Title: Entanglement entropy in conformal perturbation theory and the Einstein equation

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Abstract: <p>For a CFT perturbed by a relevant operator, the entanglement entropy of a spherical region may be computed as a perturbative expansion in the coupling. A similar perturbative expansion applies for excited states near the vacuum. I will describe a method due to Faulkner for calculating these entanglement entropies, and apply it in the limit of small sphere size. The motivation for these calculations is a recent proposal by Jacobson suggesting an equivalence between the Einstein equation and the "maximal vacuum entanglement hypothesis" for quantum gravity. This proposal relies on a conjecture about the behavior of entanglement entropies for small spheres. The calculations presented here suggest that this conjecture must be modified, but I will discuss how Jacobson's derivation still applies under the modified conjecture. </p>

Entanglement entropy in conformal perturbation theory and the Einstein equation

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AJS arXiv:1602.0xxxx

Outline

Geometry from Entanglement

Deep connections relating geometry $\leftrightarrow$ entanglement

- Area law for black hole entropy: $S_{BH} = \frac{1}{4G_N} A \leftrightarrow$ area law for entanglement entropy: $S_{EE} \propto a^{-d+2} A$

  Sorkin; Bombelli, Koul, Lee, Sorkin; Srednicki; Frolov, Novikov

- Ryu-Takayanagi formula for holographic theories: $S_\Sigma = \frac{1}{4G_N} M(\Sigma)$

  Ryu, Takayanagi; Hubeny, Rangamani, Takayanagi

- Derive linearized Einstein equation in the bulk from RT

  Lashkari, Mcdermott, Van Raamsdonk; Faulkner, Guica, Hartman, Myers, Van Raamsdonk; Swingle, Van Raamsdonk

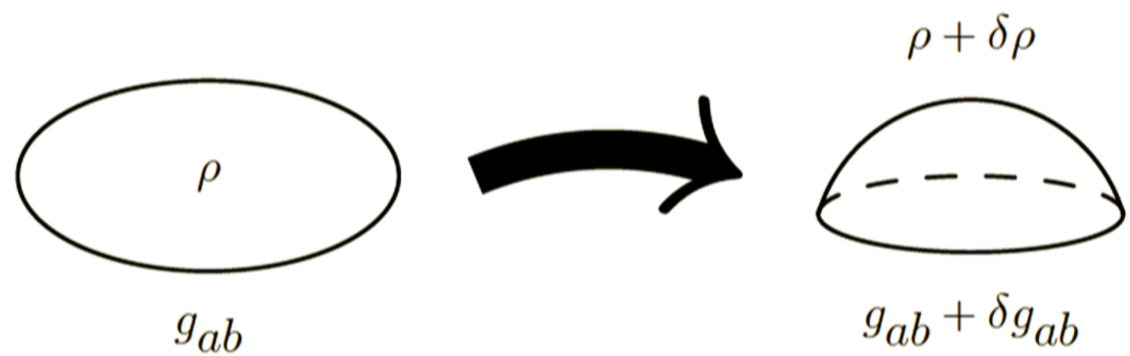
- Maximal vacuum entanglement equivalent to Einstein equation

  Jacobson; Casini, Galante, Myers; Carroll, Remmen; Varadarajan; AJS

Maximal vacuum entanglement $\Longleftrightarrow$ Einstein Equation

MVEH

The entanglement entropy of a small, geodesic ball at fixed volume is maximal in a vacuum configuration of quantum fields coupled to gravity.



$$\delta S = \delta S_{\text{UV}} + \delta S_{\text{IR}} = 0.$$
$$\delta S_{\text{UV}} = \eta \delta A \rightarrow \text{area law.}$$
$$\delta S_{\text{IR}} \rightarrow \text{EE of matter fields.}$$

Maximal vacuum entanglement $\Longleftrightarrow$ Einstein Equation

$\delta S_{\mathsf{UV}} = \eta \delta A$

- Usual area law for entanglement entropy
- η is divergent, regularization dependent
- Postulate that QG renders η finite and universal
- Will find $\eta = \frac{1}{4G_N}$ from MVEH

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Relate to curvature:

- Assume background metric is maximally symmetric, $G_{ab}^{\mathsf{MSS}} = -\Lambda g_{ab}$.
- Express metric near the center of the ball in Riemann normal coordinates.
- Change in area, holding volume fixed is

$$\delta A = -\frac{\Omega_{d-2} R^d}{d^2 - 1}(G_{00} + \Lambda g_{00})$$

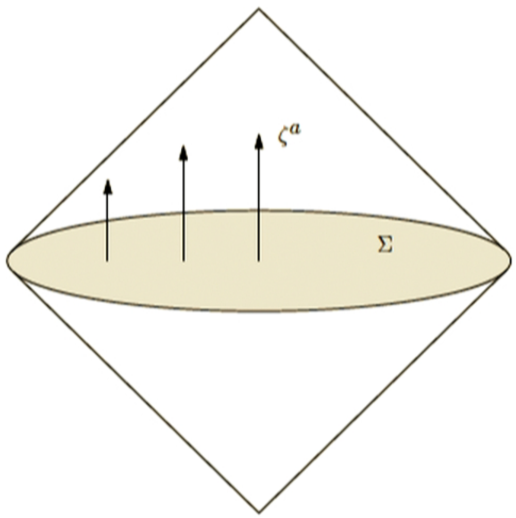
Maximal vacuum entanglement $\Longleftrightarrow$ Einstein Equation

δS_{IR}: at first order, given by first law of entanglement entropy

$$\delta S_{\mathrm{IR}} = 2\pi \delta \langle K \rangle$$

and the modular Hamiltonian is defined by

$$\rho = e^{-2\pi K}/Z$$



For a CFT,

$$K = \int_\Sigma d\Sigma^a \zeta^b T_{ab}$$
$$= \int_\Sigma d\Omega_{d-2} dr\, r^{d-2} \left(\frac{R^2 - r^2}{2R} \right) T_{00}$$

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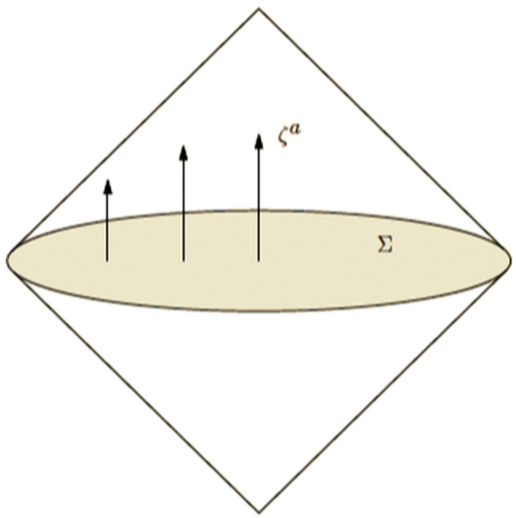

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Maximal vacuum entanglement $\Longleftrightarrow$ Einstein Equation

In the CFT case,

$$\delta S_{\mathsf{IR}} = 2\pi \frac{\Omega_{d-2} R^d}{d^2 - 1} \delta \langle T_{00} \rangle$$

Then requiring $\delta S = \delta S_{\mathsf{UV}} + \delta S_{\mathsf{IR}} = 0$ gives

$$G_{00} + \Lambda g_{00} = \frac{2\pi}{\eta} \delta \langle T_{00} \rangle$$

Impose $\delta S = 0$ at all points and in all Lorentz frames, and conservation of T_{ab} to get Einstein equation with cosmological constant.

Maximal vacuum entanglement $\Longleftrightarrow$ Einstein Equation

Non-CFT: enough if

$$\delta S_{\mathrm{IR}} = 2\pi \frac{\Omega_{d-2} R^d}{d^2 - 1} (\delta \langle T_{00} \rangle + C g_{00})$$

Then same argument gives $\Lambda(x) = C(x) + \Lambda_0$, and again recover Einstein equation.

- Requirement on C: scalar under Lorentz boosts.
- C could be state-dependent, operator-dependent.
- Will find that C depends on R in some cases: discuss later in the talk.

Focus on evaluating δS_{IR} for CFT perturbed by relevant operator.

See also Casini, Galante, Myers, arXiv:1601.00528

EE in conformal perturbation theory

Efficient technique for calculating EE of spheres developed by Faulkner arXiv:1412.5648.

- Deform CFT action $I = I_0 + \int f(x)\mathcal{O}(x)$, operator dimension Δ.
- $f(x) = g(x) + \lambda(x)$,
  g represents a theory deformation,
  λ produces excited state.
- Expand entanglement entropy perturbatively,

$$\delta S = S_g + S_\lambda + S_{g^2} + S_{g\lambda} + S_{\lambda^2} + \dots$$

- Look at terms that are $O(\lambda^1)$, any order in g.

