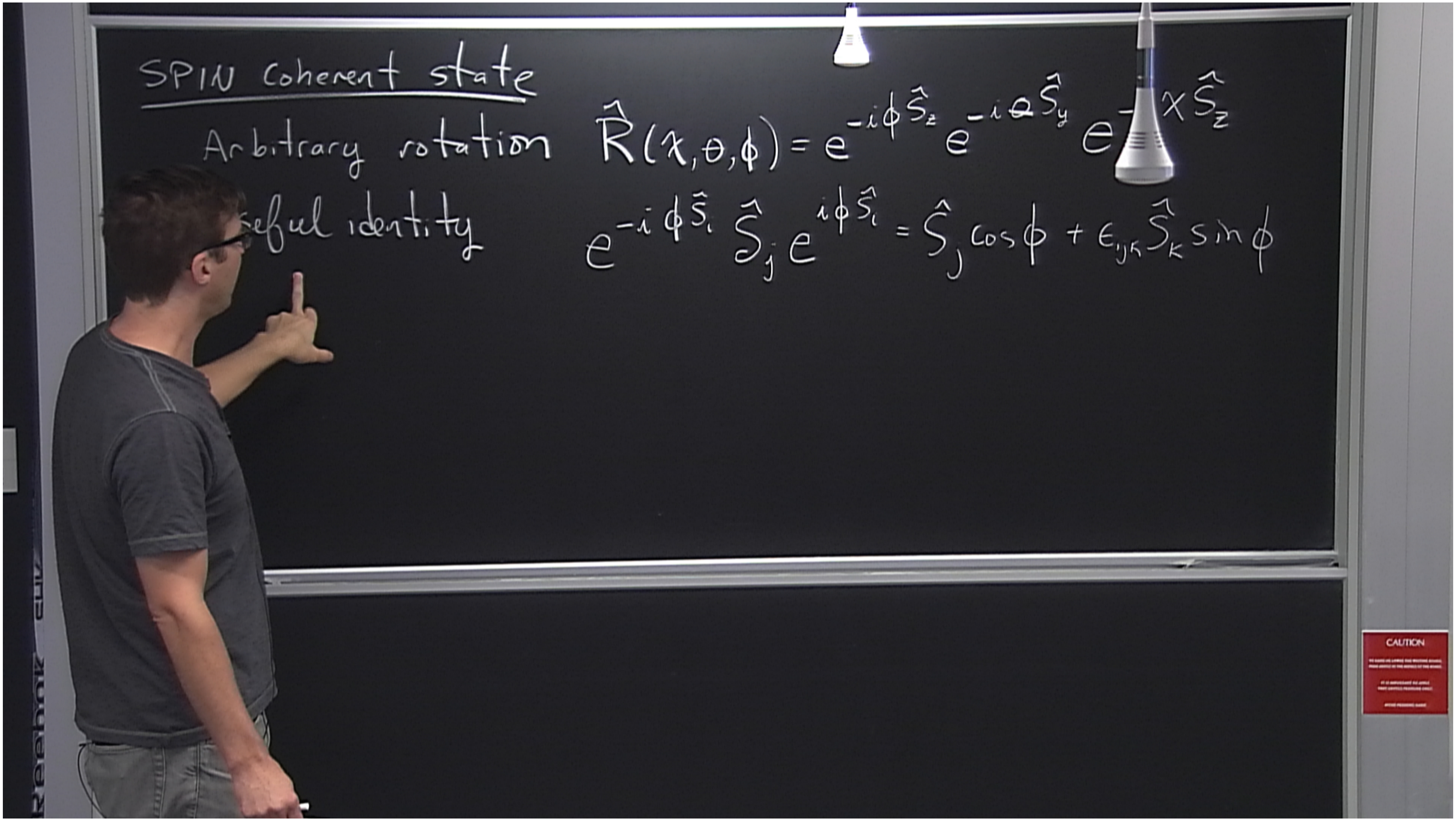
Title: PHYS 733 - Quantum Many-Body Physics (W2016) - Roger Melko - Lecture 10

Date: Feb 04, 2016 10:00 AM

URL: http://pirsa.org/16020013

Abstract:


SPIN Coherent state
Arbitrary rotation $\vec{\hat{R}}(\chi,\theta,\phi) = e^{-i\phi\hat{S}_z} e^{-i\theta\hat{S}_y} e^{-i\chi\hat{S}_z}$
seful identity $e^{-i\phi\hat{S}_i}\hat{S}_j e^{i\phi\hat{S}_i} = \hat{S}_j\cos\phi + \epsilon_{ijk}\hat{S}_k\sin\phi$
CAUTION

$$\hat{R}\hat{S}_z\hat{R}^\dagger = e^{-i\phi S_z} e^{-i\theta S_y} e^{-i\chi S_z} S_z e^{i\chi S_z} e^{i\theta S_y} e^{i\phi S_z}$$

$$= e^{-i\phi S_z} e^{-i\theta S_y} S_z e^{i\theta S_y} e^{i\phi S_z}$$

$$= e^{-i\phi S_z}\left[S_z \cos\theta + S_x \sin\theta \right] e^{i\phi S_z}$$

$$= S_z \cos\theta \, e^{-i\phi S_z} e^{i\phi S_z} + \left[e^{-i\phi S_z} S_x e^{i\phi S_z} \right] \sin\theta$$

$$\hat{R}\hat{S}_z\hat{R}^\dagger = e^{-i\phi S_z} e^{-i\theta S_y} e^{-i\chi S_z} S_z e^{i\chi S_z} e^{i\theta S_y} e^{i\phi S_z}$$

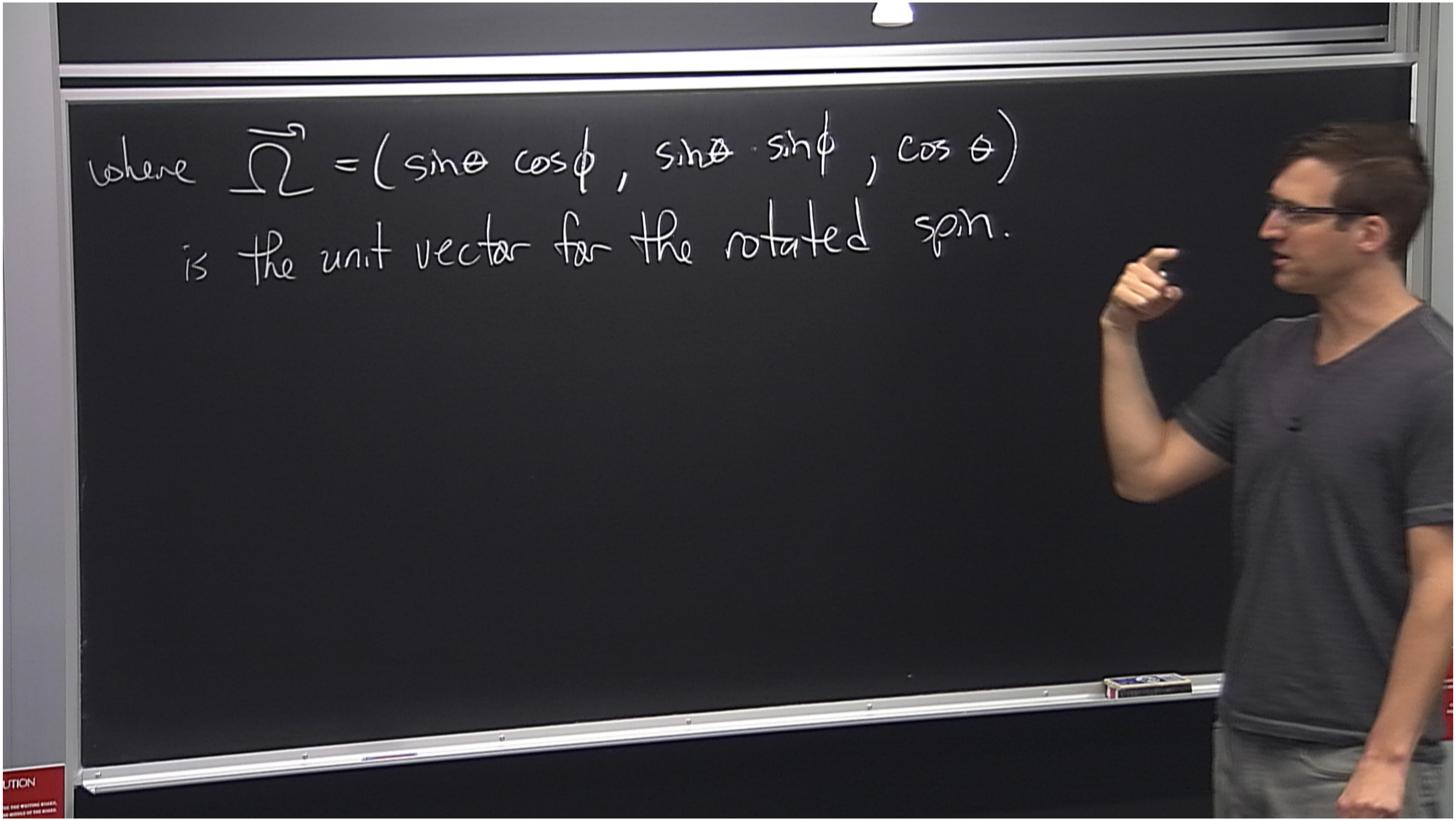
$$= e^{-i\phi S_z} e^{-i\theta S_y} S_z e^{i\theta S_y} e^{i\phi S_z}$$

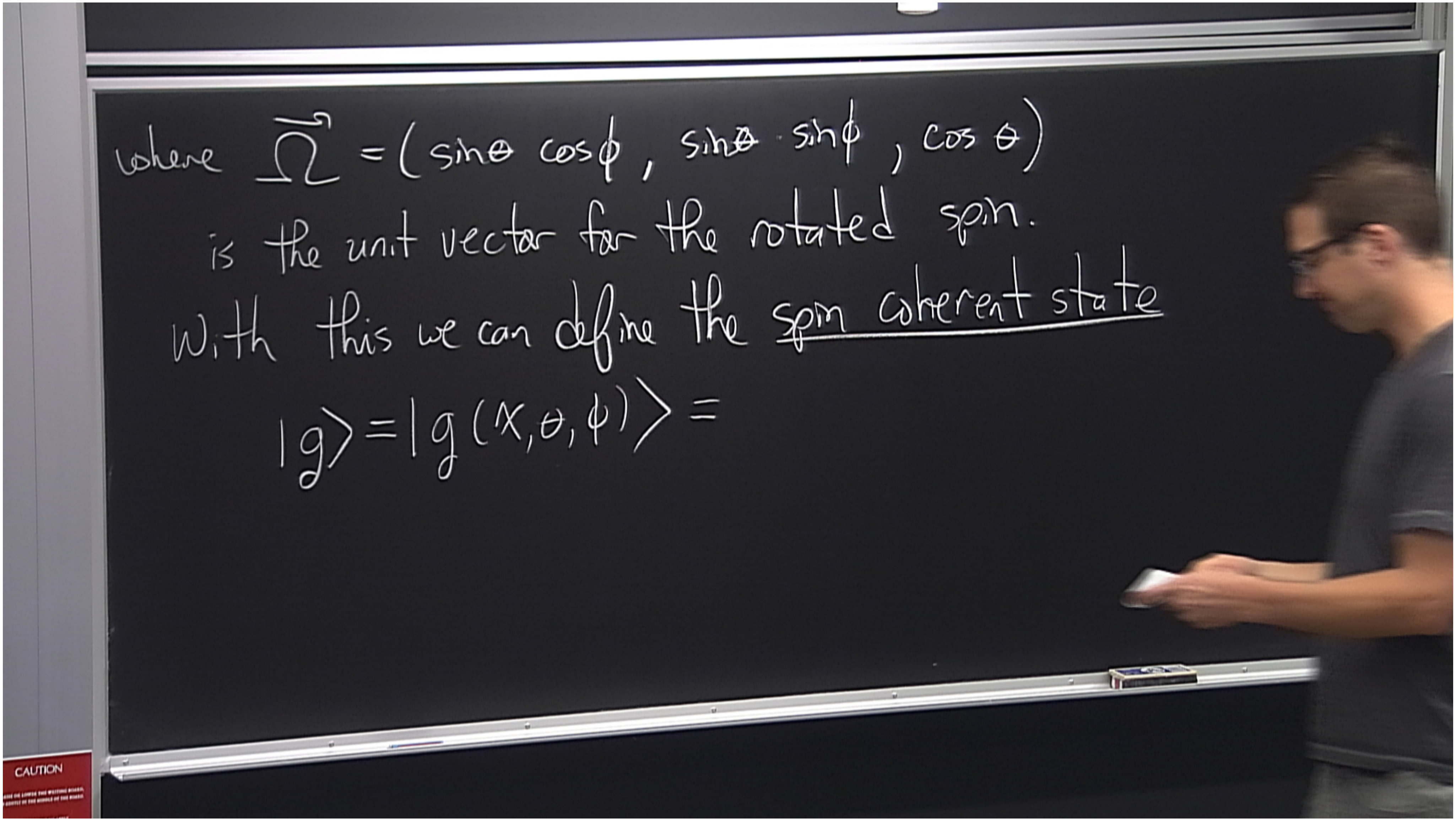
$$= e^{-i\phi S_z}\left[S_z \cos\theta + S_x \sin\theta \right] e^{i\phi S_z}$$

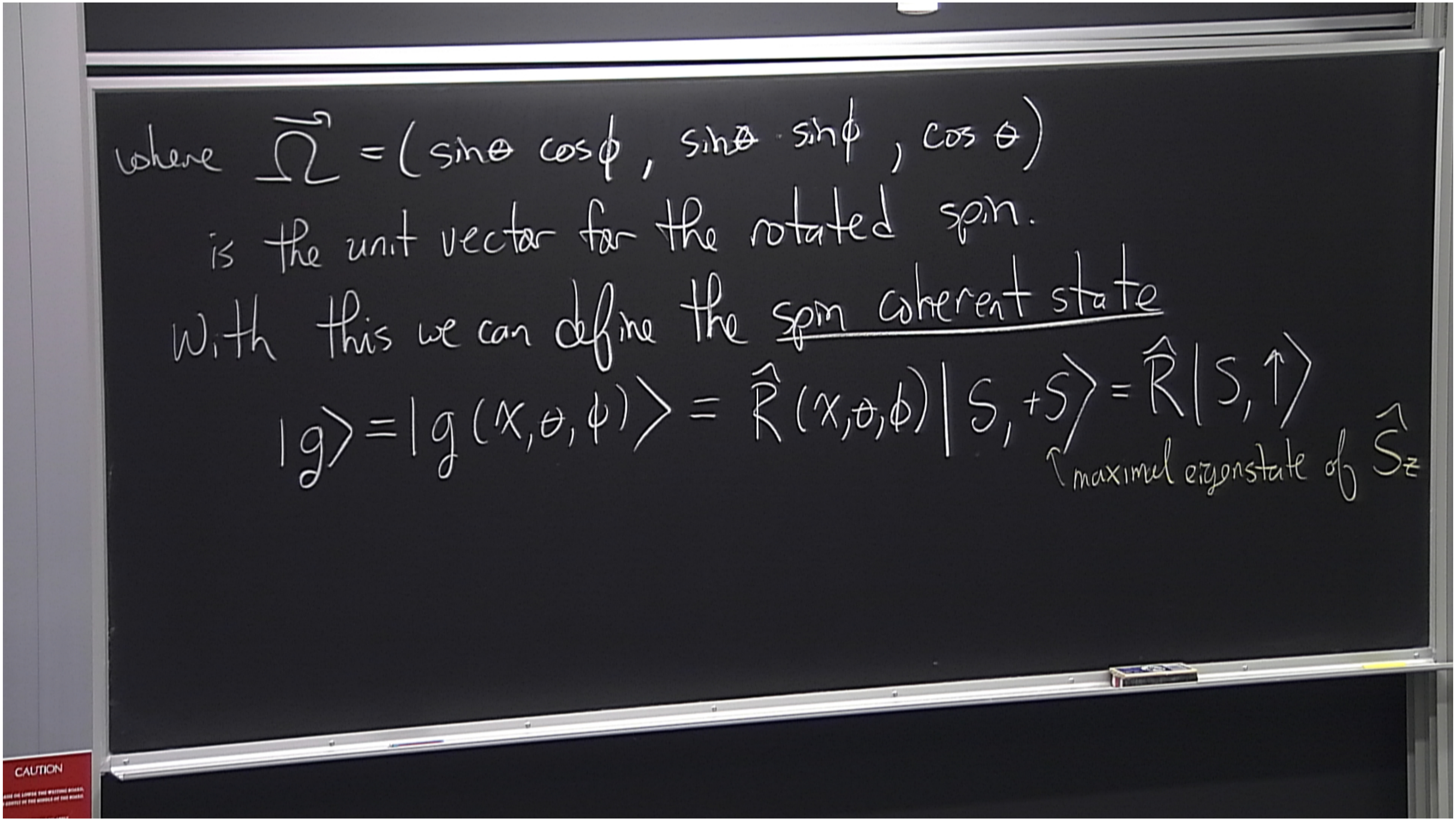
$$= S_z \cos\theta\, e^{-i\phi S_z} e^{i\phi S_z} + \left[e^{-i\phi S_z} S_x e^{i\phi S_z} \right]\sin\theta$$

