Title: On Noncontextual, Non-Kolmogorovian Hidden Variable Theories

Date: Jan 21, 2016 02:00 PM

URL: http://pirsa.org/16010073

Abstract: <p>One implication of Bell's theorem is that there cannot in general be hidden variable models for quantum mechanics that both are noncontextual and retain the structure of a classical probability space. Thus, some hidden variable programs aim to retain noncontextuality at the cost of using a generalization of the Kolmogorov probability axioms. We present a theorem to show that such programs are committed to the existence of a finite null cover for some quantum mechanical experiments, i.e., a finite collection of probability zero events whose disjunction exhausts the space of possibilities. This serves as a kind of "no-go" theorem for these alternative, or generalized, probability theories.</p>

It is well known that Bell's theorem rules out *local* hidden variable theories with determinate properties.

But some have suggested that the problem is not locality or determinate properties, but rather the use of probability theory.

We will see that Bell's theorem implicitly assumes the framework of *classical probability theory*.

Some have suggested using alternative probability theories to get around Bell's theorem.

- non-monotonic probabilities: Suppes, Hartmann
- negative probabilities: Hartle, Kronz
- complex probabilities: Srinivasan
- non-additive probabilities: Sorkin, Gudder, Dowker

Today

If one uses a noncontextual probability theory, then one is committed to a finite collection of probability zero events that exhaust all possibilities.

We call such a collection a *Finite Null Cover*.

I. Setup

II. A "No-Go" Theorem

III. Applications

IV. Discussion

Setup: Bell's Theorem and Classical Probability

I'll begin by setting up a framework for discussing Bell's theorem and probability in quantum mechanics.

Definition

A *quantum mechanical experiment* is a triple $(\mathcal{H}, \psi, S)$, where $\mathcal{H}$ is a Hilbert space, $\psi \in \mathcal{H}$ is a unit vector, and $S = \{P_1, \ldots, P_n\}$ is a sequence of projection operators on $\mathcal{H}$.

Each projection $P_i \in S$ represents a "yes-no" experiment.

The number $p_i = \langle \psi, P_i \psi \rangle$ is the probability of a "yes" outcome.

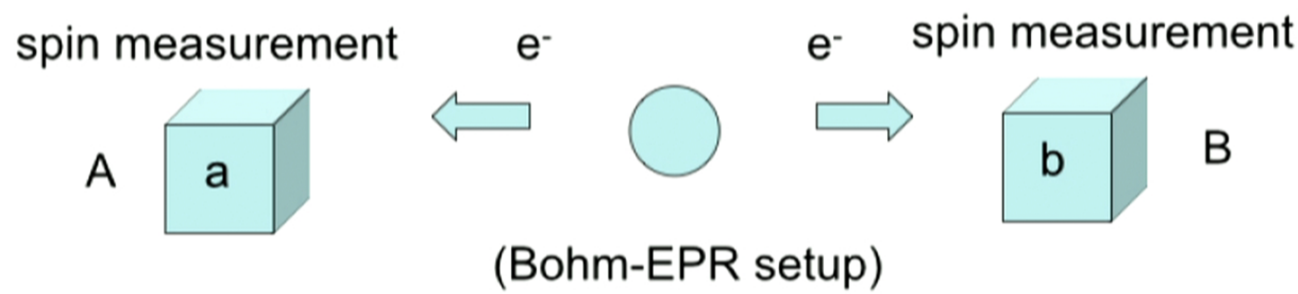
For our purposes, hidden variables specify determinate states of a system.

A determinate state is one whose properties have definite values.

We will not be concerned with dynamics.

I'll first quickly summarize a standard version of Bell's theorem.

Then I'll look at a slight variant due to Fine and Pitowsky that focuses our attention on the role of classical probability.


spin measurement
e-
e-
spin measurement
A
a
b
B
(Bohm-EPR setup)

Locality & Quasi-determinateness $\Rightarrow$ Bell's inequalities

Some quantum mechanical experiments violate Bell's inequalities, namely the EPR setup.