EE in conformal perturbation theory

Path integral representation of density matrix

$$\langle \phi_- | \rho | \phi_+ \rangle = \frac{1}{N} \int_{\substack{\phi(\Sigma_+)=\phi_+ \\ \phi(\Sigma_-)=\phi_-}} \mathcal{D}\phi \, e^{-I_0 - \int f\mathcal{O}}$$

$$= \frac{1}{Z+\delta Z} \int_{\substack{\phi(\Sigma_+)=\phi_+ \\ \phi(\Sigma_-)=\phi_-}} \mathcal{D}\phi \, e^{-I_0} \left(1 - \int f\mathcal{O} + \frac{1}{2} \iint f\mathcal{O} f\mathcal{O} - \ldots \right)$$

Viewed as evolution from Σ_+ to Σ_- with $\rho_0 = e^{-2\pi K}/Z$, gives operator expression

$$\delta\rho = -\rho_0 \int f\mathcal{O} + \frac{1}{2}\rho_0 \iint T\{f\mathcal{O} f\mathcal{O}\} - \ldots - \text{traces}$$

EE in conformal perturbation theory

Perturbative expansion of entanglement entropy

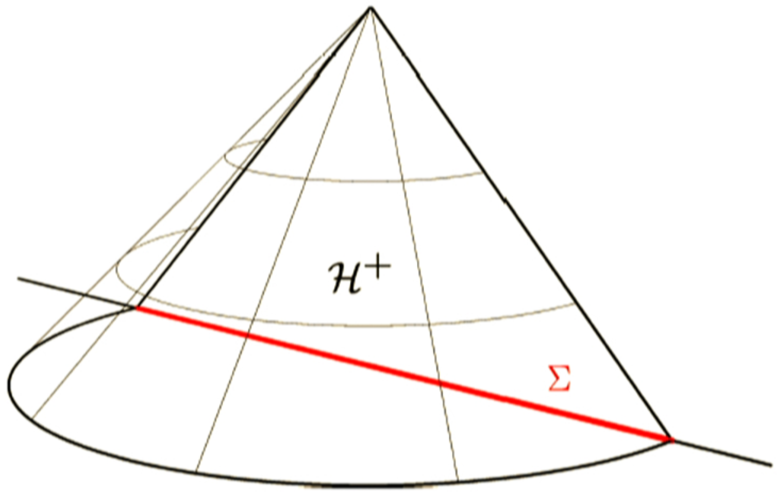
$$S = -\operatorname{Tr} \rho \log \rho$$

Logarithm involves commutators $[\rho_0, \delta\rho]$ from BCH. Instead use resolvent integral,

$$S = \int_0^\infty d\beta \left[\operatorname{Tr}\left(\frac{\rho}{\rho+\beta}\right) - \frac{1}{1+\beta}\right]$$
$$= S_0 + \operatorname{Tr} \int_0^\infty d\beta \frac{\beta}{\rho_0+\beta}\left[\delta\rho \frac{1}{\rho_0+\beta} - \delta\rho\frac{1}{\rho_0+\beta}\delta\rho\frac{1}{\rho_0+\beta} + \dots\right]$$

First term $\delta S^{(1)} = 2\pi\delta\langle K\rangle$, EE 1st law.
Second term $\delta S^{(2)}$ requires more work, but (surprisingly!) can be written holographically...

EE in conformal perturbation theory



Written as integral over AdS_{d+1} Rindler horizon,

$$\delta S^{(2)} = -2\pi \int_{\mathcal{H}^+} d\Sigma^a \xi^b T^B_{ab}$$

ϕ satisfies Klein-Gordon equation, mass $m^2 = \Delta(\Delta - d)$, stress tensor