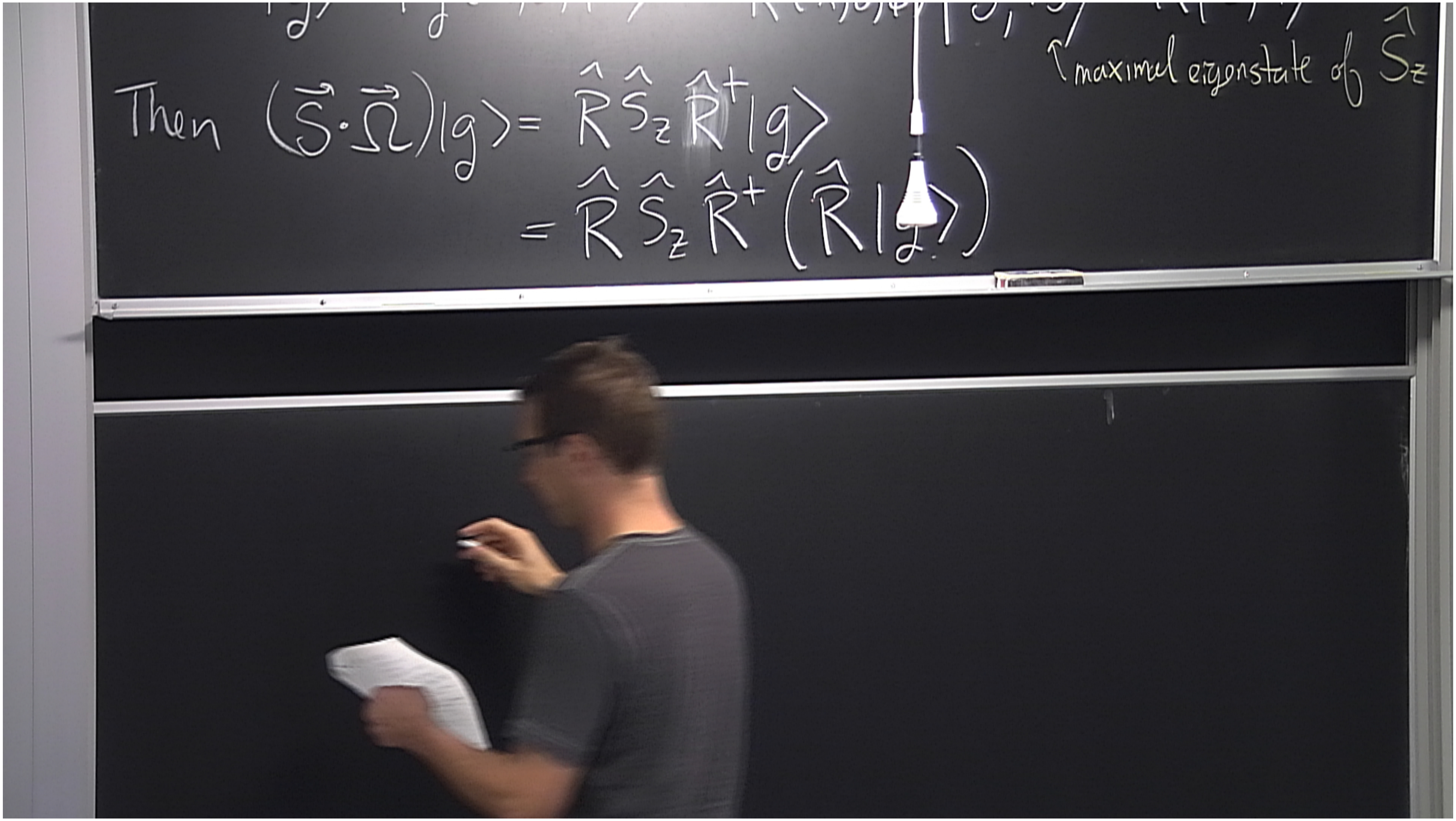
$$= S_z \cos\theta + \left[S_x \cos\phi + S_y \sin\phi \right]\sin\theta$$

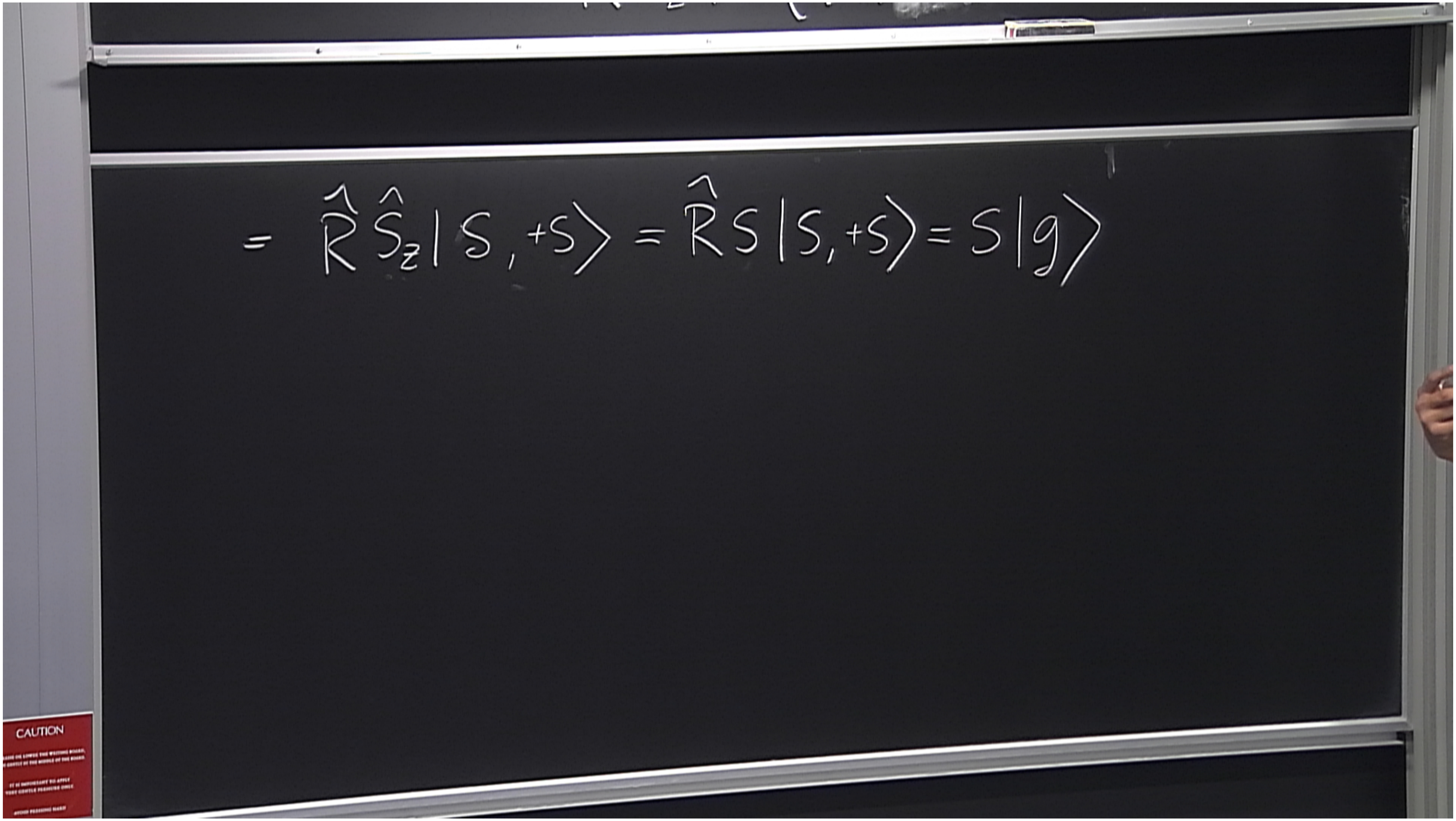
$$= \hat{S}_x \cos\phi \sin\theta + \hat{S}_y \sin\phi \sin\theta + \hat{S}_z \cos\theta = \vec{S}\cdot\vec{\Omega}$$


where $\vec{\Omega} = (\sin\theta \cos\phi, \sin\theta \cdot \sin\phi, \cos\theta)$
is the unit vector for the rotated spin.
UTION


where $\vec{\Omega} = (\sin\theta \cos\phi, \sin\theta \cdot \sin\phi, \cos\theta)$
is the unit vector for the rotated spin.
With this we can define the spin coherent state
$|g\rangle = |g(\chi, \theta, \phi)\rangle =$
CAUTION


where Ω⃗ = (sinθ cosφ, sinθ sinφ, cos θ)
is the unit vector for the rotated spin.
With this we can define the spin coherent state
|g⟩ = |g(χ,θ,φ)⟩ = R̂(χ,θ,φ)|S,+S⟩ = R̂|S,↑⟩
maximal eigenstate of Ŝz


↑ maximal eigenstate of $\hat{S}_z$
Then $(\vec{S}\cdot\vec{\Omega})|g\rangle = \hat{R}\hat{S}_z\hat{R}^\dagger|g\rangle$
$= \hat{R}\hat{S}_z\hat{R}^\dagger(\hat{R}|g\rangle)$


$= \hat{R}\hat{S}_z|s,+s\rangle = \hat{R}s|s,+s\rangle = s|g\rangle$
CAUTION

Then $(\hat{S}\cdot\hat{\Omega})|g\rangle = \hat{R}\hat{S}_z\hat{R}^{\dagger}|g\rangle$

$$= \hat{R}\hat{S}_z\hat{R}^{+}\left(\hat{R}|s,+s\rangle\right)$$

$$= \hat{R}\hat{S}_z|s,+s\rangle = \hat{R}s|s,+s\rangle = s|g\rangle$$

i.e. $|g\rangle$ is the state with the maximum spin projection along the $\vec{\Omega}$ axis.

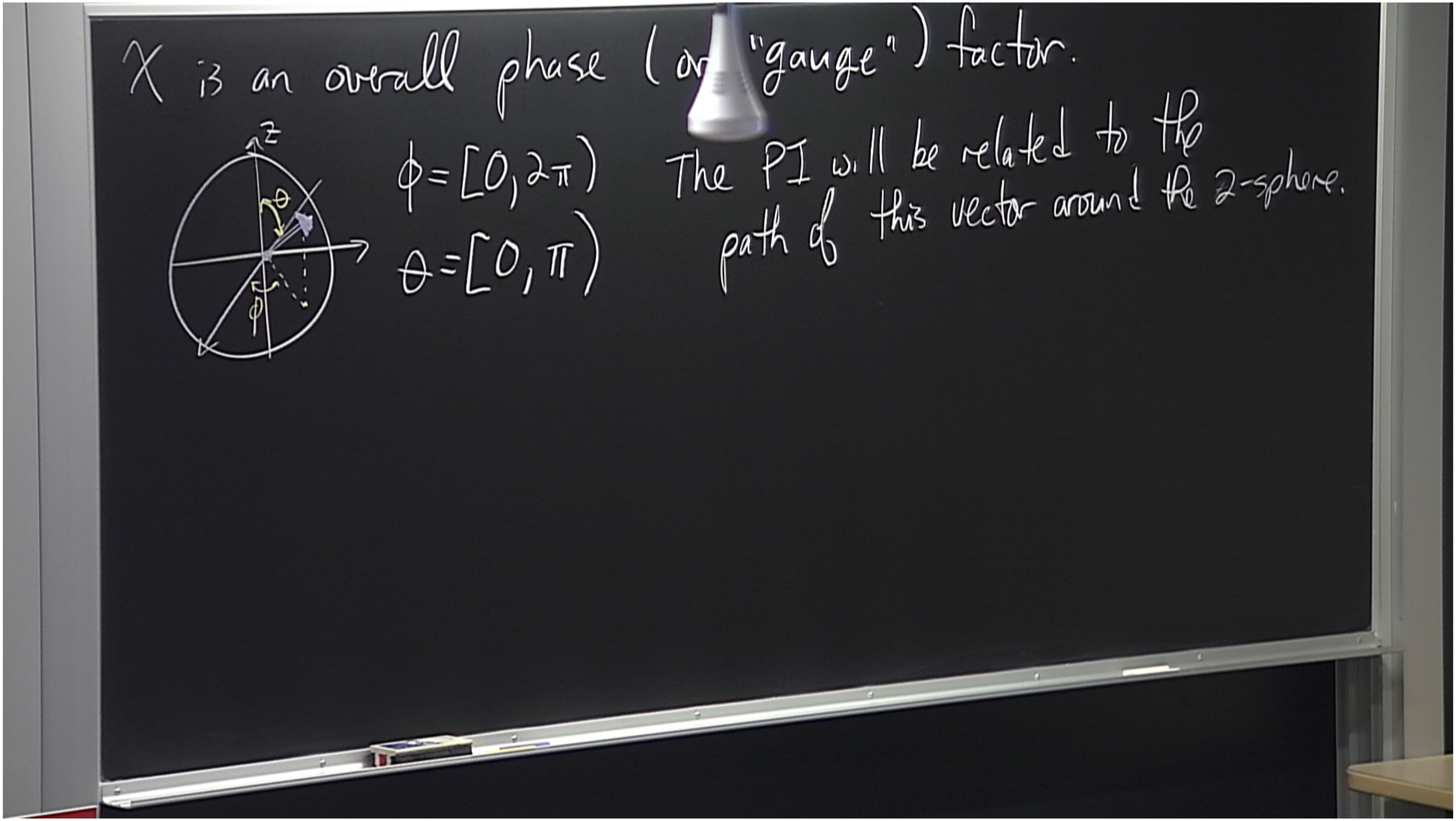
$$= \hat{R}\, \hat{S}_z\, \hat{R}^{+} \left(\hat{R} |S, +S\rangle \right)$$

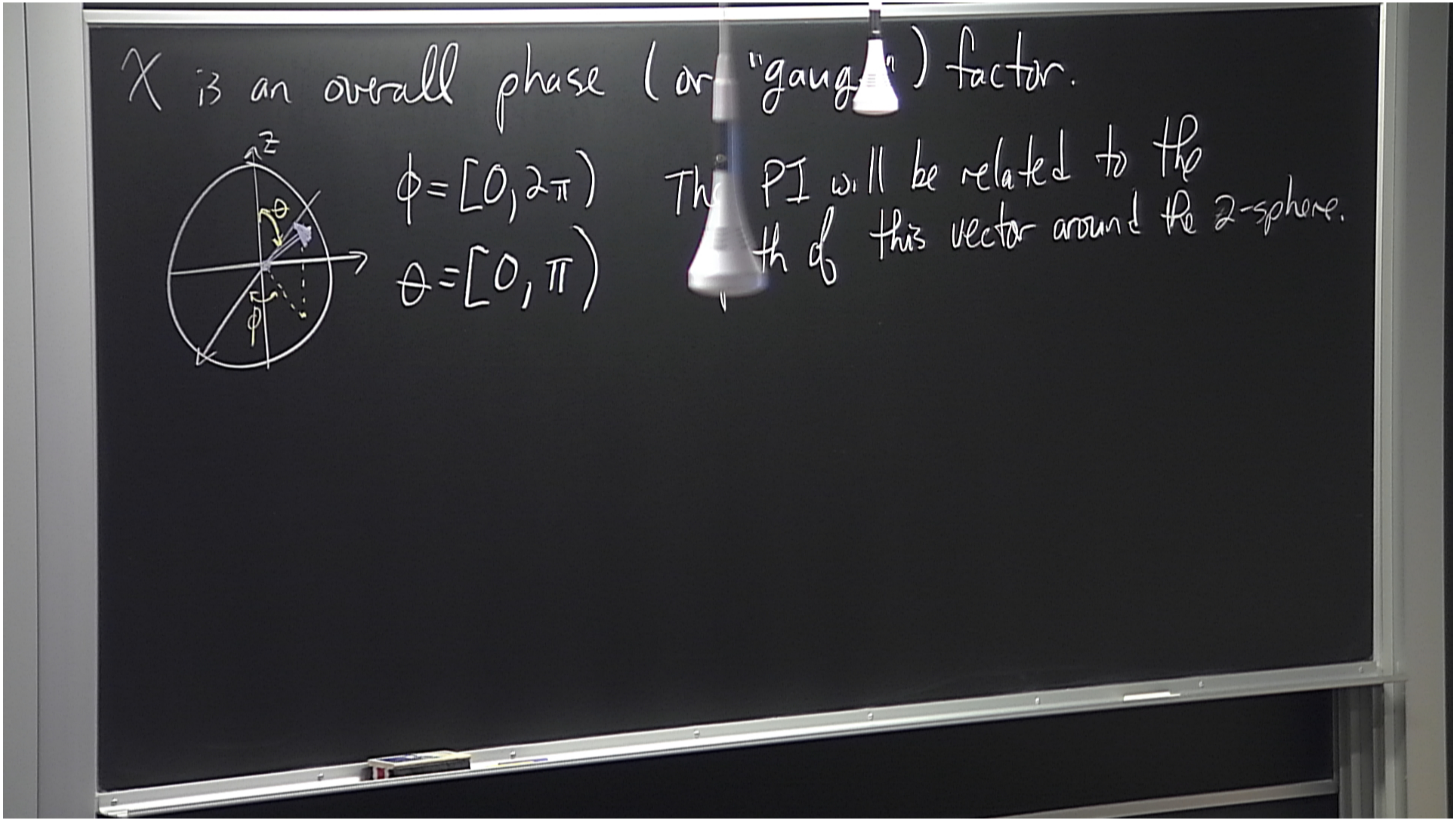
$$= \hat{R}\, \hat{S}_z |S, +S\rangle = \hat{R}\, S\, |S, +S\rangle = S |g\rangle$$

i.e. $|g\rangle$ is the state with the maximum spin projection along the $\vec{\Omega}$ axis. (Imagine $\vec{\Omega}(\tau)$ as continuous path in the PI.)

