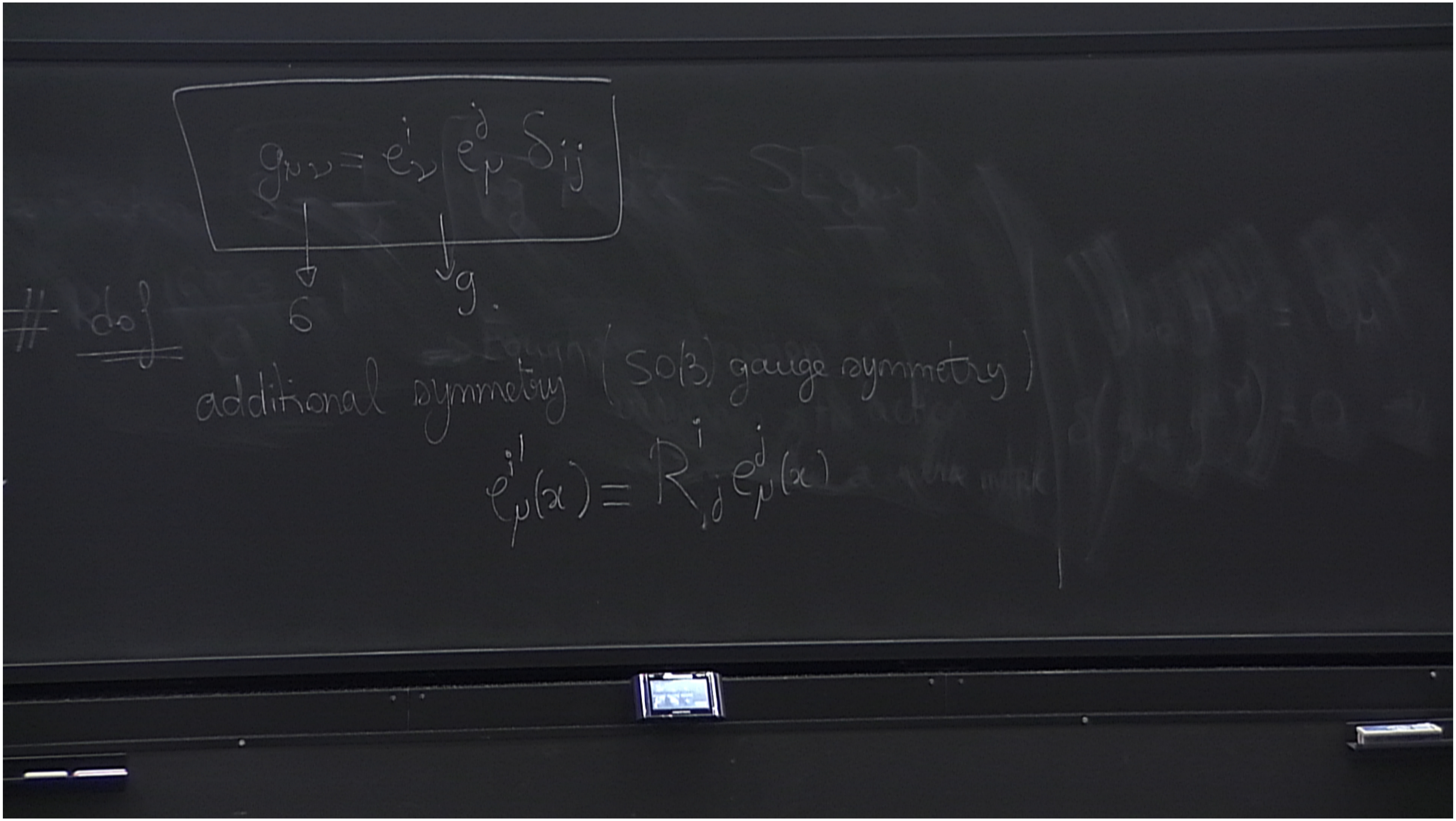
Contracting by e^{ν}_{k} $\Rightarrow$ $g_{\mu\nu} = e^{i}_{\mu} e^{j}_{\nu} \delta_{ij}$



$$\boxed{g_{\mu\nu} = e^i_\nu e^j_\mu \delta_{ij}}$$

$$\boxed{g_{\mu\nu} = e^{i}_{\nu}\, e^{j}_{\mu}\, \delta_{ij}}$$

$\#\ \underline{\underline{\text{dof}}}$ $\qquad \downarrow 6 \qquad \downarrow 9.$



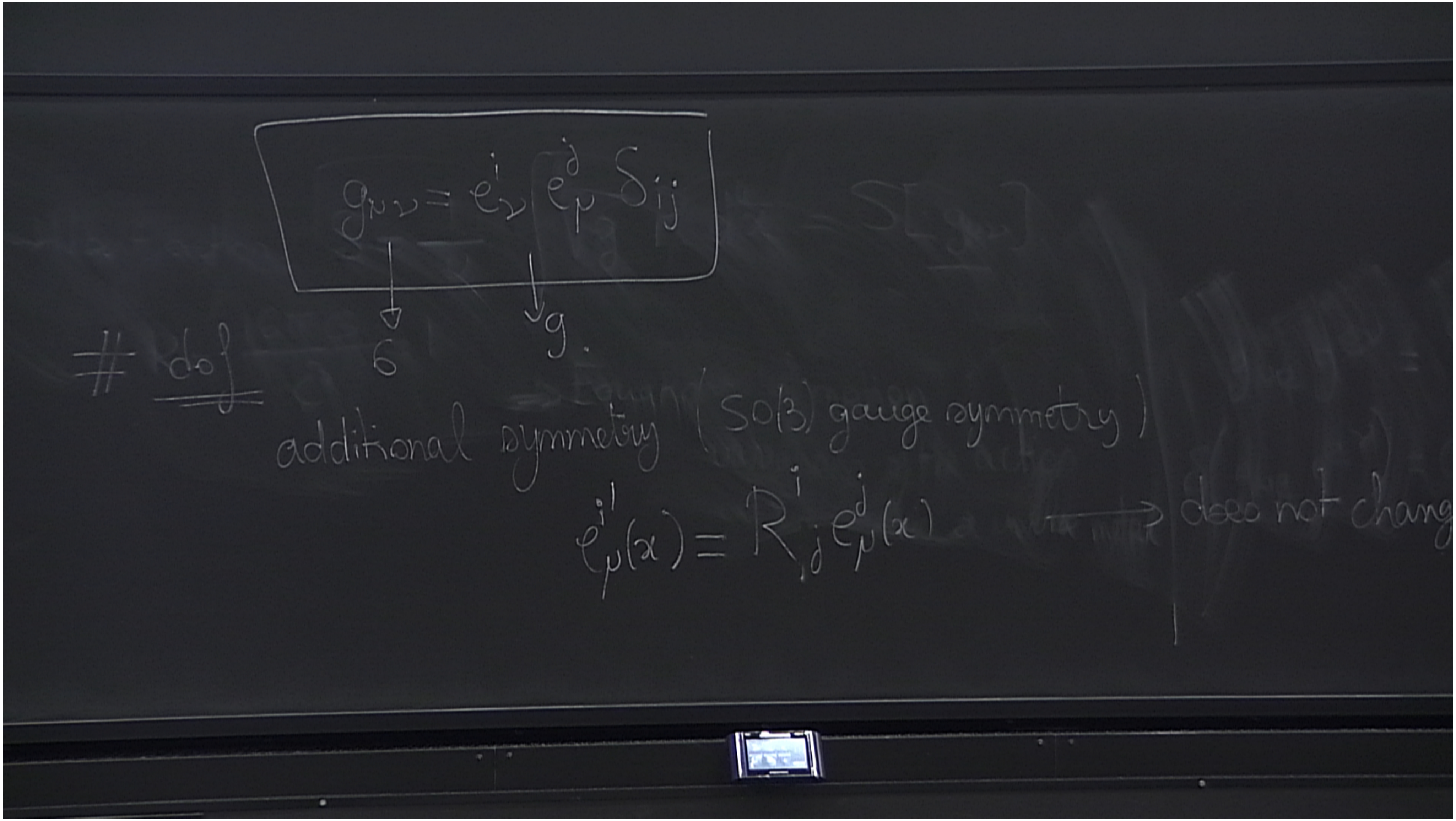
$$\boxed{g_{\mu\nu} = e^i_\nu e^j_\mu \delta_{ij}}$$

