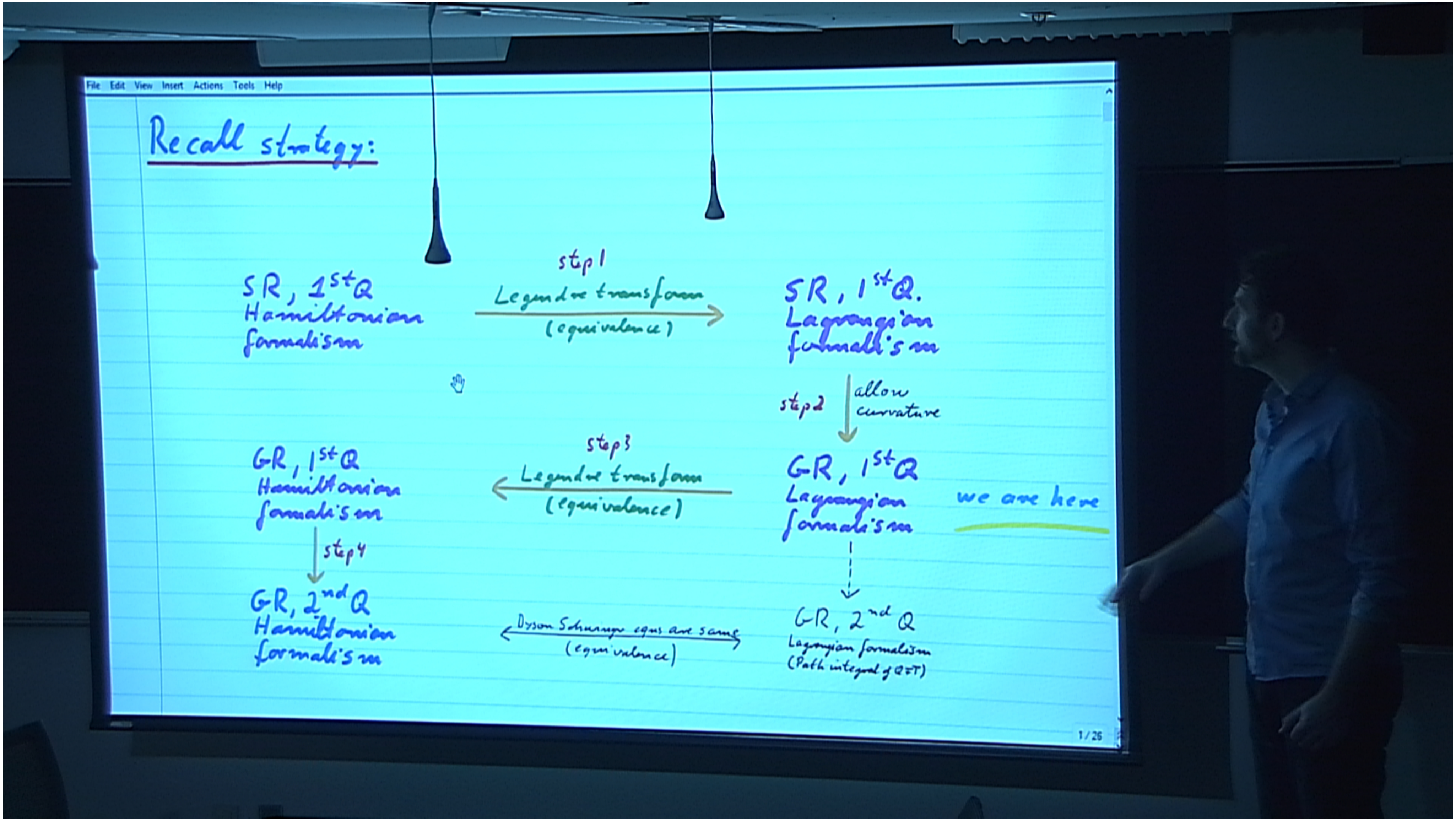
Title: Quantum Field Theory for Cosmology - Achim Kempf - Lecture 12

Date: Feb 12, 2016 01:30 PM

URL: http://pirsa.org/16020003

Abstract:


File Edit View Insert Actions Tools Help
Recall strategy:
SR, 1st Q
Hamiltonian formalism
step 1
Legendre transform
(equivalence)
SR, 1st Q.
Lagrangian formalism
step 2
allow curvature
GR, 1st Q
Hamiltonian formalism
step 3
Legendre transform
(equivalence)
GR, 1st Q
Lagrangian formalism
we are here
step 4
GR, 2nd Q
Hamiltonian formalism
Dyson Schwinger eqns are same
(equivalence)
GR, 2nd Q
Lagrangian formalism
(Path integral of QFT)
1 / 26

$$S_{KG} = \frac{1}{2}\int_{\mathbb{R}^4} \left( g^{\mu\nu}(x)\, \phi_{,\mu}(x)\, \phi_{,\nu}(x) - m^2 \phi^2(x) \right) \sqrt{|g|}\; d^4x$$

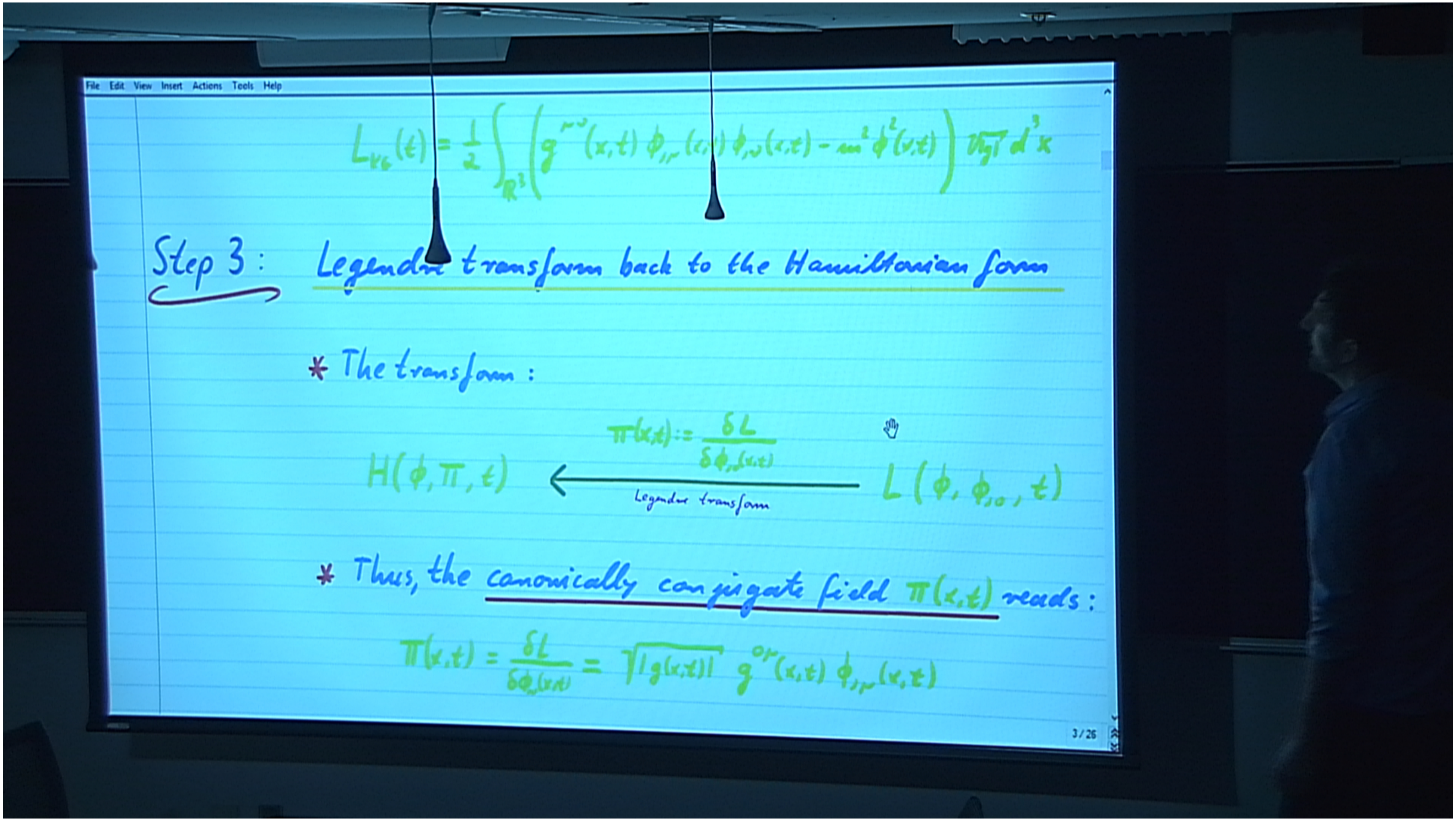
↖ we assume that the coordinate system is such, for simplicity.

It yields, via $\frac{\delta S_{KG}}{\delta \phi} = 0$ the Klein Gordon eqn:

$$\frac{1}{\sqrt{|g(x)|}} \frac{\partial}{\partial x^\mu} \left( g^{\mu\nu}(x) \sqrt{|g(x)|}\; \phi_{,\nu}(x) \right) + m^2 \phi(x) = 0 \qquad (KG)$$

* We read off the Lagrangian:

2/26


File Edit View Insert Actions Tools Help
$L_{KG}(t) = \frac{1}{2}\int_{\mathbb{R}^3}\left(g^{\mu\nu}(x,t)\,\phi_{,\mu}(x,t)\,\phi_{,\nu}(x,t) - m^2\phi^2(x,t)\right)\sqrt{|g|}\,d^3x$
Step 3: Legendre transform back to the Hamiltonian form
* The transform:
$H(\phi,\pi,t) \xleftarrow[\text{Legendre transform}]{\pi(x,t) := \frac{\delta L}{\delta\phi_{,0}(x,t)}} L(\phi,\phi_{,0},t)$
* Thus, the canonically conjugate field $\pi(x,t)$ reads:
$\pi(x,t) = \frac{\delta L}{\delta\phi_{,0}(x,t)} = \sqrt{|g(x,t)|}\; g^{0\mu}(x,t)\,\phi_{,\mu}(x,t)$
3/26

File Edit View Insert Actions Tools Help

$$L_{KG}(t) = \frac{1}{2}\int_{\mathbb{R}^3}\left(g^{\mu\nu}(x,t)\,\phi_{,\mu}(x,t)\,\phi_{,\nu}(x,t) - m^2\phi^2(x,t)\right)\sqrt{|g|}\,d^3x$$