<u>Note</u>: the angle χ does not enter into $\vec{\Omega}$

$$|g\rangle = e^{-i\phi \hat{S}_z} e^{-i\theta \hat{S}_y} e^{-i\chi \hat{S}_z} |S, +S\rangle = e^{-i\phi \hat{S}_z} e^{-i\theta \hat{S}_y} |S, +S\rangle e^{-i\chi S}$$


χ is an overall phase (or "gauge") factor.
z
θ
φ
φ = [0, 2π)
θ = [0, π)
The PI will be related to the path of this vector around the 2-sphere.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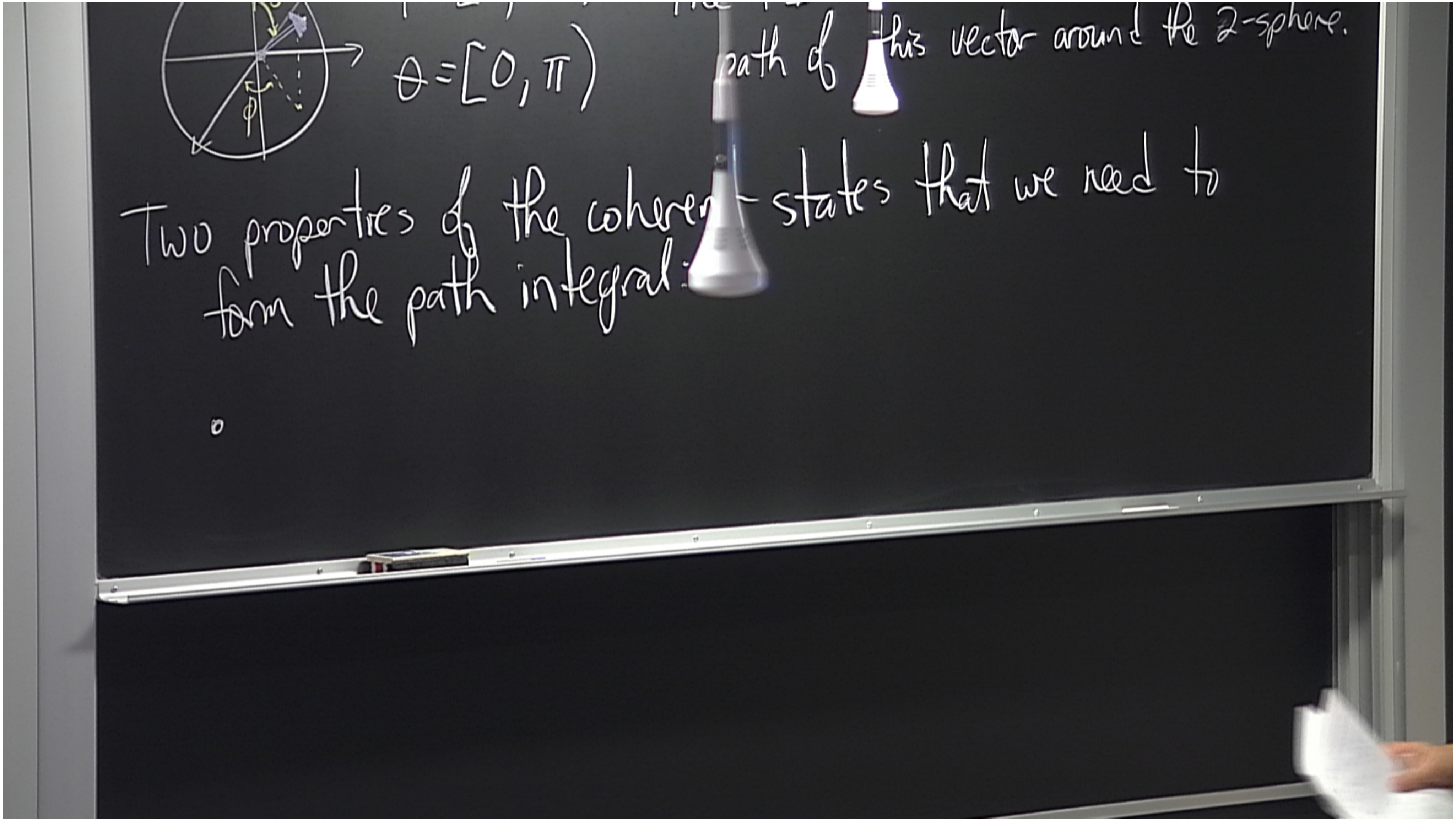
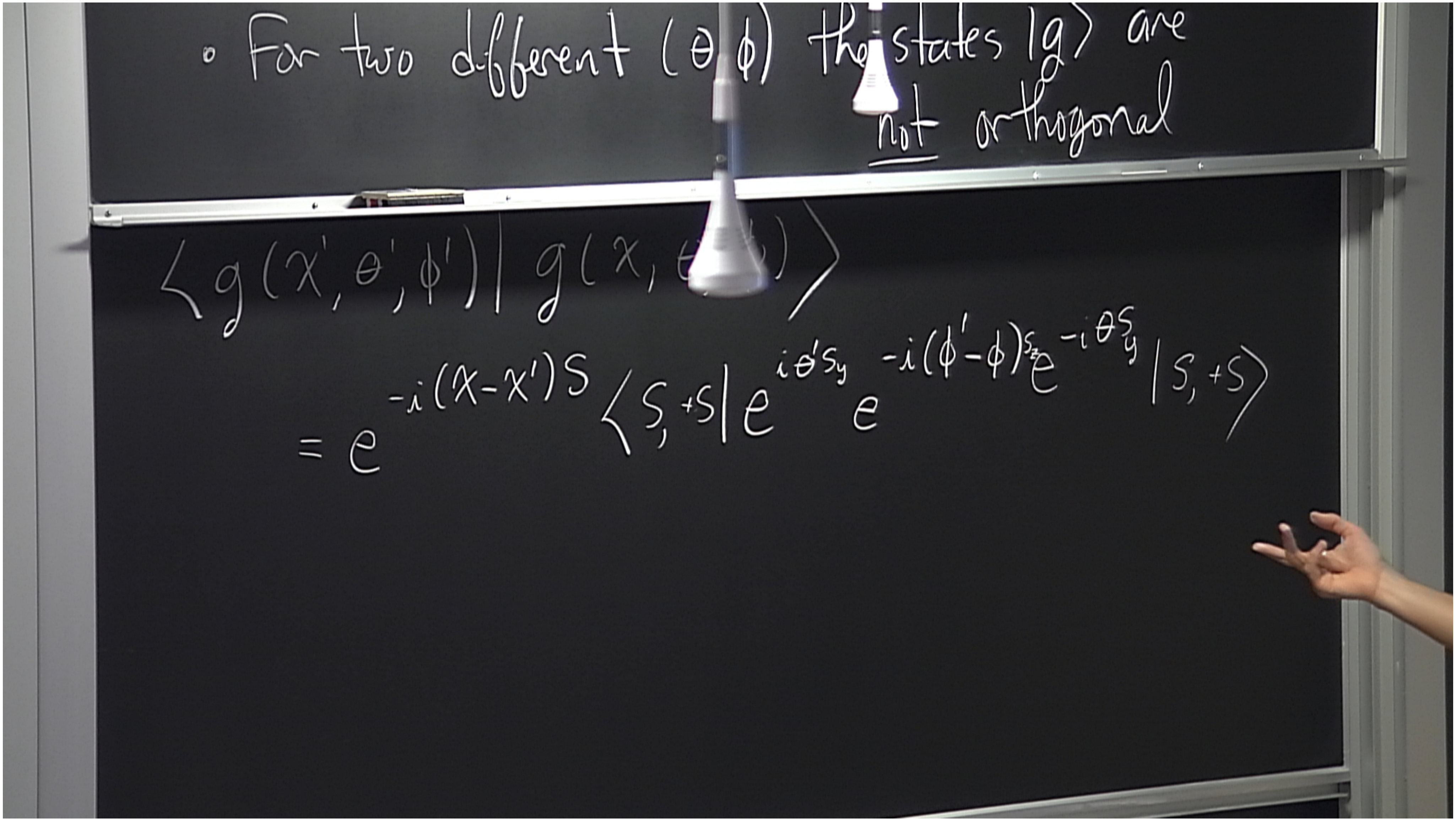
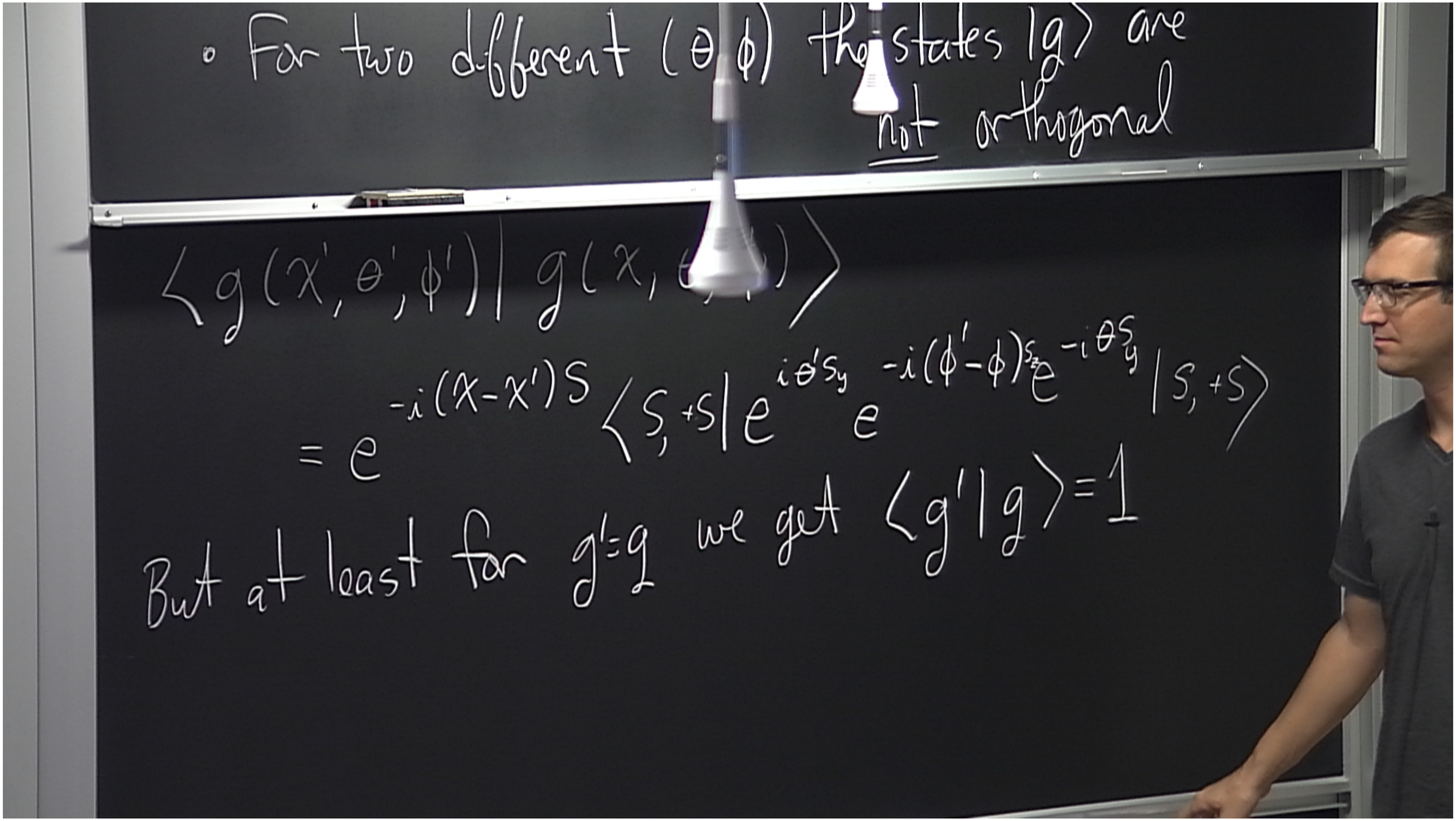
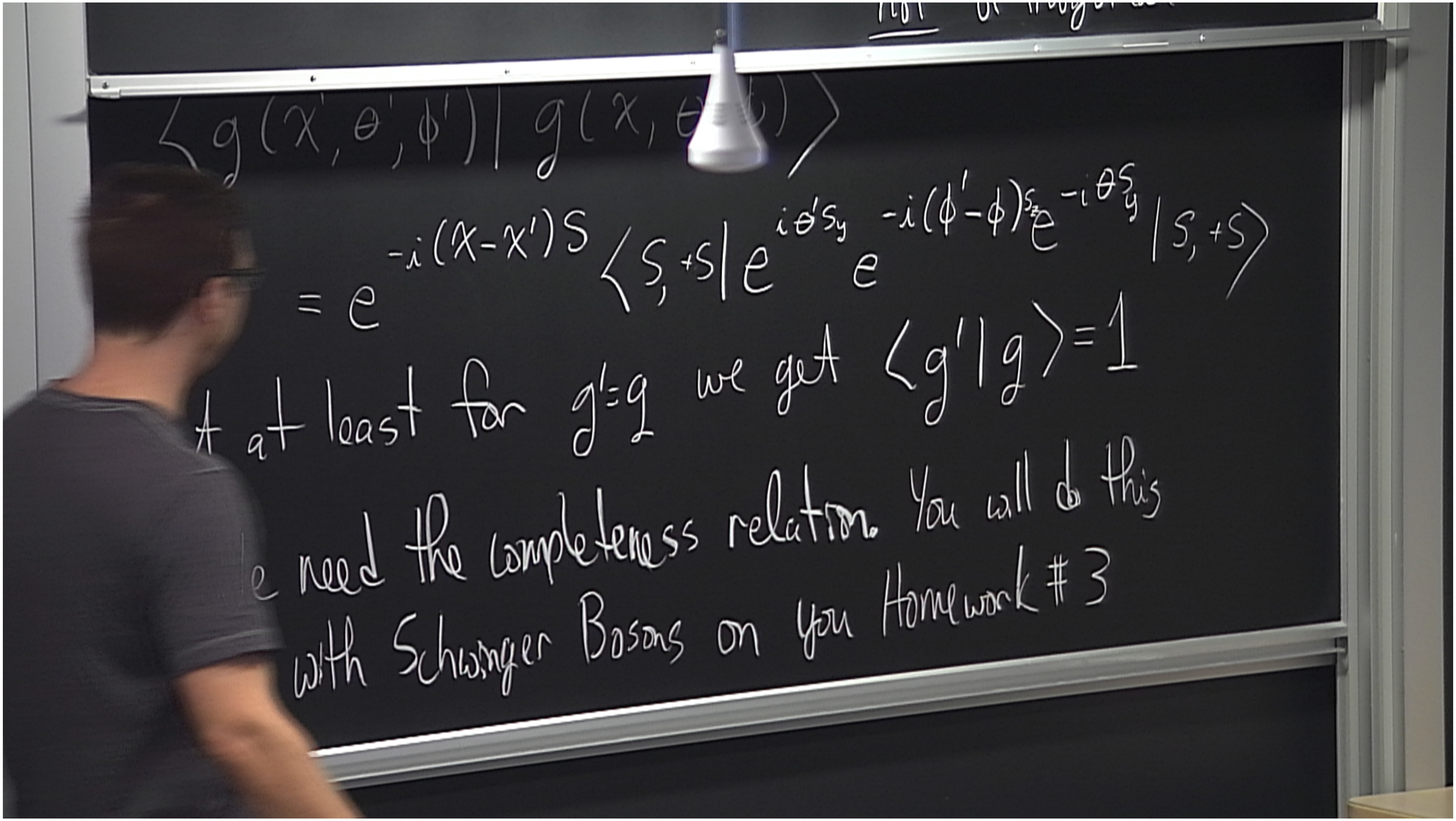

θ = [0, π)
path of this vector around the 2-sphere.
Two properties of the coherent states that we need to form the path integral:


• For two different (θ φ) the states |g⟩ are
not orthogonal
$\langle g(x', \theta', \phi') | g(x, \theta, \phi) \rangle$
$= e^{-i(x-x')S} \langle S, +S | e^{i\theta' S_y} e^{-i(\phi'-\phi)S_z} e^{-i\theta S_y} | S, +S \rangle$


• For two different (θ φ) the states |g⟩ are not orthogonal
⟨g(χ', θ', φ')| g(χ,
= e^{-i(χ-χ')S} ⟨S,+S| e^{iθ'S_y} e^{-i(φ'-φ)S_z} e^{-iθS_y} |S,+S⟩
But at least for g'=g we get ⟨g'|g⟩=1


⟨g(χ',θ',φ')| g(χ,θ,φ)⟩
= e^{-i(χ-χ')S} ⟨S,+S| e^{iθ'S_y} e^{-i(φ'-φ)S_z} e^{-iθS_y} |S,+S⟩
at least for g'=g we get ⟨g'|g⟩=1
e need the completeness relation. You will do this
with Schwinger Bosons on you Homework # 3

$$\langle g(\chi',\theta',\phi') | g(\chi,\theta,\phi) \rangle$$

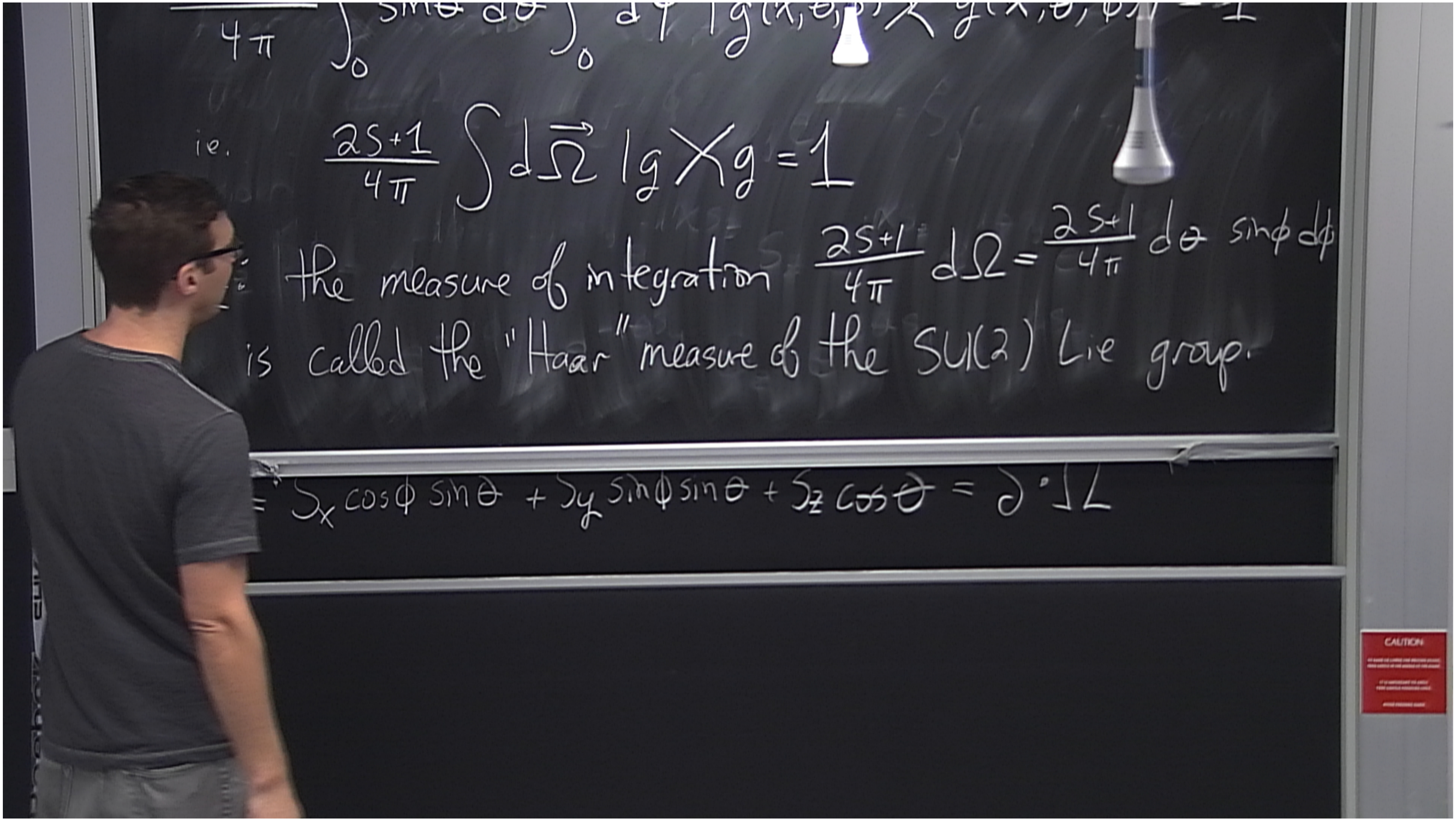
$$= e^{-i(\chi-\chi')S} \langle S,+S| e^{i\theta' S_y} e^{-i(\phi'-\phi)S_z} e^{-i\theta S_y} |S,+S\rangle$$

