Title: Symplectic duality and a presentation of the cohomology of Nakajima quiver varieties

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Abstract: <p>We will discuss a (conjectural) explicit presentation for the equivariant cohomology of Nakajima quiver varieties of type ADE. This presentation arises as a shadow of the expected symplectic duality between slices to Schubert varieties in the affine Grassmannian and Nakajima quiver varieties (a.k.a. the expected Coulomb and Higgs branches for a quiver gauge theory). </p>

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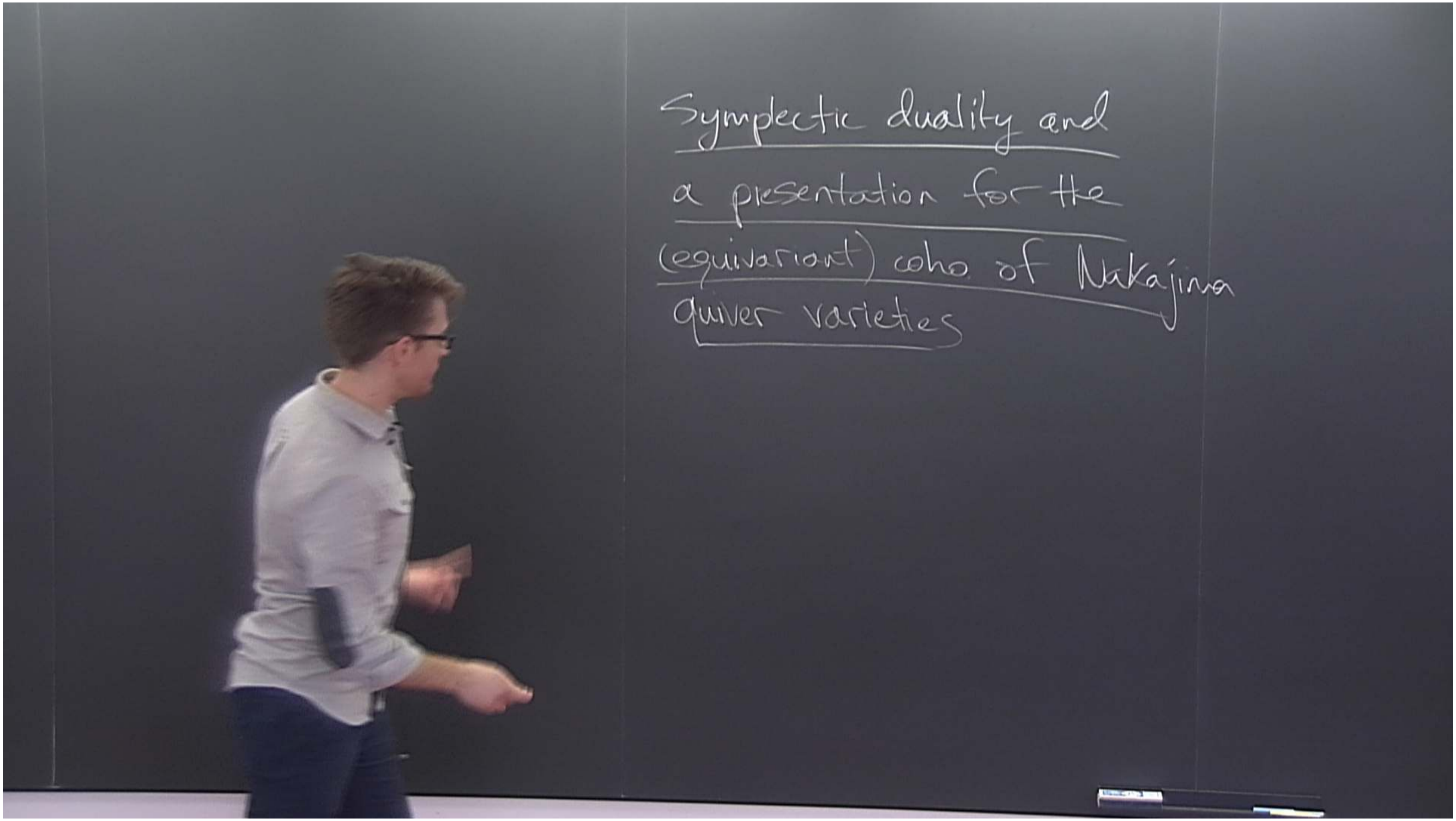
Symplectic duality and
a presentation for the
(equivariant) coho. of Nakajima
quiver varieties

Symplectic duality and a presentation for the (equivariant) coho. of Nakajima quiver varieties

Joint with J. Kamnitzer
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B. Webster
O. Yacobi.

Motivation:

3d $\mathcal{N}=4$ susy

- (complex) gauge group G
- representation N of G
- $\mu : T^*N \longrightarrow \mathrm{Lie}(G)^*$