$\downarrow$ 6 $\quad\quad \downarrow$ 9.

dof

additional symmetry (SO(3) gauge symmetry)

$$e'^i_\mu(x) = R^i{}_j e^j_\mu(x)$$



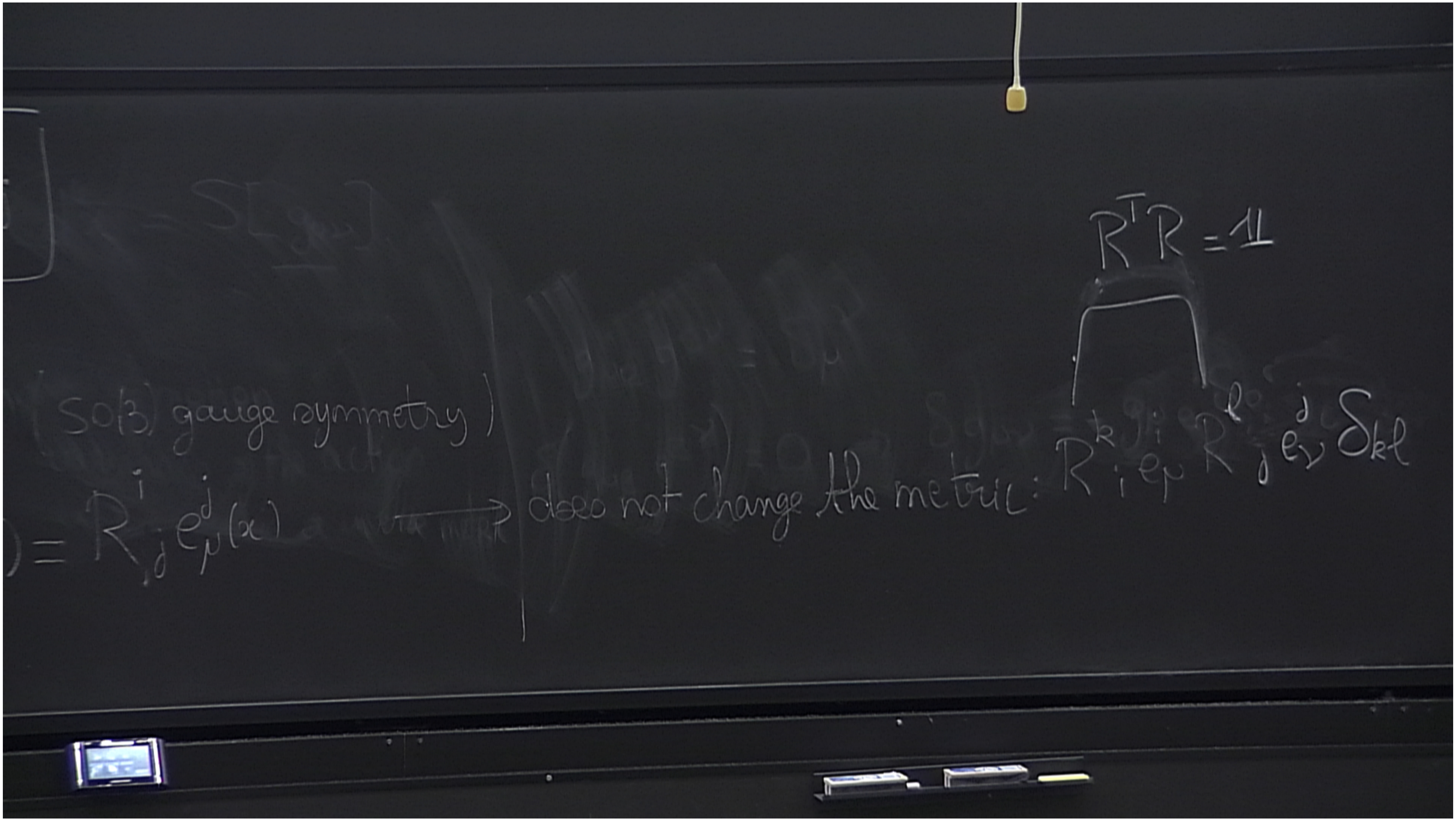
$$\boxed{g_{\mu\nu} = e^i_\nu e^j_\rho \delta_{ij}}$$

$\downarrow$ 6 $\quad\quad \downarrow$ 9.

\# dof

additional symmetry (SO(3) gauge symmetry)

$$e'^{i}_{\mu}(x) = R^i_{\ j}\, e^j_\mu(x) \longrightarrow$$ does not chang

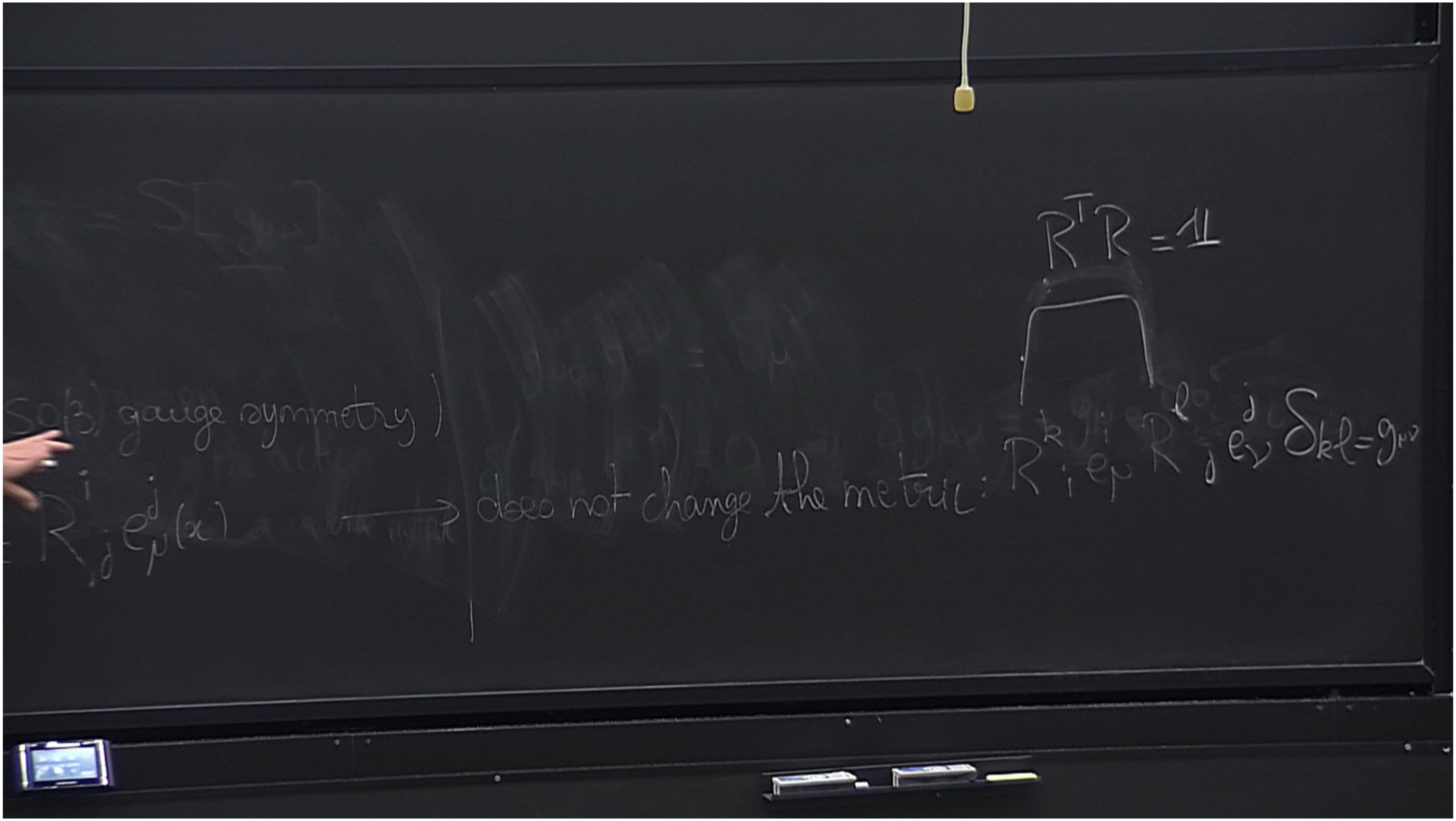


$= S[g_{\mu\nu}]$

$R^T R = \mathbb{1}$

(SO(3) gauge symmetry)