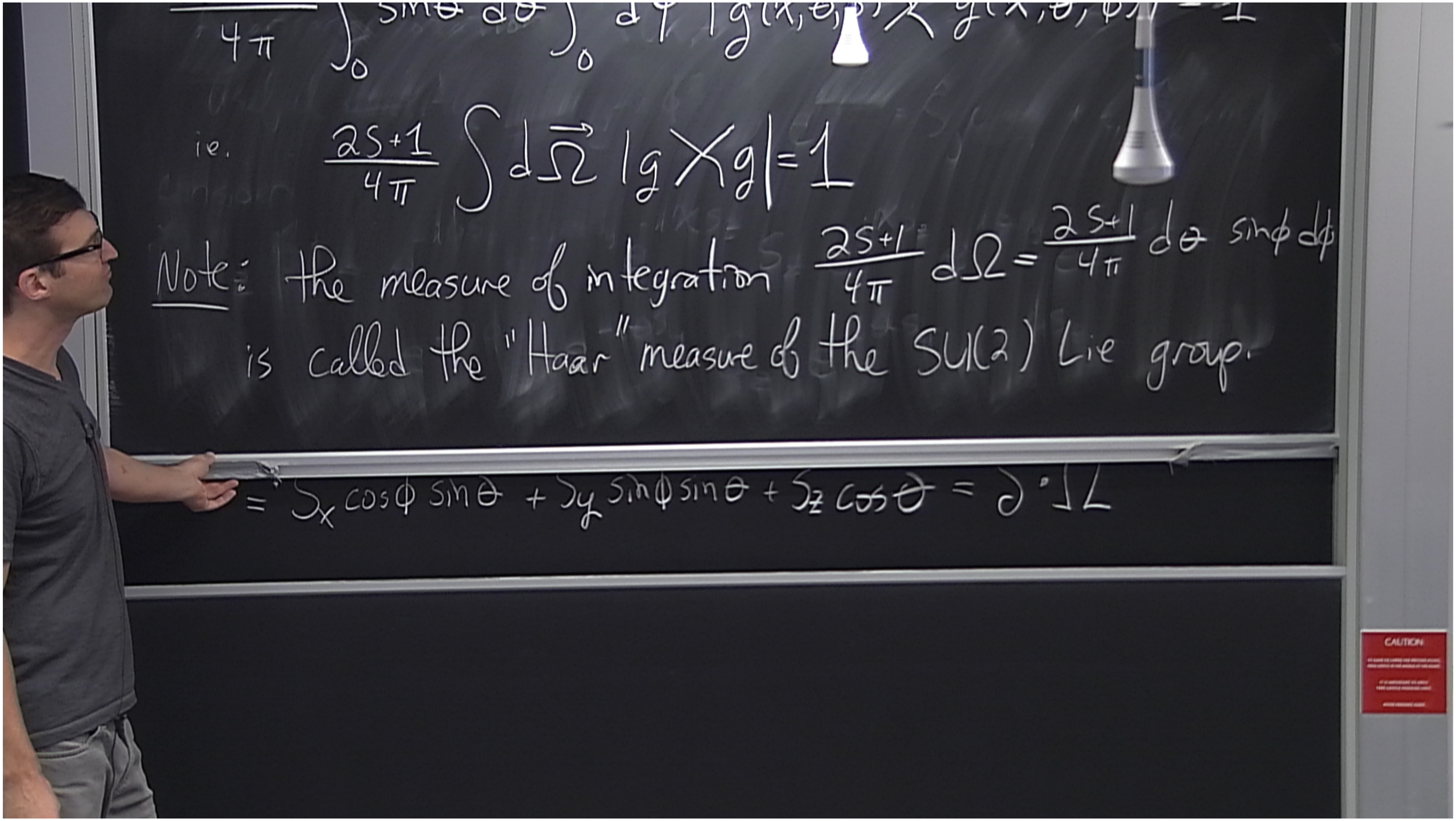
But at least for $g'=g$ we get $\langle g'|g\rangle = 1$

- We need the completeness relation. You will do this with Schwinger Bosons on you Homework # 3

$$\frac{2S+1}{4\pi}\int_0^{\pi}\sin\theta\, d\theta \int_0^{2\pi} d\phi\, |g(\chi,\theta,\phi)\rangle\langle g(\chi,\theta,\phi)| = 1$$

i.e. $$\frac{2S+1}{4\pi}\int d\vec{\Omega}\, |g\rangle\langle g| = 1$$


ie. $\frac{2S+1}{4\pi}\int d\vec{\Omega}\, |g\rangle\langle g| = 1$
the measure of integration $\frac{2S+1}{4\pi} d\Omega = \frac{2S+1}{4\pi} d\theta \sin\phi\, d\phi$
is called the "Haar" measure of the SU(2) Lie group.
$= S_x \cos\phi \sin\theta + S_y \sin\phi \sin\theta + S_z \cos\theta$

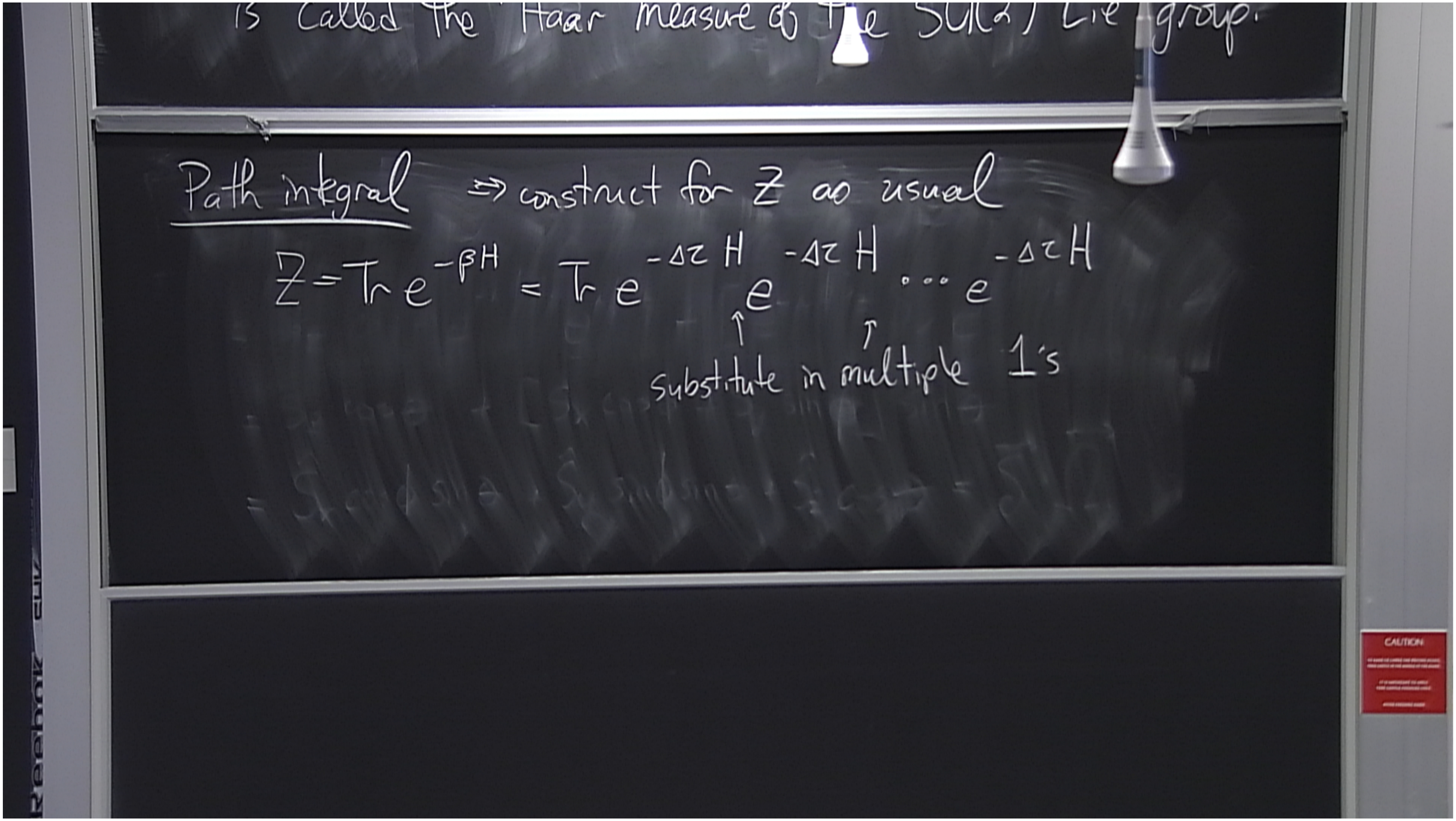


$\frac{}{4\pi}\int_0 \sin\theta\, d\theta \int_0 d\phi\, |g(x,\theta,\phi)\rangle\langle g(x,\theta,\phi)| = 1$
i.e. $\frac{2S+1}{4\pi}\int d\vec{\Omega}\, |g\rangle\langle g| = 1$
Note: the measure of integration $\frac{2S+1}{4\pi} d\Omega = \frac{2S+1}{4\pi} d\theta \sin\phi\, d\phi$
is called the "Haar" measure of the SU(2) Lie group.
$= S_x \cos\phi \sin\theta + S_y \sin\phi \sin\theta + S_z \cos\theta =$
CAUTION

is called the Haar measure of the SU(2) Lie group.

Path integral ⇒ construct for Z as usual

$$Z = \mathrm{Tr}\, e^{-\beta H} = \mathrm{Tr}\, e^{-\Delta\tau H} e^{-\Delta\tau H} \cdots e^{-\Delta\tau H}$$


is called the "Haar measure" of the SU(2) Lie group.
Path integral ⇒ construct for Z as usual
$Z = \mathrm{Tr}\, e^{-\beta H} = \mathrm{Tr}\, e^{-\Delta\tau H} e^{-\Delta\tau H} \cdots e^{-\Delta\tau H}$
substitute in multiple 1's

is called the Haar measure of the SU(2) Lie group.

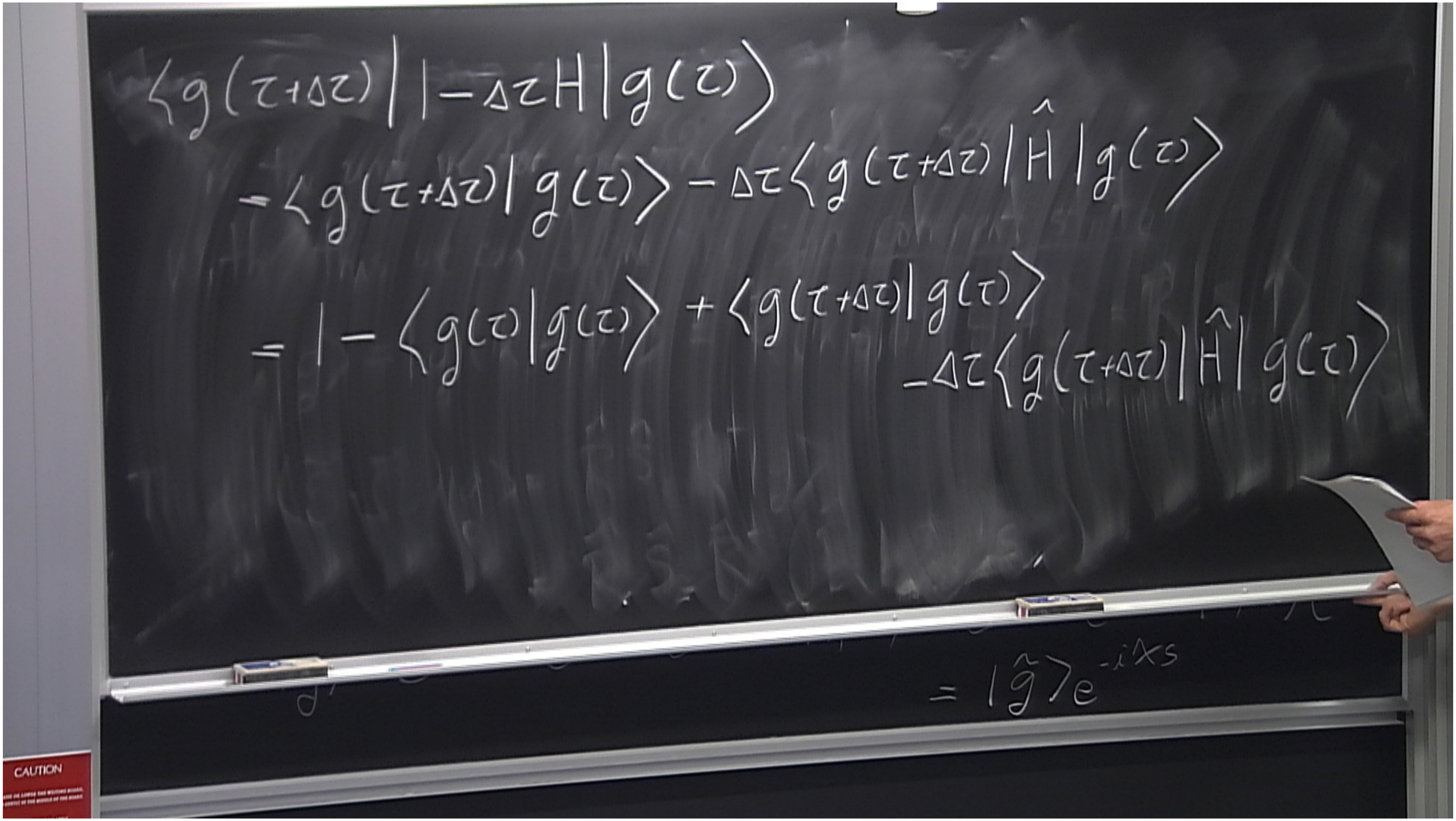
Path integral $\Rightarrow$ construct for Z as usual

$$Z = \mathrm{Tr}\, e^{-\beta H} = \mathrm{Tr}\, e^{-\Delta\tau H} e^{-\Delta\tau H} \cdots e^{-\Delta\tau H}$$

↑ substitute in multiple 1's

Thus we need to calculate $\langle g(\tau+\Delta\tau) | e^{-\Delta\tau H} | g(\tau) \rangle$

e.g. assume Hamiltonians such as $H = \vec{B}\cdot\vec{S}$ or $H = \vec{S}_i \cdot \vec{S}_j$


$\langle g(\tau+\Delta\tau) | 1 - \Delta\tau H | g(\tau) \rangle$
$= \langle g(\tau+\Delta\tau) | g(\tau) \rangle - \Delta\tau \langle g(\tau+\Delta\tau) | \hat{H} | g(\tau) \rangle$
$= 1 - \langle g(\tau) | g(\tau) \rangle + \langle g(\tau+\Delta\tau) | g(\tau) \rangle$
$- \Delta\tau \langle g(\tau+\Delta\tau) | \hat{H} | g(\tau) \rangle$
$= |\tilde{g}\rangle e^{-i\chi s}$
CAUTION

$$= 1 - \langle g(\tau)|g(\tau)\rangle + \langle g(\tau+\Delta\tau)|g(\tau)\rangle - \Delta\tau \langle g(\tau+\Delta\tau)|\hat{H}|g(\tau)\rangle$$

$$= \exp\left[\langle g(\tau+\Delta\tau)|g(\tau)\rangle - \langle g(\tau)|g(\tau)\rangle - \Delta\tau \langle g(\tau+\Delta\tau)|\hat{H}|g(\tau)\rangle\right]$$

Note: the angle χ does not enter into $\vec{\Omega}$

$$|g\rangle = e^{-i\phi\hat{S}_z} e^{-i\theta\hat{S}_y} e^{-i\chi\hat{S}_z}|S,+S\rangle = e^{-i\phi\hat{S}_z} e^{-i\theta\hat{S}_y}|S,+S\rangle e^{-i\chi S}$$

$$= |\tilde{g}\rangle e^{-i\chi S}$$

This leads to

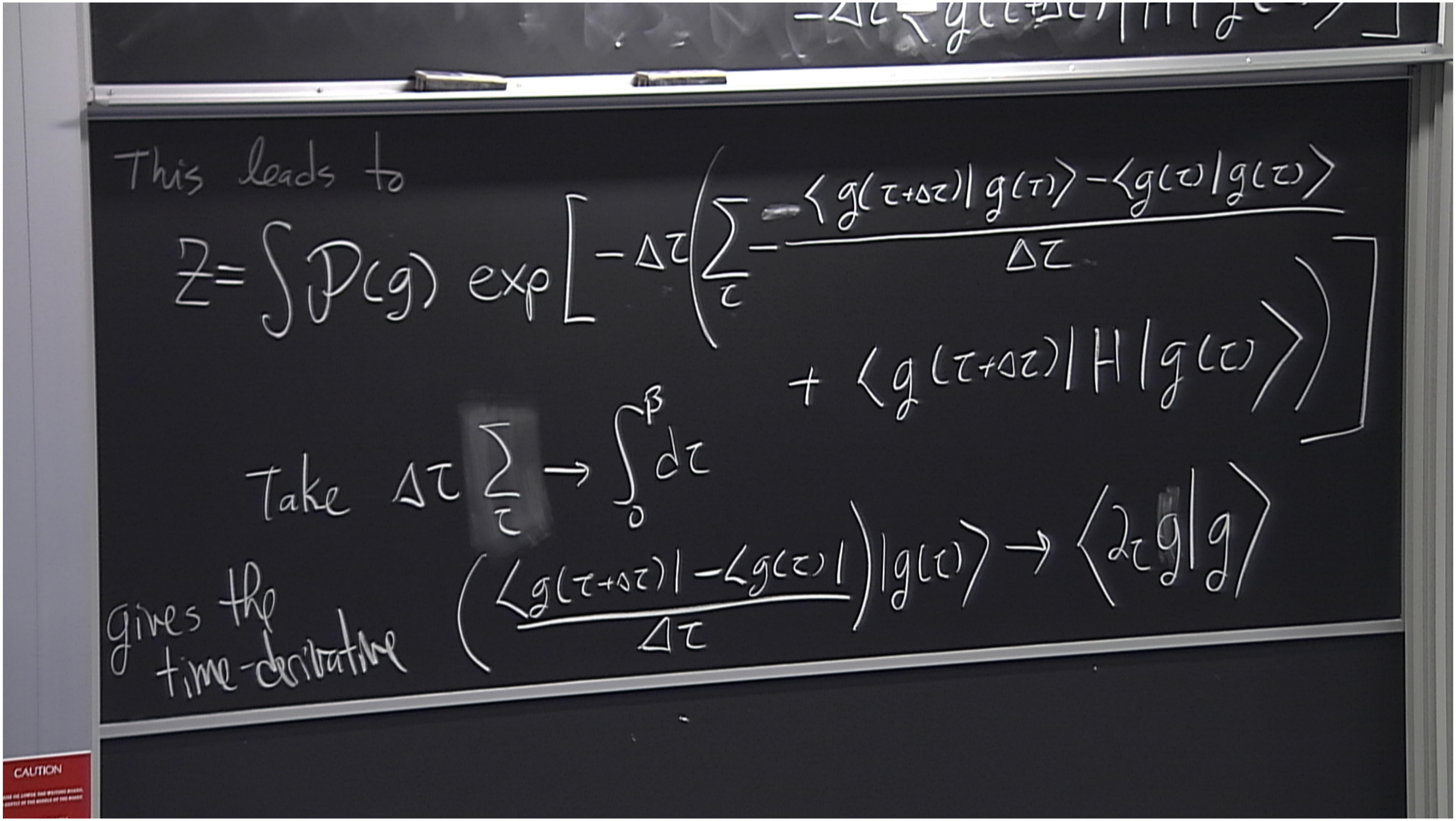
$$Z = \int \mathcal{D}(g) \exp\Big[-\Delta\tau \sum_{\tau} - \frac{\langle g(\tau+\Delta\tau) | g(\tau) \rangle - \langle g(\tau) | g(\tau) \rangle}{\Delta\tau}$$