**Step 3:** Legendre transform back to the Hamiltonian form

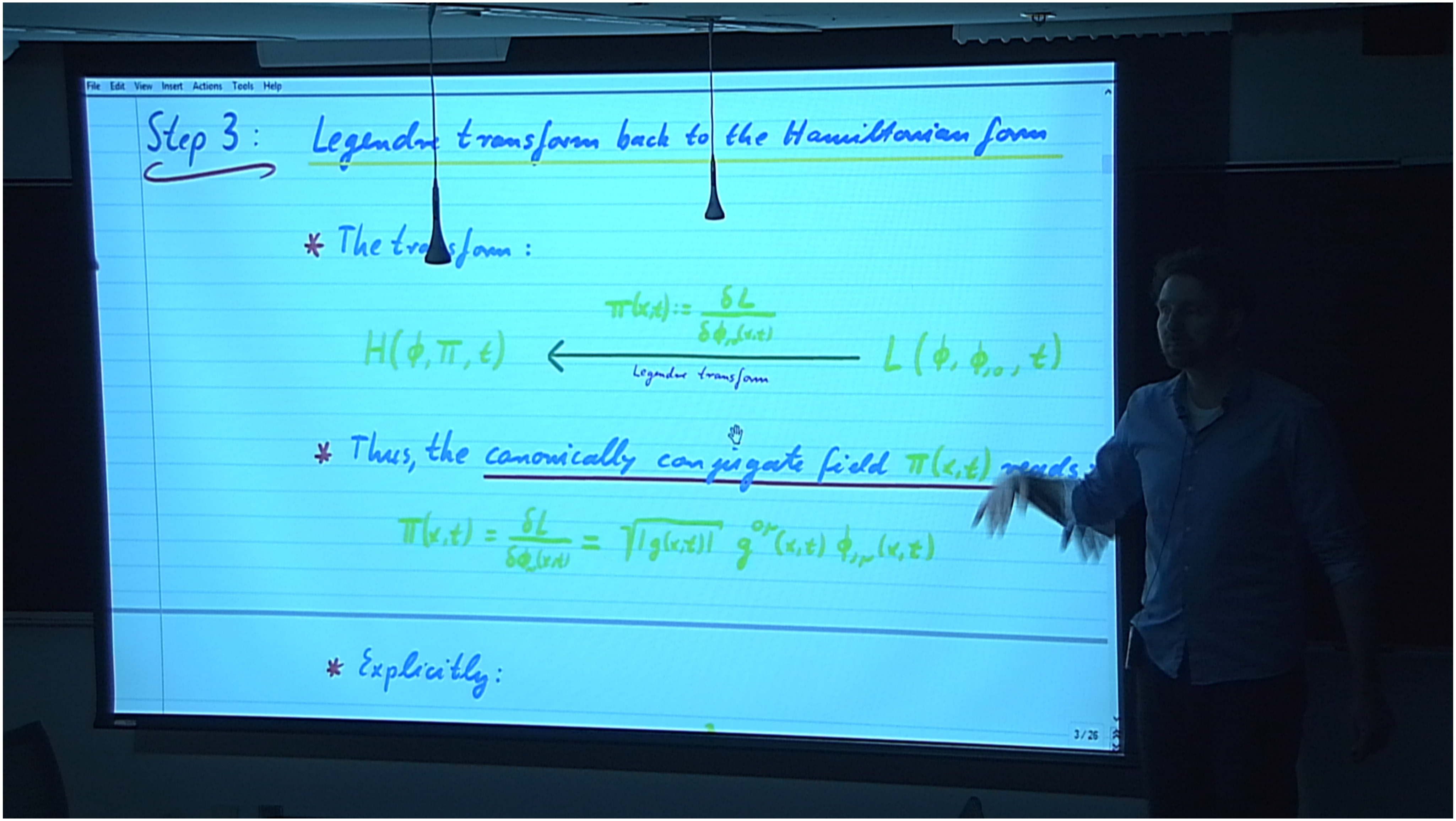
* The transform:

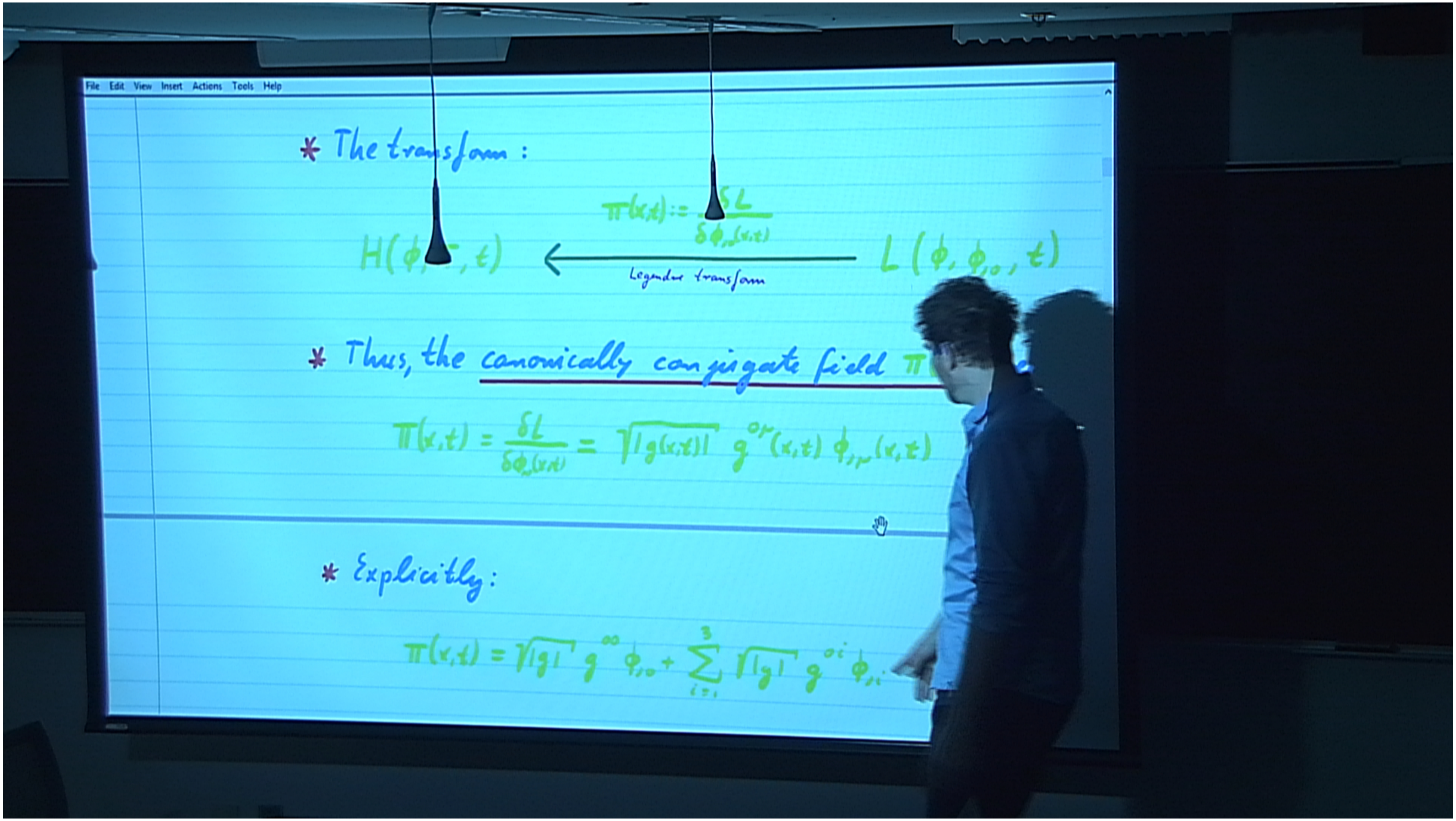
$$H(\phi,\pi,t) \xleftarrow[\text{Legendre transform}]{\pi(x,t) := \frac{\delta L}{\delta \phi_{,0}(x,t)}} L(\phi,\phi_{,0},t)$$

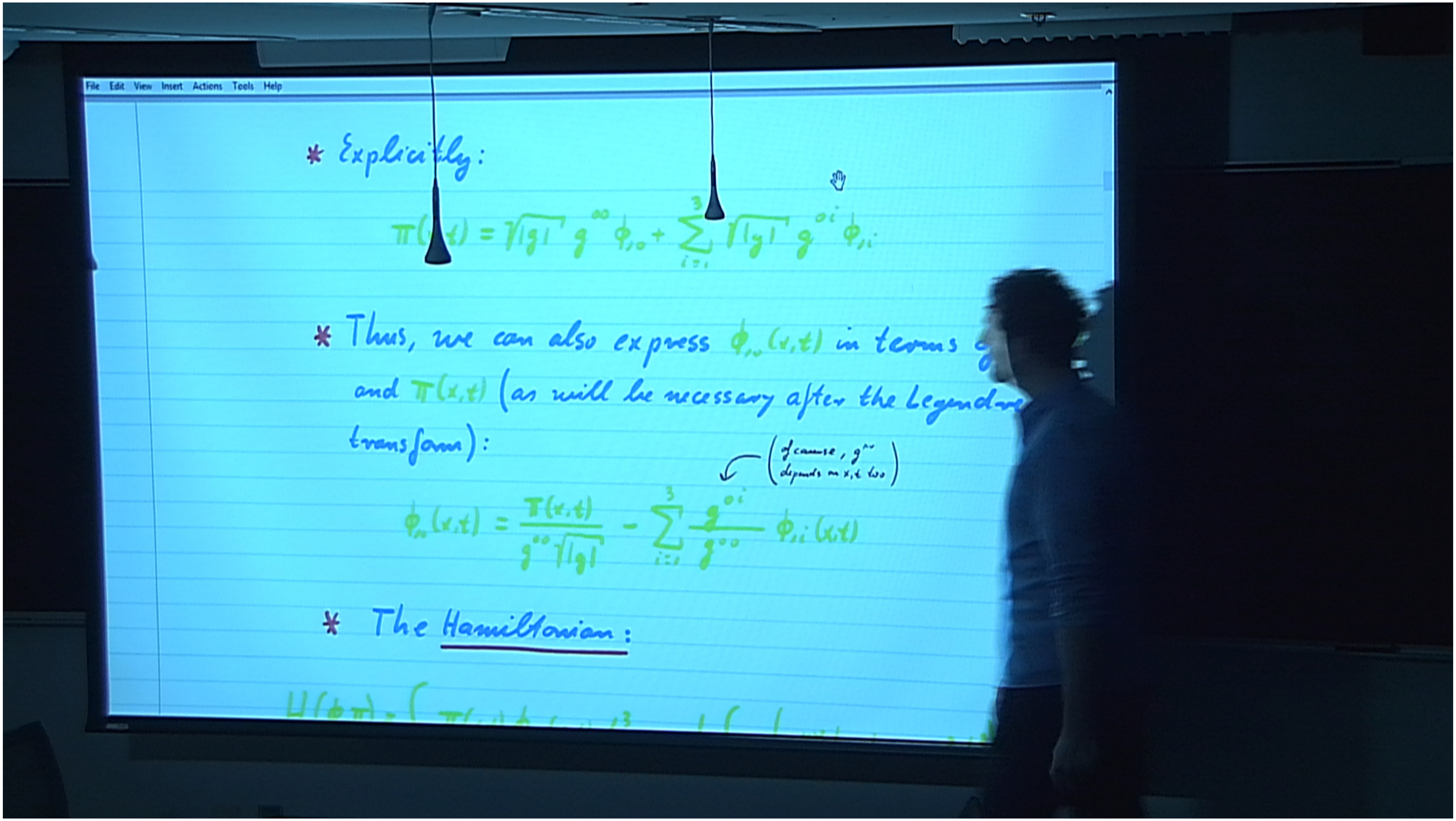
* Thus, the canonically conjugate field $\pi(x,t)$ reads:

$$\pi(x,t) = \frac{\delta L}{\delta \phi_{,0}(x,t)} = \sqrt{|g(x,t)|}\; g^{0\mu}(x,t)\,\phi_{,\mu}(x,t)$$

3 / 26


File Edit View Insert Actions Tools Help
Step 3: Legendre transform back to the Hamiltonian form
* The transform:
$\pi(x,t) := \frac{\delta L}{\delta \phi_{,0}(x,t)}$
$H(\phi, \pi, t) \xleftarrow[\text{Legendre transform}]{} L(\phi, \phi_{,0}, t)$
* Thus, the canonically conjugate field $\pi(x,t)$ reads:
$\pi(x,t) = \frac{\delta L}{\delta \phi_{,0}(x,t)} = \sqrt{|g(x,t)|}\, g^{0\mu}(x,t)\, \phi_{,\mu}(x,t)$
* Explicitly:
3/26


File Edit View Insert Actions Tools Help
* The transform :
Legendre transform
* Thus, the canonically conjugate field
* Explicitly:



File Edit View Insert Actions Tools Help
* Explicitly:
$\pi(x,t) = \sqrt{|g|}\, g^{00} \phi_{,0} + \sum_{i=1}^{3} \sqrt{|g|}\, g^{0i} \phi_{,i}$
* Thus, we can also express $\phi_{,0}(x,t)$ in terms
and $\pi(x,t)$ (as will be necessary after the Legendre
transform):
(of course, $g^{\mu\nu}$ depends on $x,t$ too)
$\phi_{,0}(x,t) = \frac{\pi(x,t)}{g^{00}\sqrt{|g|}} - \sum_{i=1}^{3} \frac{g^{0i}}{g^{00}} \phi_{,i}(x,t)$
* The Hamiltonian:

File Edit View Insert Actions Tools Help

* Thus, we can also express $\phi_{,0}(x,t)$ in terms of $\phi(x,t)$ and $\pi(x,t)$ (as will be necessary after the Legendre transform):

(of course, $g^{\mu\nu}$ depends on $x,t$ too)

$$\phi_{,0}(x,t) = \frac{\pi(x,t)}{g^{00}\sqrt{|g|}} - \sum_{i=1}^{3} \frac{g^{0i}}{g^{00}}\,\phi_{,i}(x,t) \qquad (V)$$

* The <u>Hamiltonian</u>:

$$H(\phi,\pi) = \int_{\mathbb{R}^3} \pi(x,t)\,\phi_{,0}(x,t)\,d^3x - \underbrace{\frac{1}{2}\int_{\mathbb{R}^3}\left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2\right)\sqrt{|g|}\,d^3x}_{=\,L(\phi,\,\phi_{,0}(\phi,\pi),\,t)}$$

Why don't we need a factor of $\sqrt{|g|}$ for covariance here? Because $\pi$ has it built in!