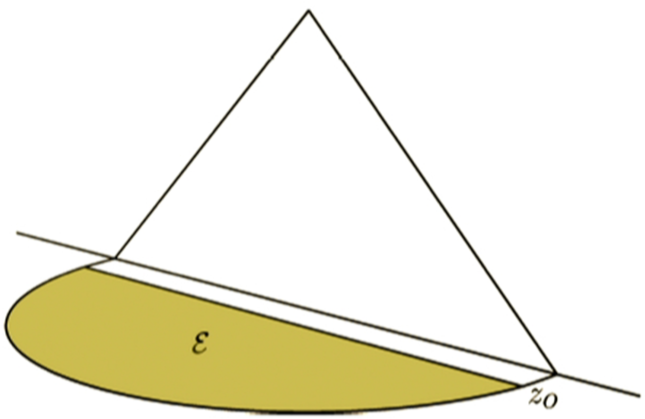
$$T^B_{ab} = \partial_a\phi\partial_b\phi - \frac{1}{2}g_{ab}(m^2\phi^2 + (\partial\phi)^2)$$

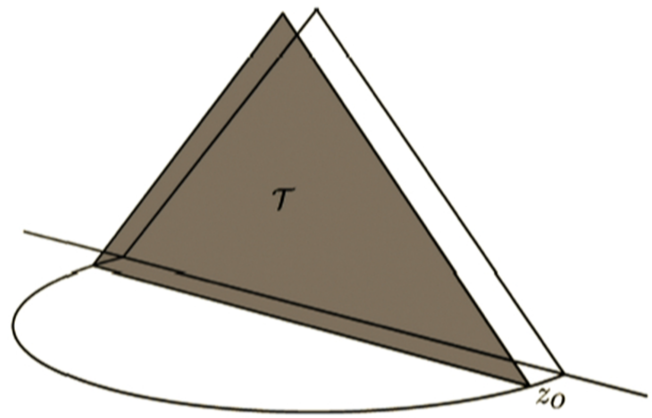
Shown to be equivalent to RT at this order

Faulkner; Faulkner, Guica, Hartman, Myers, Van Raamsdonk

EE in conformal perturbation theory

$\xi^b T^B_{ab}$ is a conserved current $\rightarrow$ deform surface to $\mathcal{E}$ and $\mathcal{T}$





- $\phi \xrightarrow{z\to 0} f z^{d-\Delta} - \frac{\langle \mathcal{O} \rangle}{2\Delta - d} z^{\Delta}$
- Finite terms
- divergence in z_0

- Finite terms $\propto \langle \mathcal{O} \rangle$
- z_0 counterterms
- counterterm canceling $\delta S^{(1)}$ stress tensor divergence

Producing excited states

Two requirements

1. ρ is Hermitian:
   When ρ defined by path integral over action $I = I_0 + \int f\mathcal{O}$,
   requires $\boxed{f(\tau) = f(-\tau)}$
   Also implies that $\partial_\tau f(0) = 0$, $\partial_\tau^3 f(0) = 0 \ldots$

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2. Expectation values are UV finite:
   At $O(\lambda^1)$, expectation value

$$\delta\langle\mathcal{O}(0)\rangle = -\int d^d x \lambda(x) \Big\langle \mathcal{O}(0)\mathcal{O}(x) \Big\rangle_0 = -\int d^d x \frac{\lambda(x) c_\Delta}{x^{2\Delta}}$$

   Diverges like $\lambda(0)\delta^{d-2\Delta}$. Subleading divergences $\sim \partial_\tau^{2n}\lambda(0)\delta^{d-2\Delta+2n}$.
   Require $\lambda(0)$, and first $2q$ τ-derivatives vanish, with

$$q = \left\lfloor \Delta - \frac{d}{2} \right\rfloor$$

EE calculations

Bulk solutions for $\phi(x)$

- ϕ satisfies a linear equation $\to \phi = \phi_g + \phi_\lambda$, theory and state deformations.
- Focus on small spheres $\to$ take $\lambda(\tau, x) = \lambda(\tau)$ spatially constant.
- Fourier decomposition $\to \lambda(\tau) = \int_0^\infty d\omega \lambda_\omega \cos \omega\tau$.
- Bulk solution for each frequency

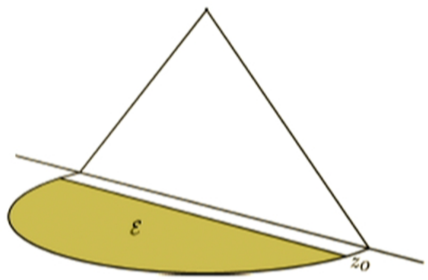
$$\phi_\omega = \lambda_\omega \left(\frac{\omega}{2}\right)^{\Delta - \frac{d}{2}} \frac{2z^{\frac{d}{2}} K_\alpha(\omega z)}{\Gamma(\Delta - \frac{d}{2})} \cos \omega\tau, \qquad \alpha = \frac{d}{2} - \Delta$$

