Title: Quantum Field Theory for Cosmology - Achim Kempf - Lecture 10

Date: Feb 05, 2016 01:30 PM

URL: http://pirsa.org/16020001

Abstract:

# QFT for Cosmology, Achim Kempf, Winter 2016, Lecture 10



## Recall:

(because the Lagrangian framework treats space and time in the same way)

* Hamiltonian formulations are suitable for quantization.
* Lagrangian formulations are suitable to achieve general relativistic covariance.

## → Strategy:

SR, $1^{st}$Q Hamiltonian formalism $\xrightarrow[\text{(equivalence)}]{\text{step 1: Legendre transform}}$ SR, $1^{st}$Q. Lagrangian formalism

SR, $1^{st}$Q. Lagrangian formalism $\xrightarrow{\text{step 2: allow curvature}}$ GR, $1^{st}$Q Lagrangian formalism

GR, $1^{st}$Q Hamiltonian formalism $\xleftarrow[\text{(equivalence)}]{\text{step 3: Legendre transform}}$ GR, $1^{st}$Q Lagrangian formalism

File Edit View Insert Actions Tools Help

## Recall:

(framework treats space and time in the same way)

* Hamiltonian formulations are suitable for quantization.
* Lagrangian formulations are suitable to achieve general relativistic covariance.

## ⤳ Strategy:

SR, $1^{st}$Q Hamiltonian formalism $\xrightarrow[\text{(equivalence)}]{\text{step 1: Legendre transform}}$ SR, $1^{st}$Q. Lagrangian formalism

SR, $1^{st}$Q. Lagrangian formalism $\xrightarrow{\text{step 2: allow curvature}}$ GR, $1^{st}$Q Lagrangian formalism

GR, $1^{st}$Q Lagrangian formalism $\xrightarrow[\text{(equivalence)}]{\text{step 3: Legendre transform}}$ GR, $1^{st}$Q Hamiltonian formalism

GR, $1^{st}$Q Hamiltonian formalism $\xrightarrow{\text{step 4}}$ GR, $2^{nd}$Q Hamiltonian formalism

GR, $1^{st}$Q Lagrangian formalism $\dashrightarrow$ GR, $2^{nd}$Q Lagrangian formalism (Path integral of QFT)

GR, $2^{nd}$Q Hamiltonian formalism $\xleftrightarrow[\text{(equivalence)}]{\text{Dyson Schwinger eqns are same}}$ GR, $2^{nd}$Q Lagrangian formalism (Path integral of QFT)

1 / 25

File Edit View Insert Actions Tools Help

We already started step 1 :

$$H[\phi,\pi,t] \quad \underset{\pi(x,t) := \frac{\delta L}{\delta \beta(x,t)} \ (T^{-1})}{\overset{\beta(x,t) := \frac{\delta H}{\delta \pi(x,t)} \ (T)}{\rightleftarrows}} \quad L[\phi,\beta,t]$$

Proposition: These equations of motion are equivalent:

Hamiltonian eqns. of motion:

$$\dot{\phi}(x,t) = \frac{\delta H[\phi,\pi,t]}{\delta \pi(x,t)} \quad (H1)$$

$$\dot{\pi}(x,t) = -\frac{\delta H[\phi,\pi,t]}{\ldots} \quad (H2)$$

Lagrangian eqns. of motion:

$$\dot{\phi}(x,t) = \beta(x,t) \quad (L1)$$

$$\frac{\delta L}{\ldots} = \frac{d}{\ldots} \frac{\delta L}{\ldots} \quad (L2)$$

2 / 25

File Edit View Insert Actions Tools Help

$$\beta(x,t) := \frac{\delta H}{\delta \pi(x,t)} \quad (T)$$

$$H[\phi,\pi,t] \quad \rightleftarrows \quad L[\phi,\beta,t]$$

$$\pi(x,t) := \frac{\delta L}{\delta \beta(x,t)} \quad (T^{-1})$$

Proposition: These equations of motion are equivalent:

Hamiltonian eqns. of motion:

$$\dot{\phi}(x,t) = \frac{\delta H[\phi,\pi,t]}{\delta \pi(x,t)} \quad (H1)$$

$$\dot{\pi}(x,t) = -\frac{\delta H[\phi,\pi,t]}{\delta \phi(x,t)} \quad (H2)$$

Lagrangian eqns. of motion:

$$\dot{\phi}(x,t) = \beta(x,t) \quad (L1)$$

$$\frac{\delta L}{\delta \phi(x,t)} = \frac{d}{dt}\frac{\delta L}{\delta \beta(x,t)} \quad (L2)$$

2 / 25

File Edit View Insert Actions Tools Help

Hamiltonian eqns. of motion:

$$\dot{\phi}(x,t) = \frac{\delta H[\phi,\pi,t]}{\delta \pi(x,t)} \quad (H1)$$

$$\dot{\pi}(x,t) = -\frac{\delta H[\phi,\pi,t]}{\delta \phi(x,t)} \quad (H2)$$

Lagrangian eqns. of motion:

$$\dot{\phi}(x,t) = \beta(x,t) \quad (L1)$$

$$\frac{\delta L}{\delta \phi(x,t)} = \frac{d}{dt}\frac{\delta L}{\delta \beta(x,t)} \quad (L2)$$

Proof: We need to show that $(H1 \wedge H2) \overset{T}{\Longleftrightarrow} (L1 \wedge L2)$.