Theorem (Bell)

There exist quantum mechanical experiments for which it is not possible to find a local, quasi-determinate Bell hidden variable theory.

Let's look at a variant to Bell's theorem that shows the role of classical probability theory.

Definition

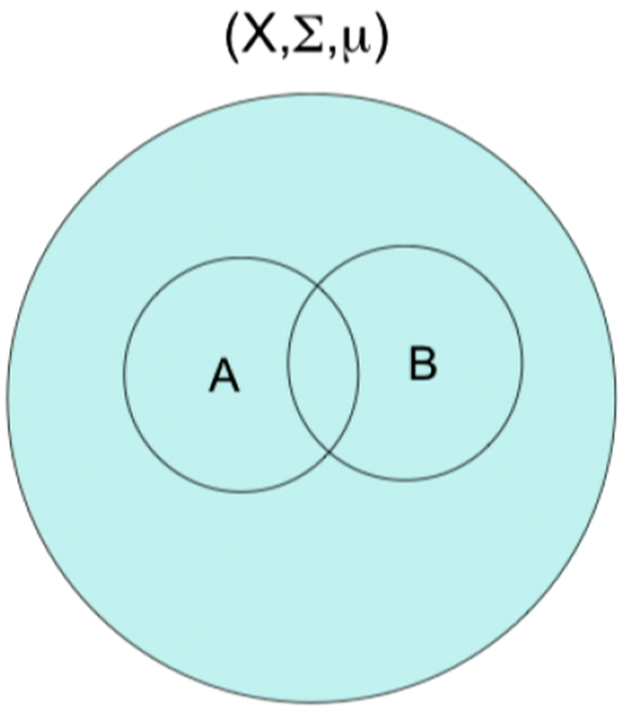
Let X be a set. A collection of subsets of $\Sigma \subseteq \mathcal{P}(X)$ is called an *algebra* if

1. $A \in \Sigma \Rightarrow (X \setminus A) \in \Sigma$, and
2. $A, B \in \Sigma \Rightarrow (A \cup B) \in \Sigma$.

Definition

A *classical probability space* is a triple (X, Σ, μ), where X is a set, Σ is an algebra of subsets of X, and $\mu : \Sigma \to \mathbb{R}$ is a real-valued function satisfying

1. $\mu(X) = 1$
2. $\mu(A) \geq 0$ for all $A \in \Sigma$, and
3. if $A \cap B = \varnothing$, then $\mu(A \cup B) = \mu(A) + \mu(B)$.


(X,Σ,μ)
A
B

Each $A \in \Sigma$ represents a proposition.

The number $\mu(A)$ is the probability that the proposition is true.

Definition

A classical probability space (X, Σ, μ) is a *representation* of a quantum mechanical experiment $(\mathcal{H}, \psi, S)$ if

1. for each $P_i \in S$, there is an $A_i \in \Sigma$ such that

$$\mu(A_i) = \langle \psi, P_i \psi \rangle$$

2. if $P_i, P_j \in S$ are compatible in the sense that $[P_i, P_j] = 0$, then

$$\mu(A_i \cap A_j) = \langle \psi, P_i P_j \psi \rangle$$

Classical probability space representation $\Rightarrow$ Bell's inequalities

Some quantum mechanical experiments violate Bell's inequalities.

Theorem (Fine/Pitowsky)

There are quantum mechanical experiments with no classical probability space representation.

How are these two "no-go" theorems related?

If a quantum experiment has a local, quasi-determinate Bell hidden variable theory, then it has a classical probability space representation.

But one can show that the "no-go" result still holds when one weakens the framework of classical probability theory.

