

Title: Quantum Field Theory for Cosmology - Achim Kempf - Lecture 9

Date: Feb 01, 2016 01:30 PM

URL: <http://www.pirsa.org/16020000>

Abstract:

# QFT for Cosmology, Achim Kempf, Winter 2016, Lecture 9

Note Title

## Mathematical preparations for QFT in curved space:

Plan today:

- Functional derivatives  $\frac{\delta F[g]}{\delta g(x)} = ?$
- Example use 1: to make the QFT Schrödinger equation well defined.
- Example use 2: to define the Functional Legendre transform.

Plan today:

□ Functional derivatives

$$\frac{\delta F[g]}{\delta g(x)} = ?$$

□ Example use 1: to make the QFT Schrödinger equation well defined.

□ Example use 2: to define the Functional Legendre transform.

□ Use both to obtain the Lagrangian formulation of QFT  
- which will be starting point for QFT on curved space.

# Functional differentiation

Recall:

a.) Differentiation of functions of one variable,  $F(u)$ :

$$\frac{dF(u)}{du} := \lim_{\varepsilon \rightarrow 0} \frac{F(u+\varepsilon) - F(u)}{\varepsilon}$$

b.) Differentiation of functions of countably many

variables,  $F(\{u_j\}_{j=1,2,3,\dots})$ :

$$\frac{\partial F(\{u_j\}_{j=1,2,\dots})}{\partial u_i} := \lim_{\varepsilon \rightarrow 0} \frac{F(u_1, \dots, u_i + \varepsilon, \dots) - F(u_1, \dots, u_i, \dots)}{\varepsilon}$$

a.) Differentiation of functions of one variable,  $F(u)$ :

$$\frac{dF(u)}{du} := \lim_{\varepsilon \rightarrow 0} \frac{F(u+\varepsilon) - F(u)}{\varepsilon}$$

b.) Differentiation of functions of countably many variables,  $F(\{u_j\}_{j=1,2,3,\dots})$ :

$$\frac{\partial F(\{u_j\}_{j=1,2,\dots})}{\partial u_i} :=$$

$$\lim_{\varepsilon \rightarrow 0} \frac{F(u_1, \dots, u_i + \varepsilon, \dots) - F(u_1, \dots, u_i, \dots)}{\varepsilon}$$

$$= \lim_{\varepsilon \rightarrow 0} \frac{F(\{u_j + \varepsilon \delta_{ij}\}_{j=1,\dots}) - F(\{u_j\}_{j=1,\dots})}{\varepsilon}$$

$$\frac{\partial F(\{u_j\}_{j=1,2,\dots})}{\partial u_i} := \lim_{\epsilon \rightarrow 0} \frac{F(u_1, \dots, u_i + \epsilon, \dots) - F(u_1, \dots, u_i, \dots)}{\epsilon}$$

$$= \lim_{\epsilon \rightarrow 0} \frac{F(\{u_j + \epsilon \delta_{ij}\}_{j=1,\dots}) - F(\{u_j\}_{j=1,\dots})}{\epsilon}$$

Definition:

c.) Differentiation of functions of uncountably many variables,  $F(\{u(x)\}_{x \in \mathbb{R}^n})$ :

Note: Since the Dirac delta is not a function but a distribution, which is only defined relative to an integral, the full definition is more technical.

$$\frac{\delta F(\{u(x)\}_{x \in \mathbb{R}^n})}{\delta u(y)} := \lim_{\epsilon \rightarrow 0} \frac{F(\{u(x) + \epsilon \delta''(x-y)\}_{x \in \mathbb{R}^n}) - F(\{u(x)\}_{x \in \mathbb{R}^n})}{\epsilon}$$

$$\lim_{\epsilon \rightarrow 0} \dots \dots \dots \epsilon$$

## Definition:

c.) Differentiation of functions of uncountably many

variables,  $F(\{u(x)\}_{x \in \mathbb{R}^n})$ :

Note: Since the Dirac delta is not a function but a distribution, which is only defined relative to an integral, the full definition is more technical.

$$\frac{\delta F(\{u(x)\}_{x \in \mathbb{R}^n})}{\delta u(y)} := \lim_{\epsilon \rightarrow 0} \frac{F(\{u(x) + \epsilon \delta''(x-y)\}_{x \in \mathbb{R}^n}) - F(\{u(x)\}_{x \in \mathbb{R}^n})}{\epsilon}$$

→ Since  $F$  is a "functional", i.e., is mapping functions to numbers

### c.) Differentiation of functions of uncountably many variables, $F(\{u(x)\}_{x \in \mathbb{R}^n})$ :

Note: Since the Dirac delta is not a function but a distribution, which is only defined relative to an integral, the full definition is more technical.

$$\frac{\delta F(\{u(x)\}_{x \in \mathbb{R}^n})}{\delta u(y)} := \lim_{\varepsilon \rightarrow 0} \frac{F(\{u(x) + \varepsilon \delta''(x-y)\}_{x \in \mathbb{R}^n}) - F(\{u(x)\}_{x \in \mathbb{R}^n})}{\varepsilon}$$

→ Since  $F$  is a "functional", i.e., is mapping functions to numbers

$$F: u \rightarrow F[u] \in \mathbb{C}$$

↑  
function

↑  
short for  $\{u(x)\}_{x \in \mathbb{R}^n}$

we call  $\frac{\delta F}{\delta u(x)}$  a functional derivative.