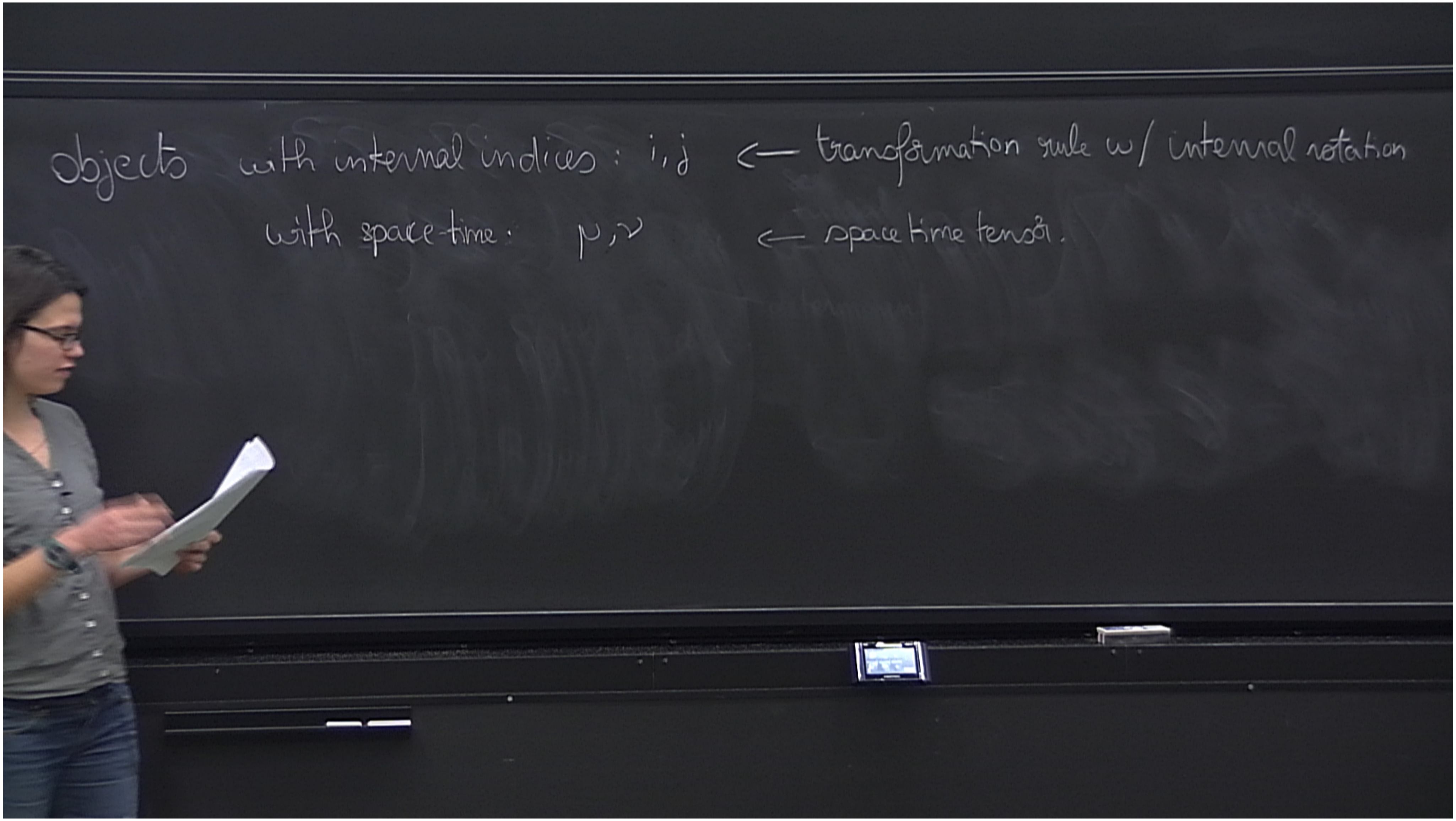
$) = R^i{}_j e^j_\mu(x) \longrightarrow$ does not change the metric: $R^k{}_i e_\mu{}^i R^l{}_j e_\nu{}^j \delta_{kl}$



$R^T R = \mathbb{1}$

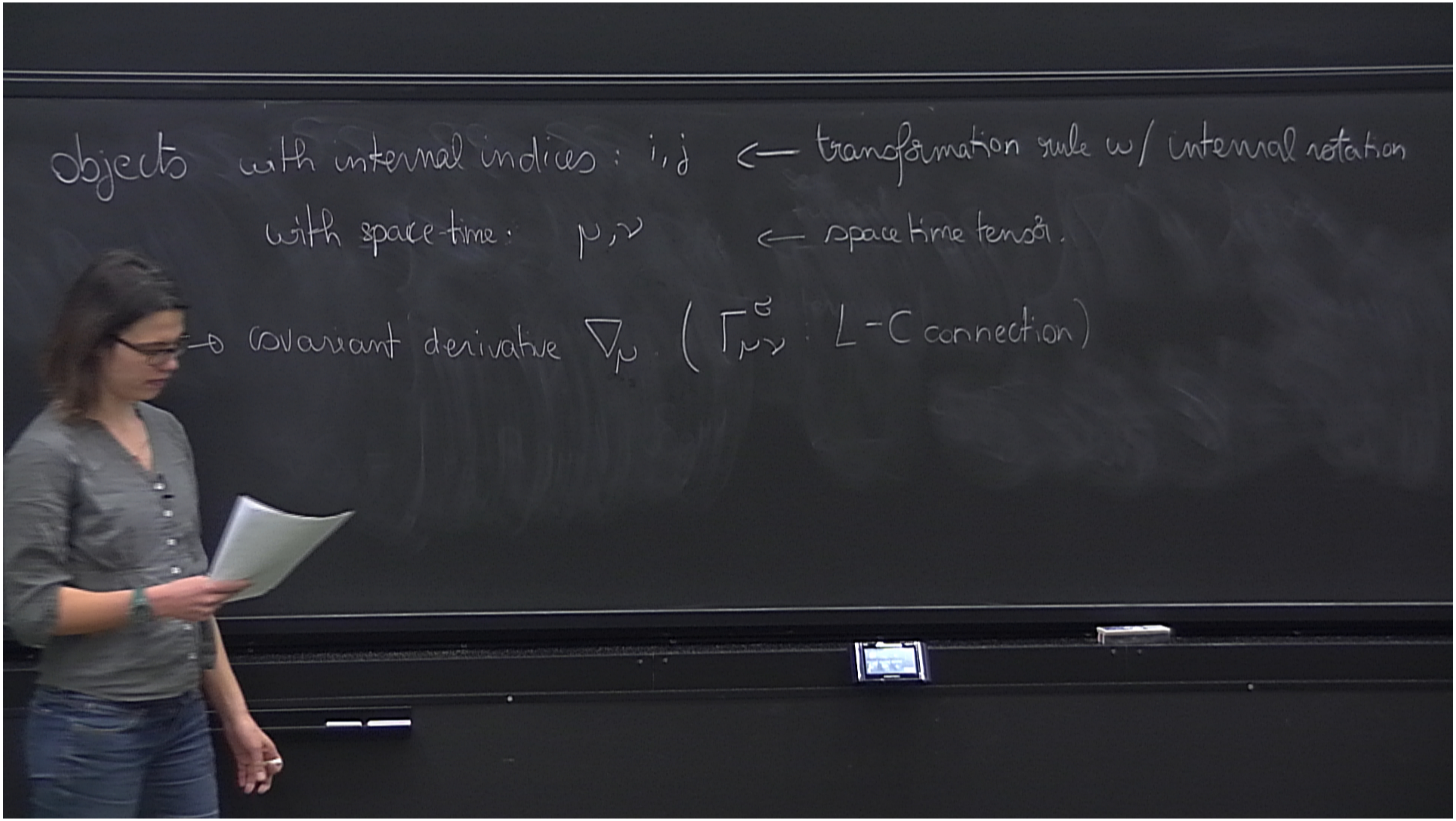
SO(3) gauge symmetry)