II. A New "No-Go" Theorem

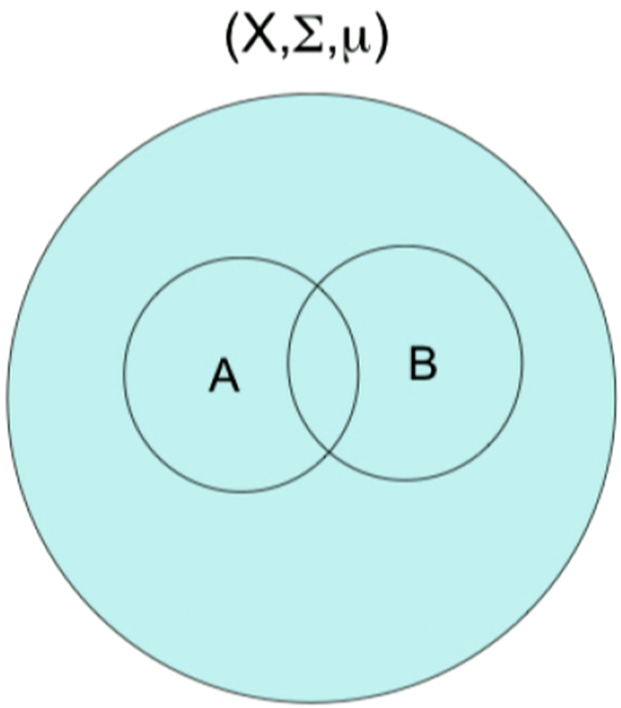
A New "No-Go" Theorem

We have seen a formulation of Bell's theorem that makes clear the role of classical probability theory.

Now I will present a result with much weaker probabilistic assumptions.

Definition

A *weak noncontextual probability space* is a triple (X, Σ, μ), where X is a set, $\Sigma \subseteq \mathcal{P}(X)$ is a collection of subsets of X, and $\mu : \Sigma \to Y$ is a Y-valued function, where $[0, 1] \subseteq Y$.


(X,Σ,μ)
A
B

Definition

A weak noncontextual probability space (X, Σ, μ) is a *representation* of a quantum mechanical experiment $(\mathcal{H}, \psi, S)$ if

1. for each $P_i \in S$, there is an $A_i \in \Sigma$ such that

$$\mu(A_i) = \langle \psi, P_i \psi \rangle$$

2. if $P_i, P_j \in S$ are compatible in the sense that $[P_i, P_j] = 0$, **then** $A_i \cap A_j \in \Sigma$ **and**

$$\mu(A_i \cap A_j) = \langle \psi, P_i P_j \psi \rangle$$

The mathematical result that fuels the current "no-go" theorem is the *Kochen-Specker theorem*.

If $P_i, P_j \in S$ are orthogonal projections, then they are *mutually exclusive*.

If $P_i, P_j, \ldots, P_k \in S$ span all of $\mathcal{H}$, then they are *exhaustive*.

If $P_i, P_j \in S$ are orthogonal projections, then they are *mutually exclusive*.

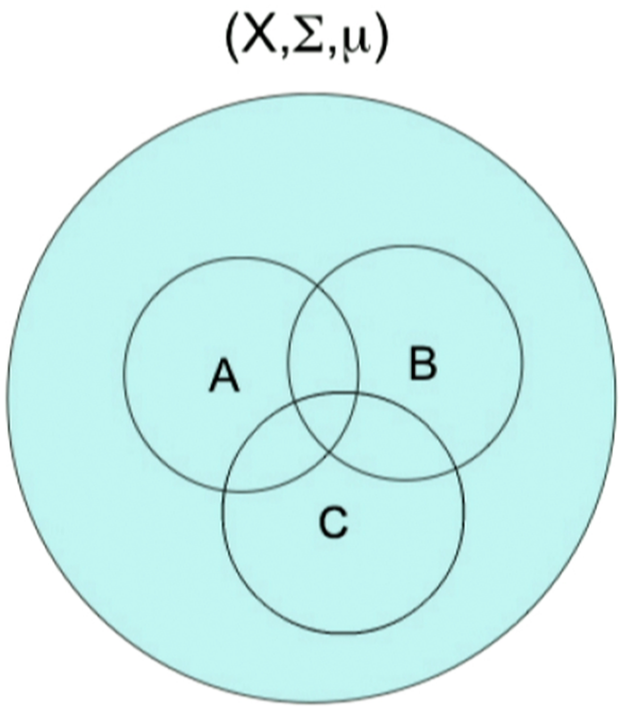
If $P_i, P_j, \ldots, P_k \in S$ span all of $\mathcal{H}$, then they are *exhaustive*.

