Title: Quantum Field Theory for Cosmology - Achim Kempf - Lecture 11

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Abstract:

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Recall the strategy:

SR, 1^{st}Q Hamiltonian formalism	step 1 Legendre transform (equivalence) →	SR, 1^{st}Q. Lagrangian formalism	we are here
		↓ step 2 allow curvature	
GR, 1^{st}Q Hamiltonian formalism	← step 3 Legendre transform (equivalence)	GR, 1^{st}Q Lagrangian formalism	
↓ step 4		↓	
GR, 2^{nd}Q Hamiltonian formalism	← Dyson Schwinger eqns are same (equivalence) →	GR, 2^{nd}Q Lagrangian formalism (Path integral of QFT)	

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Recall the strategy:

SR, 1^{st}Q
Hamiltonian
formalism

step 1
Legendre transform
(equivalence)
$\longrightarrow$

SR, 1^{st}Q.
Lagrangian
formalism

we are here

step 2 $\downarrow$ allow curvature

GR, 1^{st}Q
Hamiltonian
formalism

step 3
Legendre transform
(equivalence)
$\longleftarrow$

GR, 1^{st}Q
Lagrangian
formalism

step 4 $\downarrow$

GR, 2^{nd}Q
Hamiltonian
formalism

$\longleftrightarrow$
Dyson Schwinger eqns are same
(equivalence)

GR, 2^{nd}Q
Lagrangian formalism
(Path integral of QFT)

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Step 2 so far:

- We started with the Klein Gordon action in special relativity:

$$S[\phi] = \frac{1}{2}\int_{\mathbb{R}^4} \eta^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2 - V(\phi,\varphi_i)\, d^4x \qquad (a)$$

This formulation is correct for any inertial observer using a rectangular coordinate system, and only for those observers.

Remark: Here, $V(\phi,\varphi_i)$ is a potential. It describes how the ϕ field interacts with other fields $\{\varphi_i\}$ and with itself.

The very early universe is assumed to have been

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We obtained a formulation of the Klein Gordon action in special relativity which is the same in all coordinate systems:

$$S_{KG}[\phi] := \frac{1}{2}\int_{\mathbb{R}^4} \left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2 - V(\phi,\ell_i)\right) \overbrace{\sqrt{|g|}}^{\sqrt{-\det(g_{\mu\nu})}} d^4x \qquad (b)$$

If an inertial observer sets up a rectangular coordinate system, then the metric tensor $g_{\mu\nu}(x)$ is of course given by this constant matrix

$$g_{\mu\nu}(x) = \eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}_{\mu\nu}$$

