

Title: Symplectic duality and a presentation of the cohomology of Nakajima quiver varieties

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Abstract: <p>We will discuss a (conjectural) explicit presentation for the equivariant cohomology of Nakajima quiver varieties of type ADE. This presentation arises as a shadow of the expected symplectic duality between slices to Schubert varieties in the affine Grassmannian and Nakajima quiver varieties (a.k.a. the expected Coulomb and Higgs branches for a quiver gauge theory). </p>

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Symplectic duality and  
a presentation for the  
(equivariant) coho of Nakajima  
quiver varieties

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a presentation for the  
(equivariant) coho. of Nakajima  
quiver varieties

Joint with J. Kamnitzer  
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B. Webster  
O. Yacobi.

Motivation:

3d  $N=4$  SUSY

- (complex) gauge group  $G$
- representation  $N$  of  $G$
- $\mu: T^*N \rightarrow \text{Lie}(G)^*$

SUSY

gauge group  $g$

on  $N$  of  $g$

$\rightarrow \text{Lie}(g)^*$

Coulomb

$\mathcal{M}_C$

$\mathbb{C}^x \xrightarrow{\Theta} \pi_1(g)^\wedge$   
Hamiltonian

choice of  
def. quantization

$A_{\hbar}^*$  of  $\mathbb{C}[\mathcal{M}_C]$

Higgs

$\mathcal{M}_H = \mu^{-1}(0) // g$

GIT  $\tilde{\mathcal{M}}_H = \mu^{-1}(0) / g$

choice of  $\mathbb{C}^x \hookrightarrow \tilde{\mathcal{M}}_H$

Higgs

$$\mathcal{M}_H = \mu^{-1}(0) // G$$

GIT  $\tilde{\mathcal{M}}_H = \mu^{-1}(0) / \theta // G$

choice of  $\mathbb{C}^x \hookrightarrow \tilde{\mathcal{M}}_H$

Conjectures:

Hikita:  $\mathbb{C}[\mathcal{M}_H^x] \cong H^*(\tilde{\mathcal{M}}_H)$

Nakajima:

$$B(A_H) \cong H_{\mathbb{C}^x}^*(\tilde{\mathcal{M}}_H)$$

---

$$A = \bigoplus_{k \in \mathbb{Z}} A_k \quad \text{graded algebra}$$

$$B(A) := A_0 /$$

Higgs

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$$A = \bigoplus_{k \in \mathbb{Z}} A_k \quad \text{graded algebra}$$

$$B(A) := A_0 / \sum_{k > 0} A_{-k} A_k$$

Higgs

$$\mathcal{M}_H = \mu^{-1}(0) // G$$

GIT  $\tilde{\mathcal{M}}_H = \mu^{-1}(0) / \theta // G$

choice of  $\mathbb{C}^x \hookrightarrow \tilde{\mathcal{M}}_H$

Conjectures:

Hikita:  $\mathbb{C}[\mathcal{M}_H^*] \cong H^*(\tilde{\mathcal{M}}_H)$

Nakajima:

$$B(A_H) \cong H_{\mathbb{C}^x}^*(\tilde{\mathcal{M}}_H) \text{ over } \mathbb{C}[\hbar]$$

---

$$A = \bigoplus_{k \in \mathbb{Z}} A_k \text{ graded algebra}$$

$$B(A) := A_0 / \sum_{k > 0} A_{-k} A_k$$

A) controls weight

$l(\alpha_j)$

y action

$(, -)$

$l(\beta_j)$

Goal: Try to understand this for quiver gauge theories of type ADE

Notation: • choose orientation of Dynkin

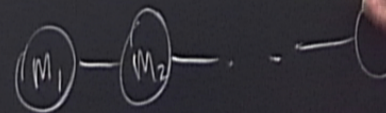
•  $G$  simple algebraic gp/ $\mathbb{C}$  of type ADE