Theorem (Kochen-Specker)

For any Hilbert space $\mathcal{H}$ with $\dim(\mathcal{H}) \geq 3$, there is a finite collection of projection operators S on it such that there is no function $f : S \to \{0, 1\}$ that assigns 1 to exactly one element of every subset of S whose elements are mutually orthogonal and span $\mathcal{H}$.

Definition

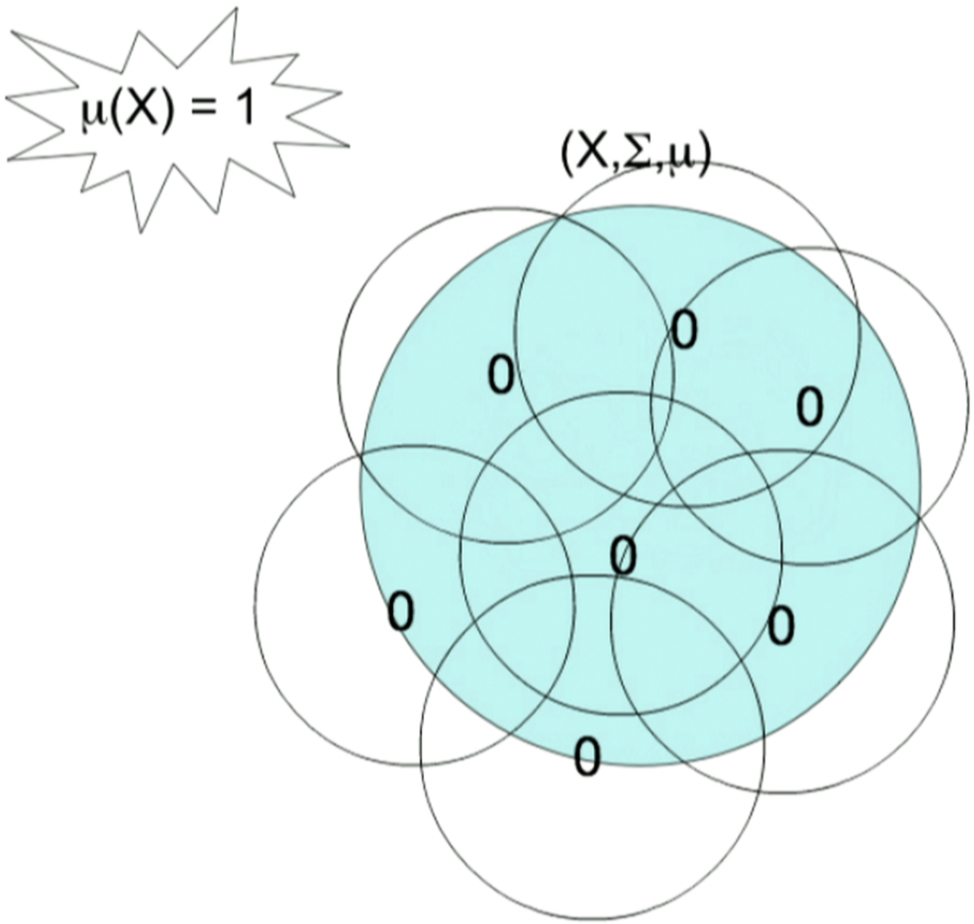
A *Kochen-Specker witness* is a quantum mechanical experiment $(\mathcal{H}, \psi, S)$ such that $\dim(\mathcal{H}) \geq 3$ and there is no function $f : S \to \{0, 1\}$ that assigns 1 to exactly one element of every subset of S whose elements are mutually orthogonal and span $\mathcal{H}$.