The case " $\Longrightarrow$ "

- Show L1: Indeed: $\dot{\phi} \overset{(H1)}{=\!=} \frac{\delta H}{\delta \pi} \overset{(T)}{=\!=} \beta$ ✓

2 / 25

File Edit View Insert Actions Tools Help

**Proof:** We need to show that $(H1 \wedge H2) \overset{T}{\Longleftrightarrow} (L1 \wedge L2)$.

The case "$\Longrightarrow$"

- Show $L1$: Indeed: $\dot{\phi} \overset{(H1)}{=\!=} \frac{\delta H}{\delta \pi} \overset{(T)}{=\!=} \beta$ ✓

- Show $L2$: Indeed:

$$\frac{d}{dt} \frac{\delta L(\phi, \beta, t)}{\delta \beta} \overset{(T^{-1})}{=\!=} \frac{d}{dt} \pi$$

$$\overset{(H2)}{=\!=} - \frac{\delta H(\phi, \pi, t)}{\delta \phi}$$

$$\overset{\text{by def. of } L}{=\!=} -\frac{\delta}{\delta \phi}\left(\int \beta(\phi,\pi)\pi \, d^3x - L(\phi, \beta(\phi,\pi), t)\right)$$

$\pi$ (T)

3 / 25

File Edit View Insert Actions Tools Help

**Proof:** We need to show that $(H1 \wedge H2) \overset{T}{\Longleftrightarrow} (L1 \wedge L2)$.

The case "$\Longrightarrow$"

- Show L1: Indeed: $\dot{\phi} \overset{(H1)}{=\!=} \frac{\delta H}{\delta \pi} \overset{(T)}{=\!=} \beta$ ✓

- Show L2: Indeed:

$$\frac{d}{dt}\frac{\delta L(\phi,\beta,t)}{\delta \beta} \overset{(T^{-1})}{=\!=} \frac{d}{dt}\pi$$

$$\overset{(H2)}{=\!=} -\frac{\delta H(\phi,\pi,t)}{\delta \phi}$$

$$\underset{\text{of } L}{\overset{\text{by def.}}{=\!=}} -\frac{\delta}{\delta\phi}\left(\int \beta(\phi,\pi)\pi\, d^3x - L(\phi,\beta(\phi,\pi),t)\right)$$

3 / 25

File Edit View Insert Actions Tools Help

□ **Show L1:** Indeed: $\dot{\phi} \overset{(H1)}{=\!=} \frac{\delta H}{\delta \pi} \overset{(T)}{=\!=} \beta$ ✓

□ **Show L2:** Indeed:

$$\frac{d}{dt}\frac{\delta L(\phi,\beta,t)}{\delta \beta} \overset{(T^{-1})}{=\!=} \frac{d}{dt}\pi$$

$$\overset{(H2)}{=\!=} -\frac{\delta H(\phi,\pi,t)}{\delta \phi}$$

$$\underset{\text{of } L}{\overset{\text{by def.}}{=\!=}} -\frac{\delta}{\delta\phi}\left(\int \beta(\phi,\pi)\pi\, d^3x - L(\phi,\beta(\phi,\pi),t)\right)$$

$$= \frac{\delta L}{\delta \phi} - \cancel{\frac{\delta \beta}{\delta \phi}\pi + \underbrace{\frac{\delta L}{\delta \beta}}_{\overset{(T)}{=}\pi}\frac{\delta \beta}{\delta \phi}}$$ ✓

File Edit View Insert Actions Tools Help

The case "$\Longleftarrow$": Exercise.

**Result so far:**

- Legendre transform to Lagrangian formulation

$\Rightarrow$ Eqns of motion can be cast in the form L1, L2, i.e.:

(Notice: Only a time derivative, no occurrence of space derivatives?) $\rightarrow$

$$\frac{\delta L}{\delta \phi(x,t)} = \frac{d}{dt} \frac{\delta L}{\delta \beta(x,t)}, \quad \beta(x,t) = \dot{\phi}(x,t)$$

**But:** How is that advantageous? These equations

4 / 25

File Edit View Insert Actions Tools Help

## Result so far:

- Legendre transform to Lagrangian formulation

$\Rightarrow$ Eqns of motion can be cast in the form L1, L2, i.e.:

(Notice: Only a time derivative, no occurrence of space derivatives?) $\rightarrow$

$$\frac{\delta L}{\delta \phi(x,t)} = \frac{d}{dt} \frac{\delta L}{\delta \beta(x,t)}, \quad \beta(x,t) = \dot{\phi}(x,t)$$

**But:** How is that advantageous? These equations still seem to treat time differently than space!

4 / 25

File Edit View Insert Actions Tools Help

# Analysis of L1, L2:

We notice: * The term $\frac{\delta L}{\delta \phi(x,t)}$ is the total derivative with respect to all occurrences of $\phi$ in $L$, including occurrences of $\frac{\partial}{\partial x_i}\phi(x,t)$ in $L$.

* Why? Because of the definition of $\frac{\delta}{\delta \phi}$:

$$\frac{\delta L}{\delta \phi(x,t)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left( L\left[\left\{\phi(x',t) + \varepsilon \delta^3(x'-x)\right\}_{x' \in \mathbb{R}^3}\right] - L\left[\left\{\phi(x',t)\right\}_{x' \in \mathbb{R}^3}\right]\right)$$

E.g.: $F[u] := \int \sin(x) \left(\frac{d}{dx} u(x)\right) dx$ Is $\frac{\delta F}{\delta u(x)} = 0$ ? No:

$= -\int \cos(x)\, u(x)\, dx$ (We assume $u(x) \to 0$ at boundaries)

5 / 25

File Edit View Insert Actions Tools Help

# Analysis of L1, L2:

We notice: * The term $\frac{\delta L}{\delta \phi(x,t)}$ is the total derivative with respect to all occurences of $\phi$ in $L$, including occurences of $\frac{\partial}{\partial x_i}\phi(x,t)$ in $L$.