$R^i{}_j e^j_\mu(x) \longrightarrow$ does not change the metric: $R^k{}_i e^i_\mu R^l{}_j e^j_\nu \delta_{kl} = g_{\mu\nu}$



objects with internal indices : i, j ← transformation rule w/ internal rotation

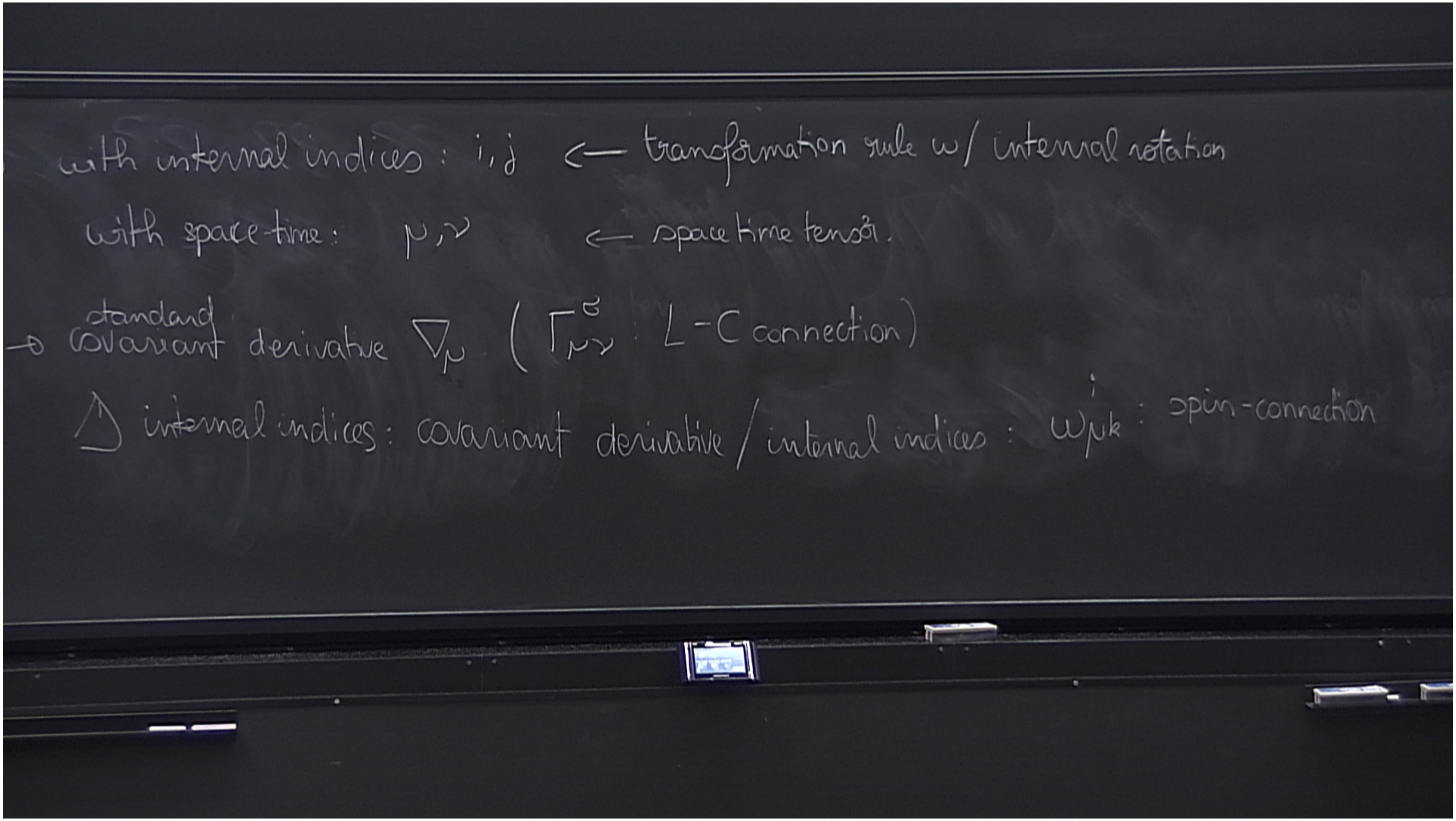
with space-time : μ, ν ← space time tensor.



objects with internal indices : i, j ← transformation rule w/ internal rotation

with space-time : μ, ν ← space time tensor.

→ covariant derivative ∇_μ ($\Gamma^{\sigma}_{\mu\nu}$: L-C connection)