Example:

$$F[u] := \int_{\mathbb{R}} \cos(x) u(x)^2 dx$$



Then:

$$\frac{\delta F}{\delta u(y)} = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left[ \int_{\mathbb{R}} \cos(x) (u(x) + \varepsilon \delta(x-y))^2 dx - \int_{\mathbb{R}} \cos(x) u(x)^2 dx \right]$$

Distribution theory would be needed. But it drops out anyway

$$= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\mathbb{R}} \cos(x) \left( u(x)^2 + \varepsilon 2u(x) \delta(x-y) + \varepsilon^2 \delta^2(x-y) - u(x)^2 \right) dx$$

$$= \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{\varepsilon} \int_{\mathbb{R}} 2u(x) \delta(x-y) \cos(x) dx$$

Example:

$$F[u] := \int_{\mathbb{R}} \cos(x) u(x)^2 dx$$



Then:

$$\frac{\delta F}{\delta u(y)} = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left[ \int_{\mathbb{R}} \cos(x) (u(x) + \varepsilon \delta(x-y))^2 dx - \int_{\mathbb{R}} \cos(x) u(x)^2 dx \right]$$

Distribution theory would be needed. But it drops out anyway

$$= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\mathbb{R}} \cos(x) \left( u(x)^2 + \varepsilon 2u(x) \delta(x-y) + \varepsilon^2 \delta^2(x-y) - u(x)^2 \right) dx$$

$$= \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{\varepsilon} \int_{\mathbb{R}} 2u(x) \delta(x-y) \cos(x) dx$$

$$\frac{\delta \hat{F}}{\delta u(y)} = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left[ \int_{\mathbb{R}} \cos(x) \left( u(x) + \varepsilon \delta(x-y) \right)^2 - \int_{\mathbb{R}} \cos(x) u(x)^2 dx \right]$$

Distribution theory would  
be needed. But it drops out anyway

$$= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\mathbb{R}} \cos(x) \left( u(x)^2 + \varepsilon^2 u(x) \delta(x-y) + \varepsilon^2 \delta^2(x-y) - u(x)^2 \right) dx$$



$$= \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{\varepsilon} \int_{\mathbb{R}} 2u(x) \delta(x-y) \cos(x) dx$$

$$= 2 \cos(y) u(y)$$

Similarly, one obtains:  $\frac{\delta}{\delta u(y)} \left( \int_{\mathbb{R}} u(x) u(x)^n dx \right) = n u(y) u(y)^{n-1}$

$$\frac{\delta \tilde{F}}{\delta u(y)} = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left[ \int_{\mathbb{R}} \cos(x) \left( u(x) + \varepsilon \delta(x-y) \right)^2 - \int_{\mathbb{R}} \cos(x) u(x)^2 dx \right]$$

Distribution theory would  
be needed. But it drops out anyway

$$= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\mathbb{R}} \cos(x) \left( u(x)^2 + \varepsilon 2 u(x) \delta(x-y) + \varepsilon^2 \delta^2(x-y) - u(x)^2 \right) dx$$



$$= \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{\varepsilon} \int_{\mathbb{R}} 2 u(x) \delta(x-y) \cos(x) dx$$

$$= 2 \cos(y) u(y)$$

Similarly, one obtains:  $\frac{\delta}{\delta} ( (u_1 \dots u_n)^m ) = (u_1 \dots u_n)^{m-1}$

Similarly, one obtains:  $\frac{\delta}{\delta u(y)} \int_R f(x) u(x)^n dx = f(y) n u(y)^{n-1}$

⇒ Functional derivatives act on polynomials  
 (and suitable power series) in  $u$  by removing  
 the integral and reducing the power in  $u$  by  
 one, as expected from ordinary derivatives.



Remark: \* Worked with  $u(x)$ .

\* Would obtain same result if we used any other continuous or discrete basis of  $L^2$ .

\* E.g. other basis (continuous):  $e^{ipx}$ , i.e. use  $\tilde{u}(p)$

\* E.g. other basis (countable):  $H_n(x)e^{-x^2}$ , i.e. use  $\tilde{u}_n$   
↑ Hermite polynomials

(and suitable power series) in  $u$  by removing the integral and reducing the power in  $u$  by one, as expected from ordinary derivatives.

Remark: \* Worked with  $u(x)$ .

\* Would obtain same result if we used any other continuous or discrete basis of  $L^2$ .

\* E.g. other basis (continuous):  $e^{ipx}$ , i.e. use  $\tilde{u}(p)$

\* E.g. other basis (countable):  $H_n(x)e^{-x^2}$ , i.e. use  $\tilde{u}_n$   
 $\vdash$  Hermite polynomials

$\Rightarrow$  Functional differentiation is, up to basis change, usual differentiation

Note: How can  $L^2[\mathbb{R}]$  have countable basis? Recall:  $L^2[\mathbb{R}]$  consists not of functions, but of equivalence classes of functions.

(and suitable power series) in  $u$  by removing the integral and reducing the power in  $u$  by one, as expected from ordinary derivatives.

Remark: \* Worked with  $u(x)$ .

\* Would obtain same result if we used any other continuous or discrete basis of  $L^2$ .

\* E.g. other basis (continuous):  $e^{ipx}$ , i.e. use  $\tilde{u}(p)$

\* E.g. other basis (countable):  $H_n(x)e^{-x^2}$ , i.e. use  $\tilde{u}_n$   
↑ Hermite polynomials

⇒ Functional differentiation is, up to basis change, usual differentiation

Note: How can  $L^2[\mathbb{R}]$  have countable basis? Recall:  $L^2[\mathbb{R}]$  consists not of functions, but of equivalence classes of functions.