SUSY

gauge group G

rep N of G

$\longrightarrow \mathrm{Lie}(G)^*$

Coulomb

$\mathcal{M}_C$ $\circlearrowleft$

$\mathbb{C}^* \xrightarrow{\theta} \pi_1(G)^\wedge$

Hamiltonian

choice of def. quantization

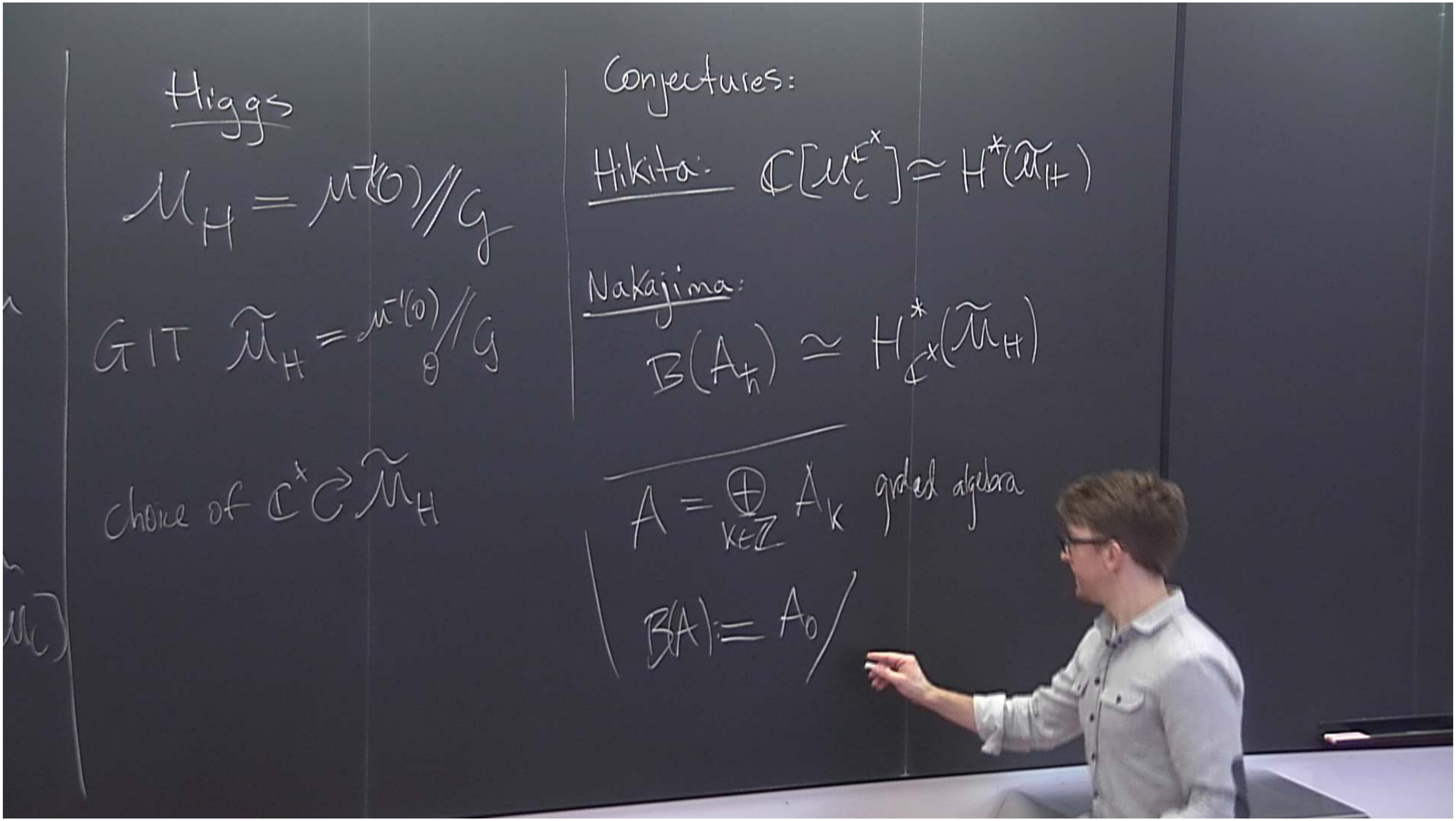
$A_\hbar$ of $\mathbb{C}[\mathcal{M}_C]$

Higgs

$\mathcal{M}_H = \mu^{-1}(0) /\!/ G$

GIT $\widetilde{\mathcal{M}}_H = \mu^{-1}(0) /\!/_{\theta} G$

choice of $\mathbb{C}^* \circlearrowright \widetilde{\mathcal{M}}_H$


Higgs
$\mathcal{M}_H = \mu^{-1}(0)/\!/G$
GIT $\widetilde{\mathcal{M}}_H = \mu^{-1}(0)/\!/_{\theta} G$
choice of $\mathbb{C}^\times \circlearrowright \widetilde{\mathcal{M}}_H$
Conjectures:
Hikita: $\mathbb{C}[\mathcal{M}_C^{\mathbb{C}^\times}] \simeq H^*(\widetilde{\mathcal{M}}_H)$
Nakajima:
$B(A_\hbar) \simeq H^*_{\mathbb{C}^\times}(\widetilde{\mathcal{M}}_H)$
$A = \bigoplus_{k\in\mathbb{Z}} A_k$ graded algebra
$B(A) := A_0/$

Higgs

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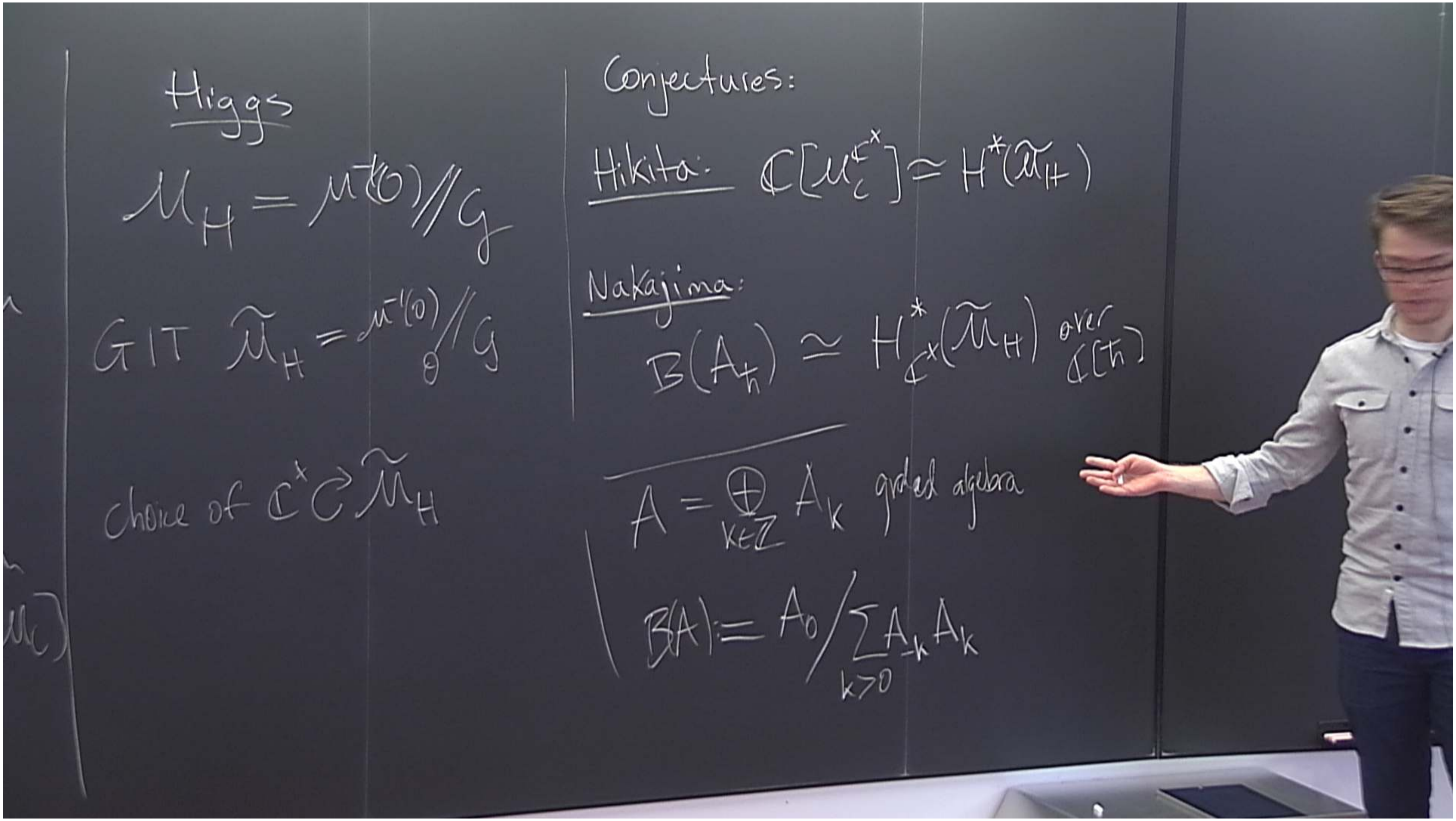
$A = \bigoplus_{k\in\mathbb{Z}} A_k$ graded algebra