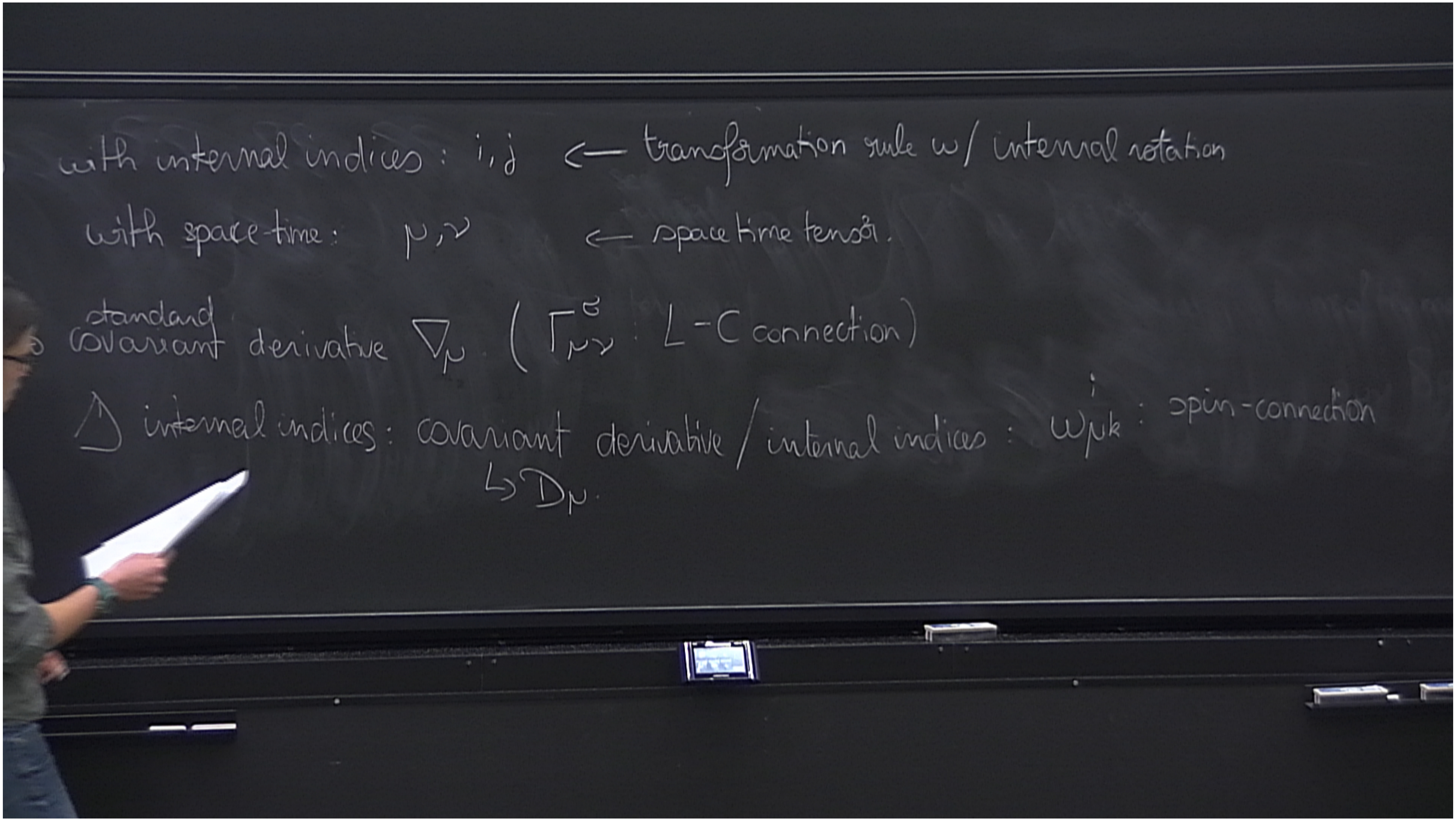


with internal indices : i, j ← transformation rule w/ internal rotation

with space-time : μ, ν ← space time tensor.

→ standard covariant derivative ∇_μ ($\Gamma_{\mu\nu}^{\sigma}$: L-C connection)

⚠ internal indices : covariant derivative / internal indices : $\omega_{\mu k}^{i}$: spin-connection



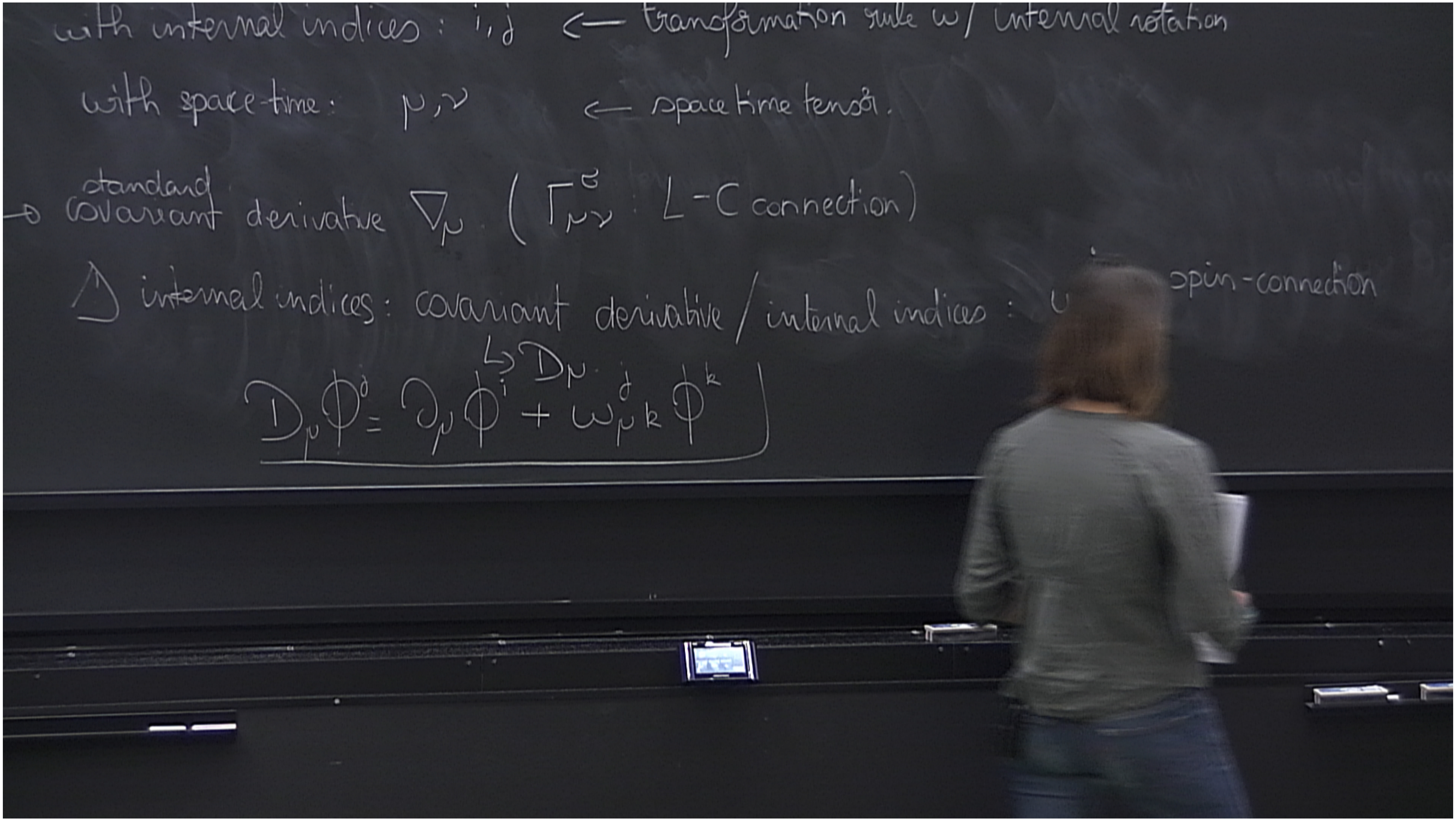
with internal indices : i, j ⟵ transformation rule w/ internal rotation

with space-time : μ, ν ⟵ space time tensor.

standard covariant derivative ∇_μ ($\Gamma^{\sigma}_{\mu\nu}$: L-C connection)

⚠ internal indices : covariant derivative / internal indices : $\omega_\mu{}^{i}{}_{k}$: spin-connection

↳ D_μ.



with internal indices : i, j ← transformation rule w/ internal rotation

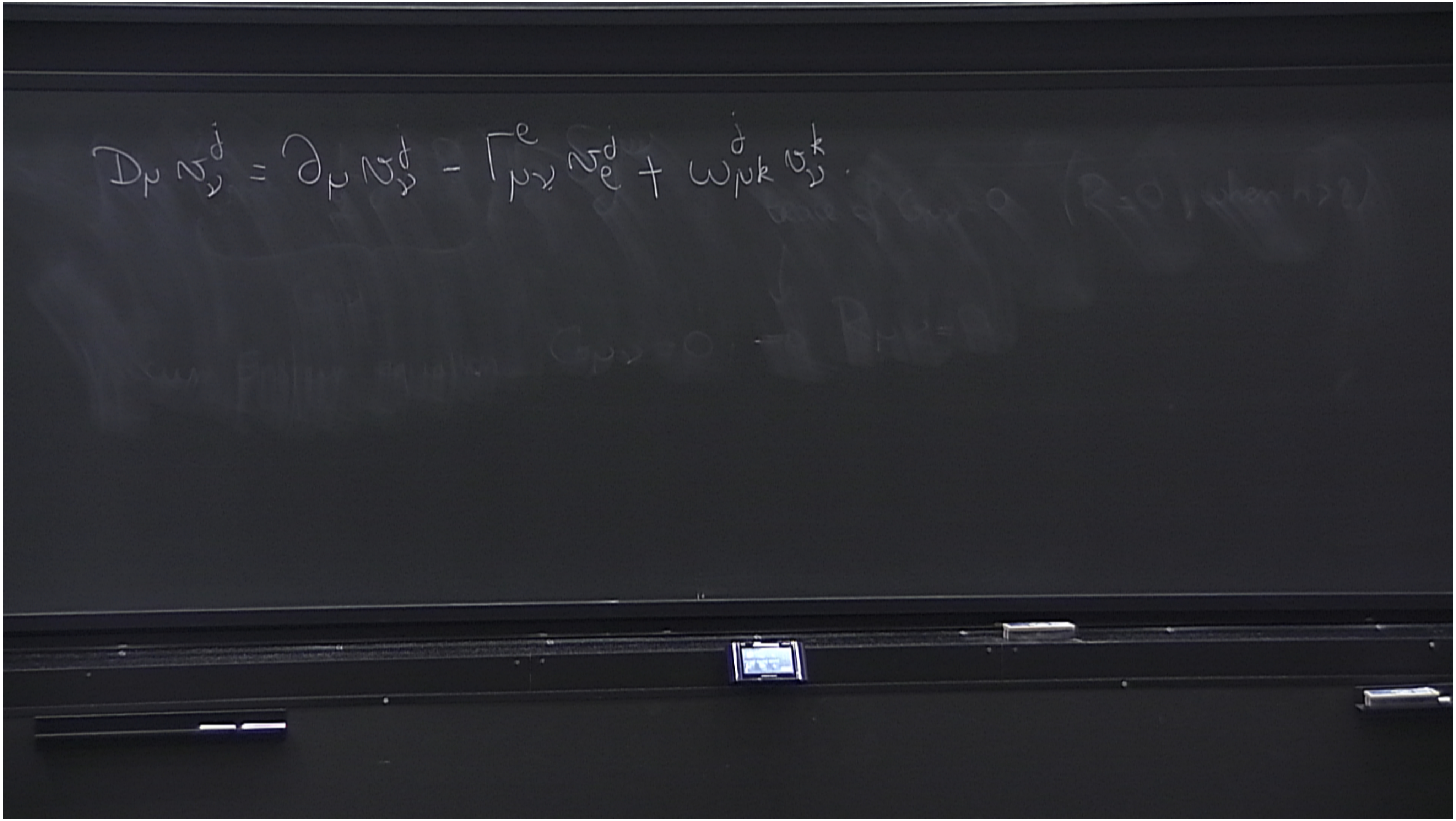
with space-time : μ, ν ← space time tensor.