* Why? Because of the definition of $\frac{\delta}{\delta\phi}$:

$$\frac{\delta L}{\delta \phi(x,t)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left( L\left[\left\{\phi(x',t) + \varepsilon \delta^3(x'-x)\right\}_{x' \in \mathbb{R}^3}\right] - L\left[\left\{\phi(x',t)\right\}_{x' \in \mathbb{R}^3}\right]\right)$$

E.g.: $F[u] := \int \sin(x)\left(\frac{d}{dx}u(x)\right)dx$ Is $\frac{\delta F}{\delta u(x)} = 0$ ? No:

$= -\int \cos(x)\, u(x)\, dx$ (We assume $u(x) \to 0$ at boundaries)

5 / 25

File Edit View Insert Actions Tools Help

We notice: * The term $\frac{\delta L}{\delta \phi(x,t)}$ is the total derivative with respect to all occurences of $\phi$ in $L$, including occurences of $\frac{\partial}{\partial x_i}\phi(x,t)$ in $L$.

* Why? Because of the definition of $\frac{\delta}{\delta \phi}$:

$$\frac{\delta L}{\delta \phi(x,t)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left( L\left[\left\{\phi(x',t) + \varepsilon \delta^3(x'-x)\right\}_{x' \in \mathbb{R}^3}\right] - L\left[\left\{\phi(x',t)\right\}_{x' \in \mathbb{R}^3}\right]\right)$$

E.g.: $F[u] := \int \sin(x)\left(\frac{d}{dx}u(x)\right)dx$ Is $\frac{\delta F}{\delta u(x)} = 0$ ? No:

$$= -\int \cos(x)\, u(x)\, dx \quad \text{(We assume } u(x) \to 0 \text{ at boundaries)}$$

$$\Rightarrow \frac{\delta F}{u(x)} = -\cos(x)$$

5 / 25

File Edit View Insert Actions Tools Help

$\Longrightarrow$ L1, L2 will contain nontrivial time and space derivatives.

$\ast$ Is there a systematic way to evaluate the derivatives with respect to $\frac{\partial\phi}{\partial x_i}$ ?

Lemma: Consider any functional $Z$ of the form:

$$Z[f] = \int \text{polynomial}\left(\frac{d}{dx} f\right) dx$$

Then: $\frac{\delta Z}{\delta f(x)} = -\frac{d}{dx}\frac{\delta Z}{\delta\left(\frac{d}{dx}f\right)}$ On the right hand side we view $\frac{d}{dx}f$ as an

6 / 25

File Edit View Insert Actions Tools Help

**Lemma:** Consider any functional $Z$ of the form:

$$Z[f] = \int \text{polynomial}\left(\frac{d}{dx} f\right) dx$$

Then: $$\frac{\delta Z}{\delta f(x)} = -\frac{d}{dx} \frac{\delta Z}{\delta\left(\frac{d}{dx} f\right)}$$

On the right hand side we view $\frac{d}{dx} f$ as an independent function.

**Example:**

Notation: $\partial_x f(x) = \frac{d}{dx} f(x)$

$$Z[f] := \int_{\mathbb{R}} \left(\partial_{x'} f(x')\right)^2 dx'$$

6 / 25

File Edit View Insert Actions Tools Help

## Example:

Notation: $\partial_x f(x) = \frac{d}{dx} f(x)$

$$Z[f] := \int_{\mathbb{R}} \left(\partial_{x'} f(x')\right)^2 dx'$$

- If we view $\partial_x f$ as an independent function, then we obtain of course:

$$\frac{\delta Z[\partial_x f]}{\delta(\partial_x f(x))} = 2\, \partial_x f(x)$$

- Our lemma claims, therefore:

$$\delta Z[f] = -\partial\, \delta Z[\partial_x f] = -2\, \partial\, \partial\, f(x)$$

File Edit View Insert Actions Tools Help

$$z[f] := \int_{\mathbb{R}} (\partial_x f(x))^2 \, dx$$

- If we view $\partial_x f$ as an independent function, then we obtain of course:

$$\frac{\delta z[\partial_x f]}{\delta(\partial_x f(x))} = 2\,\partial_x f(x)$$

- Our lemma claims, therefore:

$$\frac{\delta z[f]}{\delta f(x)} = -\partial_x \frac{\delta z[\partial_x f]}{\delta(\partial_x f(x))} = -2\,\partial_x \partial_x f(x)$$

7 / 25

File Edit View Insert Actions Tools Help

Indeed:

$$\frac{\delta}{\delta f(x)} Z[f] = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ \int_{\mathbb{R}} \left( \partial_{x'} \left( f(x') + \varepsilon \delta(x-x') \right) \right)^2 dx' - \int_{\mathbb{R}} \left( \partial_{x'} f(x') \right)^2 dx' \right]$$

$$\overset{\lim \varepsilon \to 0}{=} 2 \int_{\mathbb{R}} \left( \partial_{x'} f(x') \right) \left( \partial_{x'} \delta(x-x') \right) dx'$$

$$\underset{\text{parts}}{\overset{\text{int. by}}{=}} -2 \int_{\mathbb{R}} \left( \partial_{x'}^2 f(x') \right) \delta(x-x') \, dx' + \cancel{\text{boundary term}}$$

$$= -2 \partial_x^2 f(x)$$

8 / 25

File Edit View Insert Actions Tools Help

**Recall L2:**

$$\frac{\delta L[\phi,\beta,t]}{\delta\phi(x,t)} = \frac{d}{dt}\frac{\delta L[\phi,\beta,t]}{\delta\beta(x,t)}$$

**Use lemma:**

$$\frac{\delta L[\phi,\beta,t]}{\delta\phi(x,t)} = \frac{\delta L[\phi,\partial_1\phi,\partial_2\phi,\partial_3\phi,\beta,t]}{\delta\phi(x,t)}$$

$$-\sum_{j=1}^{3}\frac{\partial}{\partial x^j}\frac{\delta L[\phi,\partial_1\phi,\partial_2\phi,\partial_3\phi,\beta,t]}{\delta\left(\partial_j\phi(x,t)\right)}$$

$\Rightarrow$ L2 takes the form:

9 / 25

File Edit View Insert Actions Tools Help

$$\frac{\delta L[\phi,\partial_j\phi,\beta,t]}{\delta\phi(x,t)} - \sum_{j=1}^{3}\frac{\partial}{\partial x^j}\frac{\delta L[\phi,\partial_j\phi,\beta,t]}{\delta(\partial_j\phi(x,t))} = \frac{d}{dt}\frac{\delta L[\phi,\partial_j\phi,\beta,t]}{\delta\beta(x,t)}$$

Recall also L1: $\beta(x,t) = \dot{\phi}(x,t)$

$\rightsquigarrow$ One is tempted to write:

$$\frac{\delta L[\phi,\partial_j\phi,t]}{\delta\phi(x,t)} \stackrel{?}{=} \sum_{\mu=0}^{3}\frac{\partial}{\partial x^\mu}\frac{\delta L[\phi,\partial_\mu\phi,t]}{\delta(\partial_\mu\phi(x,t))} \qquad \text{with: } \partial_0 := \frac{d}{dt}$$

However:

10 / 25

$$\frac{\delta L[\phi, \partial_j\phi, \beta, t]}{\delta \phi(x,t)} - \sum_{j=1}^{3} \frac{\partial}{\partial x^j} \frac{\delta L[\phi, \partial_j\phi, \beta, t]}{\delta(\partial_j \phi(x,t))} = \frac{d}{dt} \frac{\delta L[\phi, \partial_j\phi, \beta, t]}{\delta \beta(x,t)}$$

Recall also L1: $\beta(x,t) = \dot{\phi}(x,t)$

⇝ One is tempted to write:

$$\frac{\delta L[\phi, \partial_j\phi, t]}{\delta \phi(x,t)} \stackrel{?}{=} \sum_{\mu=0}^{3} \frac{\partial}{\partial x^\mu} \frac{\delta L[\phi, \partial_\mu\phi, t]}{\delta(\partial_\mu \phi(x,t))} \qquad \text{with: } \partial_0 := \frac{d}{dt}$$

However:

Here, we must remember that here the true variable is $\beta$, and that we can set $\beta = \dot{\phi}$ only after functional differentiati...

$$\frac{\cdots}{\delta \phi(x,t)} - \sum_{j=1} \frac{\partial}{\partial x^j} \frac{\cdots}{\delta(\partial_j \phi(x,t))} = \frac{d}{dt} \frac{\cdots}{\delta \beta(x,t)}$$

**Recall also L1:** $\beta(x,t) = \dot{\phi}(x,t)$

$\rightsquigarrow$ One is tempted to write:

$$\frac{\delta L[\phi, \partial_j \phi, t]}{\delta \phi(x,t)} \overset{?}{=} \sum_{\mu=0}^{3} \frac{\partial}{\partial x^\mu} \frac{\delta L[\phi, \partial_\mu \phi, t]}{\delta(\partial_\mu \phi(x,t))} \qquad \text{with: } \partial_0 := \frac{d}{dt}$$

**However:** Here, we must remember that here the true variable is $\phi$, and that we can set $\beta = \dot{\phi}$ only after functional differentiation.

File Edit View Insert Actions Tools Help

**Ramification?** Can we use the lemma to write

$$\frac{\delta L[\phi, t]}{\delta \phi(x,t)} = 0$$

for the Euler Lagrange field equations? No!

**Because:** to apply the lemma to the derivative $\frac{\partial}{\partial t}\phi$, one would need that $L$ possesses a $t$-integration:

**Lemma:** For any functional $Z$ of the form:

$$Z[f] = \int \text{polynomial}\left(\frac{d}{dx} f\right) dx$$

11 / 25

File Edit View Insert Actions Tools Help

**Ramification:** Can we use the lemma to write

$$\frac{\delta L[\phi, t]}{\delta \phi(x,t)} = 0$$

for the Euler Lagrange field equations? No!

**Because:** to apply the lemma to the derivative $\frac{\partial}{\partial t}\phi$, one would need that $L$ possesses a $t$-integration:

**Lemma:** For any functional $Z$ of the form:

$$Z[f] = \int \text{polynomial}\left(\frac{d}{dx} f\right) dx$$

11 / 25

File Edit View Insert Actions Tools Help

# The "Action functional":

- Definition: $S[\phi] := \int_{\mathbb{R}} L[\phi,t]\, dt$

  $S[\phi]$ is called the "action of the field evolution $\phi(x,t)$"

- Then, the "Euler Lagrange field equations" are

$$\boxed{\frac{\delta S[\phi,\partial_\mu\phi]}{\delta\phi(x,t)} - \sum_{\mu=0}^{3} \frac{\partial}{\partial x^\mu} \frac{\delta S[\phi,\partial_\mu\phi]}{\delta(\partial_\mu\phi)} = 0}$$

or equivalently:

$\delta S[\phi]$ ... "The action ..."

12 / 25

File Edit View Insert Actions Tools Help

- Definition: $S[\phi] := \int_{\mathbb{R}} L[\phi, t]\, dt$

  $S[\phi]$ is called the "action of the field evolution $\phi(x,t)$"

- Then, the "Euler Lagrange field equations" are

$$\boxed{\frac{\delta S[\phi, \partial_\mu \phi]}{\delta \phi(x,t)} - \sum_{\mu=0}^{3} \frac{\partial}{\partial x^\mu} \frac{\delta S[\phi, \partial_\mu \phi]}{\delta(\partial_\mu \phi)} = 0}$$

or equivalently:

$$\boxed{\frac{\delta S[\phi]}{\delta \phi(x,t)} = 0}$$

"The action principle"

12 / 25

File Edit View Insert Actions Tools Help

or equivalently:

$$\boxed{\frac{\delta S[\phi]}{\delta \phi(x,t)} = 0}$$

"The action principle"