•  $\lambda = \sum \lambda_i \alpha_i^\vee$  dominant coweight  
 $\mu$  coweight,  $\lambda - \mu = \sum m_i \alpha_i^\vee$   
 $m_i \geq 0$

eg.  $V(\lambda) \hookrightarrow G^\vee$   
 $\mu$  a wt space in  $V(\lambda)$ .

$$G = \prod_i GL(m_i)$$

$$N = \bigoplus_{i \rightarrow j} \text{Hom}(\mathbb{C}^{m_i}, \mathbb{C}^{m_j}) \oplus \bigoplus \text{Hom}(\mathbb{C}^{m_i}, \mathbb{C}^{m_i})$$





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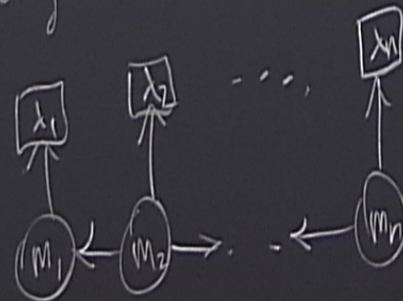
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 $m_i \geq 0$

eg.  $V(\lambda) \subset G^\vee$   
 $\mu$  a wt space in  $V(\lambda)$ .

$$G = \prod_i GL(m_i)$$

$$N = \bigoplus_{i \rightarrow j} \text{Hom}(\mathbb{C}^{m_i}, \mathbb{C}^{m_j}) \oplus \bigoplus \text{Hom}(\mathbb{C}^{m_i}, \mathbb{C}^{i'})$$



quasi-  
Thm (?)

$$\begin{pmatrix} t^k \\ t^{-k} \end{pmatrix}$$

$\tilde{M}_H =$  Nakajima quiver  
variety  $M(\lambda, \mu)$

$M_C =$  (generalized)  
slice to a Schubert  
variety in  $Gr_G$

$$Gr_{\mu}^{\lambda}$$

Affine Grassmannian

$$Gr_G = \frac{G(\mathbb{C}[[t]])}{G(\mathbb{C}[t])}$$

Schubert cells. orbits for  $G[[t]]$

$$Gr^{\lambda} = G[[t]] t^{\lambda}$$

$$\overline{Gr^{\lambda}} =$$

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$$Gr_M^\lambda$$

## Affine Grassmannian

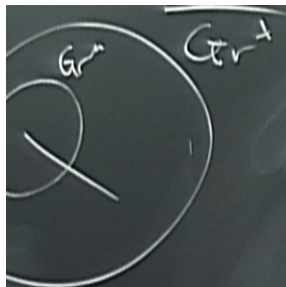
$$Gr_G = G(\mathbb{C}[[t]]) / G(\mathbb{C}[t])$$

Schubert cells. orbits for  $G[[t]]$

$$Gr^\lambda = G[[t]] t^\lambda$$

$$\overline{Gr^\lambda} = \bigcup_{\mu \leq \lambda} Gr^\mu$$

dominant



$$G_{\mu} = G_1[E^{-1}] \cdot t^{\mu}$$

$$G_1[E^{-1}] = \text{Ker}(G[E^{-1}] \xrightarrow{t \rightarrow \infty} G)$$

$$G^{\lambda} = \overline{G^{\lambda} \cap G_{\mu}}$$

$\lambda \geq \mu$  and both dominant.

$$G_1[E^{-1}] \rightarrow G_{\mu}$$

closed subvariety  $G_{\mu}$

$$\mathbb{C}[G_1[E^{-1}]] \supset \mathbb{C}[G_{\mu}]$$

$$\downarrow$$

$$\mathbb{C}[G^{\lambda}]$$

To quantize.