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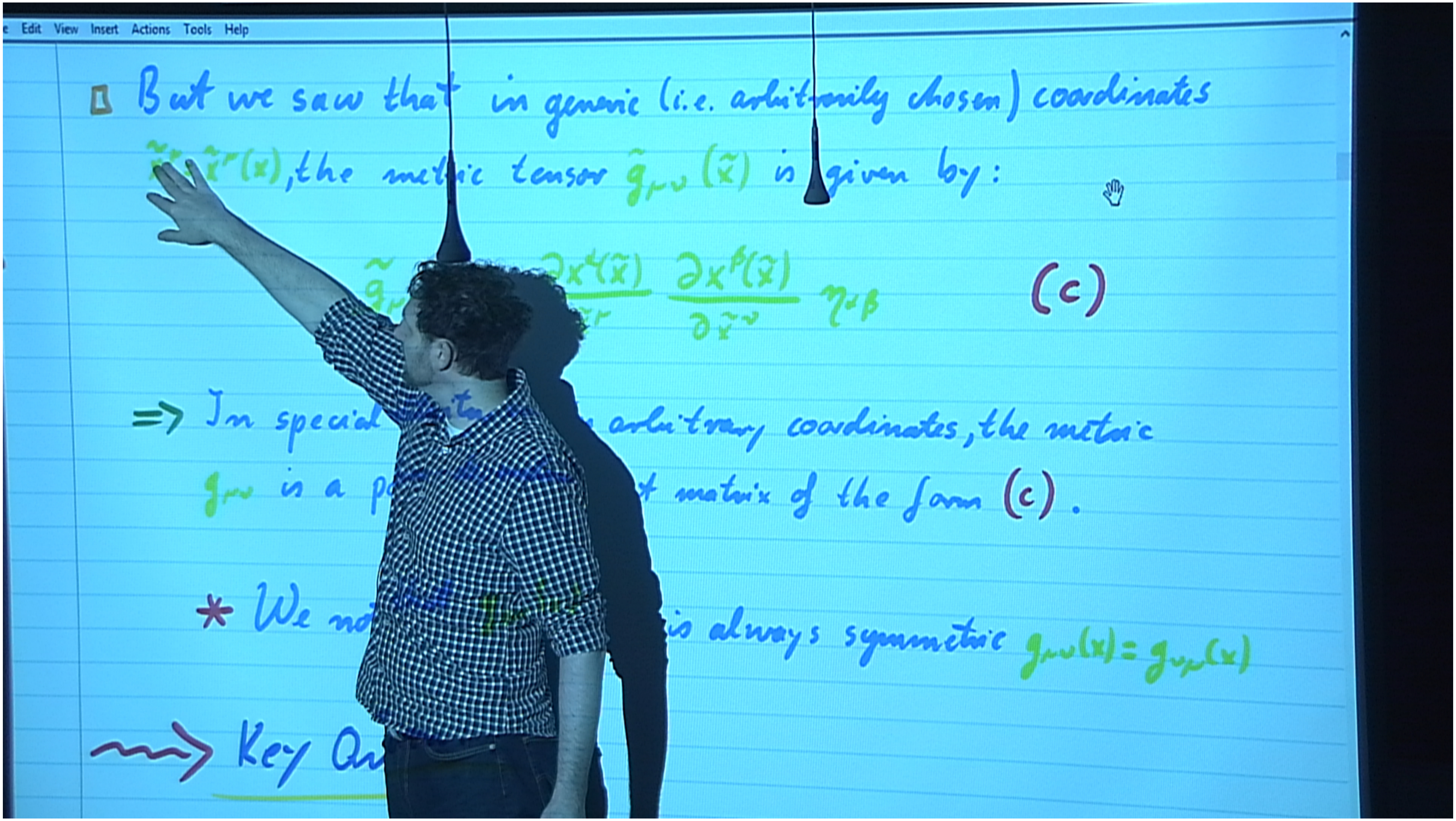
with $\sqrt{|g|} = 1$. Thus, one recovers equation (a).



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But we saw that in generic (i.e. arbitrarily chosen) coordinates
$\tilde{x}^\mu(x)$, the metric tensor $\tilde{g}_{\mu\nu}(\tilde{x})$ is given by:
$\frac{\partial x^\alpha(\tilde{x})}{\partial \tilde{x}^\mu} \frac{\partial x^\beta(\tilde{x})}{\partial \tilde{x}^\nu} \eta_{\alpha\beta}$
(c)
=> In special
arbitrary coordinates, the metric
$g_{\mu\nu}$ is a p
matrix of the form (c).
* We no
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Key Qu

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$$\tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha(\tilde{x})}{\partial \tilde{x}^\mu} \frac{\partial x^\beta(\tilde{x})}{\partial \tilde{x}^\nu} \eta_{\alpha\beta} \qquad (c)$$

$\Rightarrow$ In special relativity, in arbitrary coordinates, the metric $g_{\mu\nu}$ is a position-dependent matrix of the form (c).

* We notice that $g_{\mu\nu}(x)$ is always symmetric $g_{\mu\nu}(x) = g_{\nu\mu}(x)$

Key Question:

Do all metrics arise from $\eta_{..}$ by a change of coordinates ?

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I.e.: Can any arbitrary function obeying $g_{\mu\nu}(x) = g_{\nu\mu}(x)$ arise from $\eta_{\mu\nu}$ by changing coordinates according to (c)?

Answer: No! The others describe curved spacetimes.

$\Rightarrow$ A given spacetime can be described by any one of an equivalence class of metric functions $\{g_{\mu\nu}(x)\}$, which differ by a mere change of coordinates (i.e. which are related by a diffeomorphism).

$\leadsto$ Q: How many independent degrees of freedom D (i.e. independent functions) describe a spacetime fully?

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Key Question:

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by changing coordinates according to $g_{\mu\nu}(\tilde{x}) = \frac{\partial x^{\alpha}(\tilde{x})}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}(\tilde{x})}{\partial \tilde{x}^{\nu}} \eta_{\alpha\beta}$?

Answer: No ! The others describe "curved" spacetimes.