## Example application 1:

Schrödinger equation of QFT now well defined:

QM:  $\hat{q}_i \quad \hat{p}_i \quad i \quad t$

QFT:  $\hat{\phi}(x) \quad \hat{\pi}(x) \quad x \quad t$



QM:  $\hat{H}(t) = \sum_{i=1}^n \frac{\hat{p}_i^2}{2} + V(\hat{q}, t)$

Plays role of  $V(\hat{q}, t)$  although the first term is usually not considered to be part of the QFT's potential.



QFT:  $\hat{H}(t) = \int_{\mathbb{R}^3} \frac{\hat{\pi}(x)^2}{2} + \frac{1}{2} \hat{\phi}(x) (m^2 - \Delta) \hat{\phi}(x) + W(\hat{\phi}, t) d^3x$

[ Example:  $W(\hat{\phi}) = \frac{1}{4!} \hat{\phi}^4(x, t)$  ]

In general:  $W(\hat{\phi})$  also contains other fields

QM: Example of complete cut down to one field - see slide 1.3

## Example application 1:

Schrödinger equation of QFT now well defined:

QM:  $\hat{q}_i \quad \hat{p}_i \quad i \quad t$

QFT:  $\hat{\phi}(x) \quad \hat{\pi}(x) \quad x \quad t$



QM:  $\hat{H}(t) = \sum_{i=1}^n \frac{\hat{p}_i^2}{2} + V(\hat{q}, t)$

Plays role of  $V(\hat{q}, t)$  although the first term is usually not considered to be part of the QFT's potential.



QFT:  $\hat{H}(t) = \int_{\mathbb{R}^3} \frac{\hat{\pi}(x)^2}{2} + \frac{1}{2} \hat{\phi}(x) (m^2 - \Delta) \hat{\phi}(x) + W(\hat{\phi}, t) d^3x$

[Example:  $W(\hat{\phi}) = \frac{1}{4!} \hat{\phi}^4(x, t)$ ]

In general:  $W(\hat{\phi})$  also contains other fields

QM: Example of complete cut down time evolution due to  $\hat{H}$

QM:  $\hat{H}(t) = \sum_{j=1}^{\infty} \frac{\hat{p}_j^2}{2} + V(\hat{q}, t)$

first term is usually not considered to be part of the QFT's potential.

QFT:  $\hat{H}(t) = \int_{\mathbb{R}^3} \frac{\hat{p}(x)^2}{2} + \frac{1}{2} \hat{\phi}(x)(m^2 - \Delta) \hat{\phi}(x) + W(\hat{\phi}, t) d^3x$

Example:  $W$

In general:  $W(\delta)$

QM: Example of complete set of commuting adj. operators

QFT: Example of complete set of commuting s.adj. operators:  $\{\hat{\phi}\}$

QM: The joint eigenbasis is  $\{| \{q_i\}_{i=1}^{\infty} \rangle\}$  of the  $\{\hat{q}_i\}_{i=1}^{\infty}$ , obe

$$\hat{q}_i | \{q_i\}_{i=1}^{\infty} \rangle = q_i | \{q_i\}_{i=1}^{\infty} \rangle$$

QM:  $\hat{H}(t) = \sum_{j=1}^n \frac{\hat{p}_j^2}{2} + V(\hat{q}, t)$

first term is usually not considered  
to be part of the QFT's potential.

QFT:  $\hat{H}(t) = \int_{\mathbb{R}^3} \frac{\hat{\pi}(x)^2}{2} + \frac{1}{2} \hat{\phi}(x)(m^2 - \Delta) \hat{\phi}(x) + W(\hat{\phi}, t) d^3x$

[Example:  $W(\hat{\phi}) = \frac{1}{4!} \hat{\phi}^4(x, t)$

In general:  $W(\hat{\phi})$  also contains other fields

QM: Example of complete set of commuting s.adj. operators:  $\{\hat{q}_j\}_{j=1}^n$

QFT: Example of complete set of commuting s.adj. operators:  $\{\hat{\phi}(x)\}_{x \in \mathbb{R}^3}$

QM: The joint eigenspace  $\{| \{q_j\}_{j=1}^n \rangle\}$  of the  $\{\hat{q}_j\}_{j=1}^n$  obeys:

$$\hat{q}_i | \{q_j\}_{j=1}^n \rangle = q_i | \{q_j\}_{j=1}^n \rangle$$

QM: The joint eigenbasis  $\{|q_i\rangle\}_{i=1}^{\infty}$  of the  $\{\hat{q}_i\}_{i=1}^{\infty}$  obeys:

$$\hat{q}_i |q_j\rangle = q_i |q_j\rangle \quad \text{Hand}$$

QFT: The joint eigenbasis  $\{|\phi(x)\rangle_{x \in \mathbb{R}^3}\}$  of the  $\{\hat{\phi}(x)\}_{x \in \mathbb{R}^3}$  obeys:

$$\hat{\phi}(y) |\phi(x)\rangle_{x \in \mathbb{R}^3} = \phi(y) |\phi(x)\rangle_{x \in \mathbb{R}^3}$$

QM: Wave function of a state  $|\psi(t)\rangle \in \mathcal{H}$  in position eigenbasis:

$$\psi(\{q_i\}_{i=1}^{\infty}, t) = \langle \{q_i\}_{i=1}^{\infty} | \psi(t) \rangle \quad (\text{like } \psi(q) = \langle q | \psi \rangle)$$

QFT: Wave functional of a state  $|\Psi(t)\rangle \in \mathcal{H}$  in field eigenbasis:

QM: Wave function of a state  $|\psi(t)\rangle \in \mathcal{H}$  in position eigenbasis:

$$\psi(\{q_i\}_{i=1}^m, t) = \langle \{q_i\}_{i=1}^m | \psi(t) \rangle \quad (\text{like } \psi_q = \langle q | \psi \rangle)$$

QFT: Wave functional of a state  $|\Psi(t)\rangle \in \mathcal{H}$  in field eigenbasis:

$$\Psi[\{\phi(x)\}_{x \in \mathbb{R}^3}, t] = \langle \{\phi(x)\}_{x \in \mathbb{R}^3} | \Psi(t) \rangle$$

↑ Probability amplitude for finding function  $\phi(x)$  when measuring  $\hat{\phi}(x)$  at  $t$ .