4/26

File Edit View Insert Actions Tools Help

$$H(\phi,\pi) = \int_{\mathbb{R}^3} \pi(x,t)\,\phi_{,0}(x,t)\,d^3x - \frac{1}{2}\underbrace{\int_{\mathbb{R}^3} \left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2\right)\sqrt{|g|}\,d^3x}_{= L(\phi,\,\phi_{,0}(\phi,\pi),\,t)}$$

Why don't we need a factor of $\sqrt{|g|}$ for covariance here? Because $\pi$ has it built in!

* In $H$, one needs to express all occurring $\phi_{,0}$ in terms of the new variables $\phi$ and $\pi$, by using (V), to obtain $H(\phi,\pi,t)$.

→ **Exercise:** Calculate $H(\phi,\pi,t)$ and simplify the expression as far as possible.

4/26

* The equations of motion:

We know from the general properties of the Legendre transform that the equations of motion now take the form:

$$\frac{d}{dt}\phi(x,t) = \frac{\delta H(\phi,\pi,t)}{\delta \pi(x,t)}, \quad \frac{d}{dt}\pi(x,t) = -\frac{\delta H(\phi,\pi,t)}{\delta \phi(x,t)}$$

* Exercise: Verify that these eqns are equivalent to (KG).

We are now ready to $2^{nd}$ quantize:

5/26

File Edit View Insert Actions Tools Help

* The equations of motion:

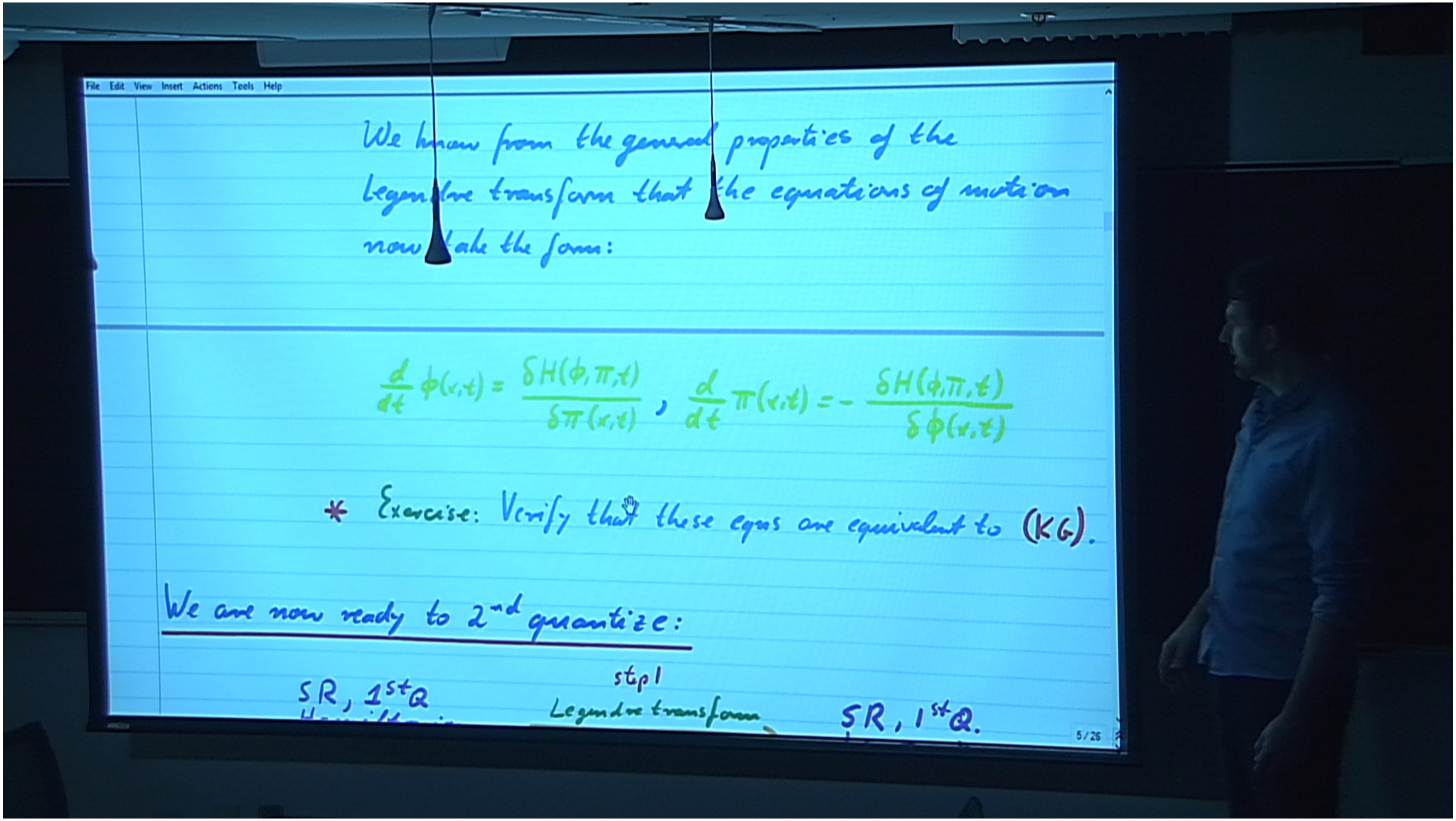
We know from the general properties of the Legendre transform that the equations of motion now take the form:

$$\frac{d}{dt}\phi(x,t) = \frac{\delta H(\phi,\pi,t)}{\delta\pi(x,t)}\ , \quad \frac{d}{dt}\pi(x,t) = -\frac{\delta H(\phi,\pi,t)}{\delta\phi(x,t)}$$

* Exercise: Verify that these eqns are equivalent to (KG).

We are now ready to 2nd quantize:

5 / 26


File Edit View Insert Actions Tools Help
We know from the general properties of the Legendre transform that the equations of motion now take the form:
$\frac{d}{dt}\phi(x,t) = \frac{\delta H(\phi,\pi,t)}{\delta \pi(x,t)}, \quad \frac{d}{dt}\pi(x,t) = -\frac{\delta H(\phi,\pi,t)}{\delta \phi(x,t)}$
* Exercise: Verify that these eqns are equivalent to (KG).
We are now ready to $2^{nd}$ quantize:
SR, $1^{st}$Q
step 1
Legendre transform
SR, $1^{st}$Q.
5/26

Solving the quantized theory is to solve:

1.) Commutation relations:

$$[\hat{\phi}(x,t), \hat{\pi}(x',t)] = i\hbar\,\delta^3(x-x')$$

$$[\hat{\phi}(x,t), \hat{\phi}(x',t)] = 0$$

$$[\hat{\pi}(x,t), \hat{\pi}(x',t)] = 0$$

(CCRs)

2.) Hermiticity:

$$\hat{\phi}^\dagger(x,t) = \hat{\phi}(x,t), \qquad \hat{\pi}^\dagger(x,t) = \hat{\pi}(x,t) \qquad \text{(HC)}$$

7/26

## 3.) Equations of motion:

In the Heisenberg picture, they are formally unchanged:

$$\frac{d}{dt}\hat{f}(\phi,\pi) = \frac{1}{i\hbar}[\hat{f}, \hat{H}] \qquad \text{for } \hat{f} = \hat{\phi}, \ \hat{f} = \hat{\pi}, \text{ etc}$$

Namely:

$$\left(\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu} g^{\mu\nu}\sqrt{|g|}\frac{\partial}{\partial x^\nu} + m^2\right)\hat{\phi}(x,t) = 0 \qquad (\text{EoM1})$$

and:

In the Heisenberg picture, they are formally unchanged:

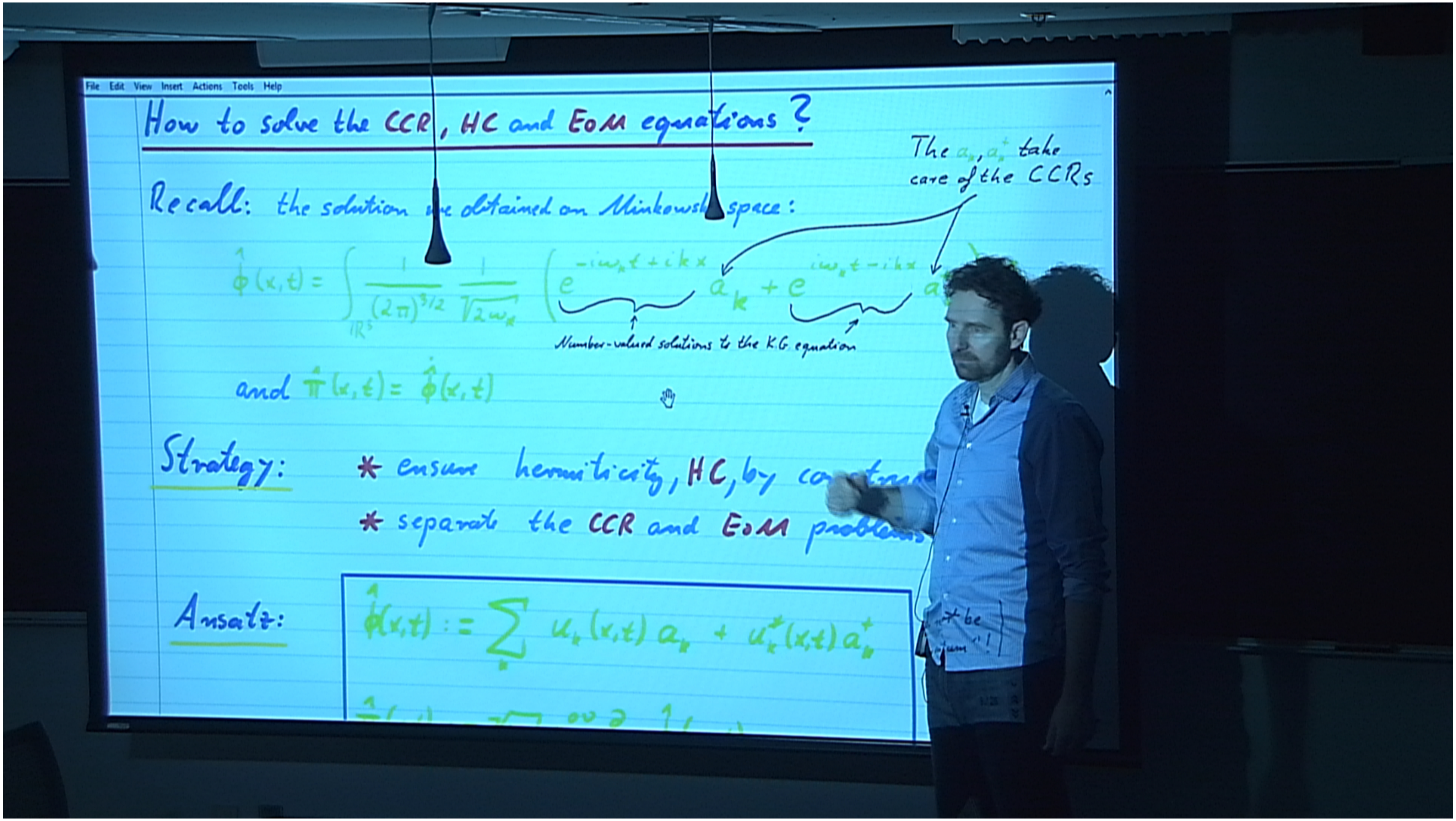
$$\frac{d}{dt}\hat{f}(\phi,\pi) = \frac{1}{i\hbar}[\hat{f}, \hat{H}] \qquad \text{for } \hat{f} = \hat{\phi}, \ \hat{f} = \hat{\pi}, \text{ etc}$$