A given spacetime can be described by any one of an equivalence class $[g]$ of metric functions $\{g_{\mu\nu}(x)\}$, which differ by a mere change of coordinates (i.e. which are related by a diffeomorphism).

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Definition: Each equivalence class $[g]$ is called a Riemannian or Lorentzian Structure, depending on the signature of the metric.

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Definition: Each equivalence class $[g]$ is called a Riemannian or Lorentzian Structure, depending on the signature of the metric.

How many Lorentzian or Riemannian structures are there?

Q: How many independent degrees of freedom D (i.e. independent functions) describe a spacetime fully?

A: In n dimensions, the metric g has n^2 component functions $g_{\mu\nu}(x)$.

Because of $g_{\mu\nu}(x) = g_{\nu\mu}(x)$, only $n(n+1)/2$ are independent.

But we can choose n functions $\tilde{x}^{\mu}(x)$ in $\tilde{g}_{..}(\tilde{x}) = \frac{\partial x^{\alpha}(\tilde{x})}{} \frac{\partial x^{\beta}(\tilde{x})}{} g_{\alpha\beta}$ ...

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$\Rightarrow$ $$\boxed{D = n(n+1)/2 - n}$$

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Curvature:

- We continue to postulate the coordinate system-independent Klein Gordon action of above:

$$S_{KG}[\phi] := \frac{1}{2}\int_{\mathbb{R}^4} \left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2 - V(\phi, \ell_i)\right)\sqrt{|g|}\, d^4x$$

- We will allow almost arbitrary metric tensors $g_{\mu\nu}(x)$, even those for which there do not exist coordinates $\tilde{x}$ in which:

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \eta_{\mu\nu} \quad \text{for all } x$$

- But we must have that, at least locally, special relativity holds!

$\Rightarrow$ Consider only $g_{\mu\nu}(x)$ for which for each x_0, there exists a change of coordinates

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- This requirement is The Equivalence Principle:

  * We postulate that gravity can always locally be eliminated:

  * We assume that if a freely falling observer in a small region sets up a rectangular coordinate system the observer will see arbitrarily small gravity effects if the region is made arbitrarily small.

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* For this to be true, any body's gravitational mass must be equal to its inertial mass, i.e. all bodies must fall equally. (Else the notion of freely falling observer) is not even well defined)

How can one identify the presence of curvature?

* Assume we are given a metric tensor

$$g_{\mu\nu}(x)$$

as an explicit matrix-valued function, in some coordinates.

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* This problem is solved in differential geometry:

Define: The "Christoffel symbol functions":

$$\Gamma^{\mu}_{\alpha\beta}(x) := \frac{1}{2} g^{\mu\nu}(x)\left(g_{\alpha\nu,\beta}(x) + g_{\beta\nu,\alpha}(x) - g_{\alpha\beta,\nu}(x)\right)$$

Define: The "Riemann Curvature Tensor":

$$R^{i}_{\ jkl}(x) := \Gamma^{i}_{lj,k}(x) - \Gamma^{i}_{kj,l}(x) + \Gamma^{s}_{lj}(x)\Gamma^{i}_{ks}(x) - \Gamma^{s}_{kj}(x)\Gamma^{i}_{ls}(x)$$

It's rôle? A space is called curved at x if the parallel

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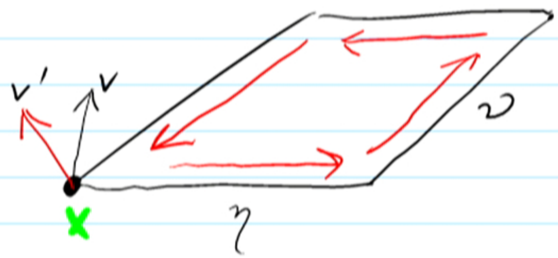
It's rôle? A space is called curved at x if the parallel transport of a vector v along an infinitesimal parallelogram returns the vector v' to x, but v' is rotated by some amount. $R^{\mu}{}_{\nu\rho}$ tells by how much:



$$\boxed{(v'-v)^a = \eta^b \nu^c R^a{}_{bcd}(x)\, v^d}$$

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Proposition:

Assume that, in a region, A, of space-time:

$$R^{\mu}{}_{\nu\rho\sigma}(x) = 0 \quad \text{for all } x \in A$$

Then and only then there exist coordinates $\tilde{x}$ so that:

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \eta_{\mu\nu} \quad \text{for all } x \in A$$

