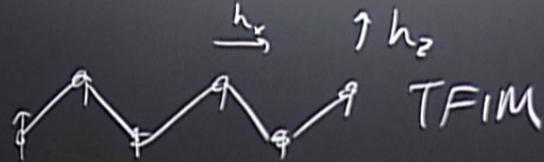
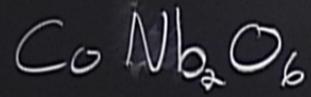


Title: PHYS 733 - Quantum Many-Body Physics (W2016) - Roger Melko - Lecture 14

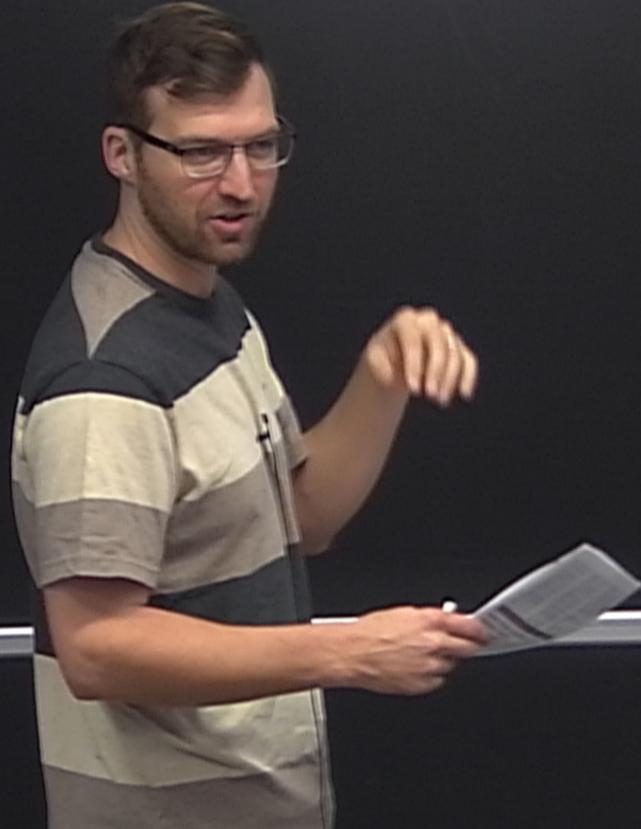
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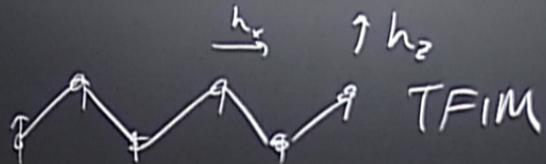
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Abstract:



$$S^{zz}(k=0, \omega)$$
$$\langle S^z(x) S^z(0) \rangle$$





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$$\langle S^z(x) S^z(0) \rangle$$

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$$1D) \quad G_0(x=0) = G(x) = e^{-|x|/\xi} \quad \Gamma = \text{mass} \neq 0$$

$$\phi(x=0) \rightsquigarrow \phi(x)$$

$$\boxed{G_0(x) \text{ diverges on } \mathbb{R} \quad \Gamma = \text{mass} = 0 \quad \Rightarrow d=1, d=2}$$

$$1D) \quad G_0(x=0) = G(x) = e^{-\frac{1}{2} m |x|} \quad m = \text{mass} \neq 0$$

$$\phi(x=0) \longleftrightarrow \phi(x) \quad \text{for } d > 2 \quad G_0(x) = \frac{\Gamma\left(\frac{d-2}{2}\right)}{4\pi^{d/2}} |x|^{2-d}$$

$G_0(x)$, d diverges \uparrow \mathbb{R} $n = \text{mass} = 0$
 $\approx d=1, d=2$

$$G_0(x) = \frac{1}{4\pi^{d/2}} |x|^{2-d}$$

Perturbation theory

- We know how to solve Gaussian theories
- these ignore interactions: consider

$$S = \int d^d x \left\{ \underbrace{\frac{1}{2} [(\partial\phi)^2 + r\phi^2]}_{S_0} + \underbrace{\frac{\lambda}{4!} \phi^4}_{S_I} \right\}$$

$$g_0(x) = \frac{1}{4\pi^{d/2}} |x|^{2-d}$$

Perturbation theory

- We know how to solve Gaussian theories
- these ignore interactions: consider

$$S = \int d^d x \left\{ \underbrace{\frac{1}{2} [(\partial\phi)^2 + r\phi^2]}_{S_0} + \underbrace{\frac{\lambda}{4!} \phi^4}_{S_I} \right\}$$

We only know how to integrate when $S_I=0$. Otherwise handle perturbatively.
Consider first the free energy of the theory. $F = -T \ln Z$

CAUTION

We only know how to integrate when $S_I = 0$. Otherwise handle perturbatively.