(X,Σ,μ)
A
B
C

Constraints

Weak Classicality If $Q \subseteq S$ contains only mutually orthogonal operators spanning $\mathcal{H}$, then $\Sigma_Q \subseteq \Sigma$ and $(X, \Sigma_Q, \mu_{|\Sigma_Q})$ is a classical probability space, where Σ_Q is the smallest algebra for X containing $\{A_i : P_i \in Q\}$.

No Finite Null Cover There is no finite collection $\{B_1, \ldots, B_m\} \subseteq \Sigma$ such that $\mu(B_i) = 0$ for all $i \in \{1, \ldots, m\}$ and $\bigcup_{i=1}^{m} B_i = X$.


μ(X) = 1
(X,Σ,μ)
0
0
0
0
0
0
0

Theorem

No Kochen-Specker witness $(\mathcal{H}, \psi, S)$ has a weak noncontextual probability space representation (X, Σ, μ) satisfying both Weak Classicality and No Finite Null Cover.

Corollary

Every Kochen-Specker witness is a quantum mechanical experiment that violates Bell's inequalities.

Kochen-Specker theorem $\Rightarrow$ Weak Probability Theorem $\Rightarrow$ Fine/Pitowsky Theorem $\Rightarrow$ Bell's Theorem

III. Applications to Alternative Probability Theories

Applications to Alternative Probability Theories

We have seen a general theorem that weakens the probabilistic assumptions of standard formulations of Bell's theorem.

Now I will show that this theorem is strong enough to rule out many alternative probability theories.

Some have suggested using alternative probability theories to get around Bell's theorem.

- non-monotonic probabilities: Suppes, Hartmann
- negative probabilities: Hartle, Kronz
- complex probabilities: Srinivasan
- non-additive probabilities: Sorkin, Gudder, Dowker

Generalized Probability Spaces

Definition

An *additive class* for a nonempty set X is a nonempty subset of the power set $\mathcal{P}(X)$ such that

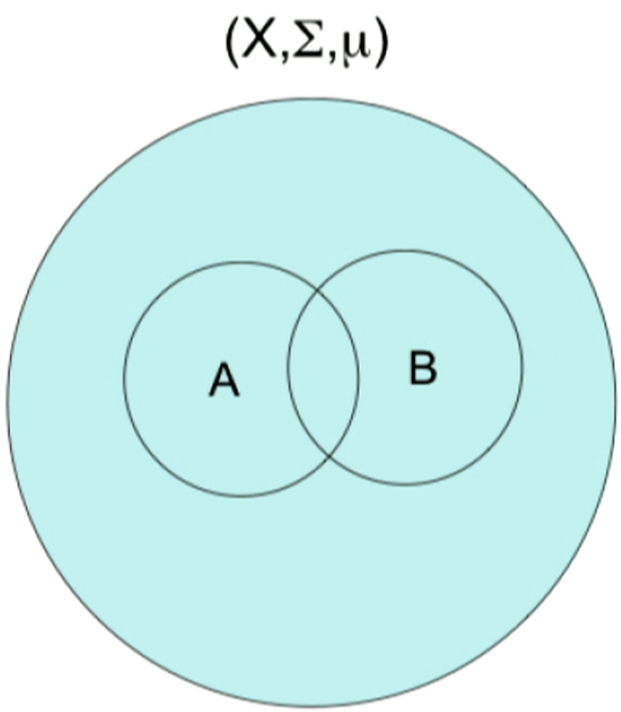
1. for all $A \in \Sigma$, $(X \setminus A) \in \Sigma$, and
2. for all **disjoint** $A, B \in \Sigma$, $A \cup B \in \Sigma$.

Generalized Probability Spaces

Definition

A *generalized probability space* is a triple (X, Σ, μ), where Σ is an **additive class** for the nonempty set X and $\mu : \Sigma \to \mathbb{R}$ is such that

1. $\mu(X) = 1$
2. $\mu(A) \geq 0$ for all $A \in \Sigma$, and
3. for all disjoint $A, B \in \Sigma$, $\mu(A \cup B) = \mu(A) + \mu(B)$.