$$-\Delta\tau \langle g(\tau+\Delta\tau)|H|g(\tau)\rangle \Big]$$

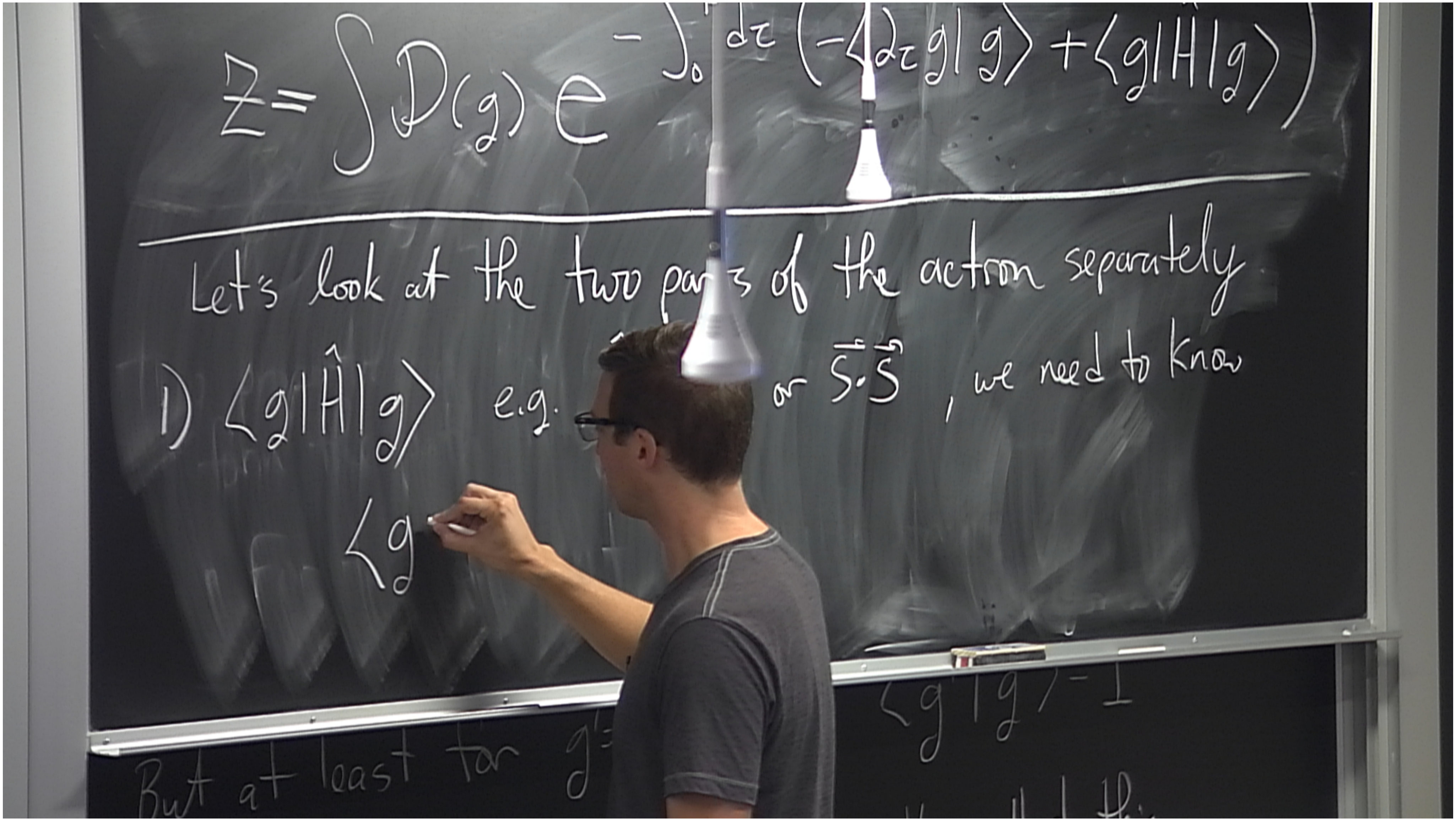
This leads to

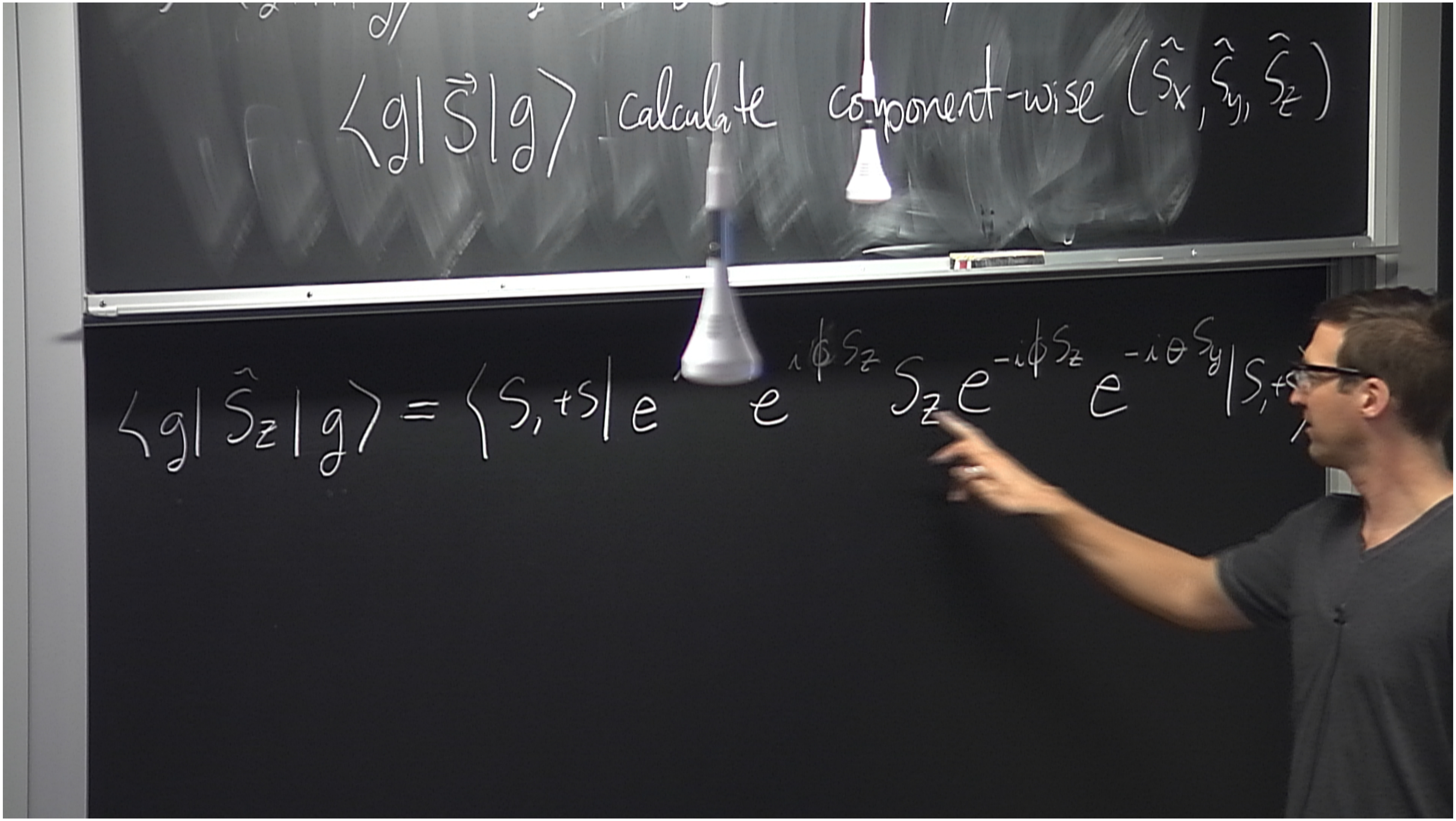
$$Z = \int \mathcal{D}(g)\, \exp\left[-\Delta\tau \left(\sum_{\tau} - \frac{\langle g(\tau+\Delta\tau)|g(\tau)\rangle - \langle g(\tau)|g(\tau)\rangle}{\Delta\tau} + \langle g(\tau+\Delta\tau)|H|g(\tau)\rangle \right)\right]$$

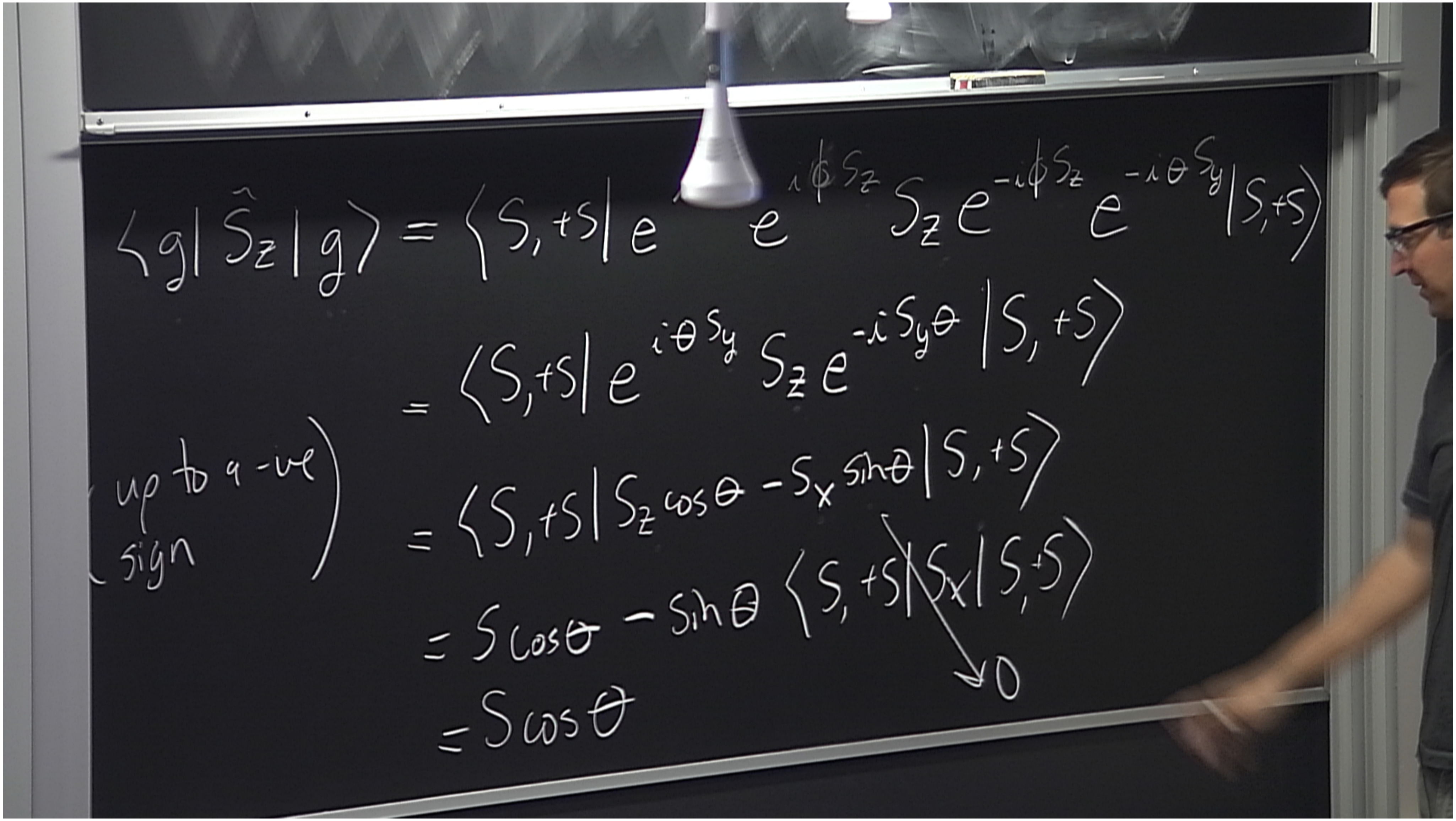
Take $\Delta\tau \sum_{\tau} \rightarrow$



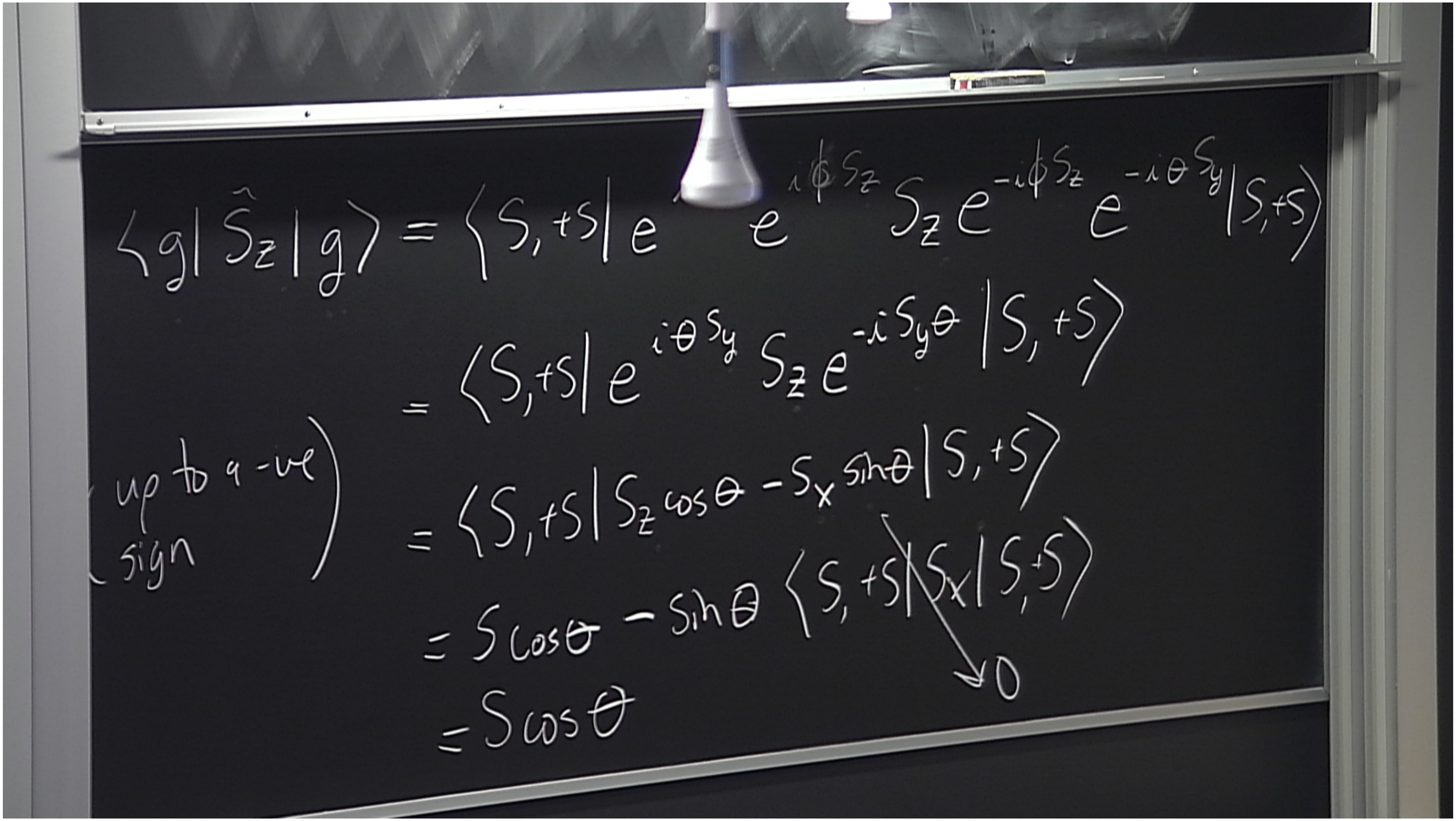
This leads to
$Z = \int \mathcal{D}(g) \exp\left[-\Delta\tau \left(\sum_\tau - \frac{\langle g(\tau+\Delta\tau) | g(\tau)\rangle - \langle g(\tau) | g(\tau)\rangle}{\Delta\tau} + \langle g(\tau+\Delta\tau) | H | g(\tau)\rangle \right) \right]$
Take $\Delta\tau \sum_\tau \to \int_0^\beta d\tau$
gives the time-derivative
$\left(\frac{\langle g(\tau+\Delta\tau)| - \langle g(\tau)|}{\Delta\tau} \right) |g(\tau)\rangle \to \langle \partial_\tau g | g \rangle$

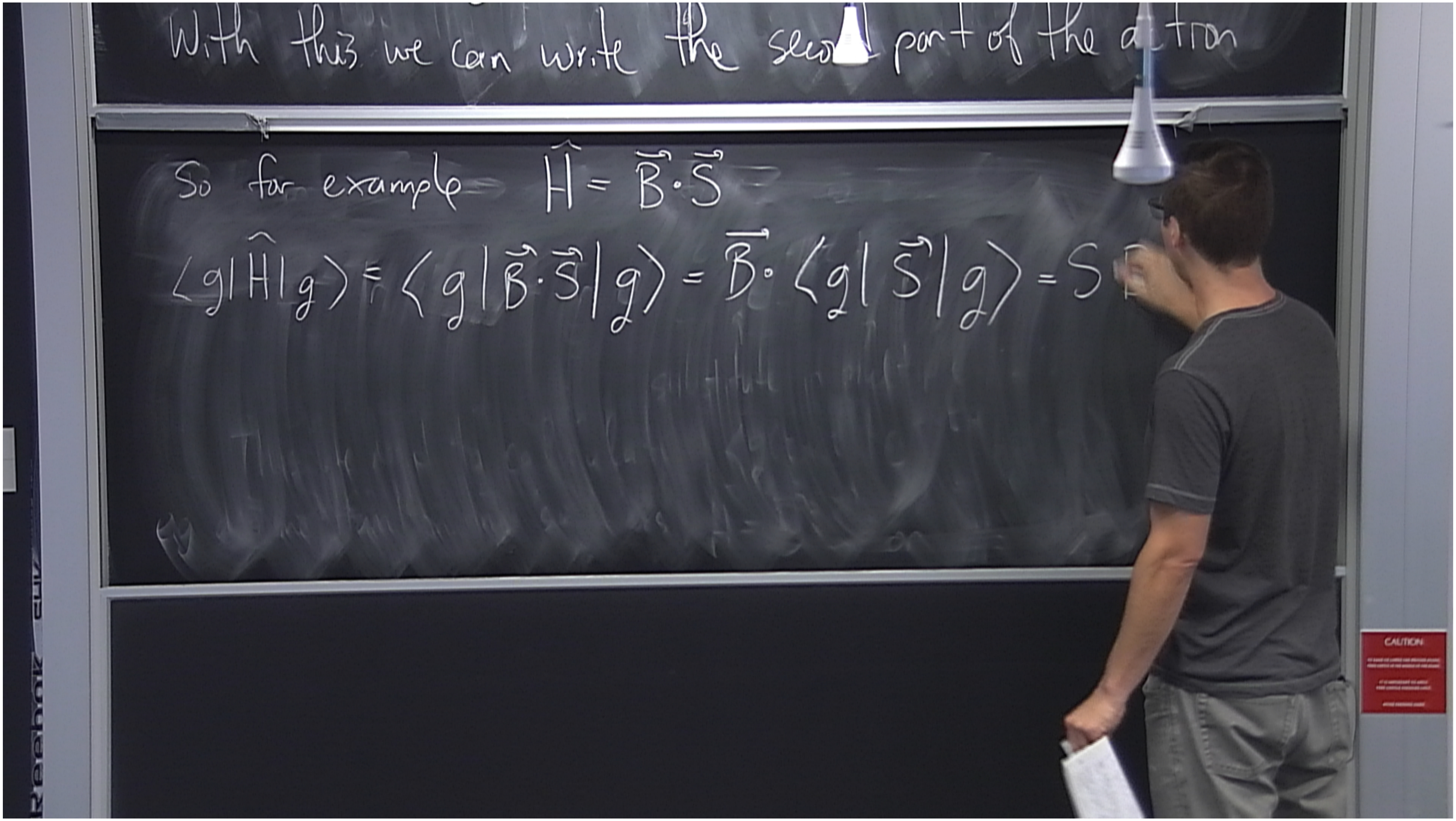

Z = ∫D(g) e^(−∫₀ dτ (−⟨∂τ g | g⟩ + ⟨g|Ĥ|g⟩))
Let's look at the two
s of the action separately
1) ⟨g|Ĥ|g⟩ e.g.
or S⃗·S⃗ , we need to know
⟨g
But at least for