Consider first the free energy of the theory. $F = -T \ln Z$

$$Z = \int \mathcal{D}[\phi] e^{-S_0 - S_I} = \underbrace{\int \mathcal{D}[\phi] e^{-S_0}}_{Z_0} \frac{\int \mathcal{D}[\phi] e^{-S_0 - S_I}}{\int \mathcal{D}[\phi] e^{-S_0}}$$

$$= Z_0 \left\langle e^{-S_I} \right\rangle_0$$

$$\langle e^{-\frac{u}{4!} \int \phi^4} \rangle_0 = \langle 1 - \frac{u}{4!} \int \phi^4 + \frac{1}{2} \left(\frac{u}{4!} \right)^2 \iint \phi^4 \phi^4 + \dots \rangle_0$$

$$= 1 - \frac{u}{4!} \int_r \langle \phi^4(r) \rangle_0 + \frac{1}{2} \left(\frac{u}{4!} \right)^2 \iint_{r, r'} \langle \phi^4(r) \phi^4(r') \rangle_0 + \dots$$

CAUTION

$$\langle e^{-\int \phi^4} \rangle_0 = \langle 1 - \int \phi^4 + \frac{1}{2} \left(\int \phi^4 \right)^2 - \dots \rangle_0$$

$$= 1 - \frac{g}{4!} \int_r \langle \phi^4(r) \rangle_0 + \frac{1}{2} \left(\frac{g}{4!} \right)^2 \int_r \int_{r'} \langle \phi^4(r) \phi^4(r') \rangle_0 + \dots$$

We know how to calculate the expectation values with Wick's Th.

$$\langle e^{-\int \phi^4} \rangle_0 = \langle 1 - \int \phi^4 + \frac{1}{2} \left(\int \phi^4 \right)^2 + \dots \rangle_0$$

$$= 1 - \frac{u}{4!} \int_r \langle \phi^4(r) \rangle_0 + \frac{1}{2} \left(\frac{u}{4!} \right)^2 \iint_{r, r'} \langle \phi^4(r) \phi^4(r') \rangle_0 + \dots$$

We know how to calculate the expectation values with Wick's Th.

e.g.) $\langle \phi^3(r) \phi(r') \rangle \Rightarrow \text{diagram} \rightarrow \Rightarrow 3G(r)$

$$\langle e^{-\int \phi^2} \rangle_0 = \langle 1 - \int \phi^2 + \frac{1}{2} \left(\int \phi^2 \right)^2 - \dots \rangle_0$$

$$= 1 - \frac{1}{4!} \int_r \langle \phi^4(r) \rangle_0 + \frac{1}{2} \left(\frac{1}{4!} \right)^2 \iint_{r, r'} \langle \phi^4(r) \phi^4(r') \rangle_0 + \dots$$

We know how to calculate the expectation values with Wick's Th.

e.g.) $\langle \phi^3(r) \phi(r') \rangle \Rightarrow 3 \langle \phi^2(r) \phi(r') \rangle \Rightarrow 3 G(r-r) G(r'-r)$
 $= 3 G(0) G(r'-r)$

$$\langle e^{-\int \phi^4} \rangle_0 = \langle 1 - \int \phi^4 + \frac{1}{2} \left(\int \phi^4 \right)^2 - \dots \rangle_0$$

$$= 1 - \frac{4}{4!} \int_r \langle \phi^4(r) \rangle_0 + \frac{1}{2} \left(\frac{4}{4!} \right)^2 \iint_{r, r'} \langle \phi^4(r) \phi^4(r') \rangle_0 + \dots$$

We know how to calculate the expectation values with Wick

e.g.) $\langle \phi^3(r) \phi(r') \rangle \Rightarrow 3 \text{ (diagram)} \Rightarrow 3 G(r-r) G(r'-r)$
 $= 3 G(0) G(r'-r)$