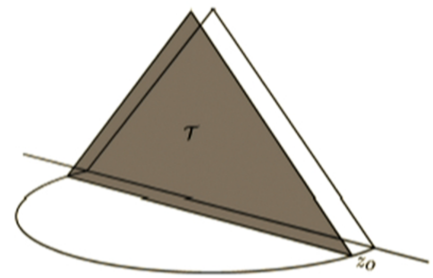
$$\xrightarrow[\tau=0]{z\to 0} \lambda_\omega z^{d-\Delta} + \beta_\omega z^\Delta$$

- Operator expectation value
$\delta\langle\mathcal{O}\rangle = \lambda_\omega = (d - 2\Delta)\beta_\omega = \frac{2\Gamma(\Delta - \frac{d}{2}+1)}{\Gamma(\Delta - \frac{d}{2})} \left(\frac{\omega}{2}\right)^{2\Delta - d}$.

EE calculations: $\Delta > \frac{d}{2}$

Bulk theory deformation $\phi_g = g z^{d-\Delta}$.
Expand ϕ_ω near $z = 0$ and fluxes at $O(\lambda^1 g^1)$





$$\int_{\mathcal{E}} d\Sigma^a \xi^b T^B_{ab}$$

$$= R^d \sum_{n=0}^{\infty} \left[\lambda_\omega R^{d-2\Delta} a_n (\omega R)^{2n} + \beta_\omega b_n (\omega R)^{2n} \right]$$

Impose $\int_0^\infty d\omega\, \omega^{2j} \lambda_\omega = 0$ for $j \le q$.
$\rightarrow$ All terms subdominant to R^d as $R \to 0$.

$\int_{\mathcal{T}} d\Sigma^a \xi^b T^B_{ab}$: Only finite term is from $t \lesssim z_0$, gives

$$-\int_\Sigma \zeta^t g \Delta \beta_\omega$$

$$= \frac{\Omega_{d-2} R^d}{d^2 - 1} \left[\frac{\Delta}{2\Delta - d} g \delta \langle \mathcal{O} \rangle \right]$$

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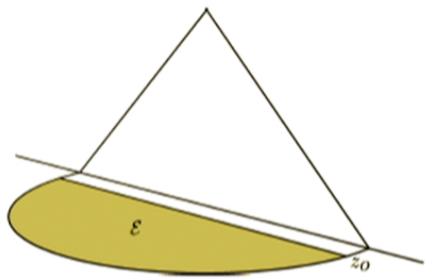

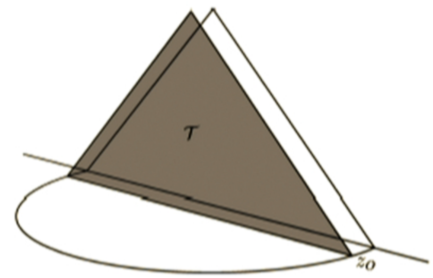

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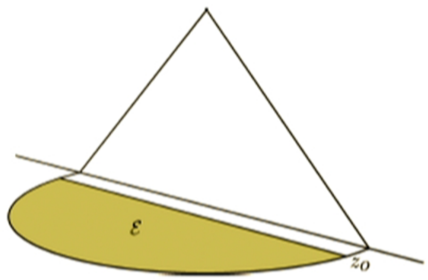

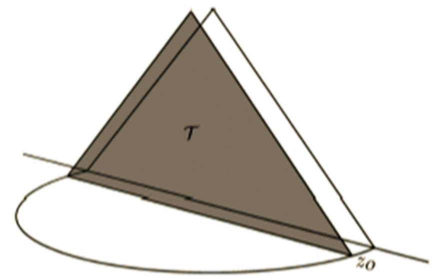

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$$- \int_\Sigma \zeta^t g \Delta \beta_\omega$$

$$= \frac{\Omega_{d-2} R^d}{d^2 - 1} \left[\frac{\Delta}{2\Delta - d} g \delta \langle \mathcal{O} \rangle \right]$$

EE calculations: $\Delta > \frac{d}{2}$

Still need the first law piece,

$$\delta S^{(1)} = 2\pi \frac{\Omega_{d-2} R^d}{d^2 - 1} \delta \langle T^0_{00} \rangle$$