Notice that the action principle, spelled out, reads:

$$0 = \frac{\delta S[\phi]}{\delta \phi(x,t)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\Big( S\big[\{\phi(x') + \varepsilon\, \delta^4(x-x')\}_{x' \in \mathbb{R}^4}\big] - S\big[\{\phi(x')\}_{x \in \mathbb{R}^4}\big]\Big)$$

12 / 25

File Edit View Insert Actions Tools Help

# The Klein Gordon action:

$$S[\phi] := \frac{1}{2}\int_{\mathbb{R}^4} (\partial_0 \phi)^2 - \sum_{j=1}^{3} (\partial_j \phi)^2 - m^2\phi^2 \, d^4x$$

- Using either the action principle or directly the Euler Lagrange field equations, one obtains indeed the Klein Gordon equation (Exercise: verify):

$$\partial_0^2\phi - \Delta\phi + m^2\phi = 0 \quad \text{i.e.} \quad (\Box + m^2)\phi(x,t) = 0$$

13 / 25

File Edit View Insert Actions Tools Help

- Using either the action principle or directly the Euler Lagrange field equations, one obtains indeed the Klein Gordon equation (Exercise: verify):

$$\partial_0^2 \phi - \Delta \phi + m^2 \phi = 0 \quad \text{i.e.} \quad (\Box + m^2)\phi(x,t) = 0$$

- **Definitions:**

  * The action's integrand is called the "**Lagrange density**" $\mathcal{L}(x,t)$:

$$S[\phi] = \int_{\mathbb{R}^4} \mathcal{L}(x,t)\, d^4x$$

  * One often formally writes:

14 / 25

File Edit View Insert Actions Tools Help

# Definitions:

* The action's integrand is called the "Lagrange density" $\mathcal{L}(x,t)$:

$$S'[\phi] = \int_{\mathbb{R}^4} \mathcal{L}(x,t)\, d^4x$$

* One often formally writes:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \sum_{\mu=0}^{3} \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = 0 \qquad (\mathcal{L})$$

* Notation often used in General Relativity:

a.) $\phi_{,\mu}(x,t) := \frac{\partial}{\partial x^\mu} \phi(x,t)$

14 / 25

File Edit View Insert Actions Tools Help

* Notation often used in General Relativity:

a.) $\phi_{,\mu}(x,t) := \frac{\partial}{\partial x^\mu}\phi(x,t)$

b.) Twice occurring indices are to be summed over (Einstein summation convention):

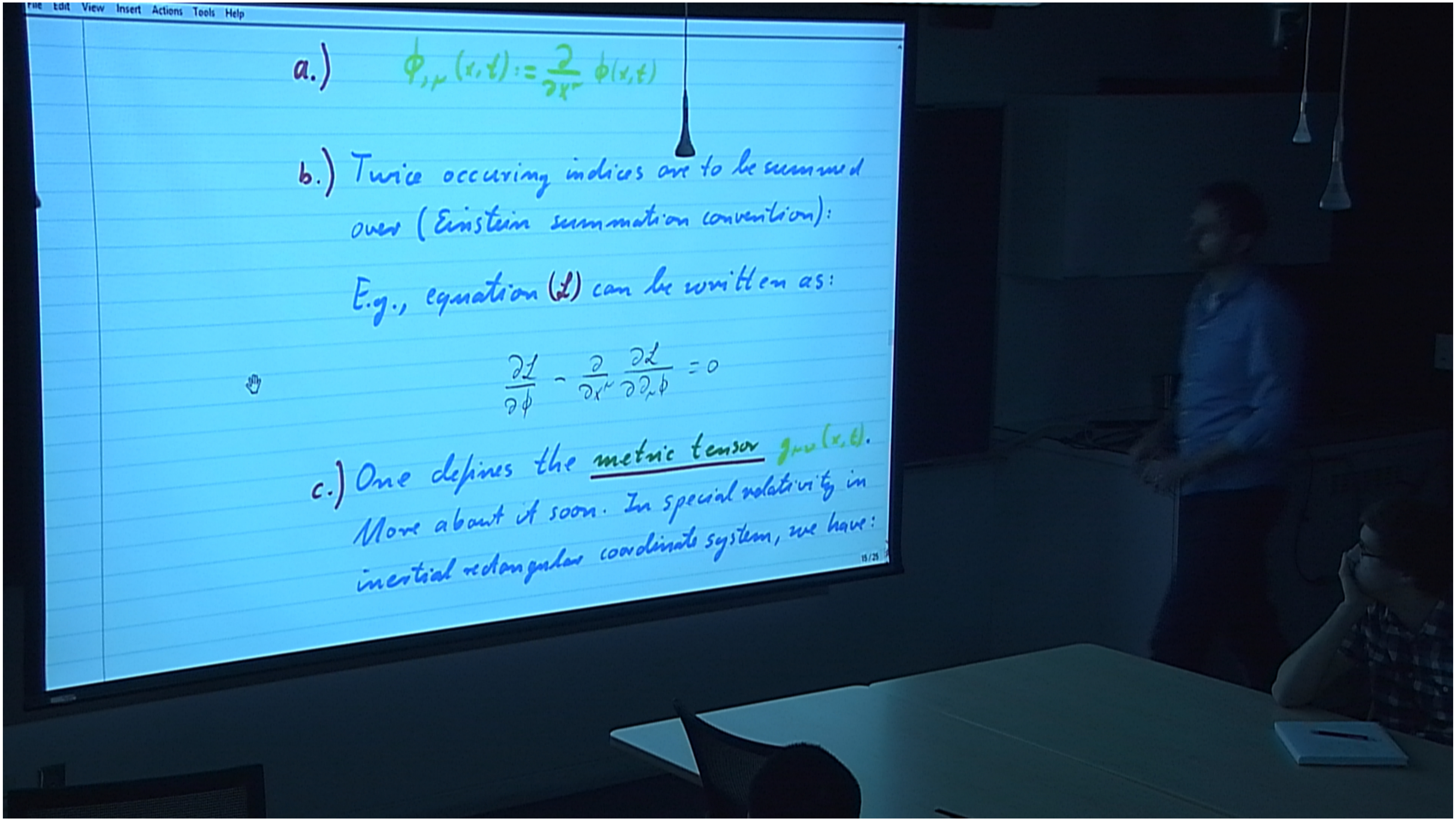
E.g., equation ($\mathcal{L}$) can be written as:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = 0$$

c.) One defines the **metric tensor** $g_{\mu\nu}(x,t)$.