$$B(A) := A_0 \Big/ \sum_{k>0} A_{-k} A_k$$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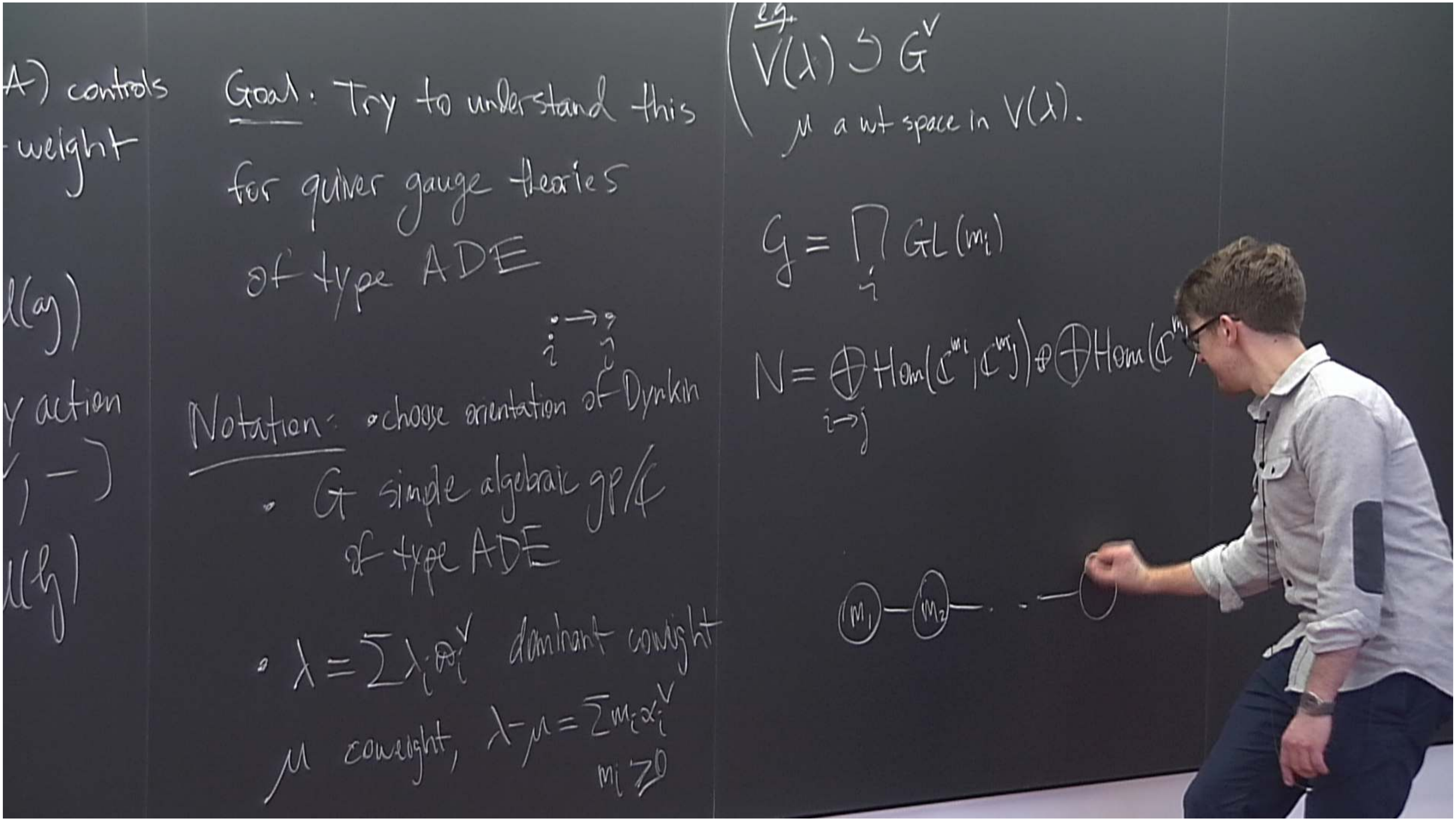
$B(A_\hbar) \simeq H^*_{\mathbb{C}^\times}(\widetilde{\mathcal{M}}_H)$ over $\mathbb{C}[\hbar]$
$A = \bigoplus_{k\in\mathbb{Z}} A_k$ graded algebra
$B(A) := A_0 / \sum_{k>0} A_{-k} A_k$


A) controls
weight
l(ay)
y action
, −]
l(h)
Goal: Try to understand this for quiver gauge theories of type ADE
$i \to j$
Notation: • choose orientation of Dynkin
• G simple algebraic gp/$\mathbb{C}$ of type ADE
• $\lambda = \sum \lambda_i \varpi_i^\vee$ dominant coweight
μ coweight, $\lambda - \mu = \sum m_i \alpha_i^\vee$, $m_i \geq 0$
e.g. $V(\lambda) \circlearrowleft G^\vee$
μ a wt space in $V(\lambda)$.
$G = \prod_i GL(m_i)$
$N = \bigoplus_{i \to j} Hom(\mathbb{C}^{m_i}, \mathbb{C}^{m_j}) \oplus \bigoplus Hom(\mathbb{C}^{m}$
m_1 — m_2 — . . . —

A) controls
weight

$\mathcal{U}(\mathfrak{g})$

y action
$(\ ,\ -)$

$\mathcal{U}(\mathfrak{h})$

<u>Goal</u>: Try to understand this for quiver gauge theories of type ADE

$\bullet \longrightarrow \bullet$
$i \qquad j$

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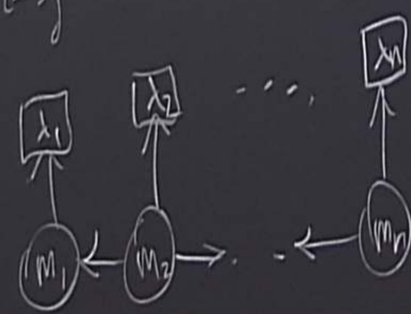

quasi-
Thm (?):

$$\begin{pmatrix} t^k & \\ & t^{-k} \end{pmatrix}$$

$\widetilde{\mathcal{M}}_H$ = Nakajima quiver variety $\mathcal{M}(\lambda,\mu)$

$\mathcal{M}_C$ = (generalized) slice to a Schubert variety in Gr_G

Gr^λ_μ

Affine Grassmannian

$$Gr_G = G(\mathbb{C}((t))) / G(\mathbb{C}[[t]])$$

Schubert cells: orbits for $G[[t]]$

$$Gr^\lambda = G[[t]]\, t^\lambda$$

$$\overline{Gr^\lambda} =$$

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$Gr^\lambda = G[[t]]\, t^\lambda$

$\overline{Gr^\lambda} = \bigcup_{\substack{\lambda \geq \mu \\ \text{dominant}}} Gr^\mu$

Gr^μ $\overline{\mathrm{Gr}^\lambda}$

$\mathrm{Gr}_\mu = G_1[t^{-1}]\, t^\mu$

$G_1[t^{-1}] = \mathrm{Ker}\left(G[t^{-1}] \xrightarrow{t=\infty} G\right)$

Def: $\mathrm{Gr}^\lambda_\mu = \overline{\mathrm{Gr}^\lambda} \cap \mathrm{Gr}_\mu$

$\lambda \geq \mu$ and both dominant.