Simplified notation:

QM:  $\psi(q, t) = \langle q | \psi(t) \rangle$

QFT:  $\Psi[\phi, t] = \langle \phi | \Psi(t) \rangle$

QM: Representation of a state  $\psi$  when  $\hat{\phi}$  is measured in a basis:

QM: Wave function of a state  $|\psi(t)\rangle \in \mathcal{H}$  in position eigenbasis:

$$\psi(\{q_i\}_{i=1}^m, t) = \langle \{q_i\}_{i=1}^m | \psi(t) \rangle \quad (\text{like } \psi_q = \langle q | \psi \rangle)$$

QFT: Wave functional of a state  $|\Psi(t)\rangle \in \mathcal{H}$  in field eigenbasis:

$$\Psi[\{\phi(x)\}_{x \in \mathbb{R}^3}, t] = \langle \{\phi(x)\}_{x \in \mathbb{R}^3} | \Psi(t) \rangle$$

↑ Hilbert space of QFT, of course  
Probability amplitude for finding function  $\phi(x)$  when measuring  $\hat{\phi}(x)$  at  $t$ .

Simplified notation:

QM:  $\psi(q, t) = \langle q | \psi(t) \rangle$

QFT:  $\Psi[\phi, t] = \langle \phi | \Psi(t) \rangle$

QM: Representation of a state  $\psi$  when  $\hat{\phi}$  is measured in a eigenbasis.

$$\Psi[\{\phi(x)\}_{x \in \mathbb{R}^3}, t] = \langle \{\phi(x)\}_{x \in \mathbb{R}^3} | \Psi(t) \rangle$$

↑  
Probability amplitude for finding function  $\phi(x)$  when measuring  $\hat{\phi}(x)$  at  $t$ .

Simplified notation:

QM:  $\psi(q, t) = \langle q | \Psi(t) \rangle$

QFT:  $\Psi[\phi, t] = \langle \phi | \Psi(t) \rangle$

QM: Representation of  $\hat{q}_i, \hat{p}_i$  obeying  $[\hat{q}_i, \hat{p}_j] = i\delta_{ij}$  in  $\hat{q}$  eigenbasis:

$$\hat{q}_i : \psi(q, t) \rightarrow q_i \psi(q, t)$$

$$\hat{p}_i : \psi(q, t) \rightarrow -i \frac{\partial}{\partial q_i} \psi(q, t)$$

QFT: Representation of  $\hat{q}_i, \hat{p}_i$  obeying  $[\hat{q}_i, \hat{p}_j] = i\delta_{ij}$  in  $\hat{q}$  eigenbasis:

QM: Representation of  $\hat{q}_i, \hat{p}_i$  obeying  $[\hat{q}_i, \hat{p}_j] = i\delta_{ij}$  in  $\hat{q}$  eigenbasis:

$$\hat{q}_i : \Psi(q, t) \rightarrow q_i \Psi(q, t)$$

$$\hat{p}_i : \Psi(q, t) \rightarrow -i \frac{\partial}{\partial q_i} \Psi(q, t)$$

QFT: Representation of  $\hat{\phi}(x), \hat{\pi}(y)$  obeying  $[\hat{\phi}(x), \hat{\pi}(y)] = i\delta^3(x-y)$  in  $\hat{\phi}$  eigenbasis:

$$\hat{\phi}(x) : \Psi[\phi, t] \rightarrow \phi(x) \Psi[\phi, t]$$

Exercise:  
Verify that  $\hat{\phi}(x), \hat{\pi}(x)$  obey the CCRs.

$$\hat{\pi}(x) : \Psi[\phi, t] \rightarrow -i \frac{\delta}{\delta \phi(x)} \Psi[\phi, t]$$

QM: Representation of  $\hat{q}_i, \hat{p}_i$  obeying  $[\hat{q}_i, \hat{p}_j] = i\delta_{ij}$  in  $\hat{q}$  eigenbasis:

$$\hat{q}_i : \Psi(q, t) \rightarrow q_i \Psi(q, t)$$

$$\hat{p}_i : \Psi(q, t) \rightarrow -i \frac{\partial}{\partial q_i} \Psi(q, t)$$

QFT: Representation of  $\hat{\phi}(x), \hat{\pi}(y)$  obeying  $[\hat{\phi}(x), \hat{\pi}(y)] = i\delta^3(x-y)$  in  $\hat{\phi}$  eigenbasis:

$$\hat{\phi}(x) : \Psi[\phi, t] \rightarrow \phi(x) \Psi[\phi, t]$$

$$\hat{\pi}(x) : \Psi[\phi, t] \rightarrow -i \frac{\delta}{\delta \phi(x)} \Psi[\phi, t]$$

Exercise:  
Verify, that  $\hat{\phi}(x), \hat{\pi}(y)$  obey the CCRs.