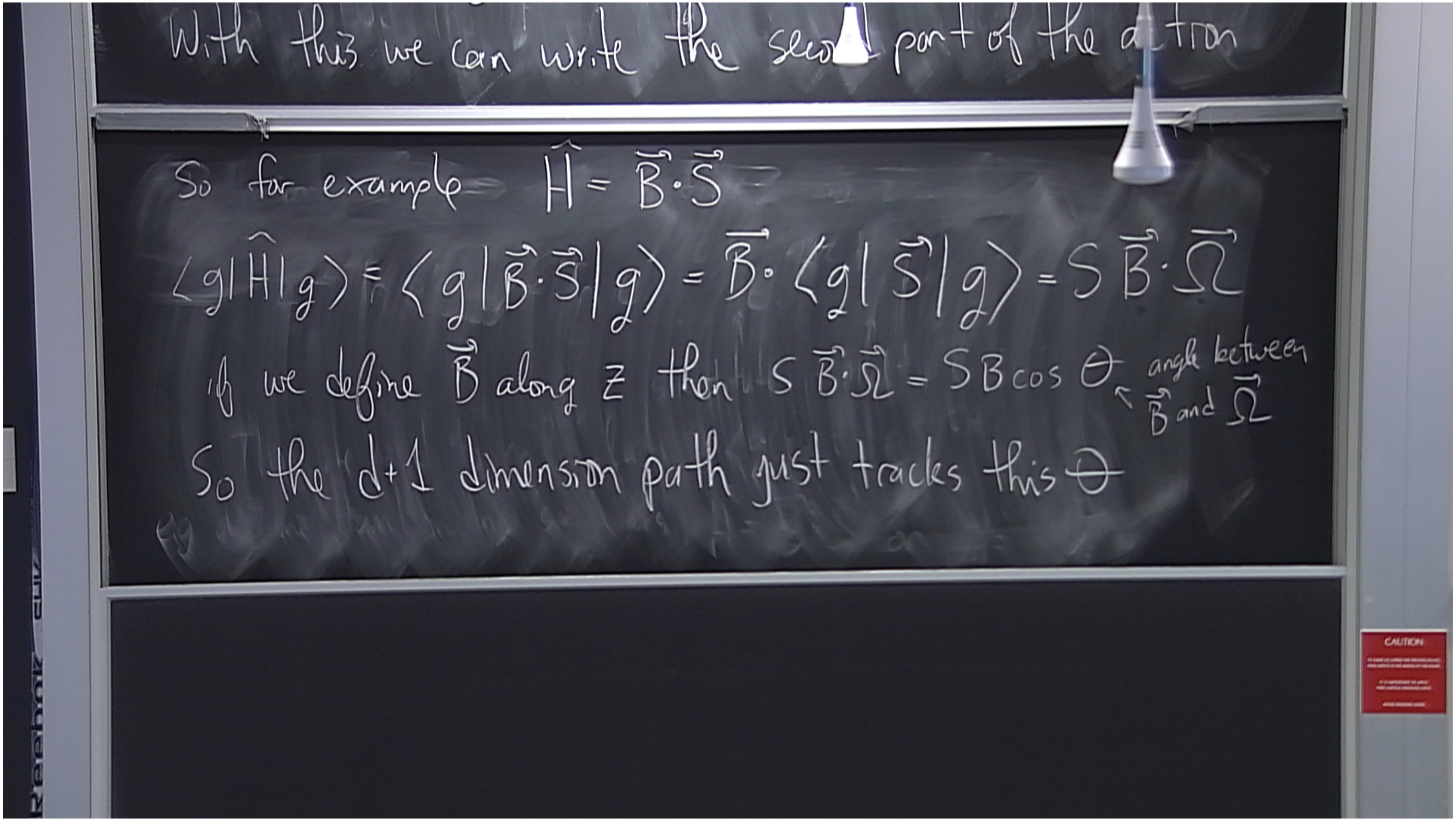

⟨g|S⃗|g⟩ calculate component-wise (Ŝx, Ŝy, Ŝz)
⟨g|Ŝz|g⟩ = ⟨s, +s| e e^{iφSz} Sz e^{-iφSz} e^{-iθSy} |s, +s⟩



$\langle g|\hat{S}_z|g\rangle = \langle S,+S| e^{i\phi S_z} S_z e^{-i\phi S_z} e^{-i\theta S_y}|S,+S\rangle$
$= \langle S,+S| e^{i\theta S_y} S_z e^{-i S_y \theta}|S,+S\rangle$
(up to a -ve sign)
$= \langle S,+S| S_z \cos\theta - S_x \sin\theta |S,+S\rangle$
$= S\cos\theta - \sin\theta \langle S,+S|S_x|S,+S\rangle$
$\to 0$
$= S\cos\theta$

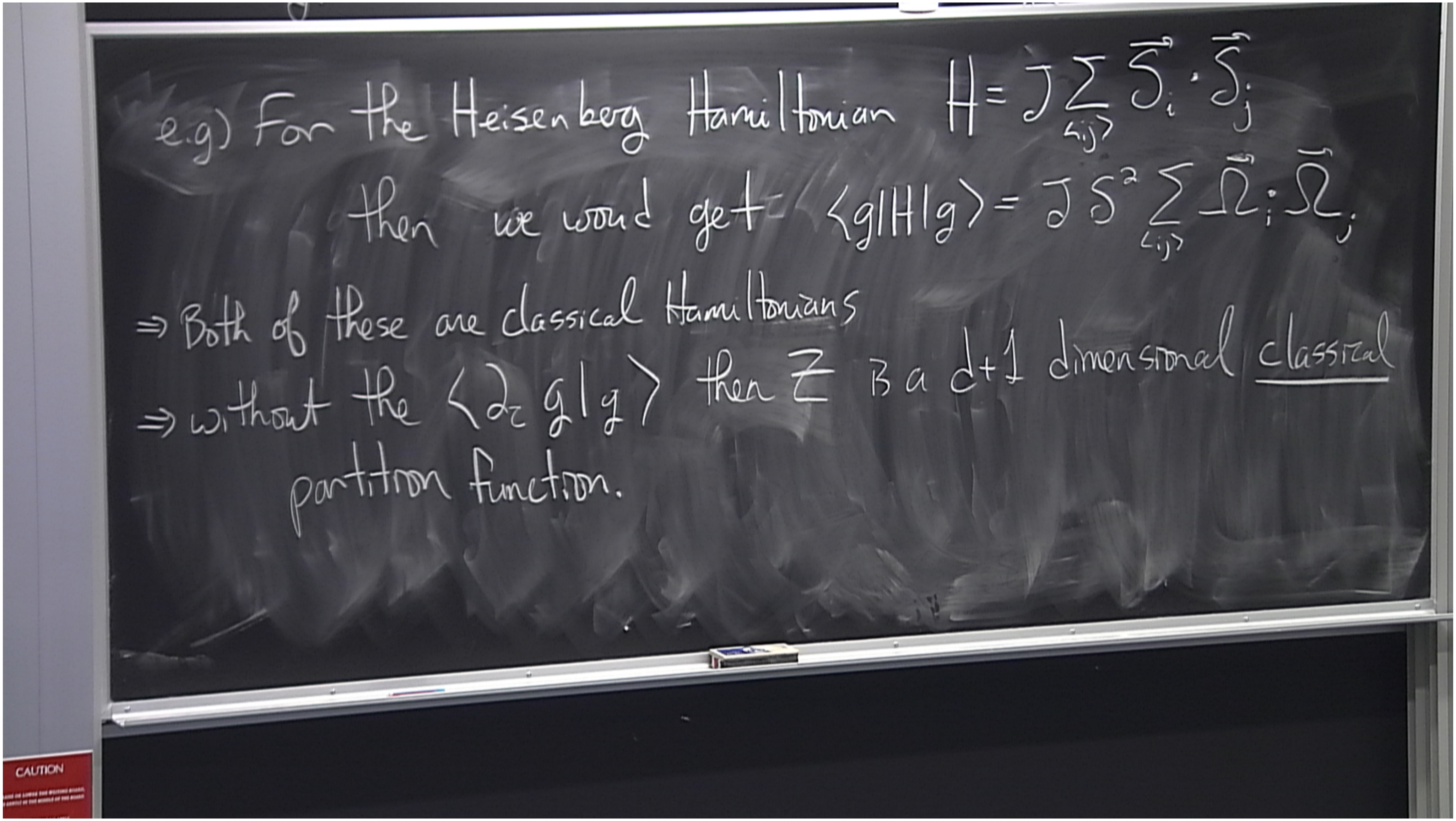

$\langle g|\hat{S}_z|g\rangle = \langle S,+S|\, e \quad e^{i\phi S_z} S_z e^{-i\phi S_z} e^{-i\theta S_y}|S,+S\rangle$
$= \langle S,+S|\, e^{i\theta S_y} S_z e^{-iS_y\theta}\,|S,+S\rangle$
(up to a -ve sign)
$= \langle S,+S|\, S_z\cos\theta - S_x\sin\theta\,|S,+S\rangle$
$= S\cos\theta - \sin\theta\,\langle S,+S|S_x|S,+S\rangle$
$\to 0$
$= S\cos\theta$


With this we can write the
part of the
So for example Ĥ = B⃗·S⃗
⟨g|Ĥ|g⟩ = ⟨g|B⃗·S⃗|g⟩ = B⃗·⟨g|S⃗|g⟩ = S
CAUTION


With this we can write the second part of the action
So for example $\hat{H} = \vec{B}\cdot\vec{S}$
$\langle g|\hat{H}|g\rangle = \langle g|\vec{B}\cdot\vec{S}|g\rangle = \vec{B}\cdot\langle g|\vec{S}|g\rangle = S\,\vec{B}\cdot\vec{\Omega}$
if we define $\vec{B}$ along z then $S\,\vec{B}\cdot\vec{\Omega} = SB\cos\theta$
angle between $\vec{B}$ and $\vec{\Omega}$
So the d+1 dimension path just tracks this θ

e.g) For the Heisenberg Hamiltonian $H = J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$

then we would get $\langle g | H | g \rangle = J S^2 \sum_{\langle ij \rangle} \vec{\Omega}_i \cdot \vec{\Omega}_j$


e.g) For the Heisenberg Hamiltonian $H = J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$
then we would get $\langle g|H|g\rangle = J S^2 \sum_{\langle ij \rangle} \vec{\Omega}_i \cdot \vec{\Omega}_j$
⇒ Both of these are classical Hamiltonians
⇒ without the $\langle \partial_\tau g | g \rangle$ then Z is a d+1 dimensional classical partition function.

⇒ without the $\langle \partial_\tau g | g \rangle$ then Z is ...

partition function.

⇒ we hope $\langle \partial_\tau g | g \rangle$ contains the QM

$$+ \langle g(\tau+\Delta\tau) | H | g(\tau) \rangle \Big) \Big]$$

Take $\Delta\tau \sum_\tau \to \int_0^\beta d\tau$

gives the time-derivative

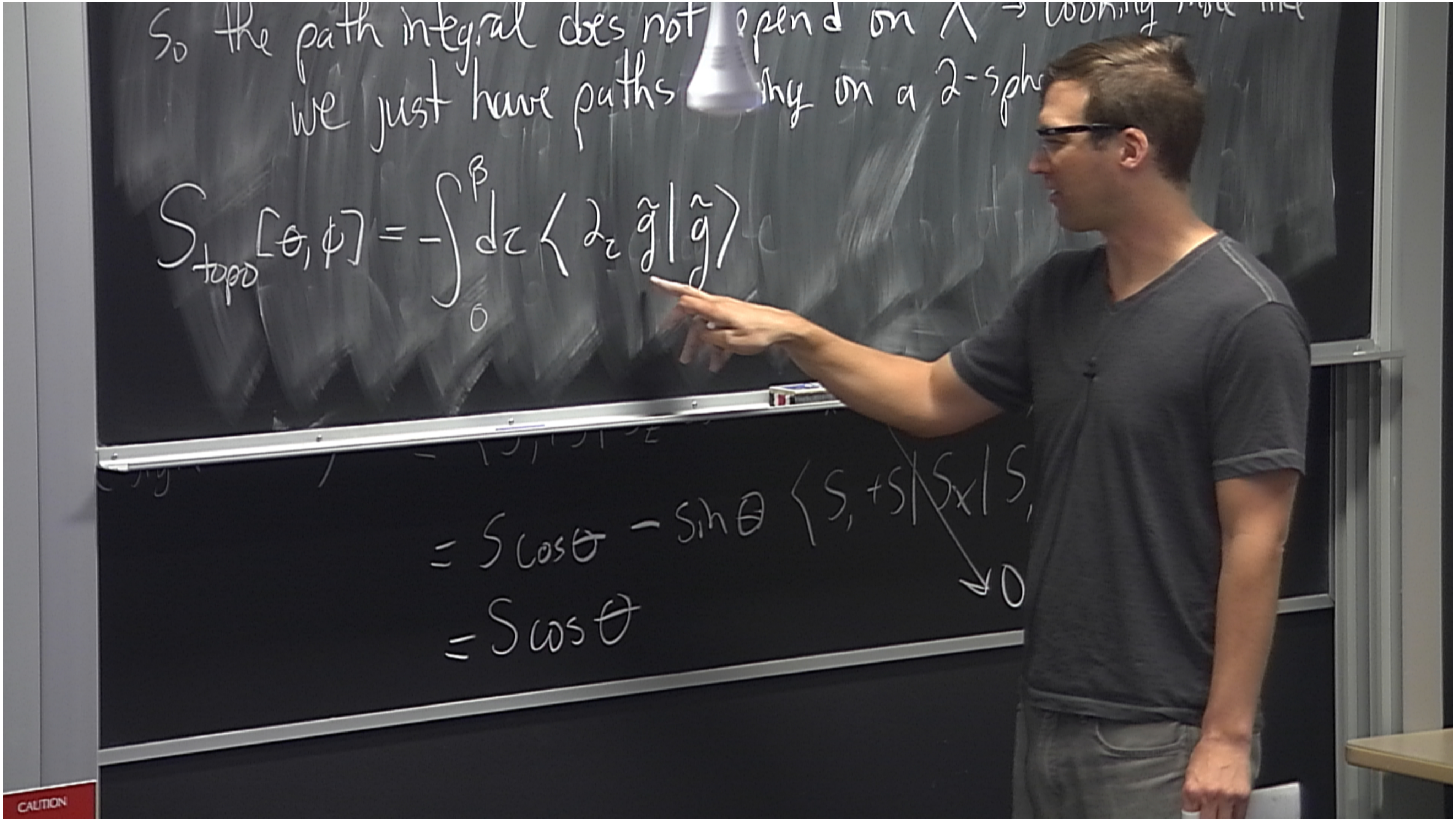
$$\left(\frac{\langle g(\tau+\Delta\tau)| - \langle g(\tau)|}{\Delta\tau} \right) |g(\tau)\rangle \to \langle \partial_\tau g | g \rangle$$