→ standard covariant derivative ∇_μ ($\Gamma^\sigma_{\mu\nu}$: L-C connection)

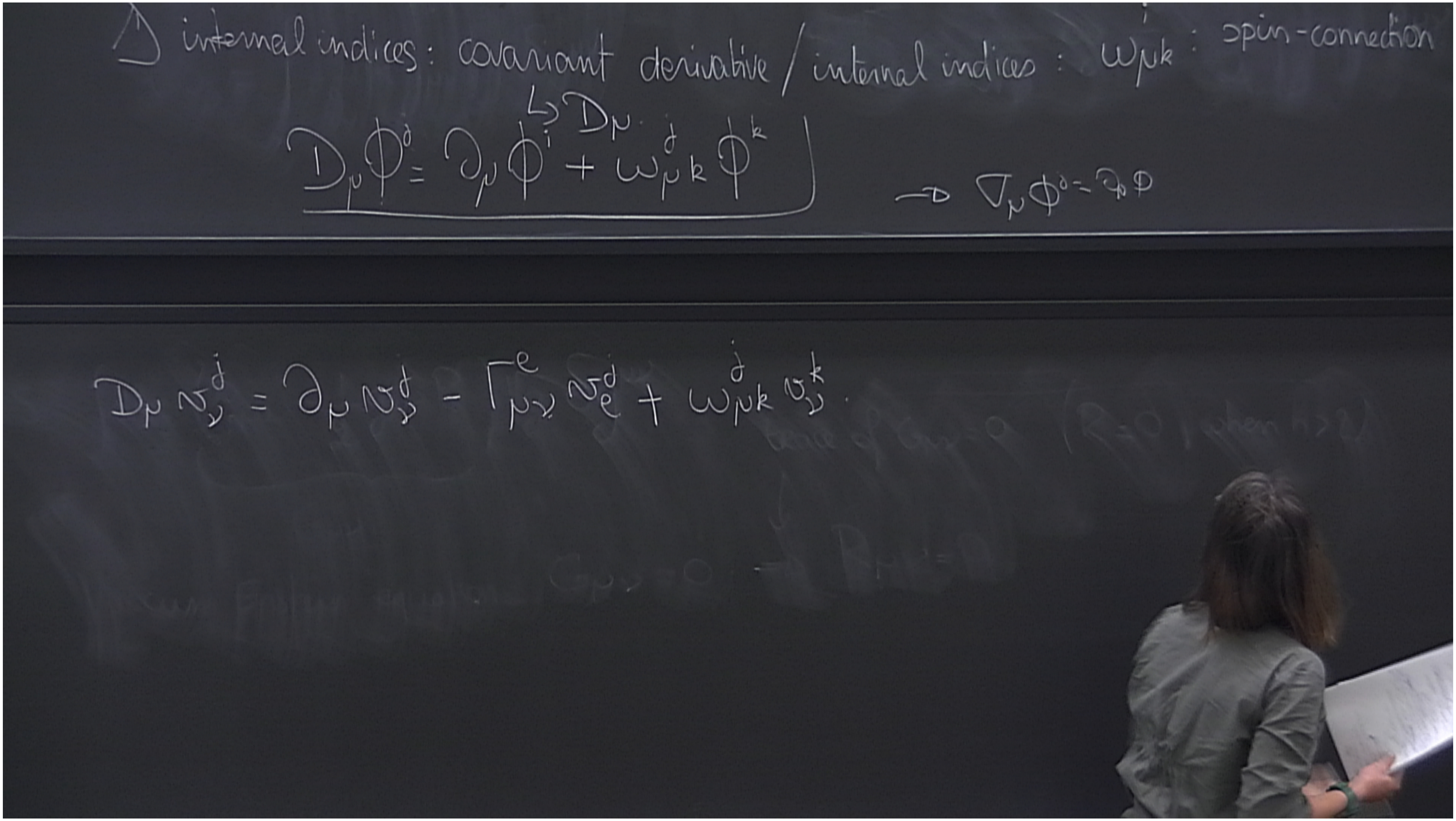
⚠ internal indices : covariant derivative / internal indices : ω [illegible] spin-connection

↳ D_μ.

$$\boxed{D_\mu \phi^j = \partial_\mu \phi^i + \omega_\mu{}^j{}_k \phi^k}$$



$$D_\mu v^j_\nu = \partial_\mu v^j_\nu - \Gamma^e_{\mu\nu} v^j_e + \omega_\mu{}^j{}_k v^k_\nu .$$