The dynamics of space-time

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The dynamics of space-time

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Which are the degrees of freedom of curvature

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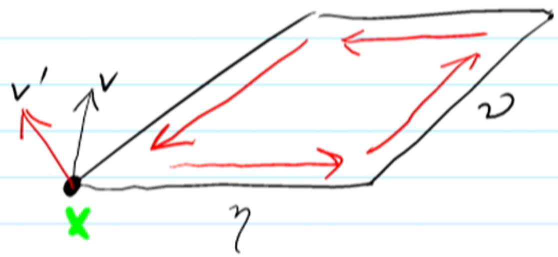

$$\boxed{(v'-v)^a = \eta^b \nu^c R^a{}_{bcd}(x)\, v^d}$$

Remark:

If the parallelogram does not even close we say that space-time has "Torsion". There is no evidence for torsion in nature.

Proposition:

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The dynamics of space-time

Problem: What are the equations of motion for spacetime's curvature?

Which are the degrees of freedom of curvature for which we have to find an equation of motion?

- We saw that the curvature of space-time is encoded in the matrix-valued metric function:

$$g_{\mu\nu}(x)$$

- However, if $g_{\mu\nu}(x)$ looks nontrivial, this can

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☐ However, if $g_{\mu\nu}(x)$ looks nontrivial, this can be for two different reasons:

1. Spacetime has little or no curvature and $g_{\mu\nu}(x)$ is nontrivial just because of an unlucky choice of coordinates.

2. Spacetime is curved, i.e., we can not make $g_{\mu\nu}(x)$ take the form $\tilde{g}_{\mu\nu}(\tilde{x}) = \eta_{\mu\nu}$ for all $\tilde{x}$ no matter which coordinates we choose.

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- Therefore, it is difficult to pinpoint in the matrix function $g_{\mu\nu}(x)$ the curvature degrees of freedom.

- And: Even the entries $R^{\mu}{}_{\nu\delta\epsilon}$ of the curvature tensor are coordinate system dependent.

Strategy:

- Use the degrees of freedom of curvature to build a scalar and therefore coordinate system independent

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$$\boxed{R_{\mu\nu}(x) := R^{i}{}_{\mu i \nu}(x)}$$

Recall: $\sum_{i=0}^{3}$ is implied

Note: Other index contractions would vanish because of antisymmetries of $R^{\mu}{}_{\nu\rho\varepsilon}(x)$ that are implied by the definition of $R^{\mu}{}_{\nu\rho\varepsilon}(x)$.

Remark:

$R_{\mu\nu}(x)$ carries strictly less information than the full Riemann curvature tensor:

* If $R_{\mu\nu}(x)=0$ it is still possible that $R^{\mu}{}_{\nu\rho\varepsilon}(x) \neq 0$!
* This happens to be the case, e.g., for gravitational waves.

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Definition: The "curvature scalar" (or "Ricci scalar")

$$R(x) := g^{\mu\nu}(x)\, R_{\mu\nu}(x)$$

Other curvature scalars:

* The simplest scalar that can be formed from the metric alone is $g^{\mu\nu}(x)\, g_{\nu\mu}(x) = 4$.

* The next simplest scalar that can be formed is the Ricci scalar $R(x)$.

* All other scalars made out of g only are composed of higher powers of the Riemann tensor $R^{\mu}{}_{\nu\delta\varepsilon}(x)$:

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The gravitational action

- A priori, the full action now reads:

other "matter" fields for e^-, quarks, photons etc

$$S_{tot}[g,\phi,e_i] = S_{KG} + S_{other} + S_{grav}$$

Klein Gordon action

with: $S_{grav}[g] := \int \left(c_0 + c_1 R(x) + c_2 R_{\mu\nu}(x) R^{\mu\nu}(x) + c_3 R^{\cdot\cdot}_{\cdot\cdot} R^{\cdot\cdot}_{\cdot\cdot} R^{\cdot\cdot}_{\cdot\cdot} + \dots \right) \sqrt{|g|}\, d^4x$

- Comparison with experiment shows evidence only for the first two terms:

$$S_{grav}[g] = -\frac{1}{16\pi G} \int \left(2\Lambda + R(x) \right) \sqrt{|g(x)|}\, d^4x$$

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"Einstein action" Newton's constant "Cosmological constant"

Remark: D. Lovelock here determined all generalizations to higher terms and higher dimensions that still possess 2nd order initial value problems.