More about it soon. In special relativity in

15 / 25


File Edit View Insert Actions Tools Help
a.) $\phi_{,\mu}(x,t) := \frac{\partial}{\partial x^\mu}\phi(x,t)$
b.) Twice occuring indices are to be summed over (Einstein summation convention):
E.g., equation ($\mathcal{L}$) can be written as:
$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = 0$
c.) One defines the metric tensor $g_{\mu\nu}(x,t)$.
More about it soon. In special relativity in
inertial rectangular coordinate system, we have:
15/25

File Edit View Insert Actions Tools Help

a.) $\phi_{,\mu}(x,t) := \frac{\partial}{\partial x^\mu}\phi(x,t)$

b.) Twice occuring indices are to be summed over (Einstein summation convention):

E.g., equation ($\mathcal{L}$) can be written as:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = 0$$

c.) One defines the **metric tensor** $g_{\mu\nu}(x,t)$. More about it soon. In special relativity in inertial rectangular coordinate system, we have:

15 / 25

File Edit View Insert Actions Tools Help

c.) One defines the **metric tensor** $g_{\mu\nu}(x,t)$. More about it soon. In special relativity in inertial rectangular coordinate system, we have:

$$g_{\mu\nu}(x,t) = \eta_{\mu\nu} := \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

- Using these definitions, the K.G. action now reads:

$$S[\phi] = \frac{1}{2} \int_{\mathbb{R}^4} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - m^2 \phi^2 \, d^4x$$

the inverse matrix to $g_{\mu\nu}$. In special relativity, both are the same: $\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

15 / 25

File Edit View Insert Actions Tools Help

$$g_{\mu\nu}(x,t) = \eta_{\mu\nu} := \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

- Using these definitions, the K.G. action now reads:

$$S[\phi] = \frac{1}{2}\int_{\mathbb{R}^4} g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2 \, d^4x$$

the inverse matrix to $g_{\mu\nu}$. In special relativity, both are the same: $\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

- The E.L. eqns read

$$\frac{\delta S[\phi,\{\phi_{,\mu}\}]}{\delta\phi(x,t)} = \partial_\mu \frac{\delta S[\phi,\{\phi_{,\mu}\}]}{\delta(\phi_{,\mu}(x,t))}$$

and yield $\quad m^2\phi = \partial_\mu g^{\mu\nu}\phi$

File Edit View Insert Actions Tools Help

$$S[\phi] = \frac{1}{2}\int_{\mathbb{R}^4} g^{\mu\nu}\,\phi_{,\mu}\,\phi_{,\nu} - m^2\phi^2\, d^4x$$

the inverse matrix to $g_{\mu\nu}$. In special relativity, both are the same: $\begin{pmatrix}1&&&\\&-1&&\\&&-1&\\&&&-1\end{pmatrix}$

- The E.L. eqns read

$$\frac{\delta S[\phi,\{\phi_{,\mu}\}]}{\delta\phi(x,t)} = \partial_\mu \frac{\delta S[\phi,\{\phi_{,\mu}\}]}{\delta(\phi_{,\mu}(x,t))}$$

and yield $\quad -m^2\phi = \partial_\mu g^{\mu\nu}\phi_{,\nu}$

i.e., of course: $\quad (\Box + m^2)\phi = 0$

We have now completed Step 1:

16 / 25

File Edit View Insert Actions Tools Help

# We have now completed Step 1:

SR, $1^{st}$Q
Hamiltonian
formalism

step 1
Legendre transform
$\longrightarrow$
(equivalence)

SR, $1^{st}$Q.
Lagrangian
formalism

$\downarrow$ step 2 — allow curvature

GR, $1^{st}$Q
Lagrangian
formalism

step 3
Legendre transform
$\longleftarrow$
(equivalence)

GR, $1^{st}$Q
Hamiltonian
formalism

$\downarrow$ step 4

GR, $2^{nd}$Q
Hamiltonian
formalism

Dyson Schwinger eqns are same
$\longleftrightarrow$
(equivalence)

GR, $2^{nd}$ Q
Lagrangian formalism
(Path integral of QFT)

17 / 25

File Edit View Insert Actions Tools Help

Now that we have a beautifully covariant Lagrangian formulation:

Step 2: How to allow for curvature of space-time?

Strategy:

A. Within special relativity, allow not just inertial rectangular coordinate systems but allow arbitrary coordinate systems.

B. Allow arbitrary coordinate system.

18 / 25

File Edit View Insert Actions Tools Help

# Step 2: How to allow for curvature of space-time?

## Strategy:

A. Within special relativity, allow not just inertial rectangular coordinate systems but allow arbitrary coordinate systems.