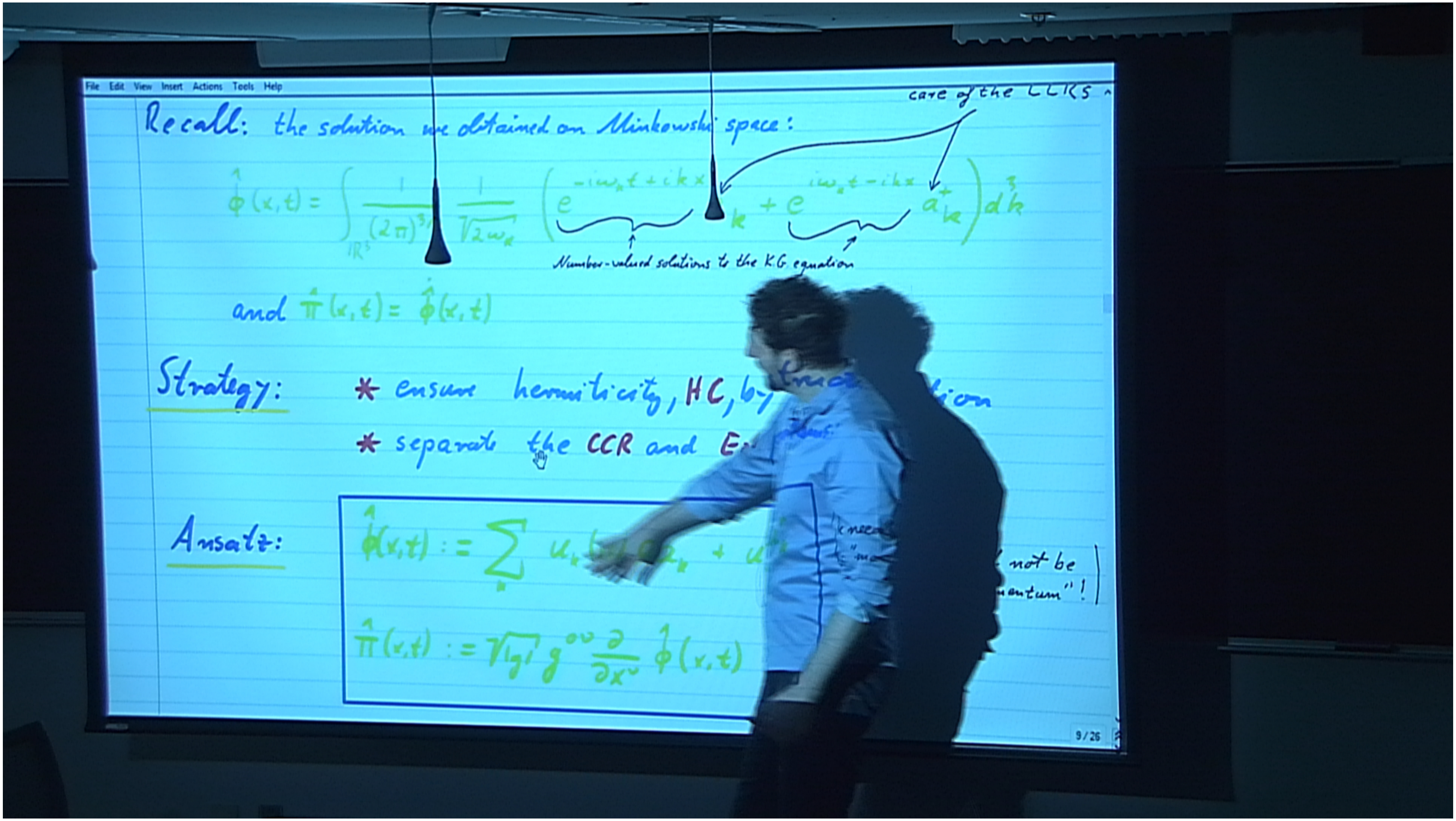
Namely:

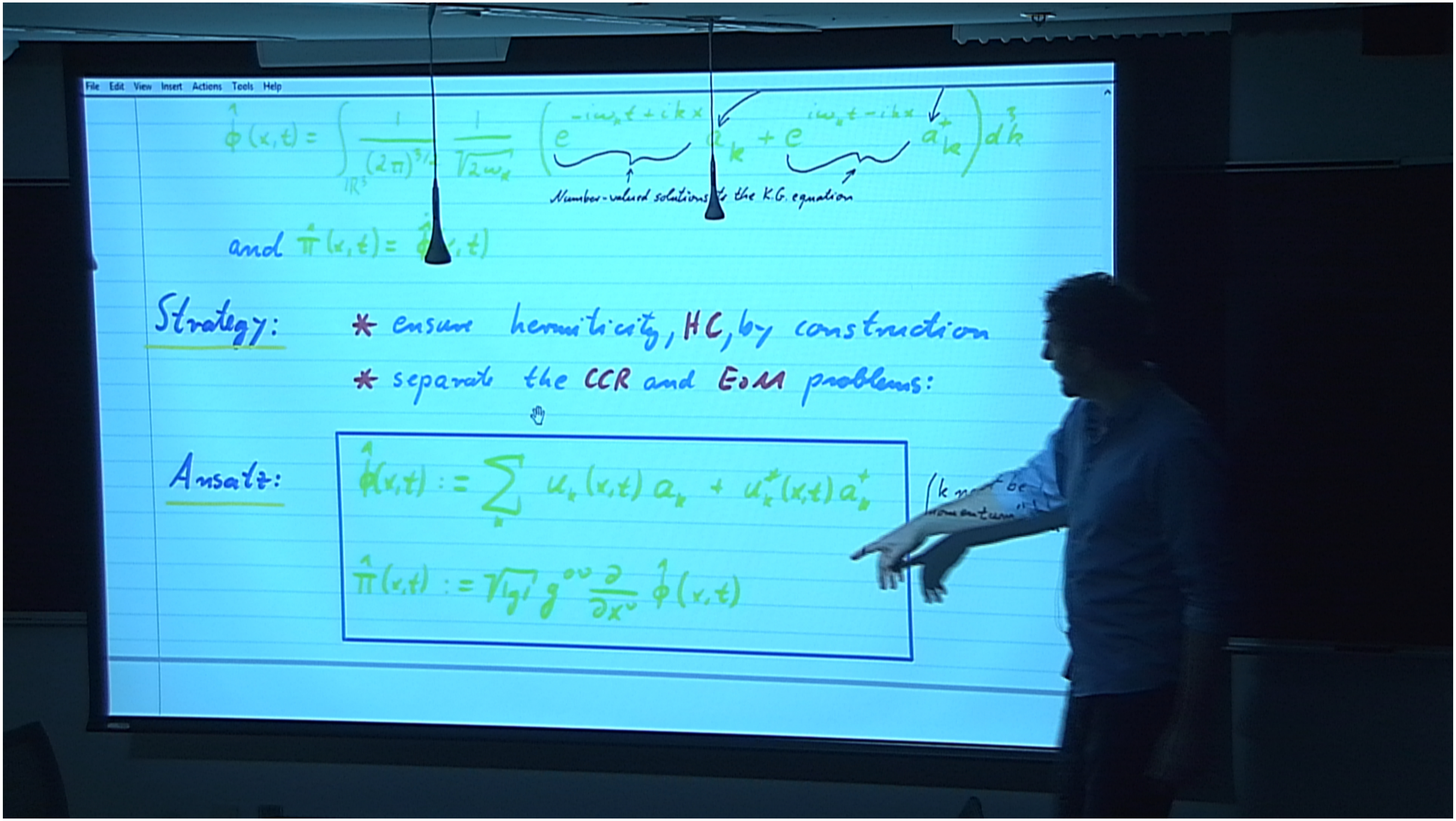
$$\left(\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu} g^{\mu\nu}\sqrt{|g|}\frac{\partial}{\partial x^\nu} + m^2\right)\hat{\phi}(x,t) = 0 \qquad (EoM1)$$

and:

$$\hat{\pi}(x,t) = \sqrt{|g|}\, g^{0\nu}\frac{\partial}{\partial x^\nu}\hat{\phi}(x,t) \qquad (EoM2)$$


File Edit View Insert Actions Tools Help
How to solve the CCR, HC and EoM equations?
The $a_k, a_k^+$ take care of the CCRs
Recall: the solution we obtained on Minkowski space:
Number-valued solutions to the K.G. equation
Strategy:
* ensure hermiticity, HC, by construction
* separate the CCR and EoM problems
Ansatz:


File Edit View Insert Actions Tools Help
Recall: the solution we obtained on Minkowski space:
Number-valued solutions to the K.G. equation
and
Strategy:
* ensure hermiticity, HC,
* separate the CCR and
Ansatz:
not be
9/26


File Edit View Insert Actions Tools Help
$\hat{\phi}(x,t) = \int_{\mathbb{R}^3} \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left( e^{-i\omega_k t + ikx} a_k + e^{i\omega_k t - ikx} a_k^+ \right) d^3k$
Number-valued solutions to the K.G. equation
and $\hat{\pi}(x,t) = \dot{\hat{\phi}}(x,t)$
Strategy:
* ensure hermiticity, HC, by construction
* separate the CCR and EoM problems:
Ansatz:
$\hat{\phi}(x,t) := \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^+$
$\hat{\pi}(x,t) := \sqrt{|g|}\, g^{0\nu} \frac{\partial}{\partial x^\nu} \hat{\phi}(x,t)$

File Edit View Insert Actions Tools Help

* The 1st equation of motion: ✓

$$\left(\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu} g^{\mu\nu}\sqrt{|g|}\frac{\partial}{\partial x^\nu} + m^2\right)\hat{\phi}(x,t) = 0 \qquad \text{(EoM1)}$$

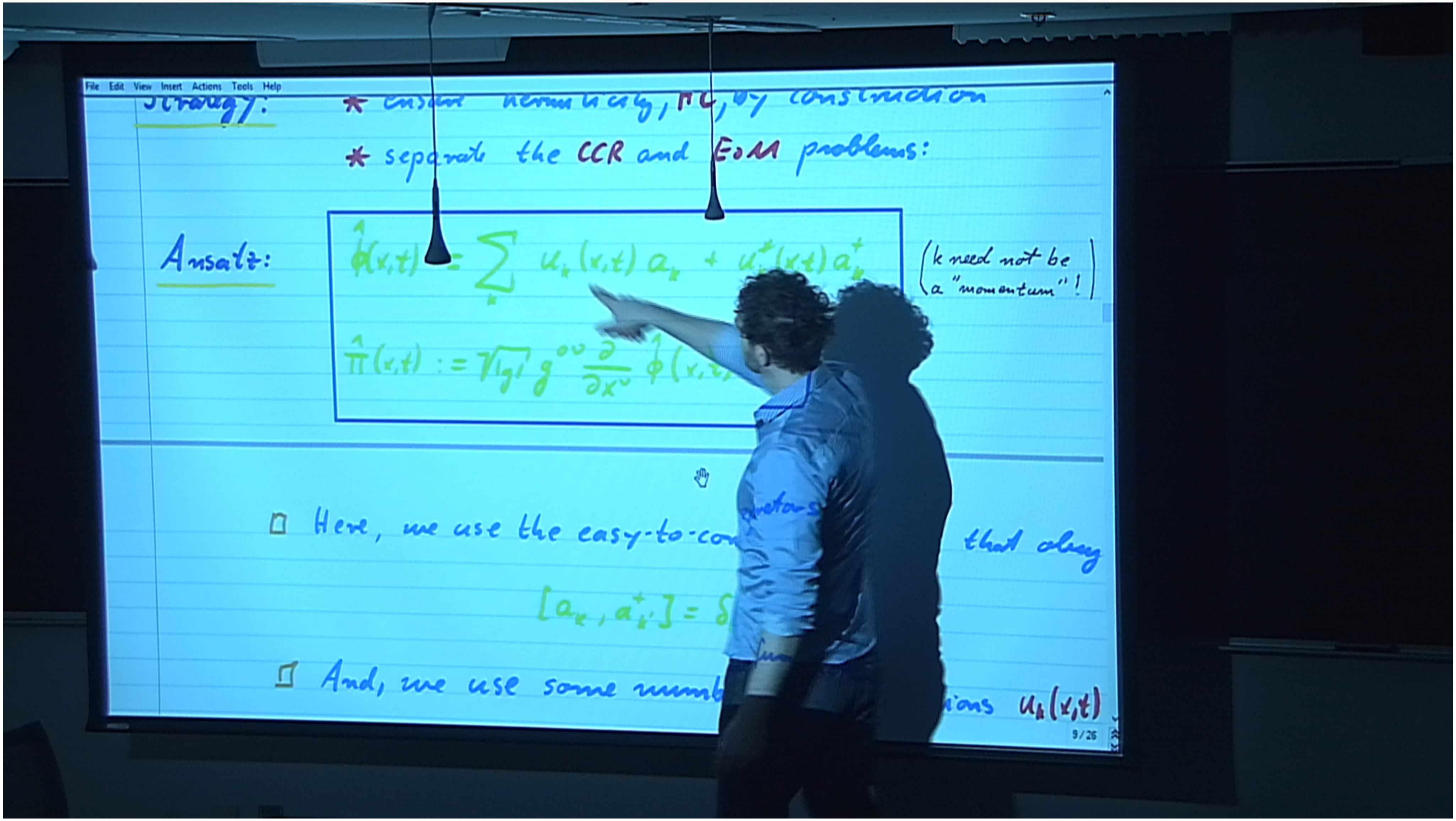
This eqn holds because in our ansatz,

$$\hat{\phi}(x,t) := \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^+$$

the $a_k$ are constant operators while the

functions $u_k(x,t)$ are assumed to solve (EoM1)