$Y = Y(\mathfrak{g})$  the  
Yangian for  $\mathfrak{g}$

$$Y \supset Y_{\mu}$$

$$\downarrow$$

$$Y_{\lambda}$$

Thm  $\mathbb{B}(Y_{\mu}^{\lambda})$

$$= \mathbb{C}[A_i^{(s)} : i \in \dots]$$

$$G_1[E^{-1}] \rightarrow G_{r,\mu}$$

closed subvariety  $G_{r,\mu}$

$$\mathbb{C}[G_1[E^{-1}]] \supset \mathbb{C}[G_{r,\mu}]$$

$$\downarrow$$

$$\mathbb{C}[G_{r,\mu}^*]$$

To quantize

$Y = Y(\mathfrak{g})$  the  
Yangian for  $\mathfrak{g}$

$$Y \supset Y_{r,\mu}$$

$$\downarrow$$

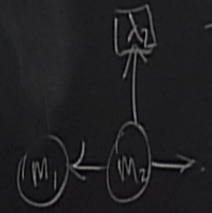
$$Y_{r,\mu}$$

Thm =  $B(Y_{r,\mu}^*)$

$$\cong \mathbb{C} \left[ \begin{matrix} \hbar \\ A_i^{(s)} : i \in I \\ \lambda \end{matrix} \right]$$

$\langle T_j^{(r)} \rangle$

for  $\mathfrak{g} = \mathfrak{sl}_r$ ,  $r > m_j$



$$\underline{\text{Thm}} \equiv B(Y_{\mu}^{\lambda})$$

$$\cong \mathbb{C} \left[ \begin{array}{c} \text{tr} \\ A_i^{(s)} \quad i \in I \\ s \geq 1 \end{array} \right]$$

$$\left\langle T_{\gamma}^{(r)} \right\rangle$$

$$\text{for } \gamma = W \otimes \omega_{\lambda}, r > m_{\gamma}$$

Aside:

Suppose  $X$  is a variety,  
gp.  $H \curvearrowright X$ .

want.  $H_H^*(X)$

$H$ -equiv

Idea: Find a collection of  $r$  v. bundles

on  $X$ ,  $\{U_{\gamma}\}$

so that  $\text{rank}(U_{\gamma}) = m_{\lambda}$

$$\underline{\text{Thm}} \equiv B(Y_{\mu}^{\lambda})$$

$$\cong \mathbb{C} \left[ \begin{matrix} A_i^{(s)} \\ \lambda \end{matrix} : \begin{matrix} i \in I \\ s \geq 1 \end{matrix} \right]$$

$$\left\langle T_{\lambda}^{(r)} \right\rangle$$

$$\text{for } \lambda = W_{\mu}^{\lambda}, r > m_{\lambda}$$

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Idea: Find a collection of  $r$  v. bundles

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Chern classes  $c_r(U_{\gamma}) = 0$  for  $r > m_{\lambda}$ .

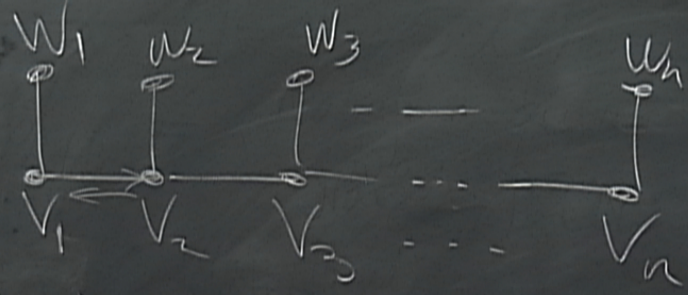
Geometric duality and

representation for the

(part) of Nakajima  
varieties

Kamnitzer  
Jacobi.

Nakajima  
Quiver variety:



each  $V_i$  and  $W_i$  gives a  
"tautological" bundle

$$\begin{array}{c} \mu^{-1}(0)/G \\ \downarrow \\ \mathcal{M}(t, \mu) \end{array}$$