B. Allow arbitrary coordinate systems and allow curvature.

18 / 25

File Edit View Insert Actions Tools Help

# A. Arbitrary coordinate systems

- Reconsider the K.G. action:

$$S[\phi] = \frac{1}{2}\int_{\mathbb{R}^4} g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2\, d^4x$$

- If we change to arbitrary coordinates

$$x^\mu \to \tilde{x}^\mu = \tilde{x}^\mu(x)$$

then: $\phi(x) \to \tilde{\phi}(\tilde{x}) = \phi(x(\tilde{x}))$

(recall that $\sum_{\nu=0}^{3}$ is implied)

19 / 25

File Edit View Insert Actions Tools Help

- If we change to arbitrary coordinates

$$x^\mu \to \tilde{x}^\mu = \tilde{x}^\mu(x)$$

then: $\phi(x) \to \tilde{\phi}(\tilde{x}) = \phi(x(\tilde{x}))$

(recall that $\sum_{\nu=0}^{3}$ is implied)

$$\frac{\partial}{\partial x^\mu}\phi(x) \to \frac{\partial}{\partial \tilde{x}^\mu}\tilde{\phi}(\tilde{x}) = \left(\frac{\partial}{\partial x^\nu}\phi(x(\tilde{x}))\right)\frac{\partial x^\nu}{\partial \tilde{x}^\mu}$$

- Therefore, if we transform

$$g^{\mu\nu}(x) \to \tilde{g}^{\mu\nu}(\tilde{x}) = \frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\frac{\partial \tilde{x}^\nu}{\partial x^\beta}g^{\alpha\beta}(x(\tilde{x}))$$

19 / 25

- If we change to arbitrary coordinates

$$x^\mu \to \tilde{x}^\mu = \tilde{x}^\mu(x)$$

then: $$\phi(x) \to \tilde{\phi}(\tilde{x}) = \phi(x(\tilde{x}))$$

(recall that $\sum_{\nu=0}^{3}$ is implied)

$$\frac{\partial}{\partial x^\mu}\phi(x) \to \frac{\partial}{\partial \tilde{x}^\mu}\tilde{\phi}(\tilde{x}) = \left(\frac{\partial}{\partial x^\nu}\phi(x(\tilde{x}))\right)\frac{\partial x^\nu}{\partial \tilde{x}^\mu}$$

- Therefore, if we transform

$$g^{\mu\nu}(x) \to \tilde{g}^{\mu\nu}(\tilde{x}) = \frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\frac{\partial \tilde{x}^\nu}{\partial x^\beta}\, g^{\alpha\beta}(x(\tilde{x}))$$

then we have that this term in the action

File Edit View Insert Actions Tools Help

$$g^{\mu\nu}(x) \longrightarrow \tilde{g}^{\mu\nu}(\tilde{x}) = \frac{\partial \tilde{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \tilde{x}^{\nu}}{\partial x^{\beta}} g^{\alpha\beta}(x(\tilde{x}))$$

then we have that this term in the action

$$g^{\mu\nu}(x)\, \phi_{,\mu}(x)\, \phi_{,\nu}(x)$$

is numerically the same in all coordinate systems:

$$g^{\mu\nu}(x)\left(\frac{\partial}{\partial x^{\mu}}\phi(x)\right)\left(\frac{\partial}{\partial x^{\nu}}\phi(x)\right) \longrightarrow \tilde{g}^{\mu\nu}(\tilde{x})\left(\frac{\partial}{\partial \tilde{x}^{\mu}}\phi(\tilde{x})\right)\left(\frac{\partial}{\partial \tilde{x}^{\nu}}\phi(\tilde{x})\right)$$

$$= g^{\alpha\beta}(x)\, \frac{\partial \tilde{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \tilde{x}^{\nu}}{\partial x^{\beta}} \frac{\partial x^{\rho}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\varepsilon}}{\partial \tilde{x}^{\nu}} \left(\frac{\partial}{\partial x^{\rho}}\phi(x)\right)\left(\frac{\partial}{\partial x^{\varepsilon}}\phi(x)\right)$$

$$= g^{\mu\nu}(x)\left(\frac{\partial}{\partial x^{\mu}}\phi(x)\right)\left(\frac{\partial}{\partial x^{\nu}}\phi(x)\right) \text{ because } \frac{\partial \tilde{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial x^{\rho}}{\partial \tilde{x}^{\mu}} = \delta^{\rho}_{\alpha}$$

20 / 25

File Edit View Insert Actions Tools Help

- Terminology:

* We say that we let $g^{\mu\nu}(x)$ transform as a contravariant tensor of rank 2.

because upper indices — because 2 indices

* With $g^{\mu\nu}(x)\, g_{\nu\beta}(x) = \delta^{\mu}{}_{\nu}$ we have

$$g_{\mu\nu}(x) \rightarrow \tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} g_{\alpha\beta}(x(\tilde{x}))$$

which is called a covariant rank 2 tensor.

- Is $S[\phi]$ now coordinate system independent?

21 / 25

File Edit View Insert Actions Tools Help

as a contravariant tensor of rank 2.

because upper indices ... because 2 indices

* With $g^{\mu\nu}(x)\, g_{\nu\rho}(x) = \delta^{\mu}{}_{\rho}$ we have

$$g_{\mu\nu}(x) \longrightarrow \tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} g_{\alpha\beta}(x(\tilde{x}))$$

which is called a covariant rank 2 tensor.

- Is $S[\phi]$ now coordinate system independent?

No, not yet!

21 / 25

File Edit View Insert Actions Tools Help

## Recall:

As $x^\mu \to \tilde{x}^\mu(x)$ the integral measure changes by a Jacobian factor:

$$\int f(x)\, d^4x \longrightarrow \int \underbrace{\tilde{f}(\tilde{x})}_{f(\tilde{x})} \underbrace{\det\left(\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right)}_{\text{a coordinate-dependent term!}} d^4\tilde{x}$$

## A compensating term is needed:

How can we modify the action $S'[\phi]$ so that:

22 / 25

File Edit View Insert Actions Tools Help

- A compensating term is needed:

How can we modify the action $S'[\phi]$ so that:

* there is no modification in cartesian coordinates
* the modification compensates the Jacobian term.