QM: Schrödinger equation:

QM: Schrödinger equation:

$$i \frac{d}{dt} \Psi(q, t) = \sum_{j=1}^n -\frac{1}{2} \frac{\partial^2}{\partial q_j^2} \Psi(q, t) + V(q, t) \Psi(q, t)$$

Recall: It is to be solved for all  $q$

QFT: Schrödinger equation:

$$i \frac{d}{dt} \Psi[\phi, t] = \int_{\mathbb{R}^3} \left( -\frac{1}{2} \frac{\delta^2}{\delta \phi(x)} + \frac{1}{2} \phi(x) (m^2 - \Delta) \phi(x) + W(\phi(x), t) \right) \Psi[\phi, t]$$

Recall: It is to be solved for all  $\phi$

Remark: With  $W$  it can be solved only perturbatively.

Exercise: Set  $W=0$ . Fourier transform to k variables in box

QM: Schrödinger equation:

$$i \frac{d}{dt} \Psi(q, t) = \sum_{j=1}^n -\frac{1}{2} \frac{\partial^2}{\partial q_j^2} \Psi(q, t) + V(q, t) \Psi(q, t)$$

Recall: It is to be solved for all  $q$

QFT: Schrödinger equation:

$$i \frac{d}{dt} \Psi[\phi, t] = \int_{\mathbb{R}^3} \left( -\frac{1}{2} \frac{\delta^2}{\delta \phi(x)} + \frac{1}{2} \phi(x) (m^2 - \Delta) \phi(x) + W(\phi(x), t) dx^3 \right) \Psi[\phi, t]$$

Recall: It is to be solved for all  $\phi$



Remark: With  $W$  it can be solved only perturbatively.

Exercise: Set  $W=0$ . Fourier transform to  $k$  variables in box

$$i \frac{d}{dt} \Psi[\phi, t] = \int_{\mathbb{R}^3} \left( -\frac{1}{2} \frac{\delta^2}{\delta \phi(x)} + \frac{1}{2} \phi(x) (m^2 - \Delta) \phi(x) + W(\phi(x), t) \right) \Psi[\phi, t]$$

Recall: It is to be solved for all  $\phi$

**Remark:** With  $W$  it can be solved only perturbatively.

**Exercise:** Set  $W=0$ . Fourier transform to  $\mathbf{k}$  variables in box regularization. Verify that the wave functional  $\Psi_0$  of the vacuum state obtained before obeys the Schr. eqn.



**Example application 2:** The functional Legendre transform!

## Example application 2: The functional Legendre transform!

□ Motivation? We will need to determine in curved space:

What becomes of:  $\hat{\pi}(x,t) = \dot{\phi}(x,t)$ ?

□ Problem? Time is preferred coordinate in Hamiltonian formalism.



\* But the formalism must be coordinate system independent to fit general relativity (GR).

\* Now, for example,  $\hat{\pi}(x,t) = \frac{d}{dt} \dot{\phi}(x,t)$  is not

the same as  $\hat{\pi}(x,\tau) = \frac{d}{d\tau} \dot{\phi}(x,\tau)$  for arbitrary  $\tau(t)$ :

! !

Strategy:

1. Transform to coordinate-independent Lagrange formalism.
2. Move from special to general relativity (GR).
3. Transform GR result back to Hamilton formalism.
4. Apply and quantization.

SR, 1<sup>st</sup>Q  
Hamiltonian  
formalism

"Legendre transform"  
equivalence

SR, 1<sup>st</sup>Q.  
Lagrangian  
formalism

GR, 1<sup>st</sup>Q  
Hamiltonian  
formalism

as outlined  
already

→ n → n

Legendre transform  
equivalence

GR, 1<sup>st</sup>Q  
Lagrangian  
formalism

↓  
↓

allow  
curvature

3. Transform GR result back to Hamilton formalism.
4. Apply 2nd quantization.

SR, 1<sup>st</sup> Q  
Hamiltonian  
formalism

"Legendre transform"  
equivalence

SR, 1<sup>st</sup> Q.  
Lagrangian  
formalism

GR, 1<sup>st</sup> Q  
Hamiltonian  
formalism

Legendre transform  
equivalence

GR, 1<sup>st</sup> Q  
Lagrangian  
formalism

GR, 2<sup>nd</sup> Q  
Hamiltonian  
formalism

Dyson Schwingr eqns are same  
equivalence

GR, 2<sup>nd</sup> Q  
Lagrangian formalism  
(Path integral of QFT)

## 4. Apply and quantization.

SR, 1<sup>st</sup>Q  
Hamiltonian  
formalism

"Legendre transform"  
equivalence

SR, 1<sup>st</sup>Q.  
Lagrangian  
formalism

GR, 1<sup>st</sup>Q  
Hamiltonian  
formalism

Legendre transform  
equivalence

GR, 1<sup>st</sup>Q  
Lagrangian  
formalism

GR, 2<sup>nd</sup>Q  
Hamiltonian  
formalism

as outlined  
already

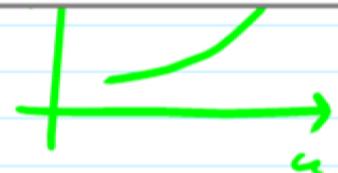
Dyson Schuring eqns are same  
equivalence

GR, 2<sup>nd</sup>Q  
Lagrangian formalism  
(Path integral of QFT)

allow  
curvature

The Legendre transform (LT).

◻ Assume given a function,  $\bar{F}(u)$ .