$G_1[t^{-1}] \longrightarrow \mathrm{Gr}_\mu$

$\cup$

closed subvariety Gr^λ_μ

$\mathbb{C}[G_1[t^{-1}]] \supset \mathbb{C}[\mathrm{Gr}_\mu]$

$\downarrow$

$\mathbb{C}[\mathrm{Gr}^\lambda_\mu]$

To quantize:

$Y = Y(\mathfrak{g})$ the Yangian for $\mathfrak{g}$

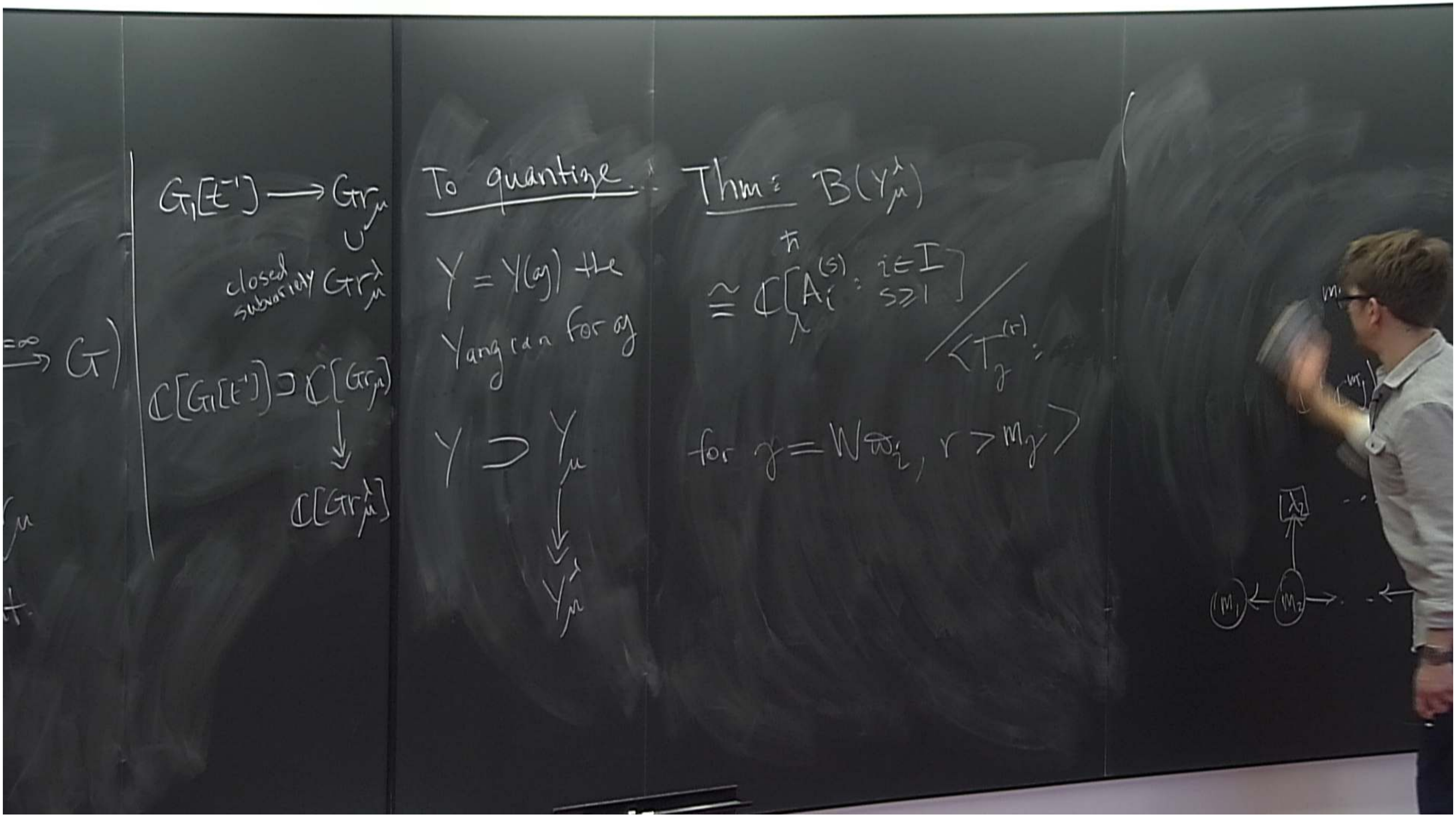
$Y \supset Y_\mu$

$\downarrow$

Y^λ_μ

Thm: $B(Y^\lambda_\mu)$

$= \mathbb{C}[A_i^{(s)} : i \in$


G1[t^-1] ⟶ Gr_μ
∪
closed subvariety Gr_μ^λ
⇒ G)
C[G1[t^-1]] ⊃ C[Gr_μ]
↓
C[Gr_μ^λ]
To quantize:
Y = Y(g) the Yangian for g
Y ⊃ Y_μ
↓
Y_μ^λ
Thm: B(Y_μ^λ)
≅ C_ℏ[A_i^(s) : i∈I, s≥1]_λ / ⟨T_γ^(r) :
for γ = Wϖ_i, r > m_γ⟩
m_1
m_2
λ_2

Thm: $B(Y^\lambda_\mu)$

$$\cong \mathbb{C}_\hbar[A_i^{(s)} : i \in I, s \geq 1] \big/ \langle T_\gamma^{(r)} : \text{for } \gamma = W\varpi_i,\ r > m_\gamma \rangle$$

Aside:
Suppose X is a variety,
gp. $H \circlearrowright X$.

Want: $H^*_H(X)$.

Idea: Find a collection of H-equiv v. bundles on X, $\{\mathcal{V}_\gamma\}$

so that $\operatorname{rank}(\mathcal{V}_\gamma) = m_\alpha$

Thm: $B_{\hbar}(Y^\lambda_\mu) \cong \mathbb{C}[A_i^{(s)} : i \in I,\ s \geq 1]_\lambda \Big/ \langle T_\gamma^{(r)} : \text{for } \gamma = W\varpi_i,\ r > m_\gamma \rangle$

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Suppose X is a variety, gp. $H \circlearrowright X$.