10/26


File Edit View Insert Actions Tools Help
* separate the CCR and EoM problems:
Ansatz:
(k need not be a "momentum"!)
Here, we use the easy-to-
that obey
And, we use some
9/26

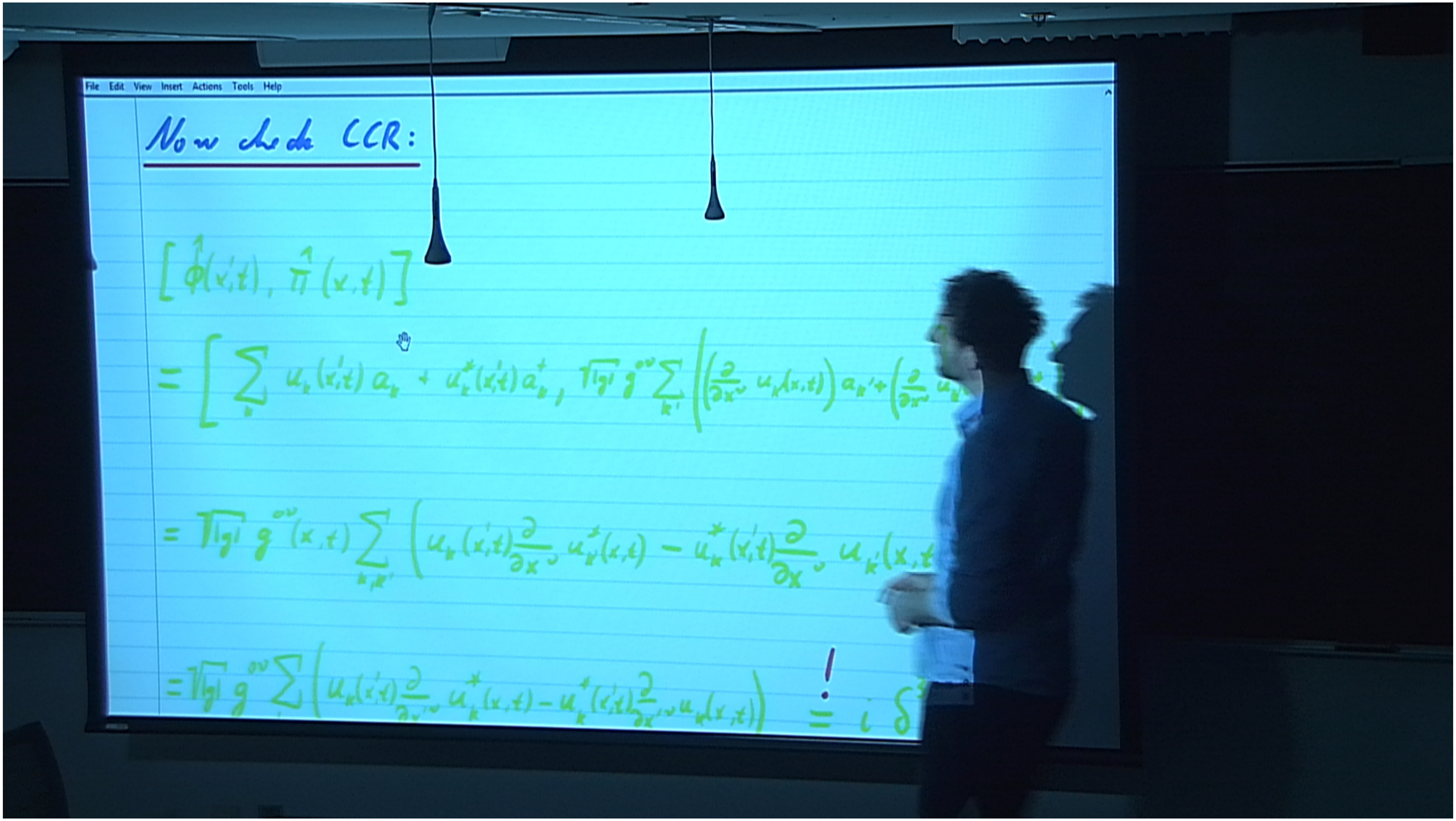
## Now check CCR:

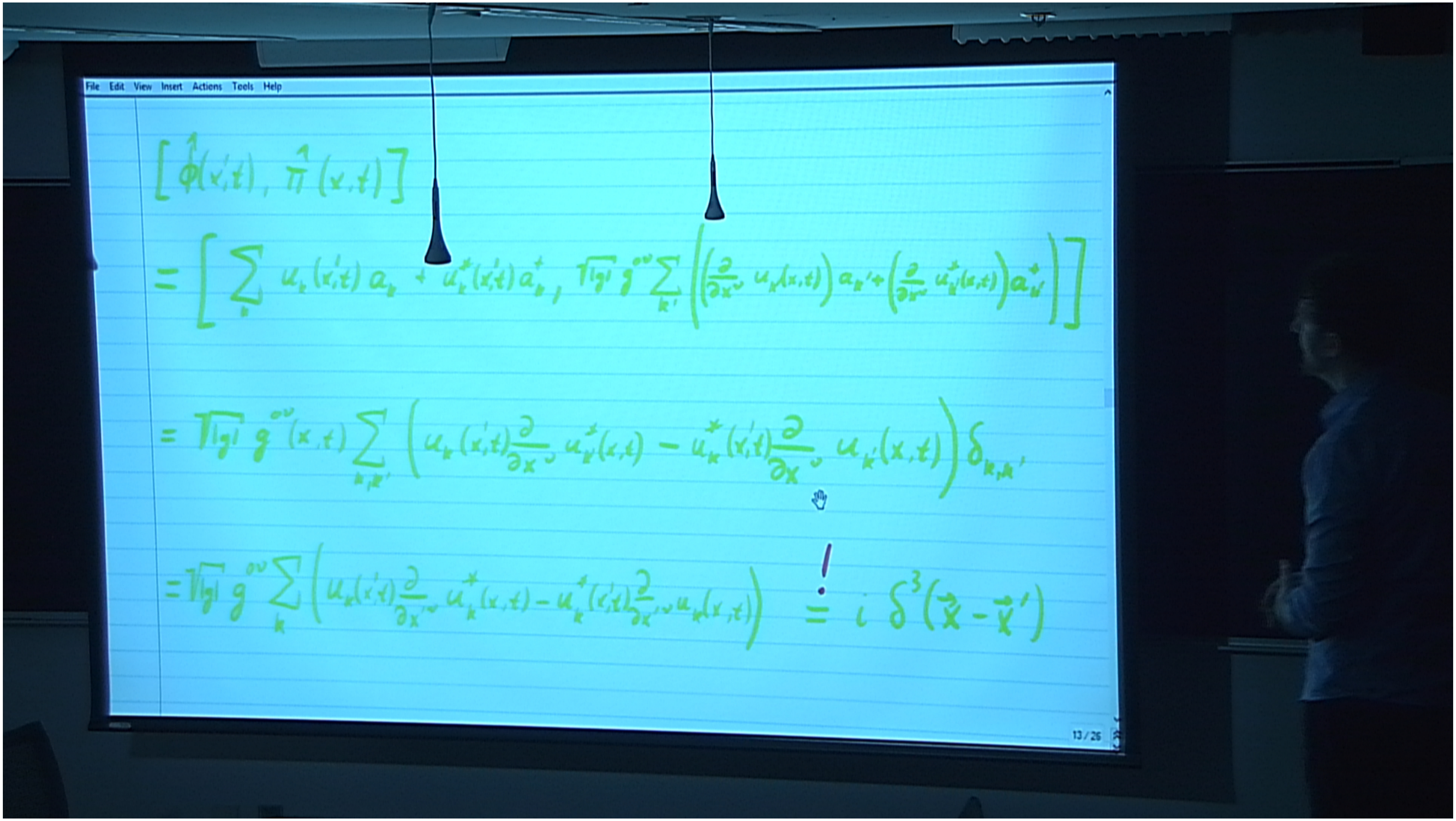
$$[\hat{\phi}(x',t), \hat{\pi}(x,t)]$$

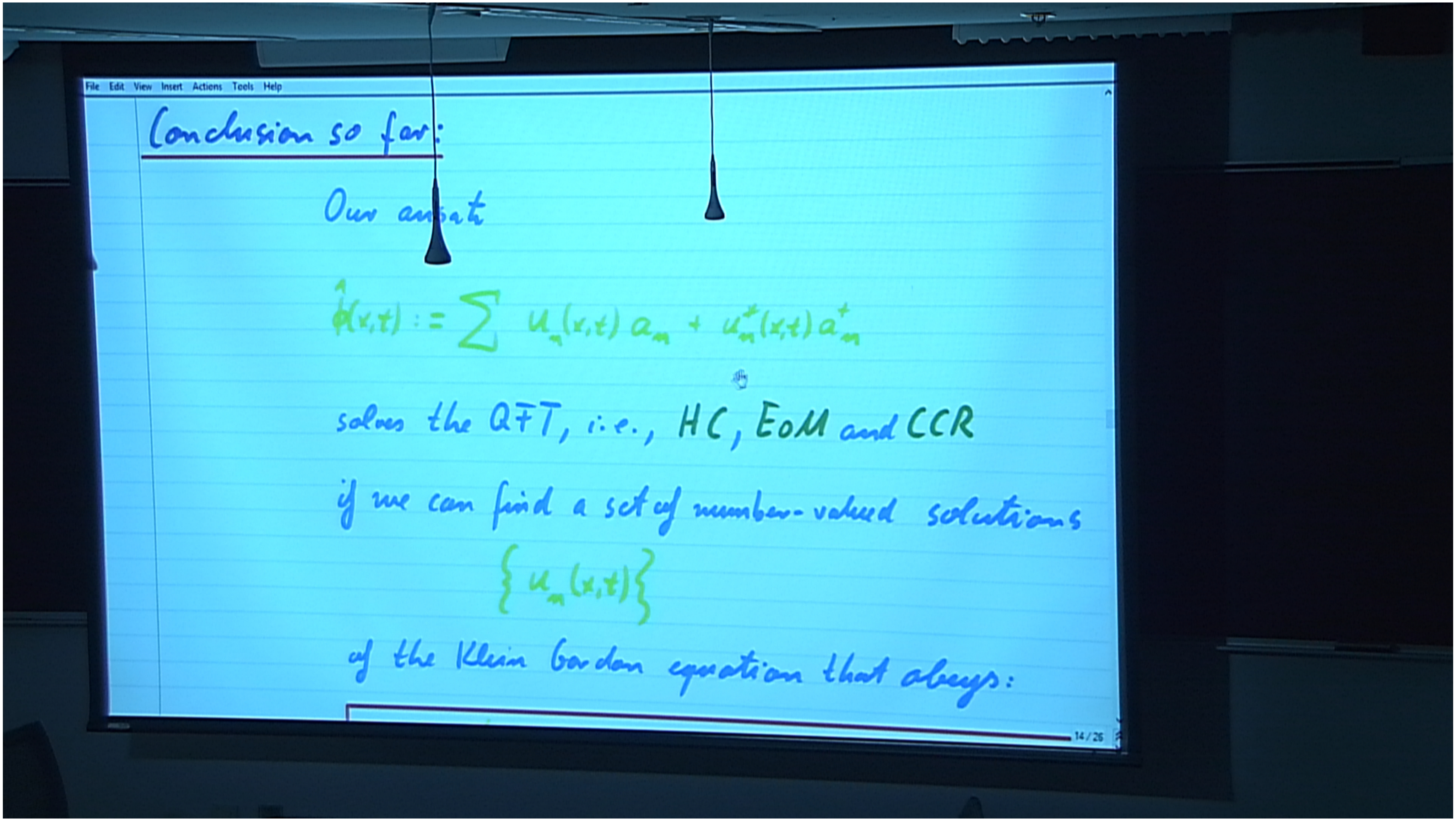
$$= \left[ \sum_k u_k(x',t) a_k + u_k^*(x',t) a_k^\dagger , \sqrt{|g|}\, g^{0\nu} \sum_{k'} \left( \left(\frac{\partial}{\partial x^\nu} u_{k'}(x,t)\right) a_{k'} + \left(\frac{\partial}{\partial x^\nu} u_{k'}^*(x,t)\right) a_{k'}^\dagger \right) \right]$$