◻ Define a new variable  $w(u)$ :

$$w(u) := \frac{d\bar{F}}{du} \quad (\text{I})$$

◻ Assume that (I) can be solved to obtain:

$$u(w)$$

(that's ok if  $\bar{F}$  is convex, say  $\bar{F}''(u) > 0$  for all  $u$ )

◻ The Legendre transform of  $\bar{F}$  is a new function,  $G$ , of  $w$ :

$$\bar{F}(u) \xrightarrow{\text{LT}} G(w)$$

◻ Namely:

$$G(w) := w u(w) - \bar{F}(u(w))$$

$u(w)$

(that's ok if  $F$  is convex, say  $F''(u) > 0$  for all  $u$ )

□ The Legendre transform of  $F$  is a new function,  $\mathfrak{f}$ , of  $w$ :

$$F(u) \xrightarrow{LT} G(w)$$

□ Namely:  $G(w) := w u(w) - F(u(w))$

## Proposition:

$$(LT)^2 = id$$

Proof:

Defining a vector field  $v(\omega) := \partial G(\omega)$

Proposition:

$$(LT)^2 = id$$

Proof:

□ Define a new variable:  $v(w) := \frac{\partial G(w)}{\partial w}$

□ In fact:

$$\begin{aligned} v(w) &= \frac{\partial}{\partial w} (w u(w) - F(u(w))) \\ &= u(w) + w \cancel{\frac{\partial u(w)}{\partial w}} - \cancel{\frac{\partial F(u(w))}{\partial u}} \cancel{\frac{\partial u(w)}{\partial w}} \\ &= u ! \end{aligned}$$

□ Therefore  $LT^2$  yields  $F(u) \xrightarrow{LT} G(w) \xrightarrow{LT} H(v)$  with:

Proposition:

$$(LT)^2 = id$$



Proof:

□ Define a new variable:  $v(w) := \frac{\partial G(w)}{\partial w}$

□ In fact:

$$\begin{aligned} v(w) &= \frac{\partial}{\partial w} (w u(w) - F(u(w))) \\ &= u(w) + w \cancel{\frac{\partial u(w)}{\partial w}} - \underbrace{\frac{\partial F(u(w))}{\partial u} \frac{\partial u(w)}{\partial w}}_{\text{II}} \\ &= u ! \end{aligned}$$

Proposition:

$$(LT)^2 = id$$



Proof:

□ Define a new variable:  $v(w) := \frac{\partial G(w)}{\partial w}$

□ In fact:

$$\begin{aligned} v(w) &= \frac{\partial}{\partial w} (w u(w) - F(u(w))) \\ &= u(w) + w \cancel{\frac{\partial u(w)}{\partial w}} - \underbrace{\frac{\partial F(u(w))}{\partial u} \frac{\partial u(w)}{\partial w}}_{\text{II}} \\ &= u ! \end{aligned}$$

## □ In fact:

$$\begin{aligned}
 v(w) &= \frac{\partial}{\partial w} (w u(w) - F(u(w))) \\
 &= u(w) + w \frac{\partial u(w)}{\partial w} - \underbrace{\frac{\partial F(u(w))}{\partial u}}_{\substack{\parallel \\ w}} \frac{\partial u(w)}{\partial w} \\
 &= u !
 \end{aligned}$$

□ Therefore  $LT^2$  yields  $F(u) \xrightarrow{LT} G(w) \xrightarrow{LT} H(v)$  with:

$$H = vw - G = vw - (wu - f) = f$$

$\uparrow$   
u from just above

### Example:

\* Consider  $f(a,b,c) := a e^{bc}$

Example:

\* Consider  $f(a, b, c) := a e^{bc}$

\* Find LT with respect to  $b$  (i.e. while treating  $a, c$  as "spectator variables":

$$f(a, b, c) \xrightarrow[b \rightarrow \beta]{LT} g(a, \beta, c)$$

\* Define  $\beta(a, b, c) := \frac{\partial f}{\partial b} = ace^{bc}$

\* Invert:  $b(a, \beta, c) = \frac{1}{c} \ln \frac{\beta}{ac}$

\* Legendre transform:  $f(a, b, c) \xrightarrow{LT} g(a, \beta, c)$

$$g(a, \beta, c) := \beta b(a, \beta, c) - f(a, b(a, \beta, c), c)$$

treating  $a, c$  as "spectator variables":

$$f(a, b, c) \xrightarrow[b \rightarrow \beta]{LT} g(a, \beta, c)$$

\* Define  $\beta(a, b, c) := \frac{\partial f}{\partial b} = ace^{bc}$

\* Invert:  $b(a, \beta, c) = \frac{1}{c} \ln \frac{\beta}{ac}$

\* Legendre transform:  $f(a, b, c) \xrightarrow{LT} g(a, \beta, c)$

$$g(a, \beta, c) := \beta b(a, \beta, c) - f(a, b(a, \beta, c), c)$$

$$g(a, \beta, c) = \frac{\beta}{c} \ln \frac{\beta}{ac} - ae^{\frac{c}{c} \ln \frac{\beta}{ac}} = \frac{\beta}{c} \ln \frac{\beta}{ac} - \frac{\beta}{c}$$

## Case of countably many variables:

I How to define

$$F(\{u_i\}) \xrightarrow{\text{LT}} G(\{w_j\}) \in \mathbb{R}$$

II Define:  $w_j := \frac{\partial F}{\partial u_j}$

III Assume we can invert to obtain:  $u_i(\{w_j\})$

IV Define:

$$u; (\{w_i\})$$

□ Define:

$$G(\{w_i\}) := \sum_i w_i u_i(\{w_i\}) - F(\{u_i(\{w_i\})\})$$

(we may also allow for spectator variables)