(X,Σ,μ)
A
B

Generalized probability spaces allow us to refrain from assigning joint probabilities to conjunctions of incompatible observables.

Corollary

No Kochen-Specker witness $(\mathcal{H}, \psi, S)$ *has a weak noncontextual hidden variable representation on a generalized probability space* (X, Σ, μ) *satisfying both Weak Classicality and No Finite Null Cover.*

This shows a sense in which generalized probability spaces cannot be used for quantum mechanics.

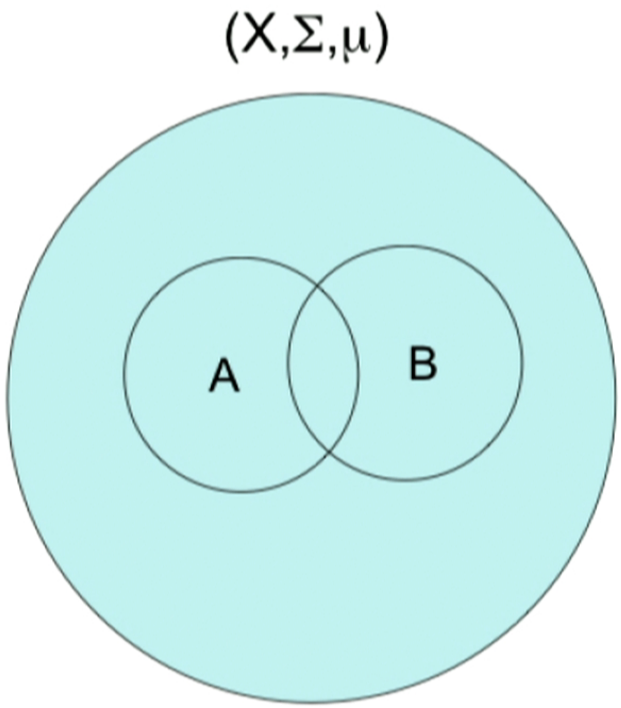
Extended Probability Spaces

Definition

A *complex probability space* is a triple (X, Σ, μ), where Σ is an algebra for the nonempty set X and $\mu : \Sigma \to \mathbb{C}$ is such that

1. $\mu(X) = 1$, and
2. for all disjoint $A, B \in \Sigma$, $\mu(A \cup B) = \mu(A) + \mu(B)$.

Negative and complex probability spaces allow us to assign probabilities beyond positive real numbers.


(X,Σ,μ)
A
B

There is a weak noncontextual hidden variable representation for the EPR-Bohm setup on both a negative and a complex probability space.

There is a weak noncontextual hidden variable representation for the EPR-Bohm setup on both a negative and a complex probability space.

Every negative or complex probability space is a weak noncontextual probability space.

Corollary

No Kochen-Specker witness $(\mathcal{H}, \psi, S)$ *has a weak noncontextual hidden variable representation on a negative or complex probability space* (X, Σ, μ) *satisfying both Weak Classicality and No Finite Null Cover.*