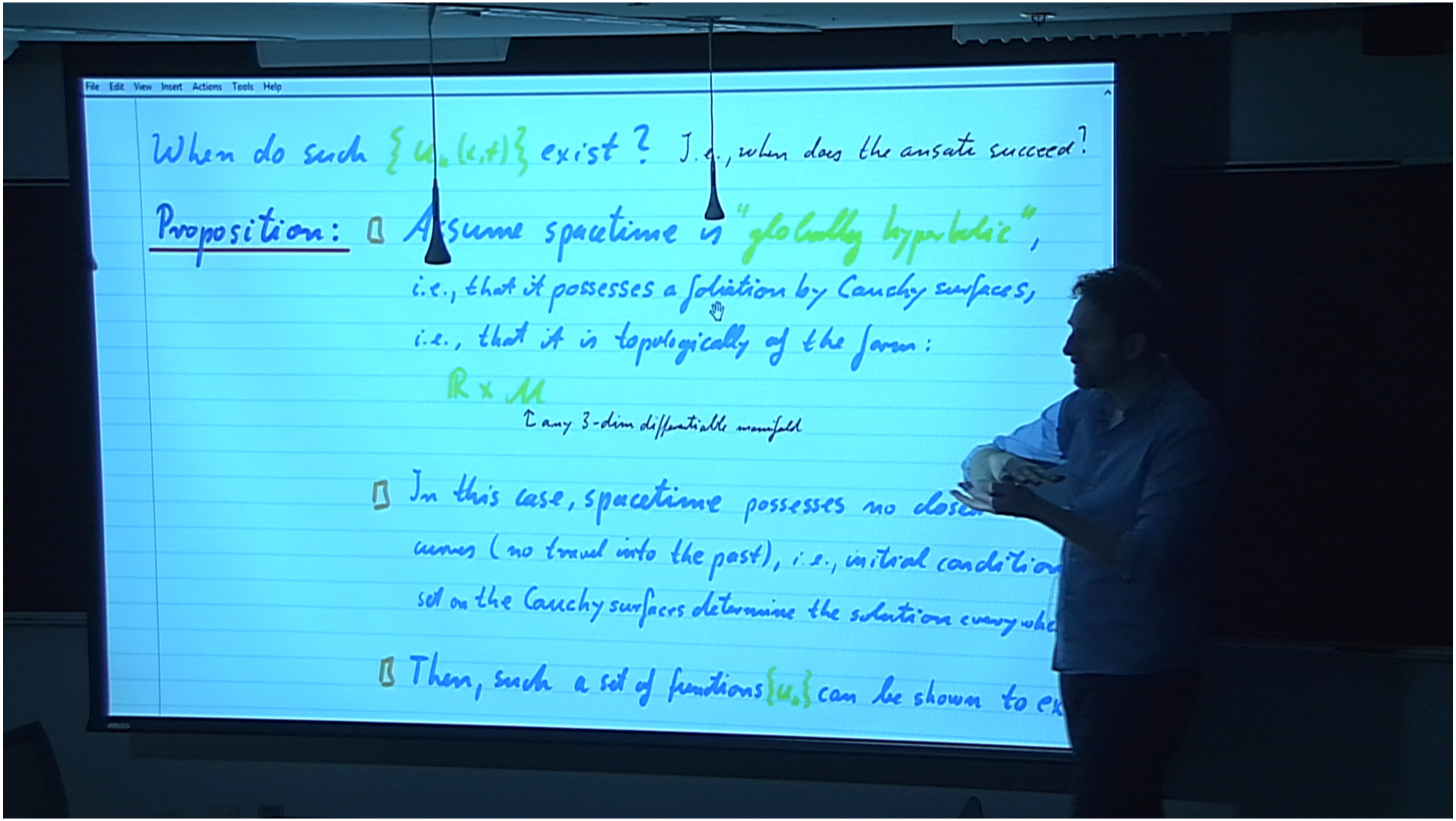
$$= \sqrt{|g|}\, g^{0\nu}(x,t) \sum_{k,k'} \left( u_k(x',t) \frac{\partial}{\partial x^\nu} u_{k'}^*(x,t) - u_k^*(x',t) \frac{\partial}{\partial x^\nu} u_{k'}(x,t) \right) \delta_{k,k'}$$

$$= \sqrt{|g|}\, g^{0\nu} \sum \left( u_k(x',t) \frac{\partial}{\partial x^\nu} u_k^*(x,t) - u_k^*(x',t) \frac{\partial}{\partial x^\nu} u_k(x,t) \right) \stackrel{!}{=} i\, \delta^3(\vec{x} - \vec{x}')$$


File Edit View Insert Actions Tools Help
Now check CCR:
$[\hat{\phi}(x',t), \hat{\pi}(x,t)]$


File Edit View Insert Actions Tools Help
$[\hat{\phi}(x',t), \hat{\pi}(x,t)]$
$= \left[\sum_k u_k(x',t)\, a_k + u_k^*(x',t)\, a_k^\dagger,\ \sqrt{|g|}\, g^{0\nu} \sum_{k'} \left(\left(\frac{\partial}{\partial x^\nu} u_{k'}(x,t)\right) a_{k'} + \left(\frac{\partial}{\partial x^\nu} u_{k'}^*(x,t)\right) a_{k'}^\dagger\right)\right]$
$= \sqrt{|g|}\, g^{0\nu}(x,t) \sum_{k,k'} \left(u_k(x',t)\frac{\partial}{\partial x^\nu} u_{k'}^*(x,t) - u_k^*(x',t)\frac{\partial}{\partial x^\nu} u_{k'}(x,t)\right) \delta_{k,k'}$
$= \sqrt{|g|}\, g^{0\nu} \sum_k \left(u_k(x',t)\frac{\partial}{\partial x^\nu} u_k^*(x,t) - u_k^*(x',t)\frac{\partial}{\partial x^\nu} u_k(x,t)\right) \overset{!}{=} i\,\delta^3(\vec{x} - \vec{x}')$
13 / 26


File Edit View Insert Actions Tools Help
Conclusion so far:
Our ansatz
$\hat{\phi}(x,t) := \sum u_n(x,t)\, a_n + u_n^*(x,t)\, a_n^+$
solves the QFT, i.e., HC, EoM and CCR
if we can find a set of number-valued solutions
$\{u_n(x,t)\}$
of the Klein Gordon equation that obeys:
14 / 26


File Edit View Insert Actions Tools Help
When do such {u_n(x,t)} exist? I.e., when does the ansatz succeed?
Proposition:
Assume spacetime is "globally hyperbolic",
i.e., that it possesses a foliation by Cauchy surfaces,
i.e., that it is topologically of the form:
ℝ × M
any 3-dim differentiable manifold
In this case, spacetime possesses no
curves (no travel into the past), i.e., initial
set on the Cauchy surfaces determine the solution
Then, such a set of functions {u_n} can be shown to

File Edit View Insert Actions Tools Help

**Proof:**

- Consider the vector space, $V$, of all real-valued solutions of the Klein Gordon equation.

- We define a bi-linear form $(\ ,\ )$ on $V$. For all $f, h \in V$:

$$(f,h) := \int_\Sigma d\Sigma_\mu \sqrt{g}\, g^{\mu\nu} \left(f \partial_\nu h - h \partial_\nu f\right)$$

↖ any spacelike hypersurface
i.e. set of points of equal time.

- Proposition: $(f,h)$ is independent of choice of $\Sigma$.
  Proof: Later (uses Stokes' theorem and K.G. equation)

16 / 26

**Theorem (Darboux):**

For any nondegenerate symplectic form $( , )$, there exists a basis $\{v_n\}$ such that, in this basis, $( , )$ takes the matrix form:

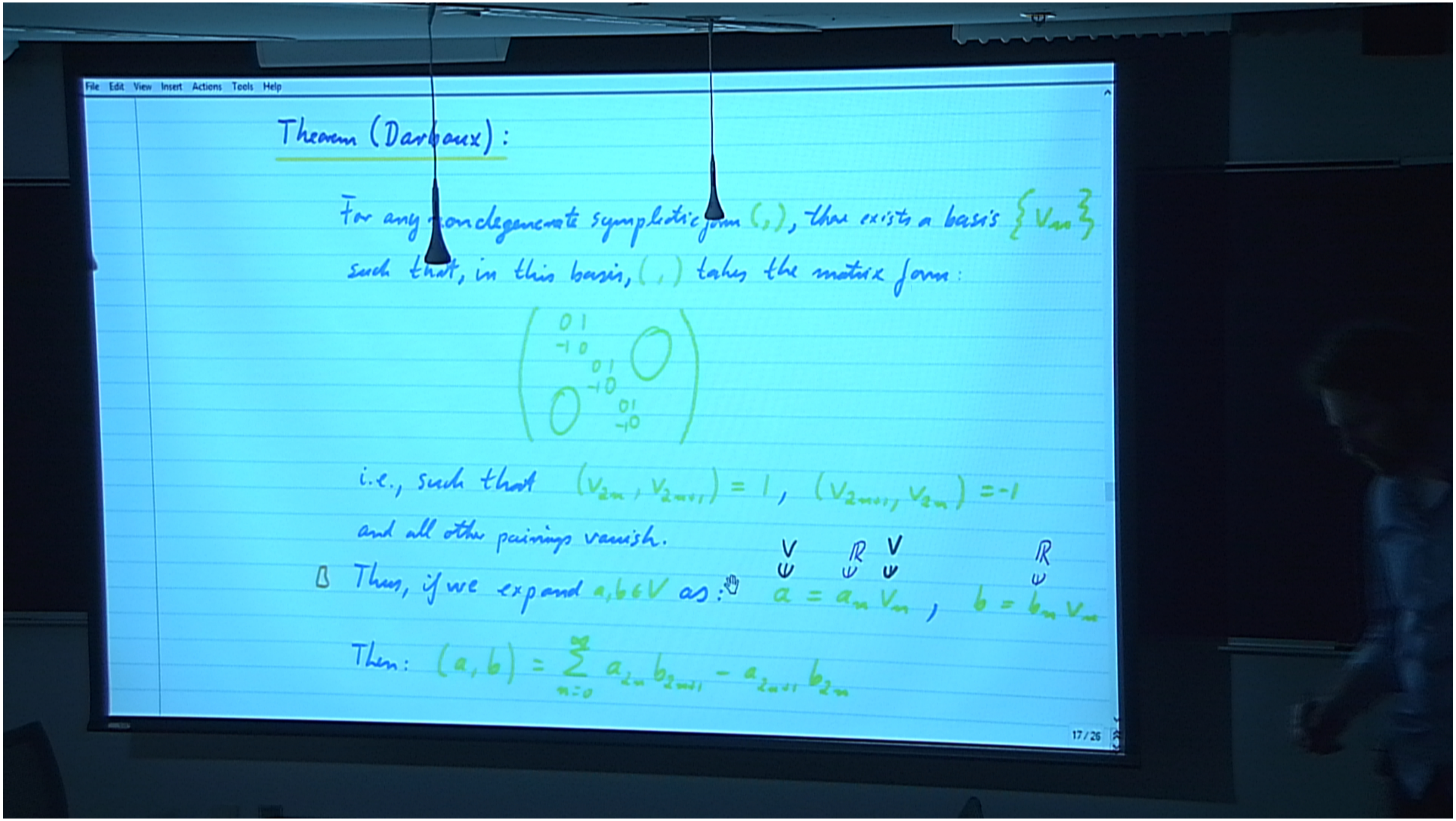
$$\begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & & & \\ & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & & 0 \\ 0 & & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & \end{pmatrix}$$

i.e., such that $(v_{2n}, v_{2n+1}) = 1$, $(v_{2n+1}, v_{2n}) = -1$

and all other pairings vanish.