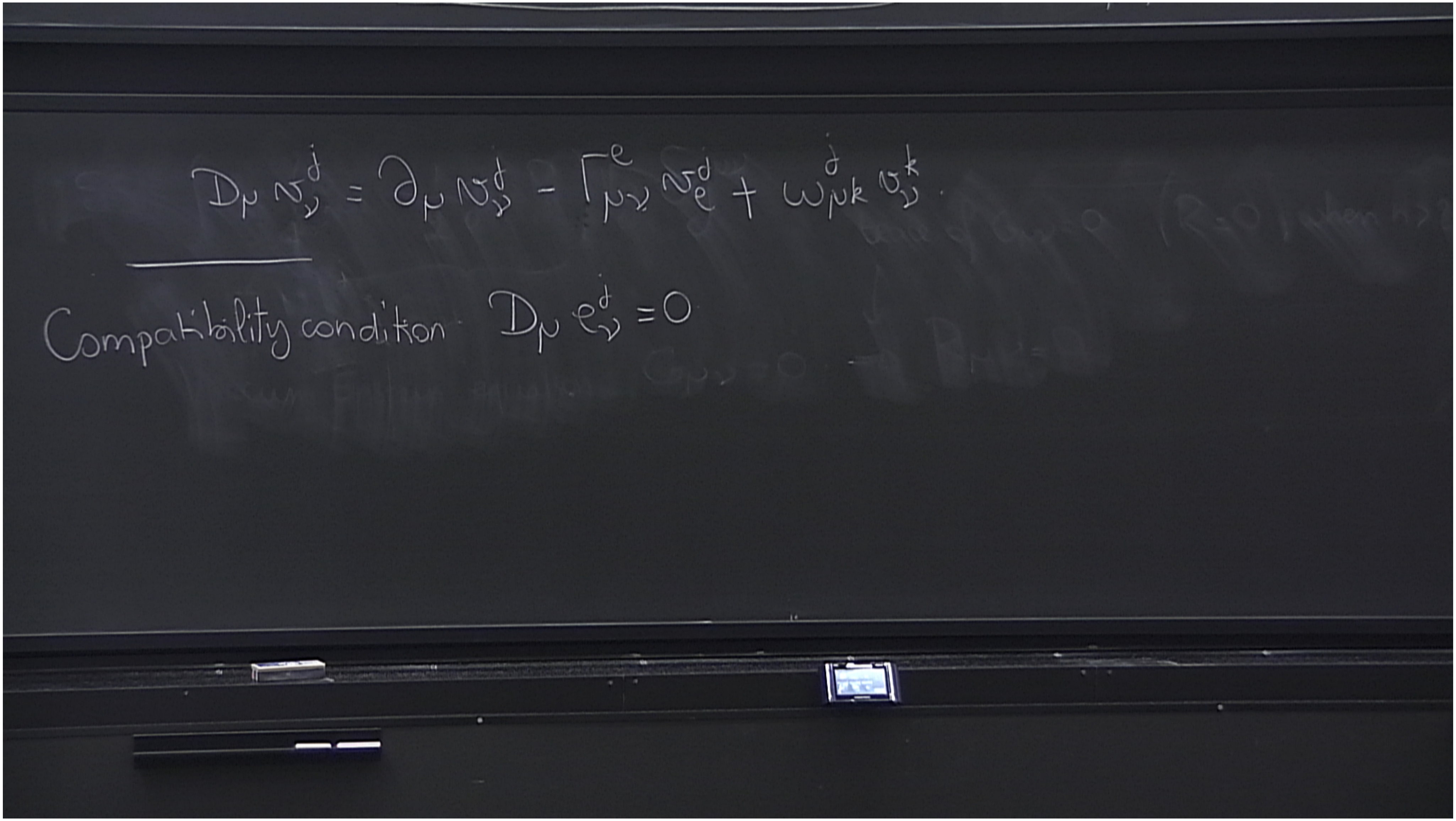
⚠ internal indices: covariant derivative / internal indices : $\omega_\mu{}^j{}_k$: spin-connection

↳ D_μ.

$$\underline{D_\mu \phi^j = \partial_\mu \phi^j + \omega_\mu{}^j{}_k \phi^k}$$

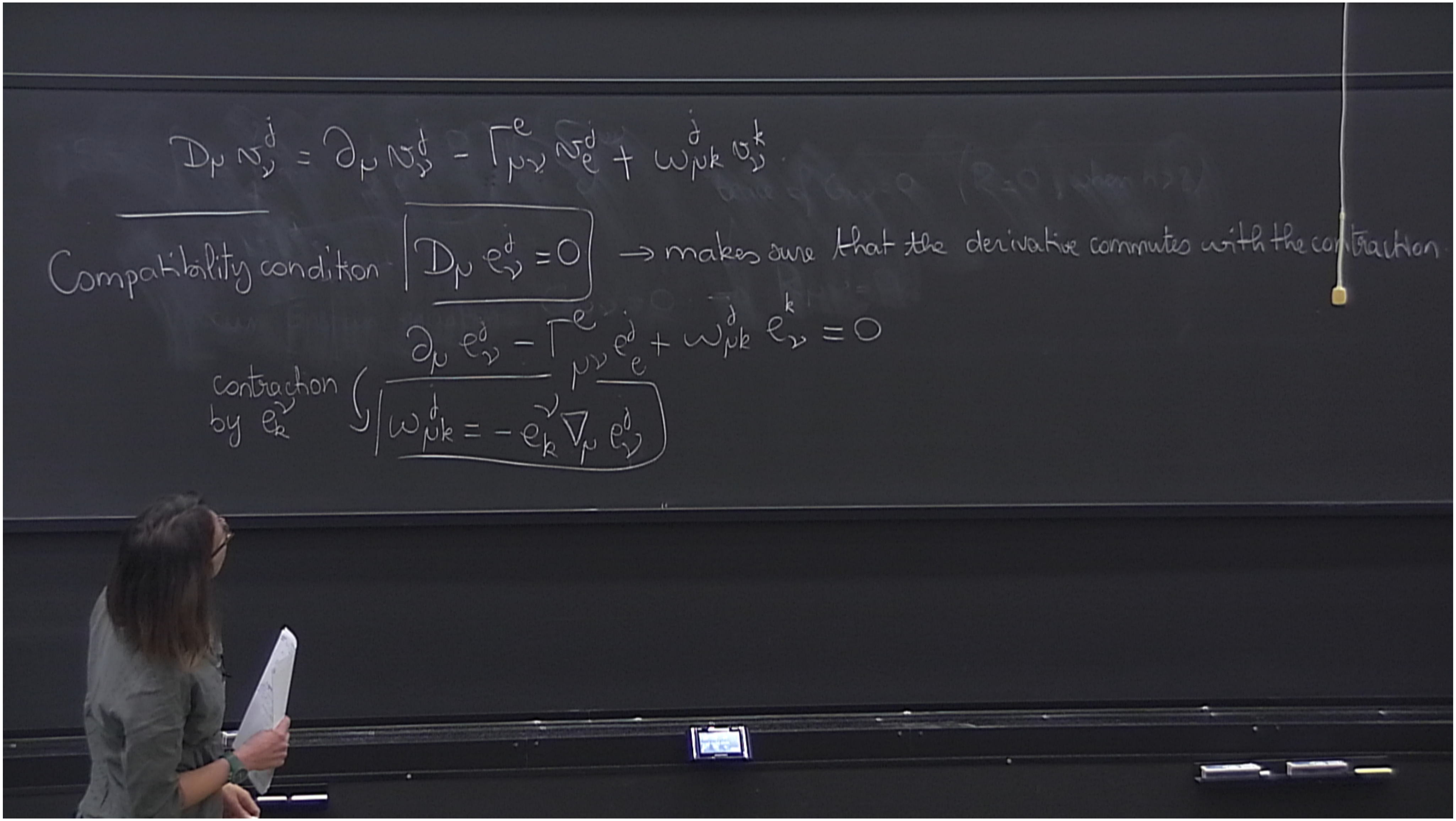
$\longrightarrow \nabla_\nu \phi^j = \partial_\nu \phi$

$$D_\mu v_\nu^j = \partial_\mu v_\nu^j - \Gamma^e_{\mu\nu} v_e^j + \omega_\mu{}^j{}_k v_\nu^k .$$



$$D_\mu v_\nu^j = \partial_\mu v_\nu^j - \Gamma^e_{\mu\nu} v_e^j + \omega_\mu{}^j{}_k v_\nu^k$$