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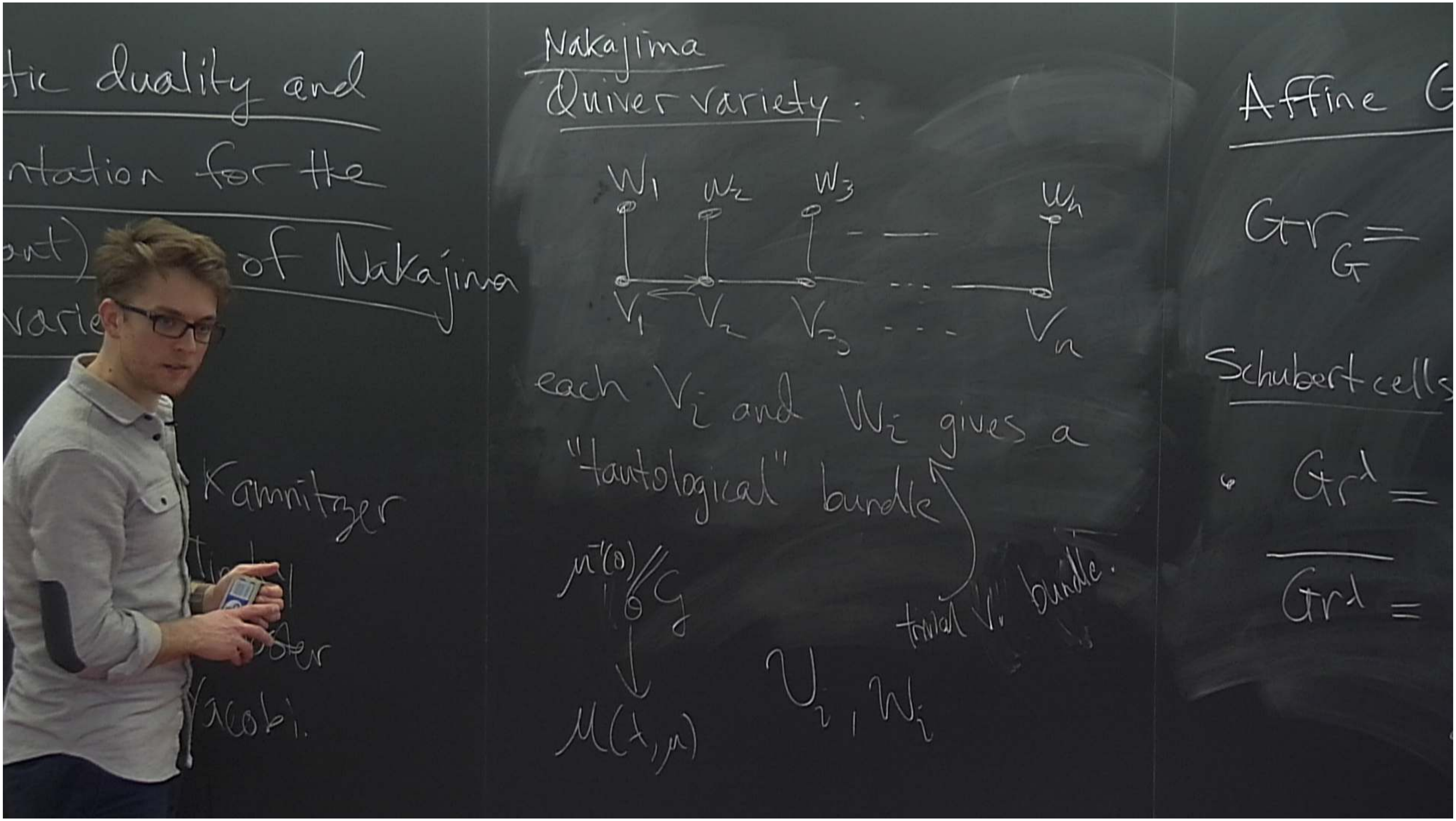
Chern classes $c_r(V_\gamma) = 0$ for $r > m_\gamma$.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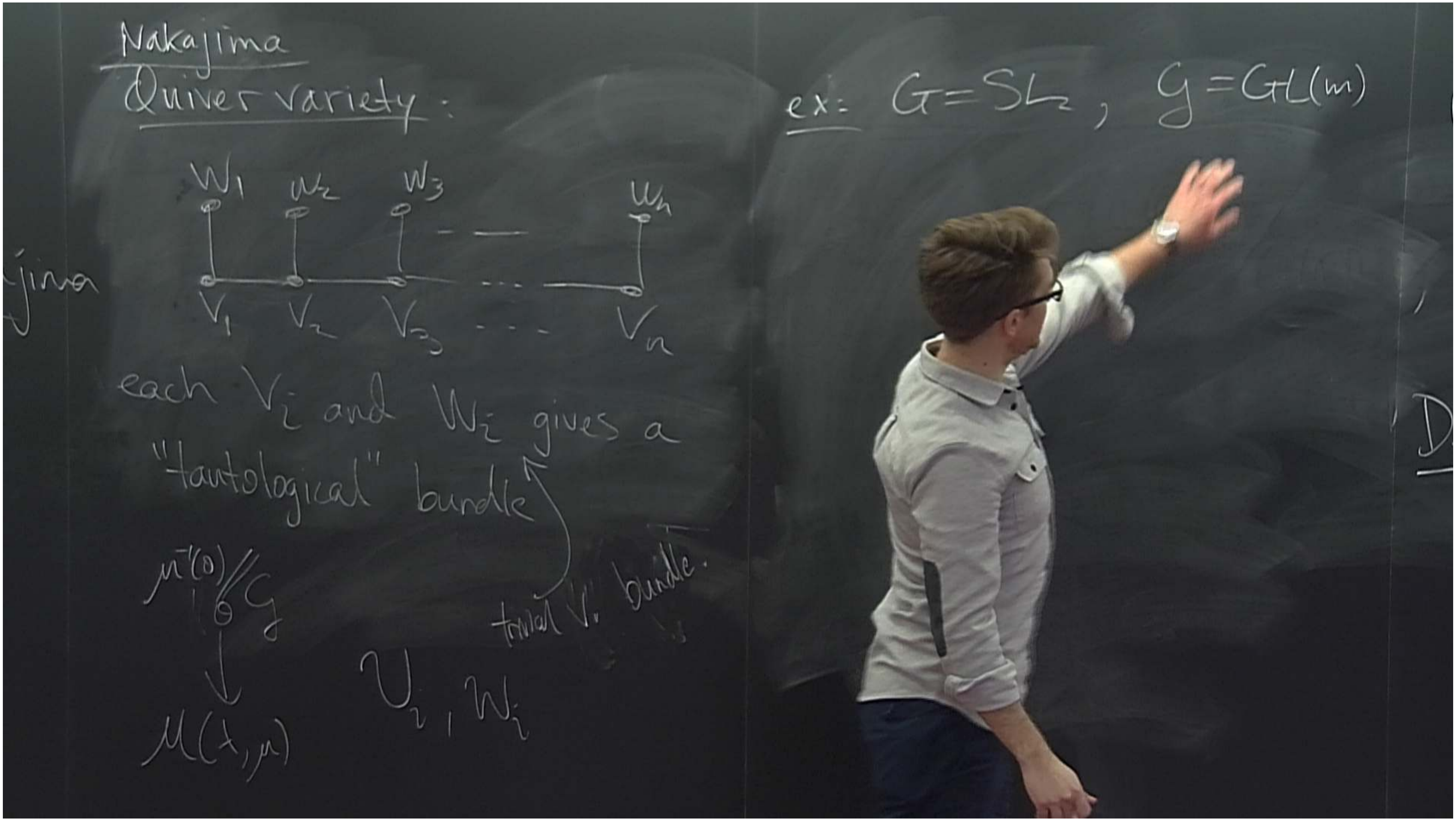
Kamnitzer
Jacobi.
Nakajima
Quiver variety:
W1 W2 W3 Wn
V1 V2 V3 Vn
each Vi and Wi gives a
"tautological" bundle
trivial Vi bundle.
μ⁻¹(0)//θG
M(λ,μ)
Vi, Wi
Affine G
GrG =
Schubert cells
Grλ =
Grλ =