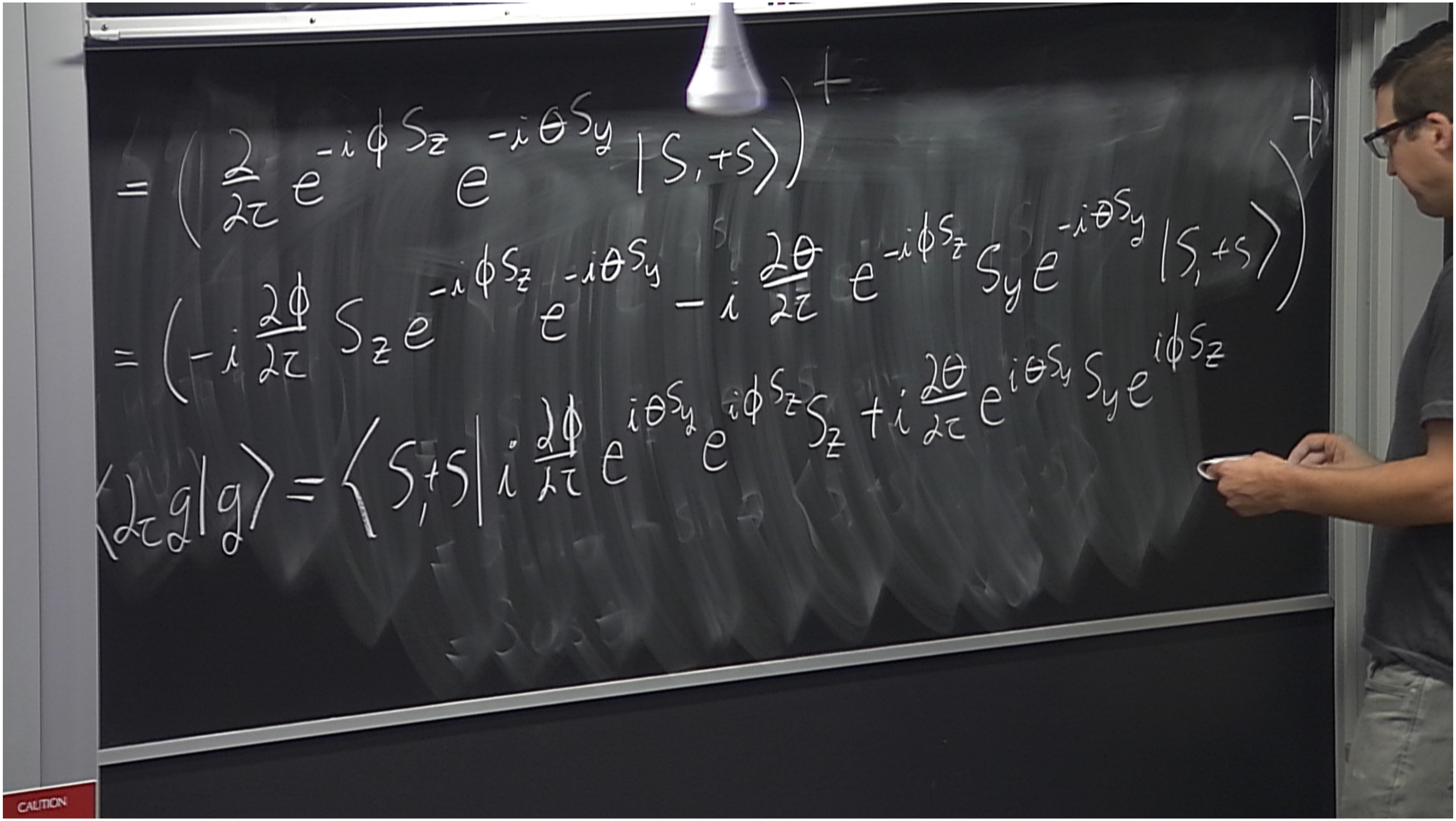
• call the 1st part of the action the "topological" piece

$$S_{topo}[\theta, \phi] = -\int_0^\beta d\tau \langle \partial_\tau g | g \rangle$$

$$= -\int_0^\beta d\tau \left\langle \frac{\partial}{\partial \tau} e^{-iS\chi} \tilde{g} \middle| \tilde{g} e^{-iS\chi} \right\rangle$$

$$= \int d\tau \left(\left\langle \frac{\partial}{\partial \tau} \tilde{g} \middle| \tilde{g} \right\rangle - iS \frac{\partial \chi}{\partial \tau} \langle \tilde{g} | \tilde{g} \rangle \right)$$

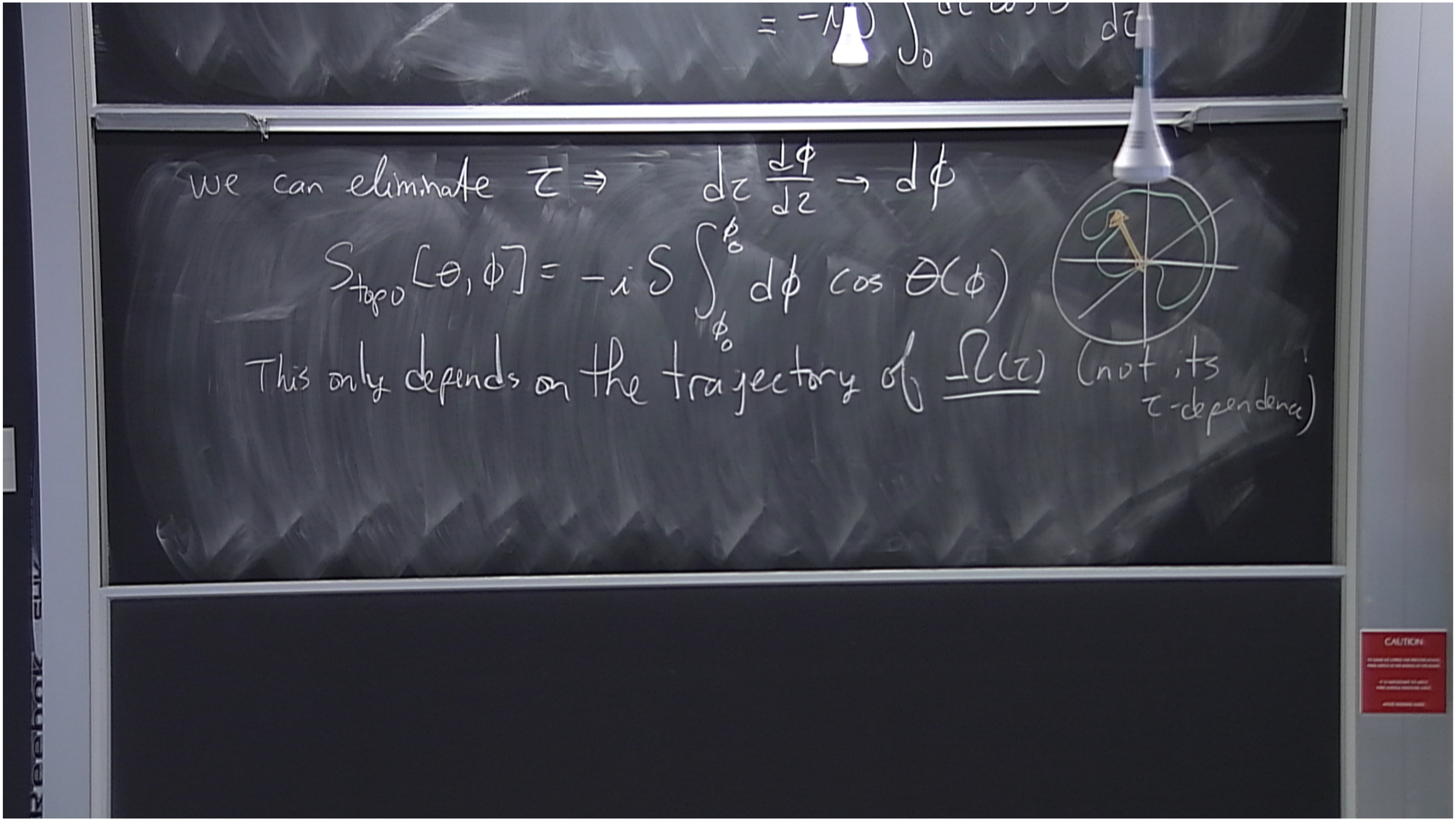

So the path integral does not depend on λ ⇒ looking more like
we just have paths on a 2-sphere
$S_{topo}[\theta,\phi] = -\int_0^\beta d\tau \langle \partial_\tau \tilde{g} | \tilde{g} \rangle$
$= S\cos\theta - \sin\theta \langle S, +S | S_x | S,$
→ 0
$= S\cos\theta$

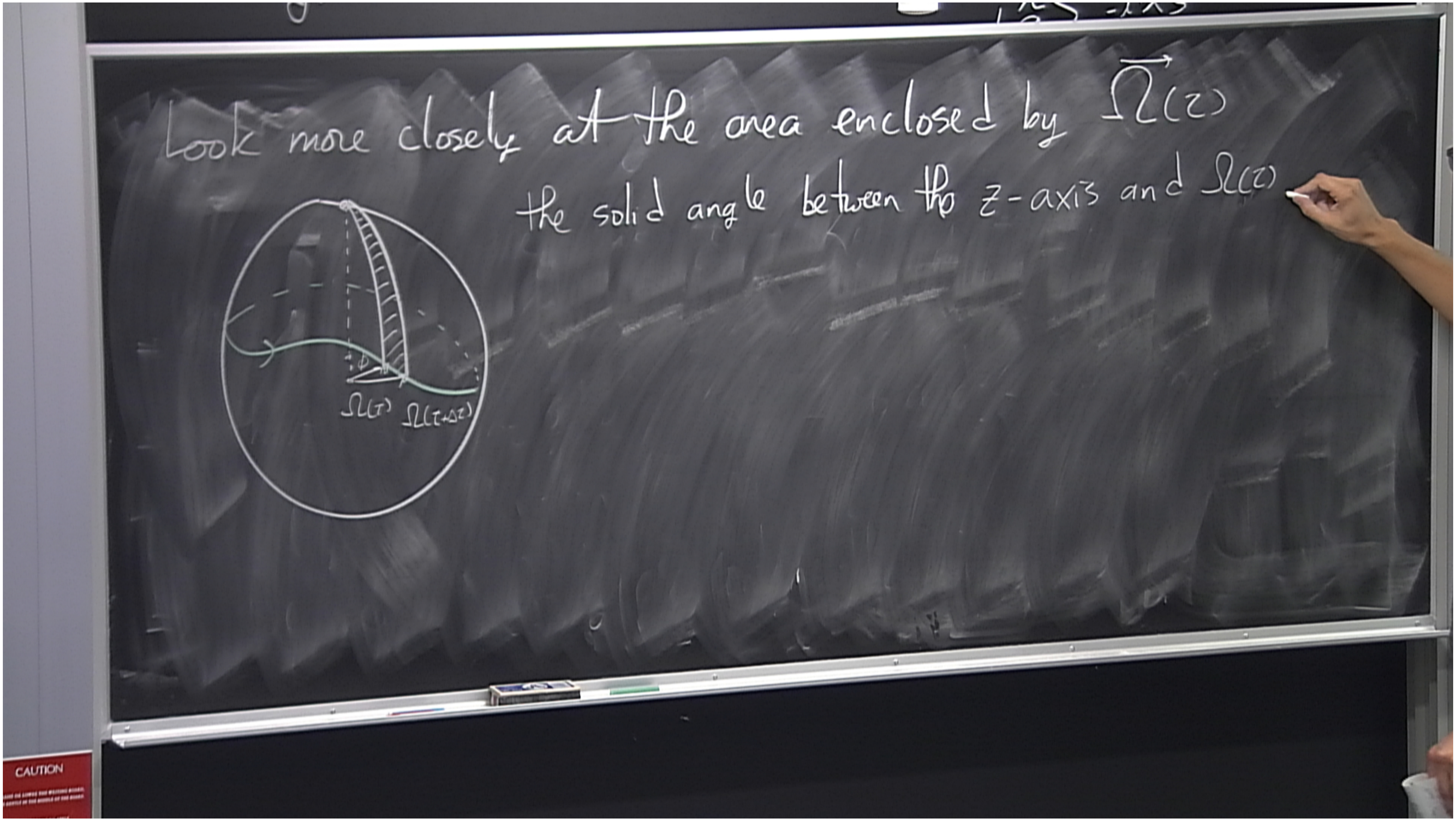

$= \left(\frac{\partial}{\partial\tau} e^{-i\phi S_z} e^{-i\theta S_y} |S,+S\rangle \right)^\dagger$
$= \left(-i \frac{\partial\phi}{\partial\tau} S_z e^{-i\phi S_z} e^{-i\theta S_y} - i \frac{\partial\theta}{\partial\tau} e^{-i\phi S_z} S_y e^{-i\theta S_y} |S,+S\rangle \right)^\dagger$
$\langle \partial_\tau g | g \rangle = \langle S,+S | i \frac{\partial\phi}{\partial\tau} e^{i\theta S_y} e^{i\phi S_z} S_z + i \frac{\partial\theta}{\partial\tau} e^{i\theta S_y} S_y e^{i\phi S_z}$
CAUTION

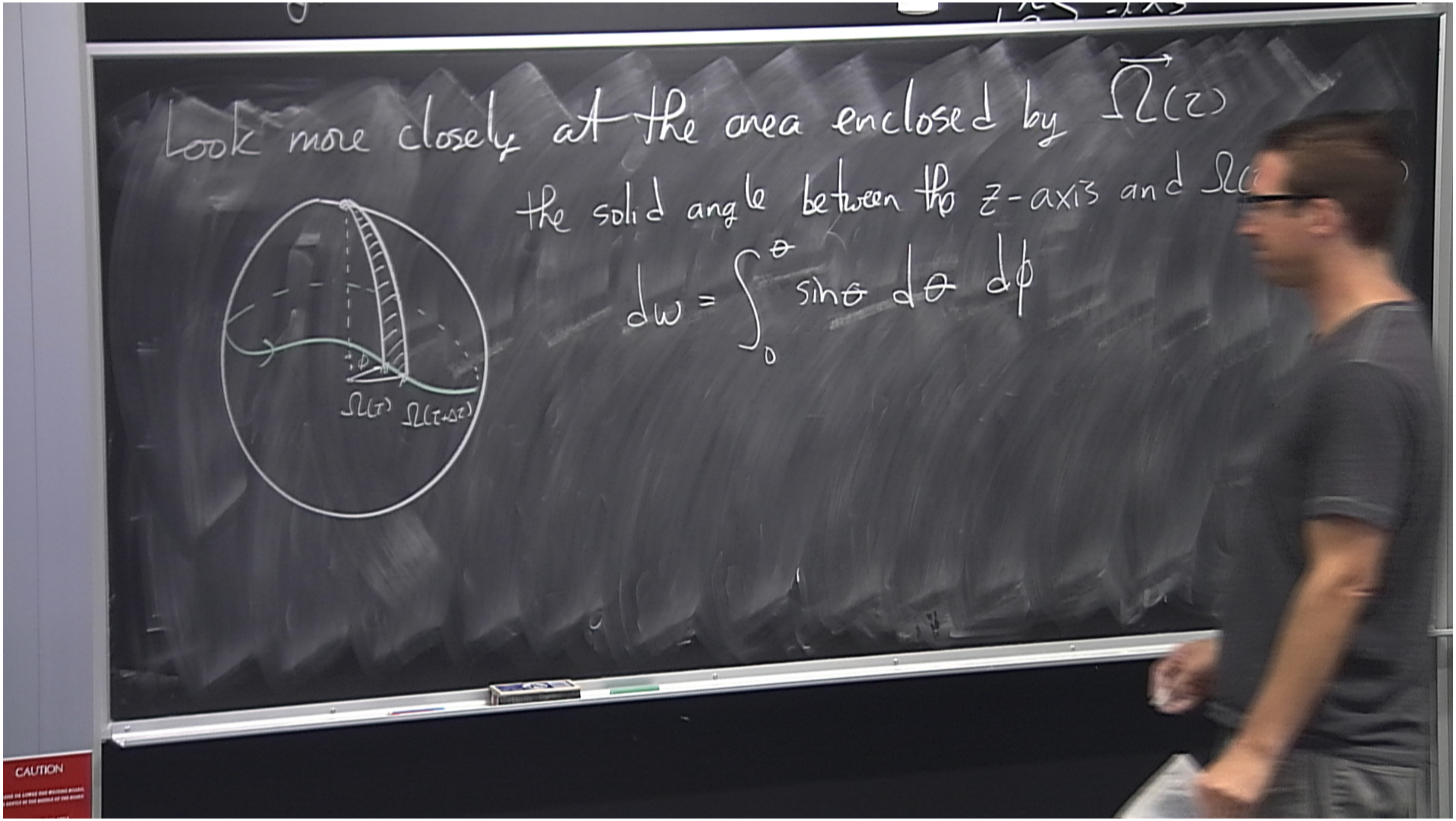
$$= \left(\frac{\partial}{\partial \tau} e^{-i\phi S_z} e^{-i\theta S_y} |S,+S\rangle \right)^\dagger$$

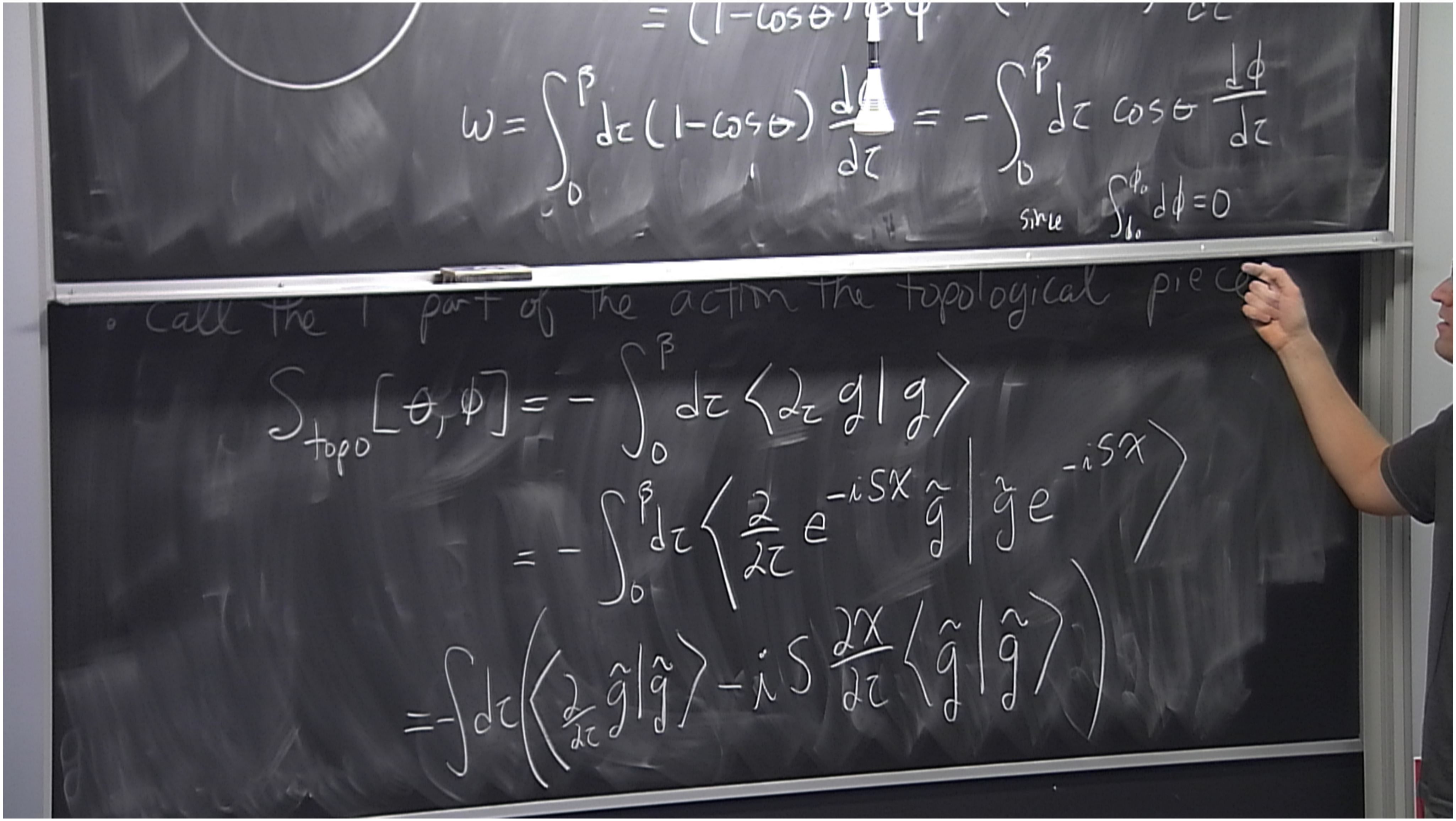
$$= \left(-i \frac{\partial \phi}{\partial \tau} S_z e^{-i\phi S_z} e^{-i\theta S_y} - i \frac{\partial\theta}{\partial\tau} e^{-i\phi S_z} S_y e^{-i\theta S_y} |S,+S\rangle \right)^\dagger$$

$$\langle \partial_\tau g | g \rangle = \langle S,+S | \left(i \frac{\partial\phi}{\partial\tau} e^{i\theta S_y} e^{i\phi S_z} S_z + i \frac{\partial\theta}{\partial\tau} e^{i\theta S_y} S_y e^{i\phi S_z} \right) \cdot \left(e^{-i\phi S_z} e^{-i\theta S_y} \right) |S,+S\rangle$$


we can eliminate τ ⇒ dτ dφ/dτ → dφ
S_topo[θ, φ] = −iS ∫_{φ0}^{φ0} dφ cos θ(φ)
This only depends on the trajectory of Ω(τ) (not its τ-dependence)