Compatibility condition $D_\mu e_\nu^j = 0$



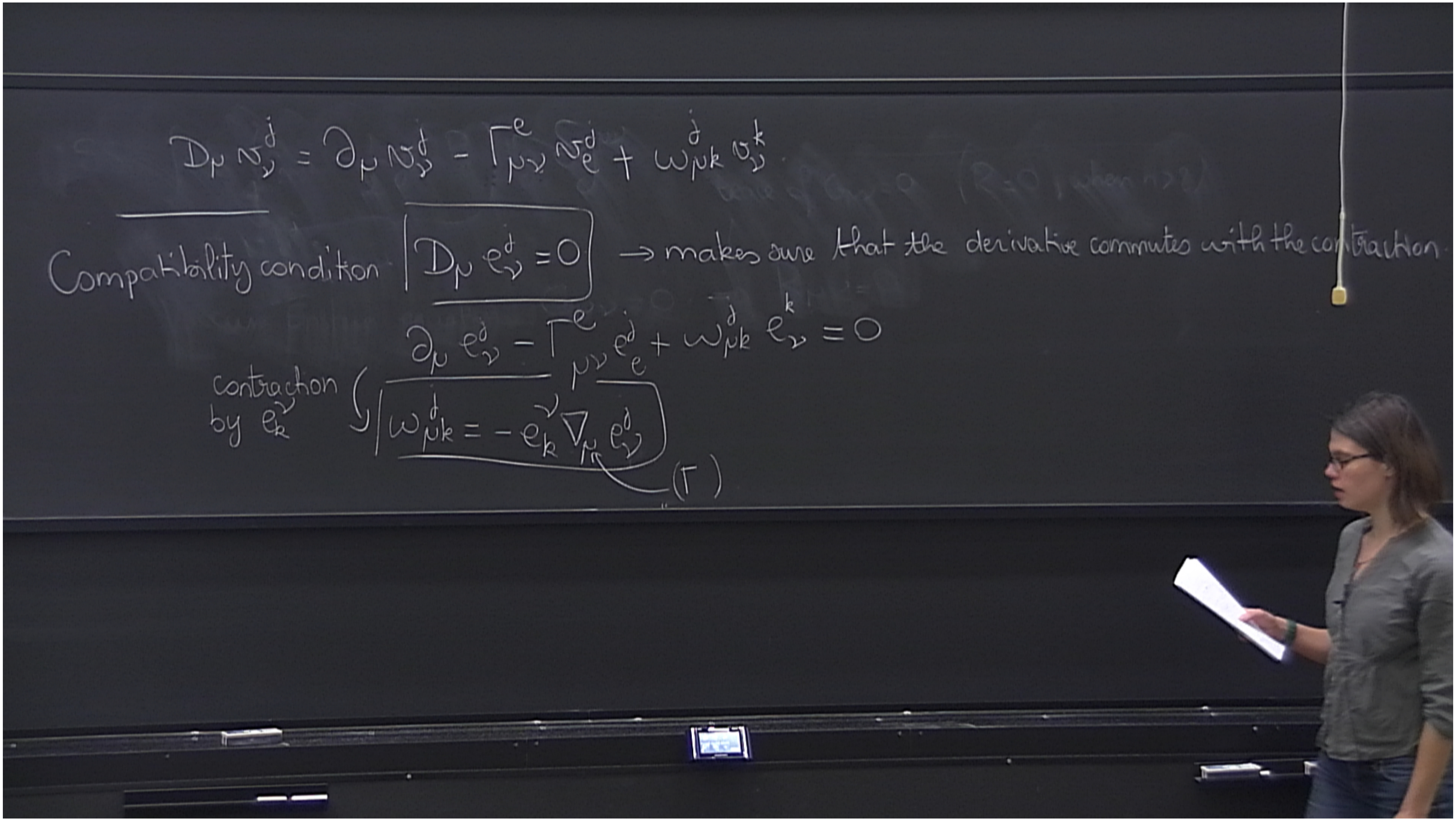
$$D_\mu v^j_\nu = \partial_\mu v^j_\nu - \Gamma^e_{\mu\nu} v^j_e + \omega^j_{\mu k} v^k_\nu$$

Compatibility condition $\boxed{D_\mu e^j_\nu = 0}$ → makes sure that the derivative commutes with the contraction.

$$\partial_\mu e^j_\nu - \Gamma^e_{\mu\nu} e^j_e + \omega^j_{\mu k} e^k_\nu = 0$$

contraction by $\overset{\nu}{e_k}$

$$\boxed{\omega^j_{\mu k} = - \overset{\nu}{e_k} \nabla_\mu e^j_\nu}$$



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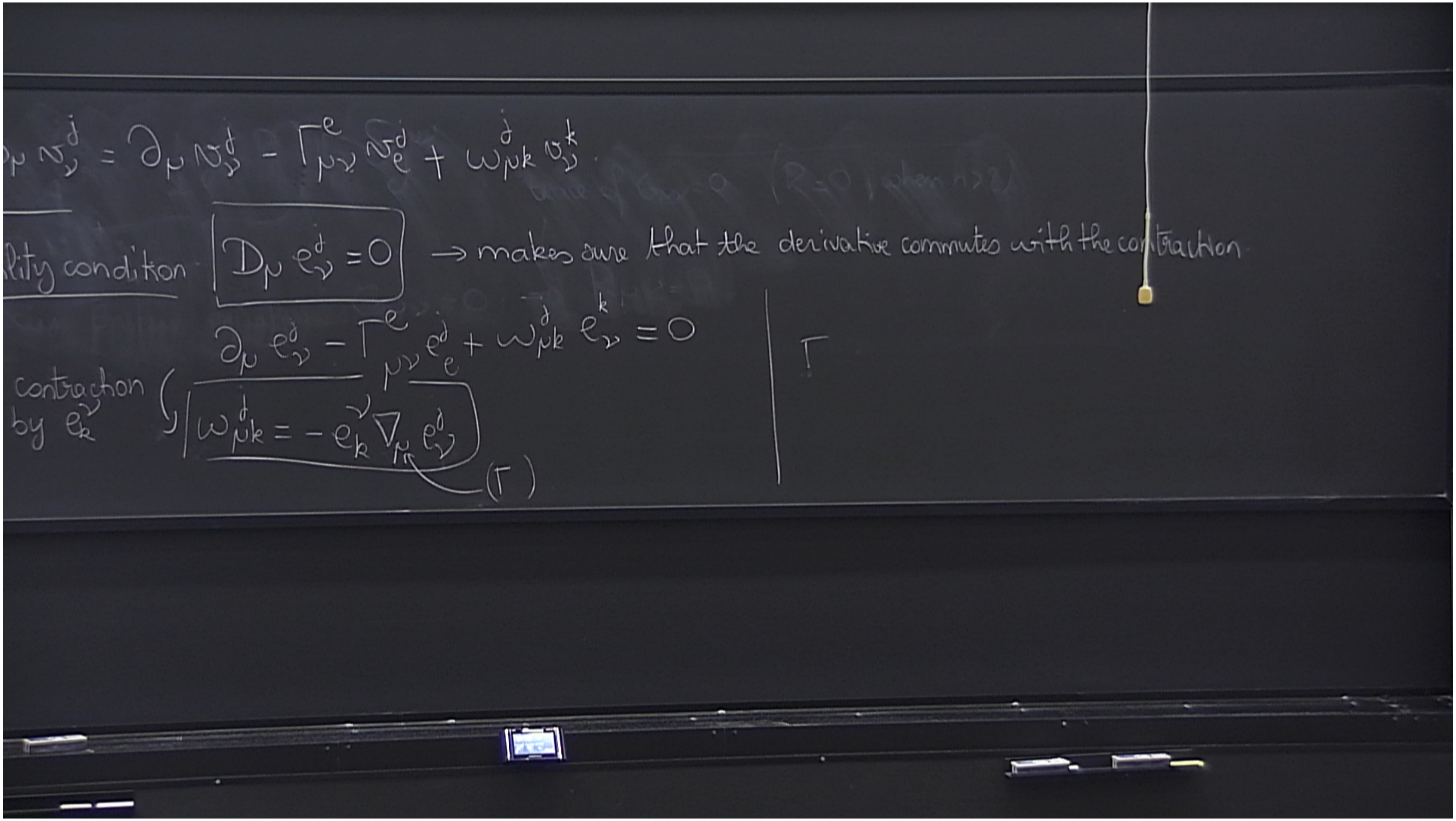
contraction by e^ν_k

$$\boxed{\omega^j_{\mu k} = - e^\nu_k \nabla_\mu e^j_\nu}$$

(Γ)



$$_\mu v^j_\nu = \partial_\mu v^j_\nu - \Gamma^e_{\mu\nu} v^j_e + \omega^j_{\mu k} v^k_\nu .$$

lity condition $\boxed{D_\mu e^j_\nu = 0}$ → makes sure that the derivative commutes with the contraction.

$$\partial_\mu e^j_\nu - \Gamma^e_{\mu\nu} e^j_e + \omega^j_{\mu k} e^k_\nu = 0$$

contraction by e^ν_k

$$\boxed{\omega^j_{\mu k} = - e^\nu_k \nabla_\mu e^j_\nu} \quad (\Gamma)$$

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