Nakajima
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W1 W2 W3 Wn
V1 V2 V3 ... Vn
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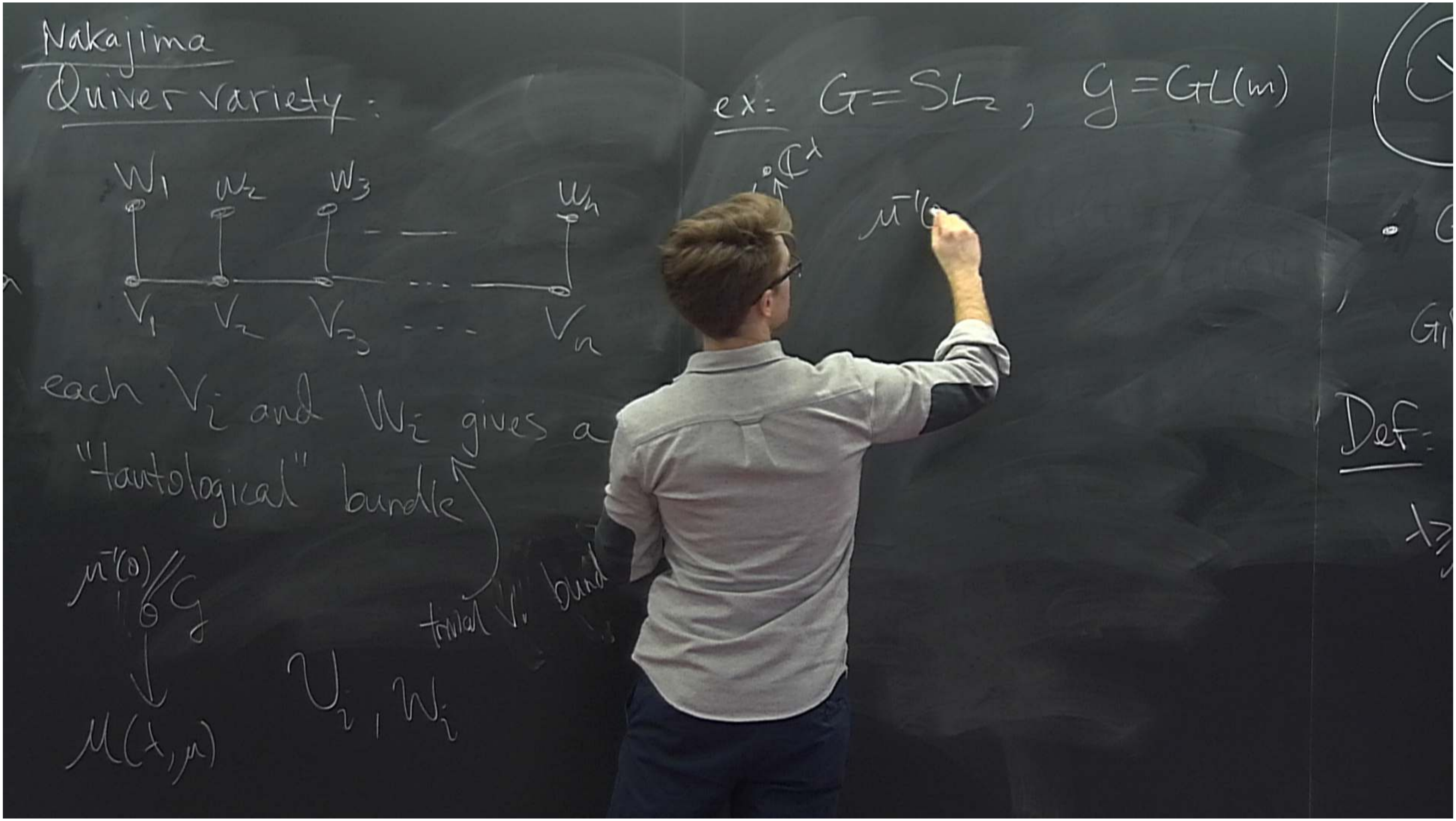
M(r,n)
Vi, Wi
ex: G=SL2, G=GL(m)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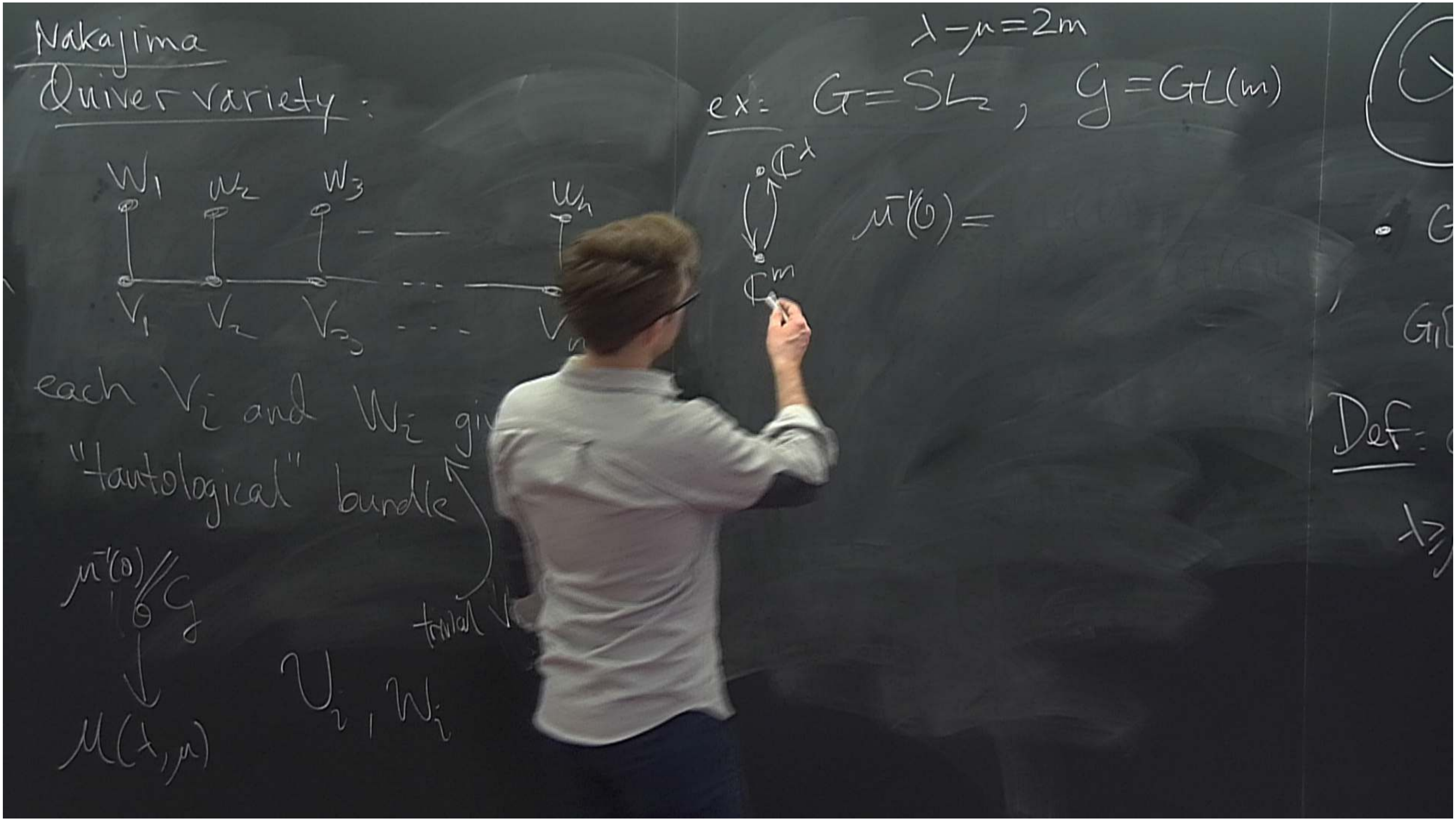
Def: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Quiver variety :
W1 W2 W3 Wn
V1 V2 V3 . . . Vn
each Vi and Wi gi
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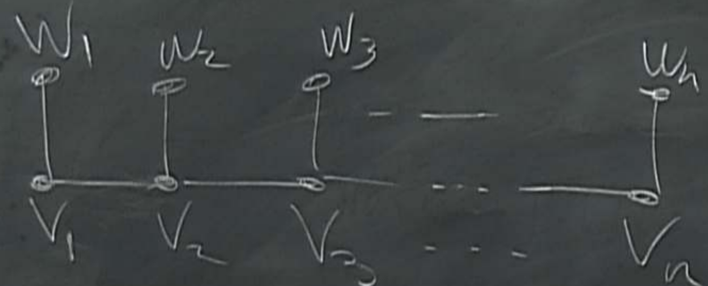
trivial
μ⁻¹(0)//G
M(λ,μ)
Vi , Wi
λ−μ=2m
ex: G=SL2 , G=GL(m)
C^λ
C^m
μ⁻¹(0) =
Def:

Nakajima

Quiver variety :



each V_i and W_i gives a "tautological" bundle

trivial V_i bundle.

$\mu^{-1}(0)/\!\!/_\theta G$

$\downarrow$

$\mathcal{M}(\lambda,\mu)$

$\mathcal{V}_i,\ \mathcal{W}_i$

$\lambda-\mu=2m$

ex: $G=SL_2$, $\mathcal{G}=GL(m)$

$\mathbb{C}^m \underset{\varepsilon}{\overset{\eta}{\rightleftarrows}} \mathbb{C}^\lambda$

$\mu^{-1}(0)^{ss} = \{(\eta,\varepsilon) : \eta\varepsilon = 0,\ \varepsilon \text{ inj.}\}$

$\downarrow GL(m)$

$\mathcal{M}(\lambda,\mu)$

$\mathcal{V},\ \mathcal{W}$

$rk=m \quad rk=\lambda$

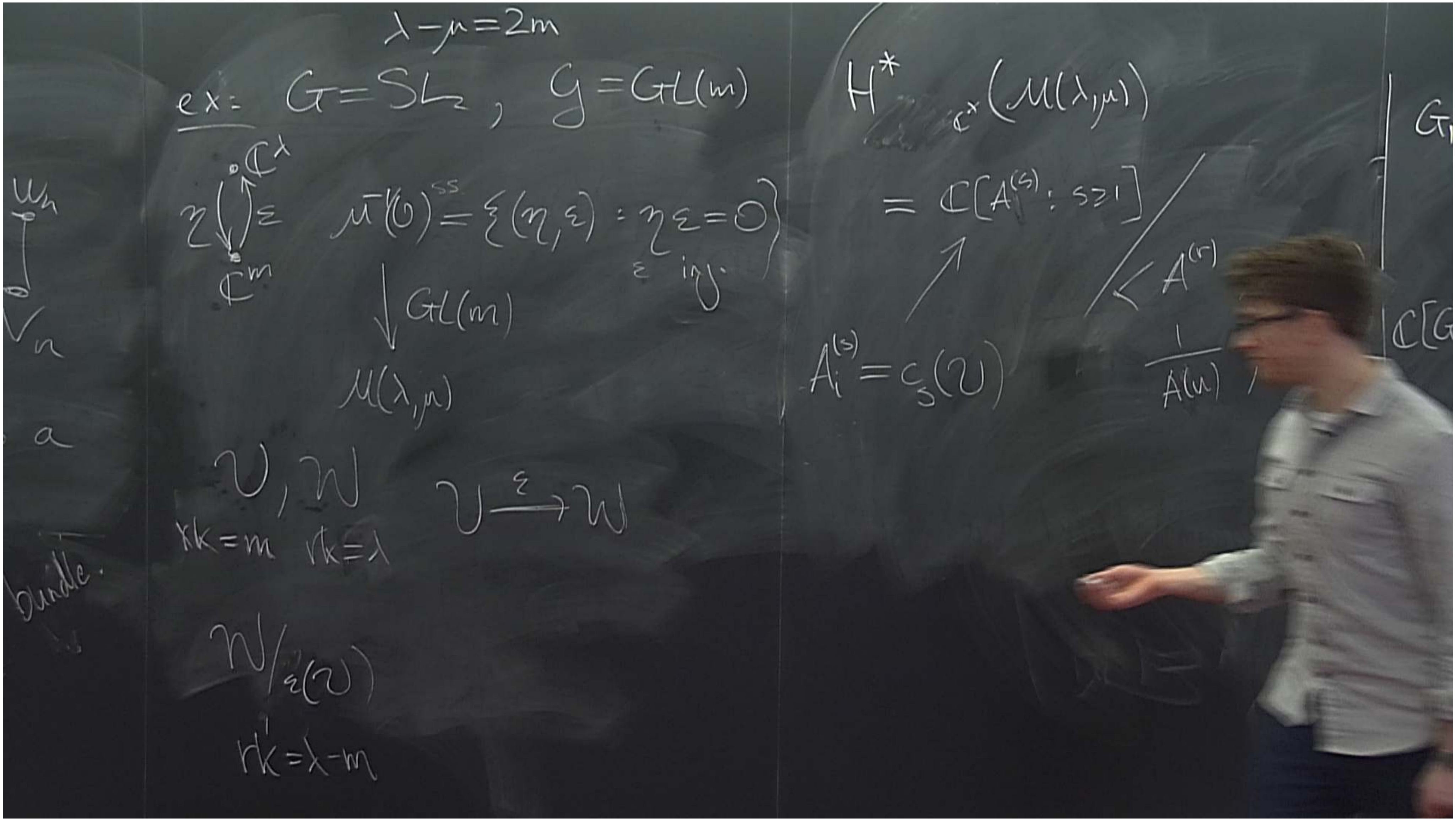
$\mathcal{V} \xrightarrow{\varepsilon} \mathcal{W}$