Look more closely at the area enclosed by Ω⃗(τ)
the solid angle between the z-axis and Ω(τ)
φ
Ω(τ)
Ω(τ+Δτ)
CAUTION


Look more closely at the area enclosed by $\vec{\Omega}(\tau)$
the solid angle between the z-axis and Ω(
$d\omega = \int_0^\theta \sin\theta \, d\theta \, d\phi$
ϕ
$\Omega(\tau)$
$\Omega(\tau+\Delta\tau)$
CAUTION

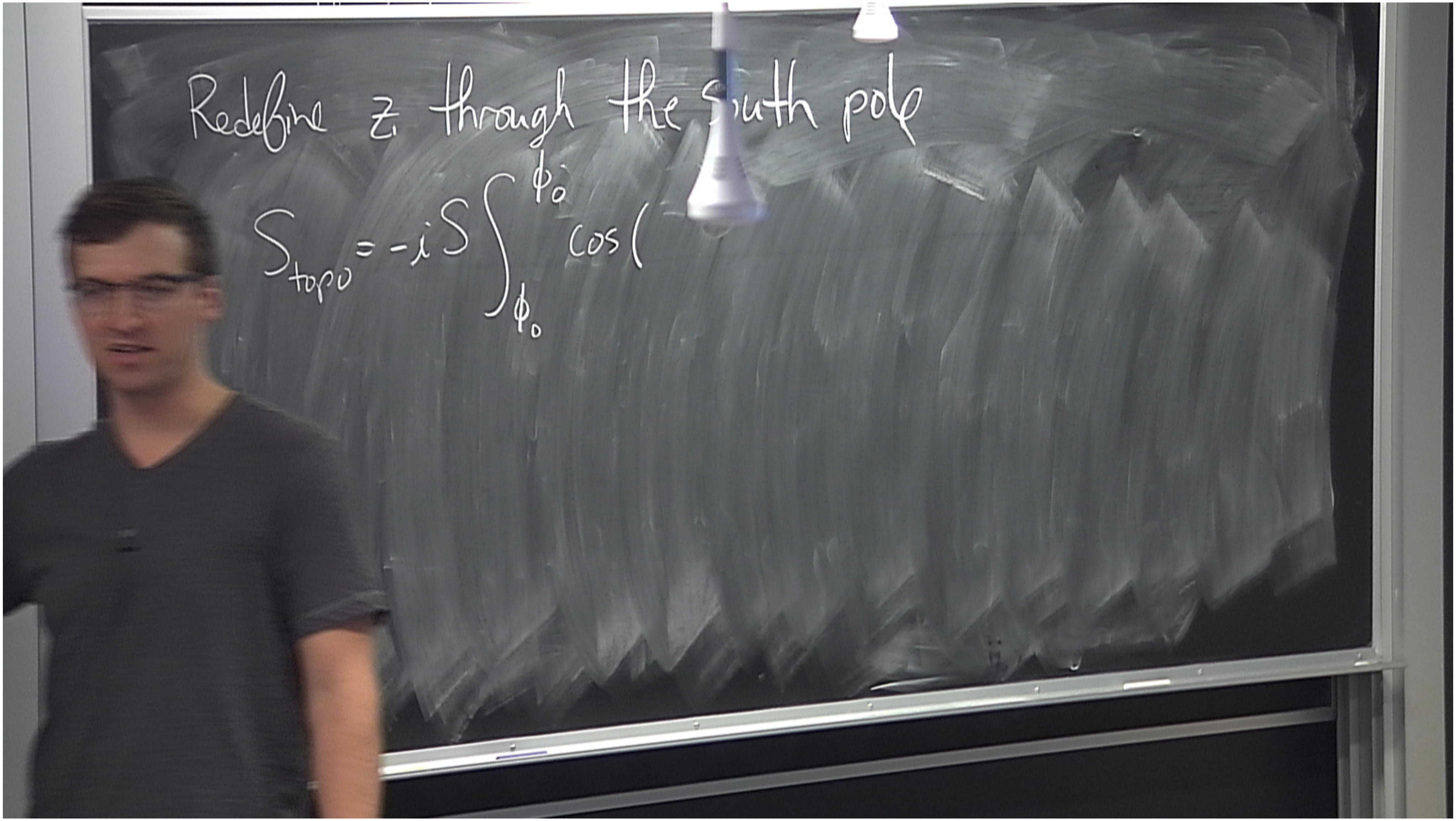

$= (1-\cos\theta)\,\partial\phi$ … $d\tau$
$\omega = \int_0^\beta d\tau\,(1-\cos\theta)\frac{d\phi}{d\tau} = -\int_0^\beta d\tau\,\cos\theta\,\frac{d\phi}{d\tau}$
since $\int_{\phi_0}^{\phi_0} d\phi = 0$
• Call the 1 part of the action the topological piece
$S_{topo}[\theta,\phi] = -\int_0^\beta d\tau\,\langle \partial_\tau g | g\rangle$
$= -\int_0^\beta d\tau\,\langle \frac{\partial}{\partial\tau} e^{-iS\chi}\tilde{g} | \tilde{g} e^{-iS\chi}\rangle$
$= -\int d\tau \left(\langle \frac{\partial}{\partial\tau}\tilde{g}|\tilde{g}\rangle - iS\frac{\partial\chi}{\partial\tau}\langle\tilde{g}|\tilde{g}\rangle\right)$

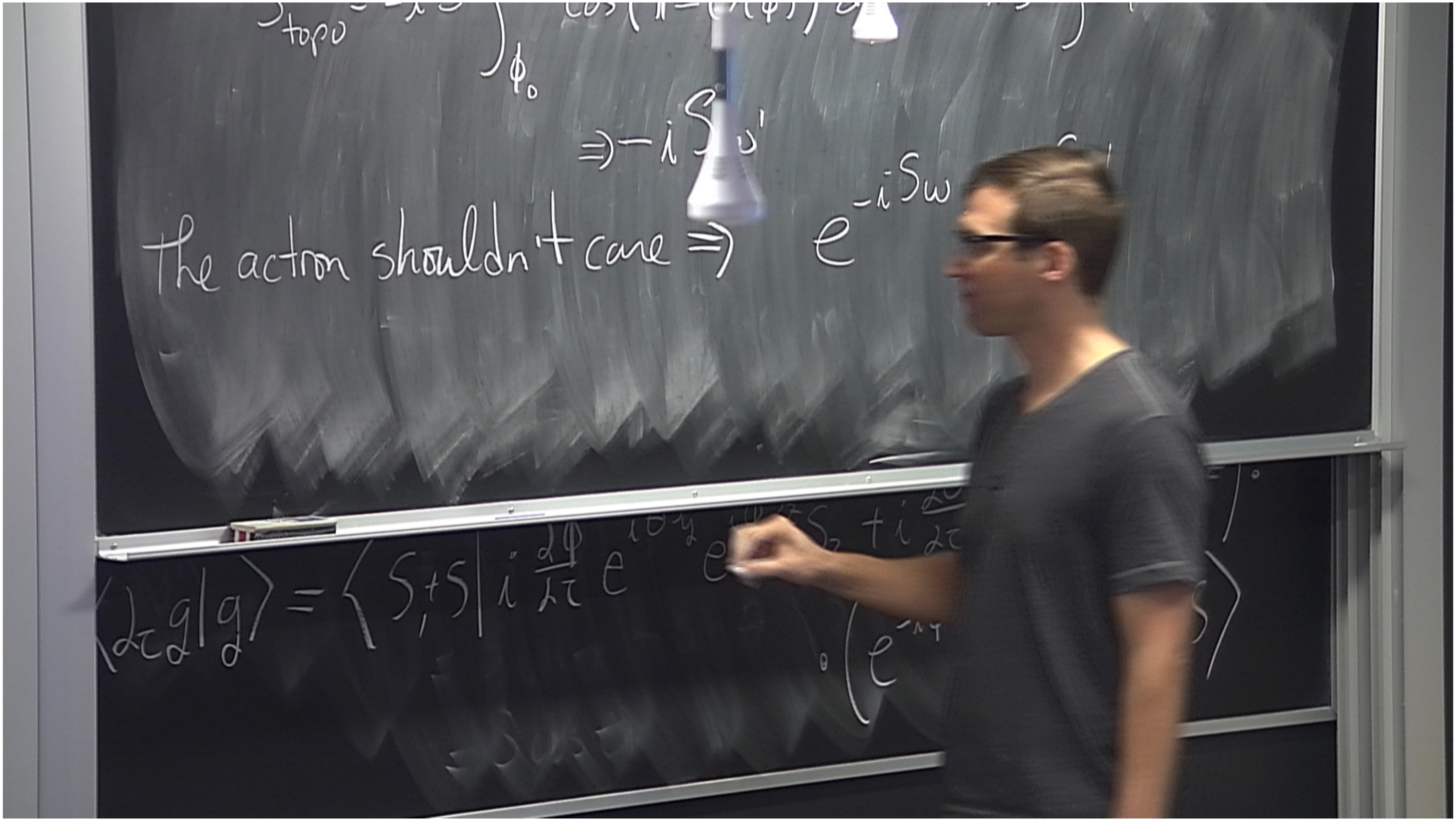
$$\omega = \int_0^T d\tau \,(1-\cos\theta)\,\frac{d\phi}{d\tau} = -\int_0^T d\tau \,\cos\theta \,\frac{d\phi}{d\tau}$$

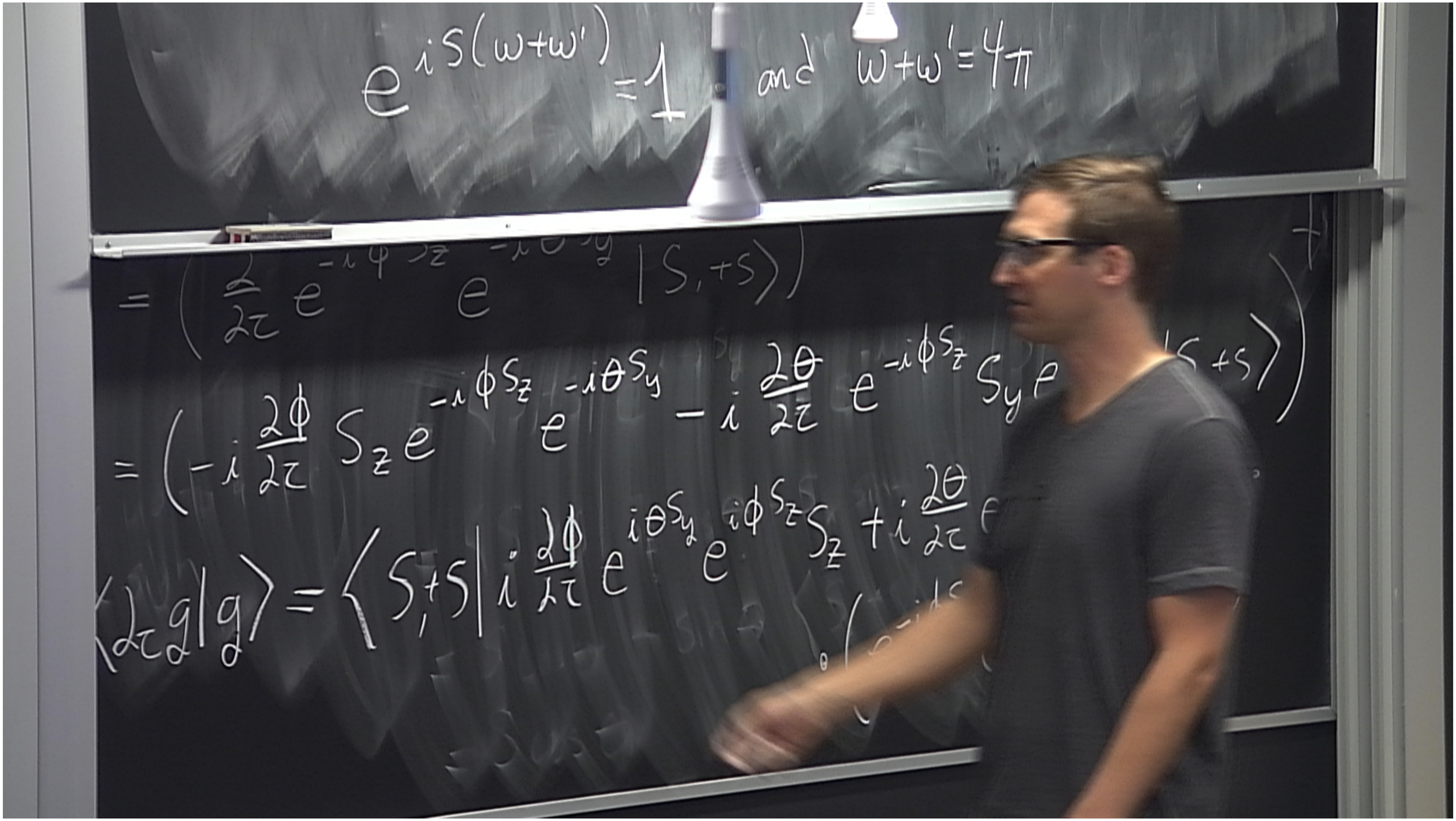
since $\int_{\phi_0}^{\phi_0} d\phi = 0$

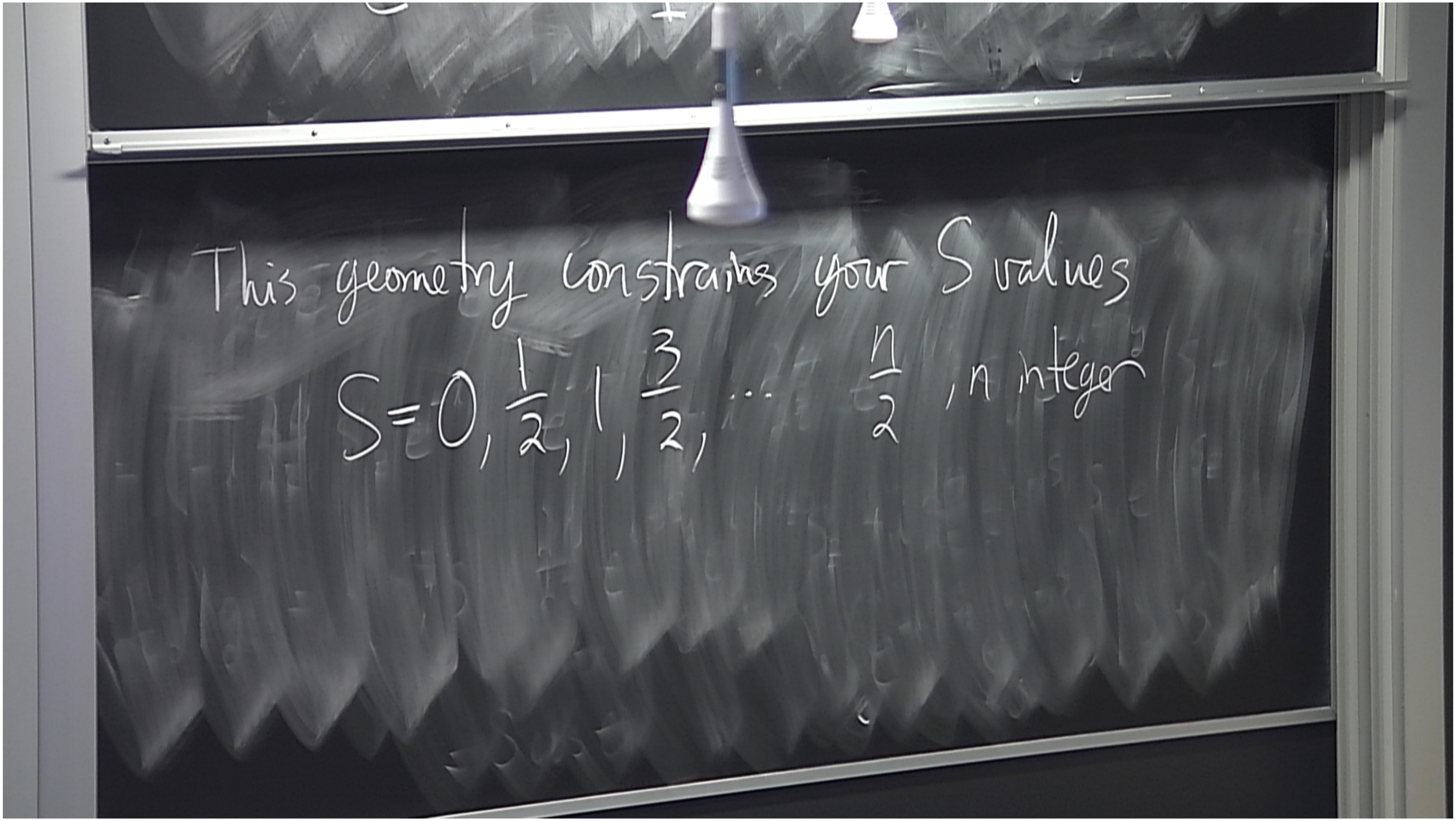
Hence the Berry phase measures the area enclosed by the path if the solid angle subtended is ω

$$S_{topo} = i S \omega$$


Redefine z_i through the South pole
$S_{topo} = -iS \int_{\phi_0}^{\phi_0'} \cos($


the action shouldn't care ⇒ e^{-iSw}

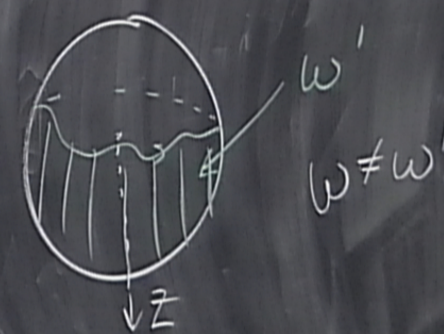

$e^{iS(\omega+\omega')}=1$ and $\omega+\omega'=4\pi$


This geometry constrains your S values
$S = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \frac{n}{2}$, n integer

since $\int_{\phi_0}^{\phi_0} d\phi = 0$

Hence the Berry phase measures the area enclosed by the path if the solid angle subtended is ω

$$S_{topo} = iS\,\omega$$



Note ω has two possible values