total  $V_i$  bundle.  
 $\mathcal{U}_i, \mathcal{W}_i$

Affine G

$$\text{Gr}_G =$$

Schubert cells

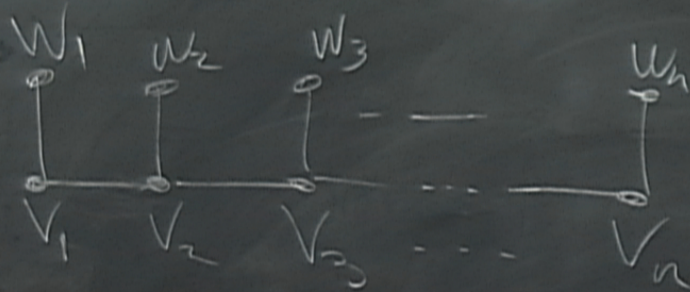
$$\text{Gr}^d =$$

$$\overline{\text{Gr}^d} =$$



Nakajima

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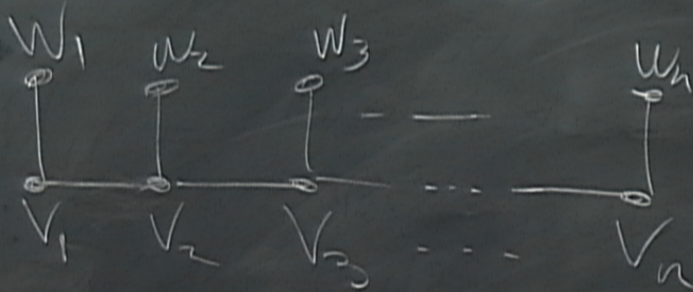
total  $V_i$  bundle  $\mathcal{U}_i, W_i$

ex:  $G = SL_2, \mathfrak{g} = \mathfrak{gl}(m)$



# Nakajima

## Quiver variety:



each  $V_i$  and  $W_i$  gives a "fantological" bundle

$$\begin{array}{c} \mu^{-1}(0) // G \\ \downarrow \\ \mathcal{M}(\lambda, \mu) \end{array}$$

$$U_i, W_i$$

trivial  $V_i$  bund

ex:  $G = SL_2, \mathfrak{g} = \mathfrak{gl}(m)$

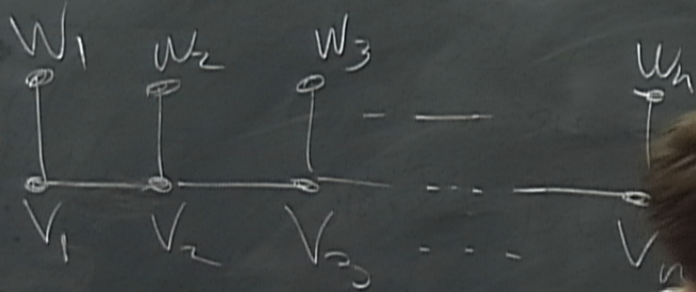
$$\mu^{-1}(0)$$

$$\mu^{-1}(0)$$

Def:

Nakajima

Quiver variety:



each  $V_i$  and  $W_i$  give  
"tautological" bundle

$$\begin{array}{c} \bar{\mu}^{-1}(0)/G \\ \downarrow \\ \mathcal{M}(\lambda, \mu) \end{array}$$

$\mathcal{U}_i, \mathcal{W}_i$

trivial  $V_i$

$$\lambda - \mu = 2m$$

ex:  $G = SL_2, \mathfrak{g} = \mathfrak{gl}(m)$

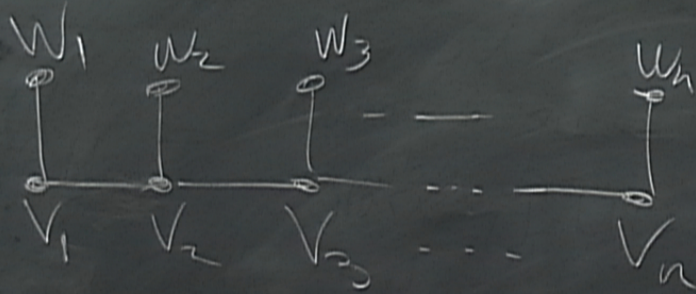


$$\bar{\mu}^{-1}(0) =$$

Def:  $\rightarrow$

# Nakajima

## Quiver variety:



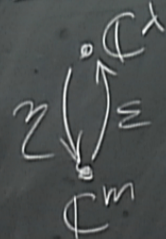
each  $V_i$  and  $W_i$  gives a "tautological" bundle