- Solution:

Modify the action to include a "Volume factor":

$$S[\phi] := \frac{1}{2}\int_{\mathbb{R}^4} \left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2\right) \sqrt{-\det(g_{\mu\nu})}\, d^4x$$

skype

- A compensating term is needed:

How can we modify the action $S'[\phi]$ so that:

- ∗ there is no modification in cartesian coordinates
- ∗ the modification compensates the Jacobian term.

- Solution:

Modify the action to include a "Volume factor":

$$S[\phi] := \frac{1}{2}\int_{\mathbb{R}^4} \left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2\right)\sqrt{-\det(g_{\mu\nu})}\, d^4x$$

File Edit View Insert Actions Tools Help

## Solution:

Modify the action to include a "Volume factor":

$$S[\phi] := \frac{1}{2}\int_{\mathbb{R}^4}\left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2\right)\overbrace{\sqrt{-\det(g_{\mu\nu})}}\, d^4x$$

## The volume factor:

* When $g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ then $\sqrt{-\det(g)} = 1$ ✓

* Lemma: When $x^\mu \to \tilde{x}^\mu(x)$ then:

short for $\sqrt{-\det(g_{\mu\nu})}$

$$\sqrt{|g|} \longrightarrow \sqrt{|\tilde{g}|} = \det\left(\frac{\partial \tilde{x}^\mu}{\partial \dots}\right)\sqrt{|g|}$$

23 / 25

File Edit View Insert Actions Tools Help

$$S[\phi] := \frac{1}{2}\int_{\mathbb{R}^4} \left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2\right) \overbrace{\sqrt{-\det(g_{\mu\nu})}}^{?}\, d^4x$$

The volume factor:

* When $g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ then $\sqrt{-\det(g)} = 1$ ✓

* Lemma: When $x^\mu \to \tilde{x}^\mu(x)$ then:

short for $\sqrt{-\det(g_{\mu\nu})}$

$$\sqrt{|g|} \longrightarrow \sqrt{|\tilde{g}|} = \det\left(\frac{\partial \tilde{x}^\mu}{\partial x^\nu}\right)\sqrt{|g|}$$

23 / 25

File Edit View Insert Actions Tools Help

## The volume factor:

* When $g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ then $\sqrt{-\det(g)} = 1$ ✓

* Lemma: When $x^\mu \rightarrow \tilde{x}^\mu(x)$ then:

short for $\sqrt{-\det(g_{\mu\nu})}$

$$\sqrt{|g|} \longrightarrow \sqrt{|\tilde{g}|} = \det\left(\frac{\partial \tilde{x}^\mu}{\partial x^\nu}\right)\sqrt{|g|}$$

## Therefore, we have now in special relativity that the action $S[\phi]$ of a field $\phi$ comes

23 / 25

File Edit View Insert Actions Tools Help

□ Therefore, we have now in special relativity that the action $S[\phi]$ of a field $\phi$ comes out the same number, independently of one's choice of coordinate system:

$$S[\phi] \longrightarrow \tilde{S}[\tilde{\phi}] = \int \tilde{\mathcal{L}} \sqrt{|\tilde{g}|}\, d^4\tilde{x}$$

$$= \int \mathcal{L} \det\left(\frac{\partial \tilde{x}}{\partial x}\right) \det\left(\frac{\partial x}{\partial \tilde{x}}\right) \sqrt{|g|}\, d^4x$$

$$= \int \mathcal{L} \det\left(\frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial \tilde{x}^\alpha}{\partial x^\nu}\right) \sqrt{|g|}\, d^4x$$

$$= \int \mathcal{L} \det\left(\delta^\mu{}_\nu\right) \sqrt{|g|}\, d^4x = \int \mathcal{L} \sqrt{|g|}\, d^4 x$$

24 / 25

File Edit View Insert Actions Tools Help

that the action $S[\phi]$ of a field $\phi$ comes out the same number, independently of one's choice of coordinate system:

$$S[\phi] \longrightarrow \tilde{S}[\tilde{\phi}] = \int \tilde{\mathcal{L}} \sqrt{|\tilde{g}|}\, d^4\tilde{x}$$

$$= \int \mathcal{L} \det\left(\frac{\partial \tilde{x}}{\partial x}\right) \det\left(\frac{\partial x}{\partial \tilde{x}}\right) \sqrt{|g|}\, d^4x$$

$$= \int \mathcal{L} \det\left(\frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial \tilde{x}^\alpha}{\partial x^\nu}\right) \sqrt{|g|}\, d^4x$$

$$= \int \mathcal{L} \det\left(\delta^\mu{}_\nu\right) \sqrt{|g|}\, d^4x = \int \mathcal{L} \sqrt{|g|}\, d^4x$$

$$= S[\phi]$$

24 / 25

File Edit View Insert Actions Tools Help

## B. How to allow curvature?

* The trivial metric $g_{\mu\nu}(x) = \eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ can look very nontrivial in generic coordinate systems: $g_{\mu\nu}(x) = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$

* But: Some metrics $g_{\mu\nu}(x)$ are not obtainable from the trivial metric by a coordinate change!

→ These metrics belong to spaces with

25 / 25

File Edit View Insert Actions Tools Help

B. How to allow curvature?

* The trivial metric $g_{\mu\nu}(x) = \eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

can look very nontrivial in generic coordinate systems: $g_{\mu\nu}(x) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$

* But: Some metrics $g_{\mu\nu}(x)$ are not obtainable from the trivial metric by a coordinate change!

→ These metrics belong to spaces with curvature. We need not change the action!

25 / 25

File Edit View Insert Actions Tools Help

B. How to allow curvature?

* The trivial metric $g_{\mu\nu}(x) = \eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$
can look very nontrivial in generic coordinate systems: $g_{\mu\nu}(x) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$

* But: Some metrics $g_{\mu\nu}(x)$ are not obtainable from the trivial metric by a coordinate change!

→ These metrics belong to spaces with curvature. We need not change the action!

25 / 25