Thus

$$\int_r \langle \phi^4(r) \rangle_0 \rightarrow \int_r \text{ (diagram) } + \dots$$



$$\begin{aligned}
\langle \phi^4(r) \phi^4(r') \rangle_0 &\Rightarrow \begin{array}{c} \diagup \diagdown \\ \times \\ \diagdown \diagup \end{array} \quad \begin{array}{c} \diagup \diagdown \\ \times \\ \diagdown \diagup \end{array} \\
&= \begin{array}{c} \diagup \diagdown \\ \circ \\ \diagdown \diagup \end{array} + \begin{array}{c} \diagup \diagdown \\ \circ \\ \diagdown \diagup \end{array} \quad [3G^2(0)]^2 \\
&= \begin{array}{c} \diagup \diagdown \\ \circ \\ \diagdown \diagup \end{array} \quad \begin{array}{c} \diagup \diagdown \\ \circ \\ \diagdown \diagup \end{array} \quad G^2(0)G^2(r'-r)
\end{aligned}$$

$$\begin{aligned}
\langle \phi^4(r) \phi^4(r') \rangle_0 &\Rightarrow \text{X}_{r'} \quad \text{X}_{r'} \\
&= \text{O}_{r'} + \text{O}_{r'} \quad [3G^2(0)]^2 \\
&= \text{O}_{r'} \text{O}_{r'} \quad G^2(0)G^2(r'-r)
\end{aligned}$$

Thus

$$\langle e^{-S_1} \rangle_0 = 1 - \frac{2}{4!} \int_r (3\infty) + \frac{1}{2} \left(\frac{2}{4!} \right)^2 \int_{r,r'} \left[9\infty + 72\infty + 24\Theta \right]$$

where

$$\int_r \infty = \int_r G^2(0) = \int d^d x G^2(0) =$$

Thus

$$\langle e^{-S_H} \rangle_0 = 1 - \frac{2}{4!} \int_r (3^\infty) + \frac{1}{2} \left(\frac{2}{4!} \right)^2 \int_{r,r'} \left[9^\infty + 72^\infty + 24\Theta \right]$$

where

$$\int_r^\infty = \int_r G^2(\omega) = \int d^d x G^2(\omega) = L^d G^2(\omega)$$

$$\int_{r,r'}^\infty = \int_r \int_{r'} G^4(\omega) = L^{2d} G^4(\omega)$$

F_0 (Gaussian case)

$$S - F - F_0 = -T \log \langle e^{-S_I} \rangle$$

F_0 (Gaussian case)

$$\begin{aligned}\delta F = F - F_0 &= -T \log \langle e^{-S_I} \rangle_0 \\ &= -T \ln \left(1 - \langle S_I \rangle_0 + \frac{1}{2} \langle S_I^2 \rangle_0 + \dots \right)\end{aligned}$$

F_0 (Gaussian case)

$$\delta F = F - F_0 = -T \log \langle e^{-S_I} \rangle_0$$

$$= -T \ln \left(1 - \langle S_I \rangle_0 + \frac{1}{2} \langle S_I^2 \rangle_0 + \dots \right)$$

use $\ln(1+\epsilon) \approx \epsilon - \frac{\epsilon^2}{2} + \frac{\epsilon^3}{3} - \frac{\epsilon^4}{4} \dots$

$$\delta F = -T \left[-\langle S_I \rangle_0 + \frac{1}{2} \left(\langle S_I^2 \rangle_0 - \langle S_I \rangle_0^2 \right) + \mathcal{O}(\epsilon^3) \right]$$

CAUTION

DO NOT TOUCH THE WRITING SURFACE.
A GLASS OF THE BRICKS OF THE BOARD.

$$\delta F = -T \left[-\langle S_I \rangle_0 + \frac{1}{2} \left(\langle S_I^2 \rangle_0 - \langle S_I \rangle_0^2 \right) + \mathcal{O}(S_I^3) \right]$$

$\begin{matrix} \infty & + & \infty \\ \infty & + & \infty \\ & + & \infty \\ & + & \infty \end{matrix}$

CAUTION

$$\delta F = -T \left[-\langle S_I \rangle_0 + \frac{1}{2} \left(\langle S_I^2 \rangle_0 - \langle S_I \rangle_0^2 \right) + \mathcal{O}(S_I^3) \right]$$

$\left[\begin{array}{c} \infty \\ \infty \end{array} \right] + \infty - \left[\begin{array}{c} \infty \\ \infty \end{array} \right]$

$$\delta F = -T \left[-\langle S_I \rangle_0 + \frac{1}{2} \left(\langle S_I^2 \rangle_0 - \langle S_I \rangle_0^2 \right) + \mathcal{O}(S_I^3) \right]$$

$$\left[\begin{array}{c} \infty \\ \infty \end{array} \right] + \infty - \left[\begin{array}{c} \infty \\ \infty \end{array} \right]$$

I

$$= 3 \ln_0(0)$$

CAUTION

DO NOT LEAN ON THE BOARD OR WRITE ON IT.
A CRACK IN THE BOARD IS THE SIGN OF THE BOARD.