Write in terms of the deformed theory stress tensor and trace,

$$T^g_{ab} = T^0_{ab} - g\mathcal{O} g_{ab}, \qquad \langle T^g \rangle = (\Delta - d) g \langle \mathcal{O} \rangle.$$

Final answer is

$$\delta S_{\lambda g} = 2\pi \frac{\Omega_{d-2} R^d}{d^2 - 1} \left(\delta \langle T^g_{00} \rangle - \frac{1}{2\Delta - d} \delta \langle T^g \rangle \right) + \text{subleading}$$

EE calculations: $\Delta < \frac{d}{2}$

New feature: IR divergence. Consider the vev

$$\langle \mathcal{O}(0) \rangle_g = -\int d^d x g(x) \Big\langle \mathcal{O}(0)\mathcal{O}(x) \Big\rangle_0 = -\int d^d x \frac{c_\Delta g(x)}{x^{2\Delta}}.$$

Cut off $g(x)$ at distance L, vev scales as $L^{d-2\Delta} \to$ divergent when $\Delta \leq \frac{d}{2}$.
L determined by the coupling $L \sim g^{\frac{1}{\Delta - d}} \to$ nonperturbative.
$R \ll L$, use IR cutoff and write everything in terms of $\langle \mathcal{O} \rangle_g$.

E.g. bulk solution on $\mathcal{E}$:

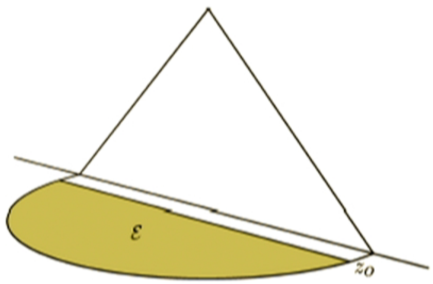
$$\phi_g = g z^{d-\Delta} - \frac{\langle \mathcal{O} \rangle_g}{2\Delta - d} z^\Delta$$

EE calculations: $\Delta < \frac{d}{2}$

State deformation solution: Keep leading terms in (ωR) expansion:

$$\phi_\omega = \lambda_\omega z^{d-\Delta} - \frac{\delta\langle\mathcal{O}\rangle}{2\Delta - d} z^\Delta$$

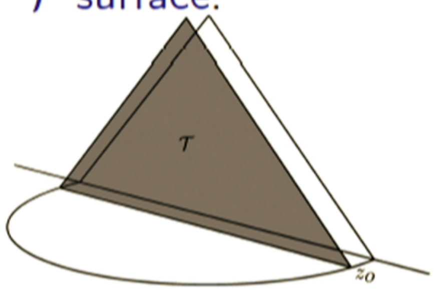
$\mathcal{E}$ surface:



$$\delta S^{(2)}_{\mathcal{E}} = -2\pi R^{2\Delta} \langle\mathcal{O}\rangle_g \delta\langle\mathcal{O}\rangle \frac{\Omega_{d-2}}{d^2-1} A(\Delta, d) + z_0\text{-div.}$$

$$A(\Delta, d) = \frac{\Delta\Gamma(\frac{d}{2}+\frac{3}{2})\Gamma(\Delta - \frac{d}{2}+1)}{(2\Delta - d)^2\Gamma(\Delta+\frac{3}{2})}$$

$\mathcal{T}$ surface:



$$\delta S^{(2)}_{\mathcal{T}} = 2\pi R^d \frac{\Omega_{d-2}}{d^2-1} \frac{\Delta g \delta\langle\mathcal{O}\rangle}{2\Delta - d} + z_0\text{-c.t.}$$

EE calculations: $\Delta < \frac{d}{2}$

Final Result:

$$\delta S_{\lambda g} = \frac{2\pi\Omega_{d-2}}{d^2-1}\left[R^d\left(\delta\langle T^g_{00}\rangle - \frac{1}{2\Delta - d}\delta\langle T^g\rangle\right) - R^{2\Delta}\langle\mathcal{O}\rangle_g\delta\langle\mathcal{O}\rangle A(\Delta, d)\right]$$

Since $\Delta < \frac{d}{2}$, when R is small enough, the second term dominates over the first.