$\square$ Thus, if we expand $a, b \in V$ as: $\overset{V}{\overset{\in}{a}} = \overset{\mathbb{R}}{\overset{\in}{a_n}} \overset{V}{\overset{\in}{v_n}}$, $b = b$

Then: $(a, b) = \sum_{n=0}^{\infty} a_{2n} b_{2n+1} - a_{2n+1} b_{2n}$



File Edit View Insert Actions Tools Help
Theorem (Darboux):
For any nondegenerate symplectic form ( , ), there exists a basis $\{v_n\}$
such that, in this basis, ( , ) takes the matrix form:
$\begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & & 0 \\ & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & \\ 0 & & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \end{pmatrix}$
i.e., such that $(v_{2n}, v_{2n+1}) = 1$, $(v_{2n+1}, v_{2n}) = -1$
and all other pairings vanish.
Thus, if we expand $a, b \in V$ as: $a = a_n v_n$, $b = b_n v_n$
Then: $(a,b) = \sum_{n=0}^{\infty} a_{2n} b_{2n+1} - a_{2n+1} b_{2n}$
17 / 26

i.e., such that $(v_{2n}, v_{2n+1}) = 1$, $(v_{2n+1}, v_{2n}) = -1$

and all other pairings vanish.

□ Thus, if we expand $a, b \in V$ as: $\overset{V \ni}{a} = \overset{\mathbb{R} \ni}{a_n}\,\overset{V \ni}{v_n}$, $\quad b = \overset{\mathbb{R} \ni}{b_n}\,v_n$

Then: $(a,b) = \sum_{n=0}^{\infty} a_{2n}\, b_{2n+1} - a_{2n+1}\, b_{2n}$

□ Now assume we picked such a basis $\{v_n\}$ in $V$.

□ Recall: $V$ = space of real-valued solutions of K.G. eqn.

- What is a natural product $\langle \, , \, \rangle$ on $\overline{V}$?

- On $\overline{V}$ we define:

$$\langle f, h\rangle := i\int_{\Sigma} d\Sigma_\mu \sqrt{g}\, g^{\mu\nu}\left(f^*\partial_\nu h - (\partial_\nu f^*)h\right)$$

- Then, $( \, , \, )$ yields:

$$\langle u_n, u_m\rangle = -\delta_{n,m}, \quad \langle u_n^*, u_m^*\rangle = +\delta_{n,m}, \quad \langle u_n, u_m^*\rangle = 0 \qquad (I)$$

Exercise: verify this.

- What is a natural product $\langle\, ,\, \rangle$ on $\bar{V}$?

- On $\bar{V}$ we define:

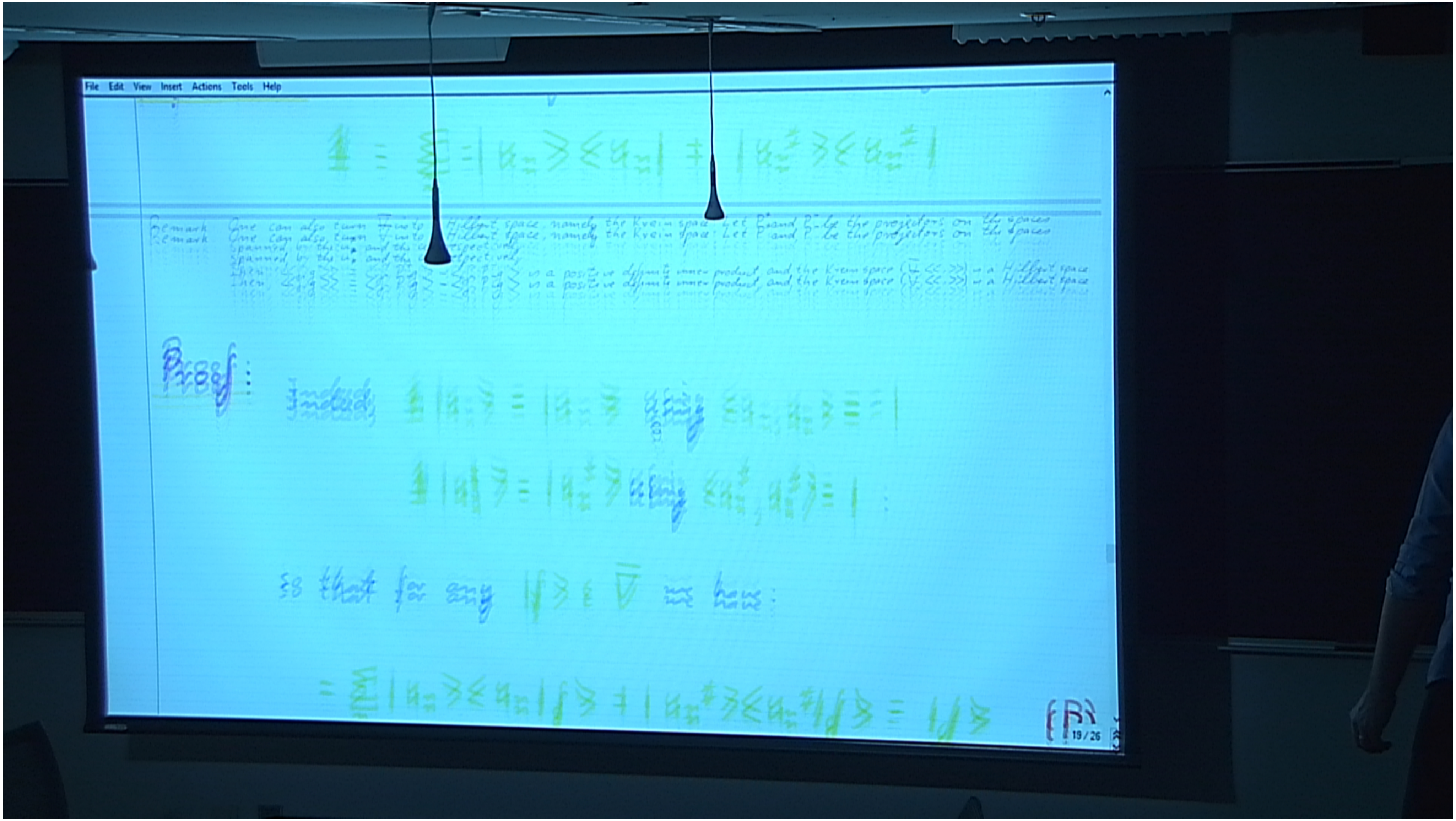
$$\langle f, h\rangle = i \int_{\Sigma} d\Sigma_\mu \sqrt{g}\, g^{\mu\nu}\left(f^* \partial_\nu h - (\partial_\nu f^*) h\right)$$

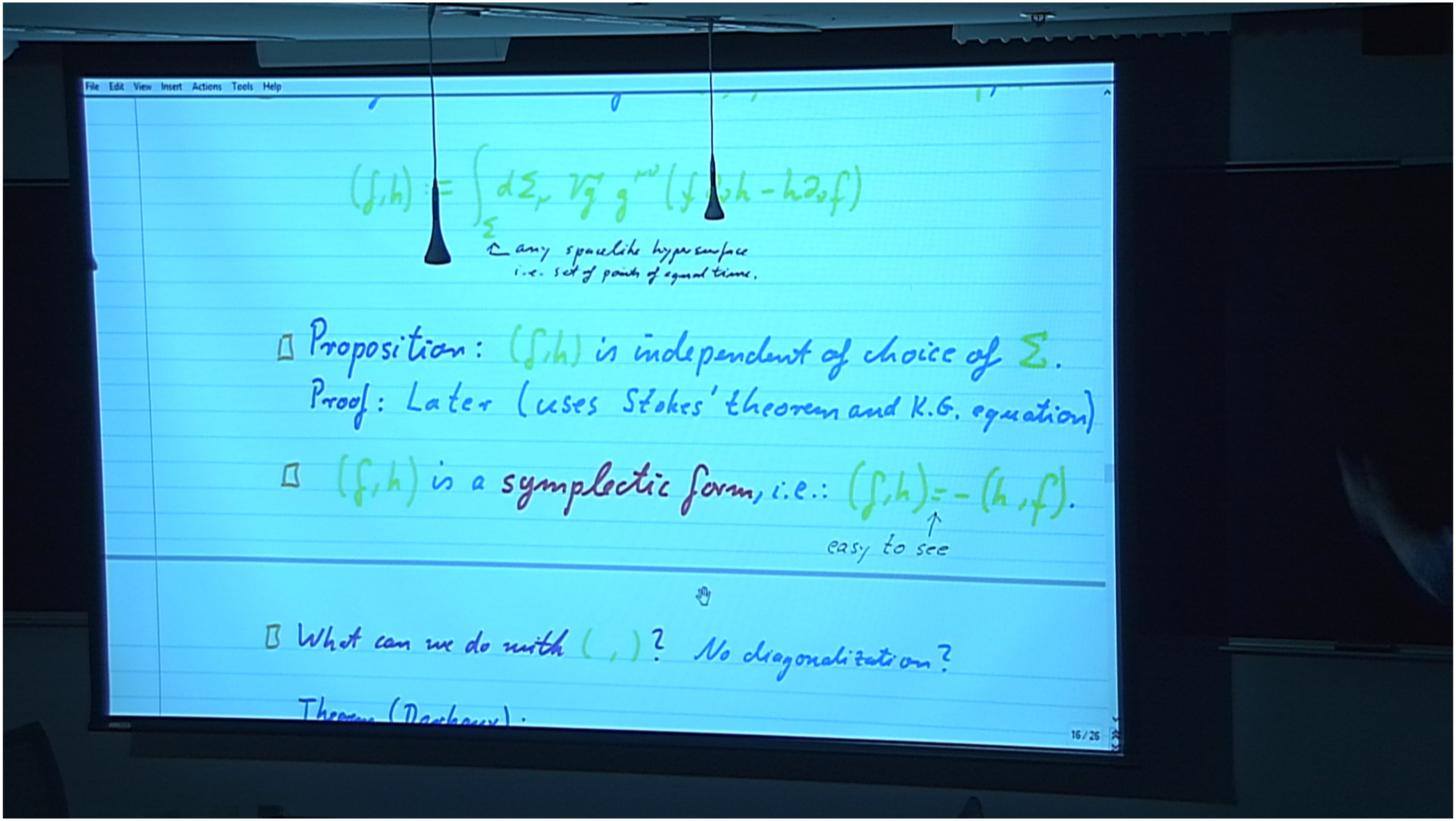
- Then, $(\, ,\, )$ yields:

$$\langle u_n, u_m\rangle = -\delta_{n,m}, \quad \langle u_n^*, u_m^*\rangle = +\delta_{n,m}, \quad \langle u_n, u_m^*\rangle = 0 \qquad (I)$$

Exercise: verify this.