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total  $V_i$  bundle  $\rightarrow$   
 $\mathcal{U}_i, \mathcal{W}_i$

$$\lambda - \mu = 2m$$

ex:  $G = SL_2, \mathfrak{g} = \mathfrak{gl}(m)$



$$\mu^{-1}(0)^{ss} = \{ (\eta, \epsilon) = \eta \epsilon = 0 \}$$

$\epsilon \text{ inj.}$

$$\downarrow G$$

$$\mathcal{M}(\lambda, \mu)$$

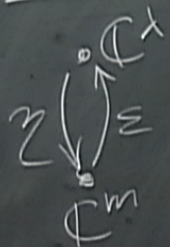
$$\begin{array}{c} \mathcal{U}, \mathcal{W} \\ \text{rk} = m \quad \text{rk} = \lambda \end{array}$$

$$\mathcal{U} \xrightarrow{\epsilon} \mathcal{W}$$

$$\begin{array}{c} \mathcal{W} / \epsilon(\mathcal{U}) \\ \text{rk} = \lambda - m \end{array}$$

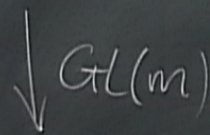
$$\lambda - \mu = 2m$$

ex:  $G = SL_2, \mathfrak{g} = \mathfrak{gl}(m)$



$$\bar{u}(0)^{ss} = \left\{ (\eta, \varepsilon) = \eta \varepsilon = 0 \right\}$$

$\varepsilon = i\eta$



$$\mathcal{M}(\lambda, \mu)$$

$$\mathcal{U}, \mathcal{W}$$

$$rk = m \quad rk = 1$$

$$\mathcal{U} \xrightarrow{\varepsilon} \mathcal{W}$$

$$\mathcal{W}/\varepsilon(\mathcal{U})$$

$$rk = \lambda - m$$

$$H^* \text{ex}^*(\mathcal{M}(\lambda, \mu))$$

$$= \mathbb{C}[A^{(s)} : s \geq 1]$$

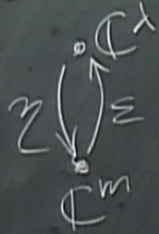
$$A_i^{(s)} = c_s(\mathcal{U})$$

$$\mathbb{C}[A^{(r)}]$$

$$\frac{1}{A^{(r)}}$$

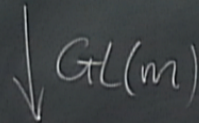
$$\lambda - \mu = 2m$$

ex:  $G = SL_2, \mathfrak{g} = \mathfrak{gl}(m)$



$$\bar{\mu}(0) = \{(\eta, \varepsilon) : \eta \varepsilon = 0\}$$

$\varepsilon \text{ inj.}$



$$\mathcal{U}(\lambda, \mu)$$

$$\mathcal{U}, \mathcal{W}$$

$$rk = m \quad rk = \lambda$$

$$\mathcal{U} \xrightarrow{\varepsilon} \mathcal{W}$$

$$c = \frac{1}{a} (w/\varepsilon(v)) =$$

$$w/\varepsilon(v)$$

$$rk = \lambda - m$$

$$= \sum_{r \geq 0} (-\frac{1}{a})^r c_r (w/v)$$

$$T^*(\mathfrak{gl}(m, \lambda))$$

$$H^* \text{ ex } (\mathcal{U}(\lambda, \mu))$$

$$= C[A^{(s)} : s \geq 1]$$

$$\langle A^{(r)}, r > m \rangle$$

$$A_i^{(s)} = c_s(v)$$

$$[u^t] A(u), t > \lambda - m$$

$$A(u) = \sum \bar{u}^r A^{(r)}$$