In fact, one can show in general that
 \mathcal{Z}_F = "sum of all connected diagrams"
 (can be proven with a "replica trick" calculation)



$$\langle \phi^4(n) \rangle_0 \rightarrow \left[\text{diagram} + \text{diagram} \right] = 3 G_0^2(0)$$

CAUTION

Physically, if the disconnected diagrams did not

vanish, the free energy would not be proportional to L^d (volume), but some higher power.

CAUTION

DO NOT TOUCH THE WRITING BOARD
A CAUTION IS THE SYMBOL OF THE BOARD

Physically, if the disconnected diagrams did not

vanish, the free energy would not be proportional to L^d (volume), but some higher power.

CAUTION

DO NOT TOUCH THE WALLS OR BOARD.
A CRACK IN THE WALL OF THE BOARD

Physically, if the disconnected diagrams did not

vanish, the free energy would not be proportional to L^d (volume), but some higher power (unphysical).

CAUTION

DO NOT TOUCH THE BOARD SURFACE.
A GLASS OF THE BOARD IS THE BOARD.

Physically, if the disconnected diagrams did not vanish, the free energy would not be proportional to L^d (volume), but some higher power (unphysical).

Correlation functions: we can also develop a P.T.

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}[\phi] O e^{-S_0} e^{-S_1}$$

Physically, if the disconnected diagrams did not

vanish, the free energy would not be proportional to L^d (volume), but some higher power (unphysical)

Correlation functions: we can also develop a P.T.

$$\langle O \rangle = \frac{Z_0}{Z_0} \frac{1}{Z} \int \mathcal{D}[\phi] O e^{-S_0} e^{-S_1} = \frac{Z_0}{Z} \left\langle O e^{-S_1} \right\rangle_0$$

$$N = \langle \phi(r_1) \phi(r_2) \rangle_0 + \left(\frac{-2u}{4!} \right) \langle \phi(r_1) \phi(r_2) \phi^4(r) \rangle$$

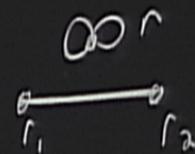
$$N = \langle \phi(r_1) \phi(r_2) \rangle_0 + \left(\frac{-u}{4!} \right) \langle \phi(r_1) \phi(r_2) \phi^4(r) \rangle_0 + \mathcal{O}(u^2)$$

first term $\langle \phi(r_1) \phi(r_2) \rangle_0 = \text{---} = G_0(r_1 - r_2)$

Second term:



- 3 ways to get



- 4 \cdot 3 ways to get



$$N = \text{---} + \text{---} + \text{---} + \mathcal{O}(u^2)$$

Diagram 1: A horizontal line with two vertices labeled r_1 and r_2 .

Diagram 2: A horizontal line with two vertices labeled r_1 and r_2 , and a loop attached to the middle vertex labeled r .

Diagram 3: A horizontal line with two vertices labeled r_1 and r_2 , and a self-energy loop (two circles) attached to the middle vertex labeled r .

$$\mathbb{D} = 1 + \text{---} + \mathcal{O}(u^2) \leftarrow \text{similar to the free energy}$$

Diagram: A horizontal line with a loop attached to the middle vertex.

$$\stackrel{\text{so}}{\langle \phi(r_1) \phi(r_2) \rangle} = \text{---}$$

Diagram: A simple horizontal line connecting two vertices.

now expand the denominator $\frac{1}{1+\epsilon} = 1 - \epsilon + \epsilon^2 - \epsilon^3 + \dots$

$$\langle \phi(r_1) \phi(r_2) \rangle = \left(\frac{1}{r} + 12 \frac{1}{r^3} + 3 \frac{1}{r^5} \right) (1 - 3\epsilon + \dots)$$

The combined expansion of N and D cancels terms like $(3 \frac{1}{r^5}) - (\frac{1}{r}) (3\epsilon) = 0$

now expand the denominator $\frac{1}{1+\epsilon} = 1 - \epsilon + \epsilon^2 - \epsilon^3 + \dots$

$$\langle \phi(r_1) \phi(r_2) \rangle = \left(\frac{1}{2} + 12 \frac{\epsilon}{2} + 3 \frac{\epsilon^2}{2} \right) (1 - 3\epsilon + \dots)$$

The combined expansion of N and D cancels terms like

$$\left(3 \frac{\epsilon^2}{2} \right) - \left(\frac{1}{2} \right) (3\epsilon) = 0$$

exercise: look at the $\left(\frac{4}{4!} \right)^2$ term - $\frac{N}{D}$