Case of uncountably many variables:

□ How to define

$$F[\{u(x)\}_{x \in \mathbb{R}^n}] \xrightarrow{LT} G[\{w(x)\}_{x \in \mathbb{R}^n}] ?$$

Case of uncountably many variables:

□ How to define

$$\mathcal{F}\left[\{u(x)\}_{x \in \mathbb{R}^n}\right] \xrightarrow{LT} \mathcal{G}\left[\{w(x)\}_{x \in \mathbb{R}^n}\right] ?$$

□ Define:

$$w(x) := \frac{\delta \mathcal{F}}{\delta u(x)}$$



□ Assume we can solve to obtain:

$$u(x, \{w(x')\}_{x' \in \mathbb{R}^n})$$

□ Define:

$$\mathcal{G}\left[\{w(x)\}_{x \in \mathbb{R}^n}\right] := \int_{\mathbb{R}^n} w(x) u(x, \{w(x')\}_{x' \in \mathbb{R}^n}) dx - \overline{\mathcal{F}}\left[\{u(x, \{w(x')\}_{x' \in \mathbb{R}^n})\}\right]$$

## Application to CM:

\* Assume the Hamiltonian  $H(q, p)$  is given.

\* Hamilton equations for arbitrary  $f(q, p)$ :

$$\dot{f}(q, p) = \{ f(q, p), H(q, p) \}$$

Recall: Poisson bracket  
 $\{q, p\} = 1$

See my notes to AMATH673:

Dirac showed: Quantization consists in keeping the Poisson bracket definition and the Hamilton equations unchanged while allowing  $q, p$  noncommutativity in such a way that the Poisson algebra structure stays. This fixes noncommutativity to be  $\hat{q}\hat{p} - \hat{p}\hat{q} = i\hbar$  and  $\{\hat{f}_1, \hat{f}_2\} = \frac{i\hbar}{\ell^2} \{f_1, f_2\}$

\* From this, one can prove the eqns of motion for  $q, p$ :

$$\dot{q} = \frac{\partial H(q, p)}{\partial p}, \quad \dot{p} = -\frac{\partial H(q, p)}{\partial q} \quad (\text{EoM})$$

\* Legendre transform:

The "Lagrangian"

## Application to CM:

\* Assume the Hamiltonian  $H(q, p)$  is given.

\* Hamilton equations for arbitrary  $f(q, p)$ :

$$\dot{f}(q, p) = \{f(q, p), H(q, p)\}$$

Recall: Poisson bracket  
 $\{q_i, p_j\} = i\hbar$

See my notes to AMATH673:

Dirac showed: Quantization consists in keeping the Poisson bracket definition and the Hamilton equations unchanged while allowing  $q, p$  noncommutativity in such a way that the Poisson algebra structure stays. This fixes noncommutativity to be  $\hat{q}\hat{p} - \hat{p}\hat{q} = i\hbar$  and  $\{\hat{q}_i, \hat{q}_j\} = \frac{i}{\hbar} \epsilon_{ijk} \hat{p}_k$

\* From this, one can prove the eqns of motion for  $q, p$ :

$$\dot{q} = \frac{\partial H(q, p)}{\partial p}, \quad \dot{p} = -\frac{\partial H(q, p)}{\partial q} \quad (\text{EoM})$$

\* Legendre transform:

The "Lagrangian"

$$q = \frac{\partial H(q,p)}{\partial p}, \quad p = -\frac{\partial H(q,p)}{\partial q} \quad (\text{EoM})$$

\* Legendre transform:

$$H(q,p) \xrightarrow{LT} L(q,b) \quad (q \text{ is spectator})$$

The "Lagrangian"

\* Example:  $H(q,p) := \frac{p^2}{2} + V(q)$

$$b := \frac{\partial H(q,p)}{\partial p} \stackrel{\text{EoM}}{=} \dot{q} \quad \leftarrow \text{Notice: This arose due to } \frac{p^2}{2} \text{ term.}$$

$$\Rightarrow L(q,b) = L(q,\dot{q}) = \dot{q} p(q,\dot{q}) - H(q, p(q,\dot{q}))$$

Proposition:

The equations of motion (EoM) now take the form:

\* Example:  $H(q, p) := \frac{p^2}{2} + V(q)$

$$b := \frac{\partial H(q, p)}{\partial p} \stackrel{L.T.}{=} \dot{q} \quad \leftarrow \text{Notice: This arose due to } \frac{p^2}{2} \text{ term.}$$

$$\Rightarrow L(q, b) = L(q, \dot{q}) = \dot{q} p(q, \dot{q}) - H(q, p(q, \dot{q}))$$

### Proposition:

The equations of motion (EoM) now take the form:

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \quad (\text{Euler-Lagrange equation})$$

### Proof: Exercise

Example:  $H = \frac{p^2}{2m} + \frac{\omega^2}{2} q^2 \longleftrightarrow L[q, b] = \frac{1}{2} \dot{q}^2 - \frac{\omega^2}{2} q^2$

Example:

$$H = \frac{p^2}{2m} + \frac{\omega^2}{2} q^2 \quad \xleftarrow{LT} \quad L[q, b] = \frac{1}{2} \dot{q}^2 - \frac{\omega^2}{2} q^2$$

$$\dot{q} = p, \dot{p} = -\omega^2 q$$

$$-\omega^2 q = \ddot{q}, \quad b = \dot{q}$$

✓ classical (not conformal) field theory

## Application to CFT:

□ Assume Hamiltonian  $H(\phi, \pi)$  is given.