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The equations of motion

The action principle is to require that the action be

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(See exercise in Mukhanov's text) →

$$\boxed{R_{\mu\nu}(x) - \frac{1}{2} g_{\mu\nu}(x) R(x) + \Lambda g_{\mu\nu}(x) = +8\pi G\, T_{\mu\nu}(x)}$$

$$\sim \frac{\delta S'_{grav}}{\delta g_{\mu\nu}(x)} \qquad \sim - \frac{\delta(S_{other} + S'_{KG})}{\delta g_{\mu\nu}(x)}$$

* Here, $T_{\mu\nu}(x)$ is the "Energy Momentum Tensor".

Neglecting the contribution by the $e_i(x)$, one obtains:

$$T^{(K.G.)}_{\mu\nu}(x) = \phi_{,\mu}(x)\,\phi_{,\nu}(x) - g_{\mu\nu}(x)\left(\frac{1}{2} g^{\alpha\beta}(x)\, \phi_{,\alpha}(x)\, \phi_{,\beta}(x) - \overbrace{V(\phi(x))}^{\text{mass term } m^2\phi^2 \text{ included}}\right)$$

* **Quantization:** To quantize the Einstein equation

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* **Quantization:** To quantize the Einstein equation is difficult for many reasons:

- For example, it is difficult to separate the curvature degrees of freedom from mere artifacts

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curvature degrees of freedom from mere artifacts of the choice of the coordinate system.

- Also, the Einstein equation is highly nonlinear.
- So far, all attempts have run into severe difficulties, even perturbative approaches.

⇝ **This course:** 1.) We will here initially consider known classical solutions $g_{\mu\nu}(x)$ and quantize only $\phi(x)$.

2.) Then, we will quantize linear perturbations of the metric.

c) Require: $\frac{\delta}{\delta\phi(x)} S_{tot}[g,\phi] = 0$

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c) Require: $\frac{\delta}{\delta\phi(x)} S_{tot}[g,\phi] = 0$

- Since ϕ occurs only in S_{KG} we have, equivalently:

$$\frac{\delta S_{KG}}{\delta\phi(x)} = 0$$

- Recall S_{KG}:

Example of a potential

$$S_{KG}[\phi] = \frac{1}{2}\int_{\mathbb{R}^4} \left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2 - \underbrace{\lambda\phi^4}\right)\sqrt{|g|}\, d^4x$$

- Apply the Euler Lagrange equations:

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Next: Step 3 in

SR, 1^{st}Q
Hamiltonian
formalism

step 1
Legendre transform
(equivalence)

SR, 1^{st}Q.
Lagrangian
formalism

step 2

allow
curvature

GR, 1^{st}Q
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GR, 1^{st}Q
Hamiltonian
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step 4

GR, 2^{nd}Q
Hamiltonian
formalism

Dyson Schwinger eqns are same
(equivalence)

(PI)

GR, 2^{nd} Q
Lagrangian formalism
(Path integral of QFT)

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step 5

GR, 1^{st}Q Hamiltonian formalism ← Legendre transform (equivalence) — GR, 1^{st}Q Lagrangian formalism

step 4 ↓   ↓ (PI)

GR, 2^{nd}Q Hamiltonian formalism ⟷ Dyson Schwinger eqns are same (equivalence) ⟷ GR, 2^{nd}Q Lagrangian formalism (Path integral of QFT)

Comment on step (PI): 2^{nd} quantization with path integral

- Assume a fixed spacetime is chosen and we are given its metric $g_{\mu\nu}(x)$ in some arbitrary coordinate system

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- Consider, e.g., the vacuum expectation value of $\phi(\vec{x},t)\,\phi(\vec{x}',t)$, i.e., the correlation function of field amplitudes:

$$G(\vec{x},t,\vec{x}',t) := \langle 0|\, \hat{\phi}(\vec{x},t)\, \hat{\phi}(\vec{x}',t)\, |0\rangle$$

- We will later see how to calculate it using commutation relations etc.

- With Feynman we also get it from the path integral:

Advantages:

1) Quick derivation of Feynman rules

2) Manifestly covariant.

$$G(\vec{x},t,\vec{x}',t) = N \int \phi(\vec{x},t)\,\phi(\vec{x}',t)\, e^{\frac{i}{\hbar} S_{Kl}[\phi,g]}\, D[\phi]$$

Problems:

1) Ill defined ...

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