18/26


File Edit View Insert Actions Tools Help
Remark One can also turn V into a Hilbert space, namely the Krein space. Let P+ and P- be the projectors on the spaces spanned by the un and the un# respectively.
Then
is a positive definite inner product and the Krein space
is a Hilbert space
Proof:
Indeed
so that for any
we have:
19/26


File Edit View Insert Actions Tools Help
$(f,h) := \int_\Sigma d\Sigma_\mu \sqrt{g}\, g^{\mu\nu} (f \partial_\nu h - h \partial_\nu f)$
any spacelike hypersurface
i.e. set of points of equal time.
Proposition: $(f,h)$ is independent of choice of $\Sigma$.
Proof: Later (uses Stokes' theorem and K.G. equation)
$(f,h)$ is a symplectic form, i.e.: $(f,h) = -(h,f)$.
easy to see
What can we do with $( \, , \, )$? No diagonalization?
16/26

File Edit View Insert Actions Tools Help

**Proof:** Indeed, $\mathbb{1}|u_n\rangle = |u_n\rangle$ using $\langle u_n, u_n\rangle = -1$

$\mathbb{1}|u_n^*\rangle = |u_n^*\rangle$ using $\langle u_n^*, u_n^*\rangle = 1$ .

so that for any $|f\rangle \in \bar{V}$ we have:

$$-\sum_n |u_n\rangle\langle u_n|f\rangle + |u_n^*\rangle\langle u_n^*|f\rangle = |f\rangle \qquad \text{(P)}$$

Writing this out, we will now show that it yields (W), i.e.:

$$\boxed{\sqrt{|g|}\, g^{0\nu} \sum_k \left( u_k(x,t) \frac{\partial}{\partial x'^\nu} u_k^*(x',t) - u_k^*(x,t) \frac{\partial}{\partial x'^\nu} u_k(x',t) \right) = i\,\delta^3(x-x')}$$

20 / 26

**Proof:**

Indeed, $\mathbb{1}|u_n\rangle = |u_n\rangle$ using $\langle u_n, u_n\rangle = -1$

$\mathbb{1}|u_n^*\rangle = |u_n^*\rangle$ using $\langle u_n^*, u_n^*\rangle = 1$.

so that for any $|f\rangle \in \bar{V}$ we have:

$$-\sum_n |u_n\rangle\langle u_n|f\rangle + |u_n^*\rangle\langle u_n^*|f\rangle = |f\rangle \qquad \text{(P)}$$

Writing this out, we will now show that it yields (W), i.e.:

$$\boxed{\sqrt{|\gamma|}\, g^{0\nu} \sum_k \left( u_k(x,t) \frac{\partial}{\partial x'^\nu} u_k^*(x',t) - u_k^*(x,t) \frac{\partial}{\partial x'^\nu} u_k(x',t) \right) = i\,\delta^3(x-x')}$$

reads:

$$\sum_n u_n(x,t)\, i\int_\Sigma d^3x' \sqrt{g}\, g^{0\nu}\left(u_n^* \partial_\nu f - (\partial_\nu u_n^*) f\right)$$

$$-\sum_n u_n^*(x,t)\, i\int_\Sigma d^3x' \sqrt{g}\, g^{0\nu}\left(u_n \partial_\nu f - (\partial_\nu u_n) f\right) = f(x,t)$$

Now, interchanging $\sum_n$ and $\int_\Sigma$ yields $\forall f \in \overline{V}$:

$$i\int_\Sigma d^3x \sqrt{|g(x)|}\, g^{0\nu}(x) \sum_n \Big( u_n(x,t)\, u_n^*(x',t)\,\partial_{x',\nu} - u_n(x,t)\,\partial_{x',\nu} u_n^*(x',t)$$

File Edit View Insert Actions Tools Help

- Now choose an arbitrary function $g(x')$.

- Then there exists a solution $f(x',t)$ of the Klein Gordon equation obeying:

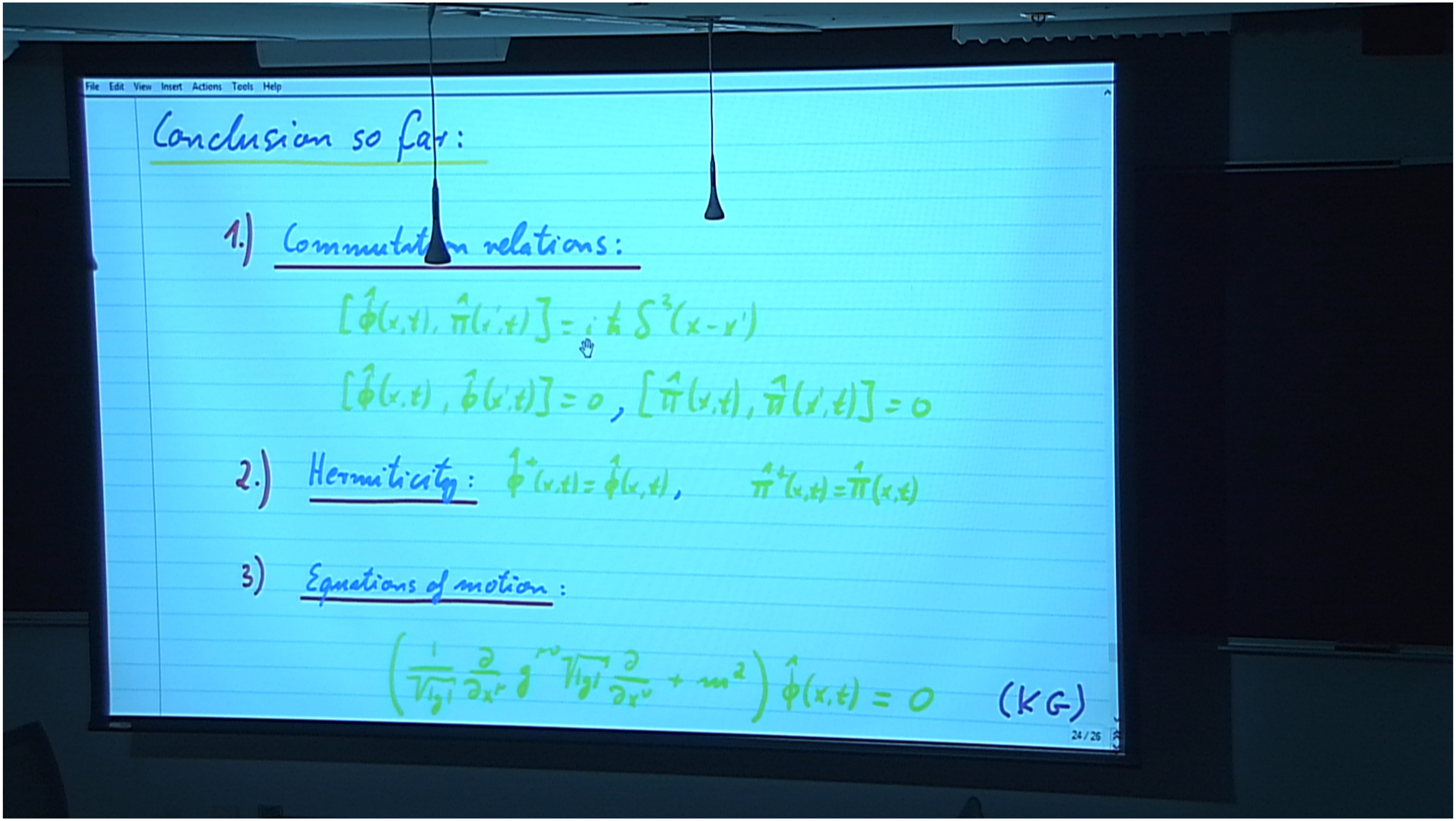
1) $f(x',t_0) = g(x')$
2) $g^{\nu\nu}(x',t_0)\,\partial_{x'^\nu} f(x',t_0) = 0$

(Because the $2^{nd}$ order K.G. equation on a globally hyperbolic spacetime has a well-defined Cauchy problem)

- Therefore, $(*)$ yields, for all choices of $g(x)$:

$$i\int_\Sigma d^3x' \sqrt{|g(x')|}\, g^{0\nu}_{(x')} \sum_n \left( -u_n(x,t)\,\partial_{x'^\nu} u_n^*(x',t) + u_n^*(x,t)\,\partial_{x'^\nu} u_n(x',t) \right) g(x') = g(x) \quad \forall g(x)$$

23 / 26


File Edit View Insert Actions Tools Help
Conclusion so far:
1.) Commutation relations:
$[\hat{\phi}(x,t), \hat{\pi}(x',t)] = i\hbar\, \delta^3(x-x')$
$[\hat{\phi}(x,t), \hat{\phi}(x',t)] = 0, \quad [\hat{\pi}(x,t), \hat{\pi}(x',t)] = 0$
2.) Hermiticity: $\hat{\phi}^+(x,t) = \hat{\phi}(x,t), \quad \hat{\pi}^+(x,t) = \hat{\pi}(x,t)$
3) Equations of motion:
$\left(\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^\mu} g^{\mu\nu} \sqrt{|g|} \frac{\partial}{\partial x^\nu} + m^2\right) \hat{\phi}(x,t) = 0$ (KG)
24 / 26

File Edit View Insert Actions Tools Help

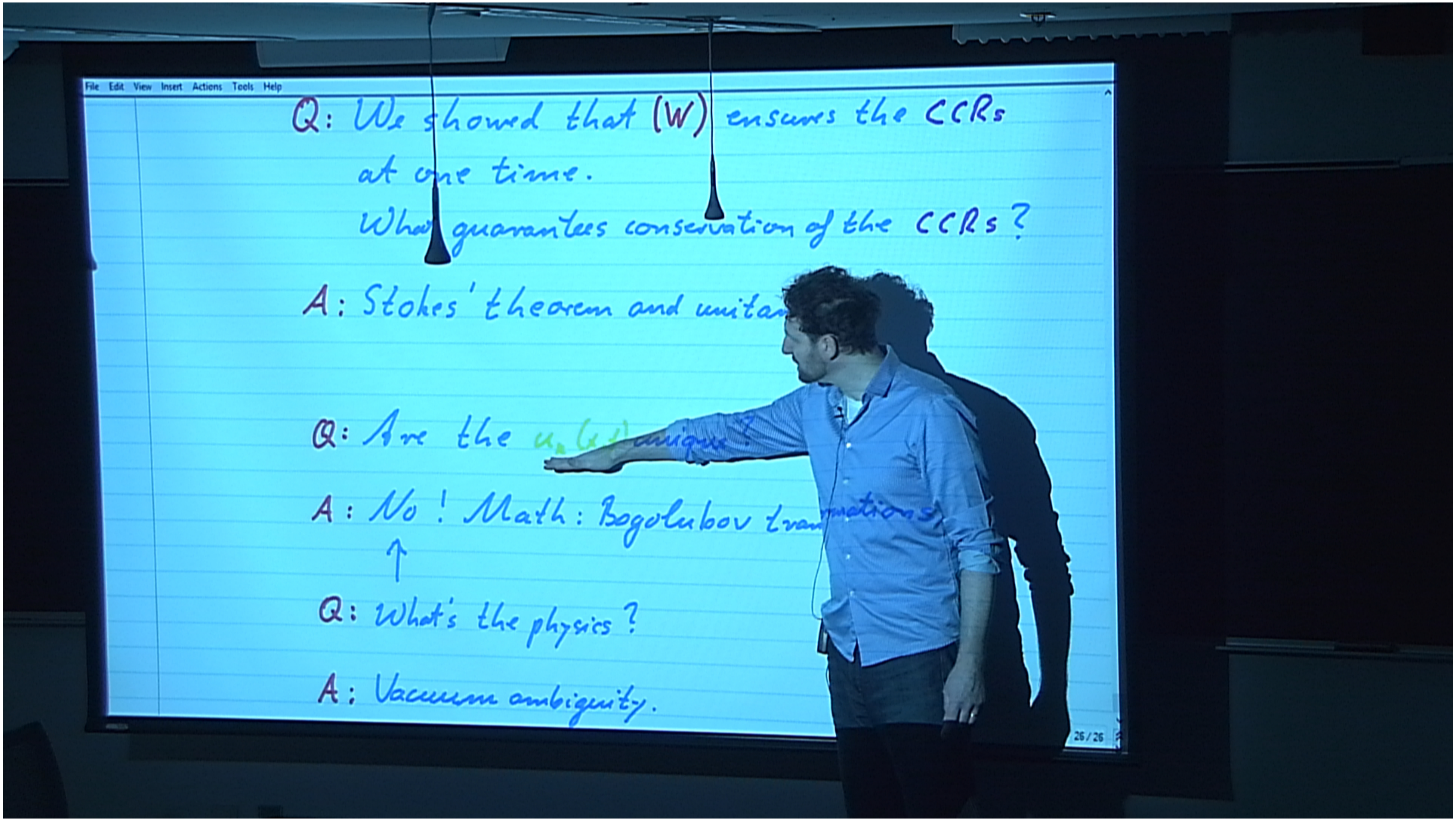
on any globally hyperbolic spacetime:

$$\boxed{\begin{aligned}\hat{\phi}(x,t) &:= \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^+ \\ \hat{\pi}(x,t) &:= \sqrt{|g|}\, g^{0\nu} \frac{\partial}{\partial x^\nu} \hat{\phi}(x,t)\end{aligned}}$$

where $[a_k, a_{k'}^+] = \delta(k-k')$ and where the $u_k(x,t)$ are number-valued solutions to (KG) which also obey (R1):

$$\boxed{\sqrt{|g|}\, g^{0\nu}(x) \sum_k \left( u_k(x,t) \frac{\partial}{\partial x'^\nu} u_k^*(x',t) - u_k^*(x,t) \frac{\partial}{\partial x'^\nu} u_k(x',t) \right) = i\,\delta^3(x-x')}$$

25 / 26


File Edit View Insert Actions Tools Help
Q: We showed that (W) ensures the CCRs
at one time.
What guarantees conservation of the CCRs?
A: Stokes' theorem and unita
Q: Are the unique?
A: No! Math: Bogolubov tra
Q: What's the physics?
A: Vacuum ambiguity.
26 / 26