□ Hamilton equation for arbitrary  $f(\phi, \pi)$ :

$$\dot{f}(\phi, \pi, x, t) = \{f(t, \pi, x, t), H(\phi, \pi)\}$$

with:  $\{\phi(x, t), \pi(x', t)\} = \delta^3(x - x')$

✓ classical (not conformal) field theory

## Application to CFT:

- Assume Hamiltonian  $H(\phi, \pi)$  is given.
- Hamilton equation for arbitrary  $f(\phi, \pi)$ :

$$\dot{f}(\phi, \pi, x, t) = \{f(t, \bar{\pi}, x, t), H(\phi, \pi)\}$$

with:  $\{ \phi(x, t), \pi(x', t) \} = \delta^3(x - x')$

- This yields the eqns of motion:

$$\dot{\phi}(x, t) = \frac{\delta H}{\delta \pi(x, t)} \quad \dot{\pi}(x, t) = - \frac{\delta H}{\delta \phi(x, t)} \quad (EoM)$$

- $\rightarrow$   $\text{Lagr.} \rightarrow \text{Tens. el. ...}$

□ Assume Hamiltonian  $H(\phi, \pi)$  is given.

□ Hamilton equation for arbitrary  $f(\phi, \pi)$ :

$$\dot{f}(\phi, \pi, x, t) = \{f(t, \pi, x, t), H(\phi, \pi)\}$$

with:  $\{\phi(x, t), \pi(x', t)\} = \delta^3(x - x')$

□ This yields the eqns of motion:

$$\dot{\phi}(x, t) = \frac{\delta H}{\delta \pi(x, t)} \quad \dot{\pi}(x, t) = - \frac{\delta H}{\delta \phi(x, t)} \quad (\text{EoM})$$

□ Legendre Transform:

$$H(\phi, \pi) \xrightarrow{\text{LT}} L(\phi, \dot{\phi})$$

↓  
 spectator

□ Example:  $H := \int \frac{1}{2} \pi(x, t)^2 + V(\phi(x)) d^3x$

$$\begin{aligned} S(x, t) &:= \frac{\delta H}{\delta \pi(x, t)} \\ &= \dot{\phi}(x, t) \end{aligned}$$

Thus:

← Notice: this is because of  
the particular  $\pi^2$  term in  $H$ .  
On curved space it will be  
different.

$$L(\phi, \pi) = L(\phi, \dot{\phi})$$

$$= \int_{\mathbb{R}^3} \dot{\phi}(x, t) \pi(\phi, \dot{\phi}, x, t) d^3x - H(\phi, \pi(\phi, \dot{\phi}, x, t))$$

Proposition: The eqns of motion (EoM) are equivalent to:

□ Example:  $H := \int \frac{1}{2} \pi(x,t)^2 + V(\phi(x)) d^3x$

$$S(x,t) := \frac{\delta H}{\delta \pi(x,t)}$$

$$= \dot{\phi}(x,t)$$

← Notice: this is because of  
the particular  $\pi^2$  term in  $H$ .  
On curved space it will be  
different.

Thus:

$$L(\phi, \pi) = L(\phi, \dot{\phi})$$



$$= \int_{\mathbb{R}^3} \dot{\phi}(x,t) \pi(\phi, \dot{\phi}, x, t) d^3x - H(\phi, \pi(\phi, \dot{\phi}, x, t))$$

Proposition: The eqns of motion (EoM) are equivalent to:

$$SL - d SL$$

Exercise: Check

Proposition: The eqns of motion (EoM) are equivalent to:

$$\frac{\delta L}{\delta \phi(x,t)} = \frac{d}{dt} \frac{\delta L}{\delta \dot{\phi}(x,t)}$$

Exercise: Check

Euler Lagrange eqn.

Example:

$$H(\phi, \pi) = \int_{\mathbb{R}^3} \frac{\pi^2(x,t)}{2} + \frac{1}{2} \phi(x,t) (m^2 - \Delta) \phi(x,t) d^3x$$

yields:  $\dot{\phi}(x,t) = \pi(x,t)$        $\ddot{\pi}(x,t) = (-m^2 + \Delta) \phi(x,t)$

i.e.:  $\ddot{\phi} - \Delta \phi + m^2 \phi = 0$       K.G. eqn.

Example:

$$H(\phi, \pi) = \int_{\mathbb{R}^3} \frac{\pi^2(x,t)}{2} + \frac{1}{2} \phi(x,t)(m^2 - \Delta) \phi(x,t) d^3x$$

yields:  $\dot{\phi}(x,t) = \pi(x,t)$        $\ddot{\pi}(x,t) = (-m^2 + \Delta) \phi(x,t)$

i.e.:  $\ddot{\phi} - \Delta \phi + m^2 \phi = 0$       K.G. eqn.

After Legendre transform:

$$L(\phi, \dot{\phi}) = \int_{\mathbb{R}^3} \frac{\dot{\phi}^2(x,t)}{2} - \frac{1}{2} \phi(x,t)(m^2 - \Delta) \phi(x,t) d^3x$$

yields directly:  $-(m^2 - \Delta) \phi = \ddot{\phi}$



$$e^{iB(\omega)} = \int_{\mathbb{R}} e^{i\omega t} e^{-iA(t)} dt$$

largest contrib:  $\frac{d}{dt}(i\omega t - iA(t)) = 0$

$$B(\omega) \approx i\omega t - A(t) \Big|_{\omega = \frac{dA}{dt}}$$

for  $\omega = \frac{dA(t)}{dt}$