$\mathcal{W}/\varepsilon(\mathcal{V})$

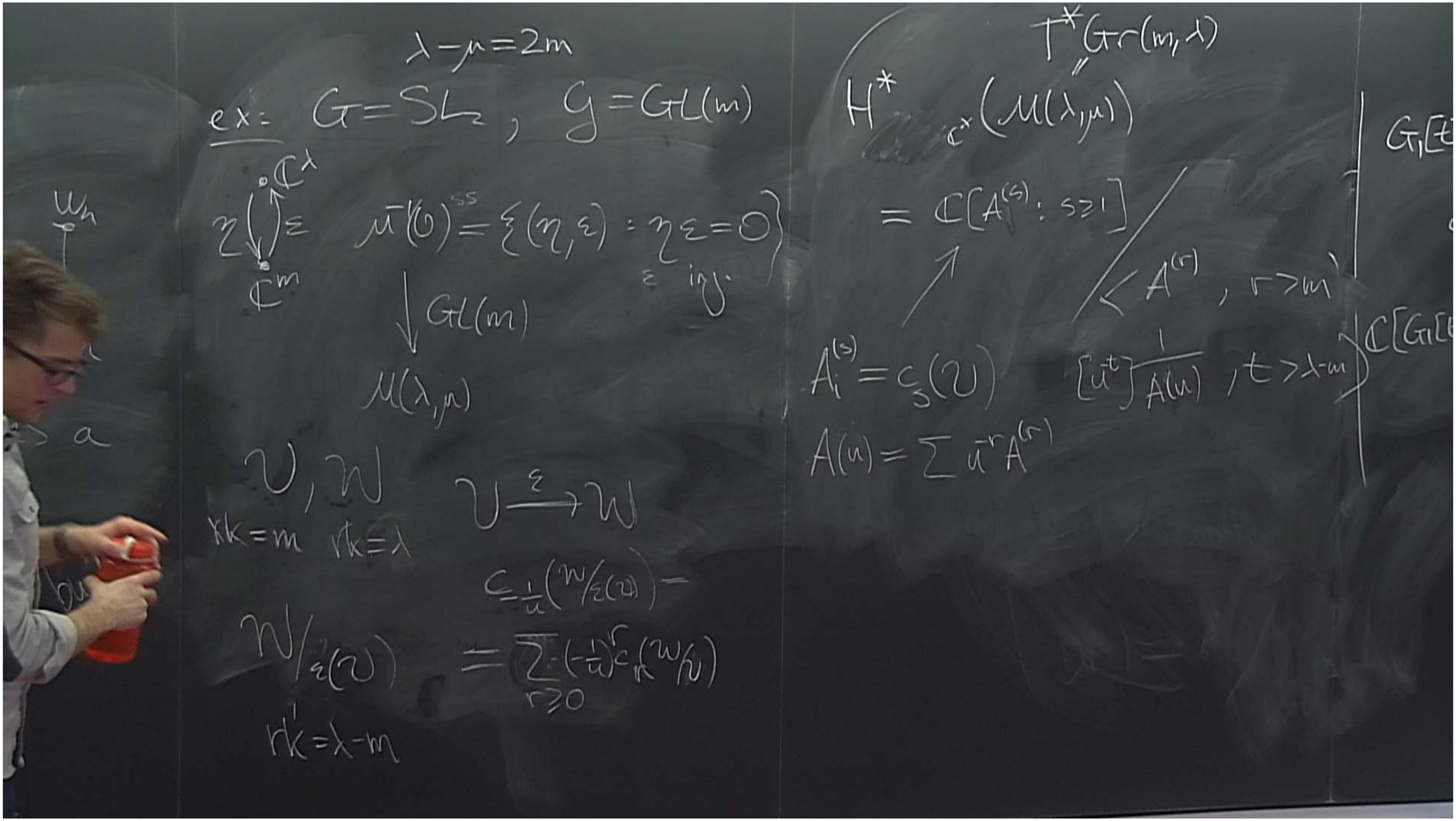
$rk=\lambda-m$

H^*_{GL}

$=$



λ − μ = 2m
ex: G = SL₂ , G = GL(m)
μ⁻¹(0)^ss = {(η, ε) : ηε = 0}
ε inj.
GL(m)
M(λ, μ)
V, W
rk = m
rk = λ
V →ε W
W/ε(V)
rk = λ − m
H*_C* (M(λ, μ))
= C[A_i^(s) : s ≥ 1]
A_i^(s) = c_s(V)
A^(r)
1/A(u)
bundle.



λ − μ = 2m
ex: G = SL_2 , G = GL(m)
μ^{-1}(0)^{ss} = {(η, ε) : ηε = 0, ε inj.}
GL(m)
M(λ, μ)
V, W
rk = m rk = λ
V --ε--> W
W/ε(V)
rk = λ − m
c_{-1/u}(W/ε(V)) = Σ_{r≥0} (−1/u)^r c_r(W/V)
T*Gr(m, λ)
H*_{C*}(M(λ, μ))
= C[A^{(s)} : s ≥ 1] / < A^{(r)}, r > m
[u^t] 1/A(u), t > λ−m >
A_i^{(s)} = c_s(V)
A(u) = Σ u^{-r} A^{(r)}
C[G_1[t