EE calculations: $\Delta = \frac{d}{2}$

New feature: Renormalization scale.
For the vev

$$\langle \mathcal{O}(0)\rangle_g = -\int d^d x \frac{g c_\Delta}{x^d} = -g c_\Delta \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}\int \frac{d\tau}{\tau}$$

Logarithmically divergent. Point-splitting regulator (cutoff for $|\tau| < \delta$) also needs a cutoff at renormalization scale $|\tau| \geq \mu^{-1}$. Gives

$$\langle \mathcal{O}\rangle_g^{\text{div.}} = -g c_\Delta \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} 2\log \mu\delta$$

Renormalized vev with IR cutoff subtracts off this divergence, giving

$$\langle \mathcal{O}\rangle_g^{\text{ren.}} = -g c_\Delta \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} 2\log \mu L$$

EE calculations: $\Delta = \frac{d}{2}$

Remainder of the calculation proceeds as before. Result:

$$\delta S_{\lambda g} = \frac{2\pi \Omega_{d-2} R^d}{d^2-1} \left[\delta \langle T^g_{00} \rangle + \delta \langle T^g \rangle \left(\frac{2}{d} - \frac{1}{2} H_{\frac{d+1}{2}} + \log \frac{\mu R}{2} \right) - \frac{d}{2} \langle \mathcal{O} \rangle_g \delta \langle \mathcal{O} \rangle \right]$$

- μ dependence cancels between $\log \frac{\mu R}{2}$.
- Expression in terms of IR cutoff L has no μ ambiguity.
- $R^d \log R$ term dominates as $R \to 0$.

Discussion

Summary of results

- Calculated entanglement entropy in CFT perturbed by relevant $\mathcal{O}$, first order change relative to the vacuum.
- Extended Faulkner's calculation to $\Delta \leq \frac{d}{2}$ when $R \ll L$, answer depends on nonperturbative vev $\langle \mathcal{O} \rangle_g$.
- For $\Delta \leq \frac{d}{2}$, the $\delta \langle T^g_{00} \rangle$ term is subdominant as $R \to 0$.

Discussion: Implications for Einstein Equation

Conjectured form of δS_{IR}

$$\delta S_{\text{IR}} = 2\pi \frac{\Omega_{d-2} R^d}{d^2 - 1} \left(\delta \langle T_{00} \rangle + C g_{00} \right)$$

- C must transform as a scalar $\rightarrow$ supported by these calculations. State is stationary on times scales $\sim R$, boosted state will be characterized by same operator expectation values.
- C contains a term $\sim R^{2\Delta - d}$ (or $\log R$), which dominates at small R when $\Delta \leq \frac{d}{2}$.
- Proposal (Jacobson): Allow local curvature scale $\Lambda(x)$ to be R-dependent.

Discussion: Implications for Einstein Equation

$\Lambda = \Lambda(R)$:
When $\Delta < \frac{d}{2}$, as $R \to 0$,

$$\Lambda(R) = \frac{2\pi}{\eta} C \sim \ell_P^{d-2} R^{2\Delta - d} \delta\langle \mathcal{O} \rangle^2$$

Require $\Lambda(R) R^2 \ll 1$ to justify flat space modular Hamiltonian, neglecting higher curvature.

$$\Rightarrow \frac{R}{\ell_P} \ll \left(\frac{1}{\ell_P^{2\Delta} \delta\langle \mathcal{O} \rangle^2} \right)^{\frac{1}{2\Delta - d + 2}}$$

Require $\Lambda(R) \ell_P^2 \gg 1$ to avoid strong QG effects

$$\Rightarrow \frac{R}{\ell_P} \gg \left(\ell_P^{2\Delta} \delta\langle \mathcal{O} \rangle^2 \right)^{\frac{1}{d - 2\Delta}}$$

Wide range of R values satisfying these.

Discussion: Future work

- Investigate Lorentz transformations more thoroughly

  Rosenhaus, Smolkin; Faulkner, Leigh, Parrikar

- Higher order corrections: may still be possible holographically since $\mathcal{O}$ three-point function is fixed by conformal invariance.

  Holographic calculation: Casini, Galante, Myers

- Address IR divergences more thoroughly, perhaps in simplified cases (e.g. free field theories, 2D models).

  Casini, Huerta; Blanco, Casini; Casini, Galante, Myers; Zamolodchikov;...

- Higher curvature corrections to Einstein equation: Higher order expansion in RNC, shape deformations of entangling surface.

  Rosenhaus, Smolkin; Faulkner, Leigh, Parrikar