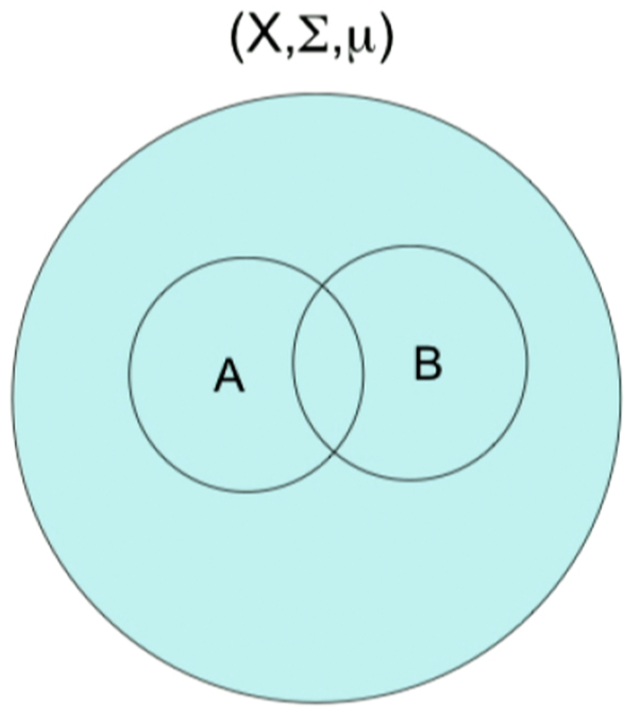
Non-additive Probability Spaces

Definition

An *upper (lower) probability space* is a triple (X, Σ, μ), where Σ is an algebra for the nonempty set X and $\mu : \Sigma \to \mathbb{R}$ is such that

1. $\mu(X) = 1$,
2. $\mu(A) \geq 0$ for all $A \in \Sigma$, and
3. for all disjoint $A, B \in \Sigma$, $\mu(A \cup B) \leq (\geq) \mu(A) + \mu(B)$.

Upper (lower) probability spaces allow us to assign probabilities as intervals rather than precise numbers.


(X,Σ,μ)
A
B

Non-additive Probability Spaces

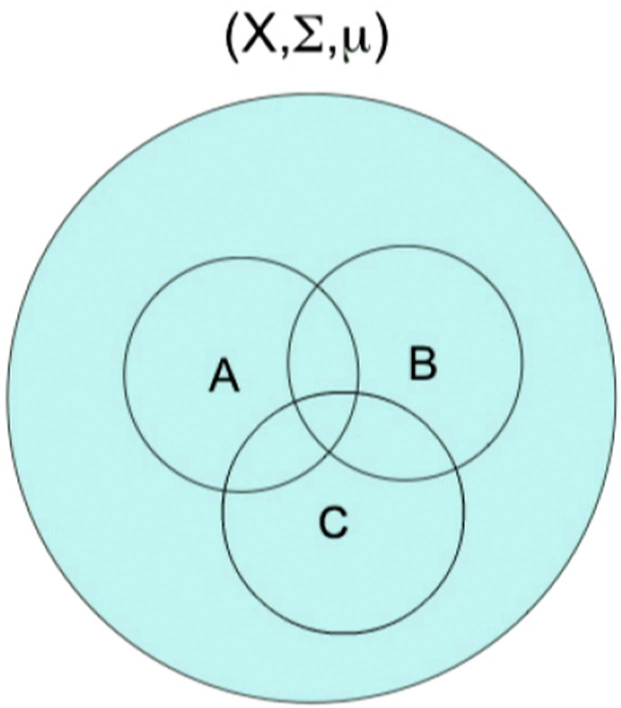
Definition

A *quantum measure space* is a triple (X, Σ, μ), where Σ is an algebra for the nonempty set X and $\mu : \Sigma \to \mathbb{R}$ is such that

1. $\mu(X) = 1$,
2. $\mu(A) \geq 0$ for all $A \in \Sigma$
3. for all pairwise disjoint $A, B, C \in \Sigma$,

$$\mu(A \cup B \cup C) = \mu(A \cup B) + \mu(B \cup C) + \mu(A \cup C) - \mu(A) - \mu(B) - \mu(C)$$

Quantum measure spaces allow us to assign probabilities with pairwise interference.


(X,Σ,μ)
A
B
C

Every upper (lower) probability space and quantum measure space is a weak noncontextual probability space.

Corollary

No Kochen-Specker witness $(\mathcal{H}, \psi, S)$ *has a weak noncontextual hidden variable representation on an upper (lower) probability space or a quantum measure space* (X, Σ, μ) *satisfying both Weak Classicality and No Finite Null Cover.*

This shows a sense in which upper (lower) probability spaces and quantum measure spaces cannot be used for quantum mechanics.

IV. Discussion

Discussion

We have seen a general theorem that rules out many alternative probability theories for quantum mechanics.

But how are we to understand the constraints that go into this theorem?

Constraints

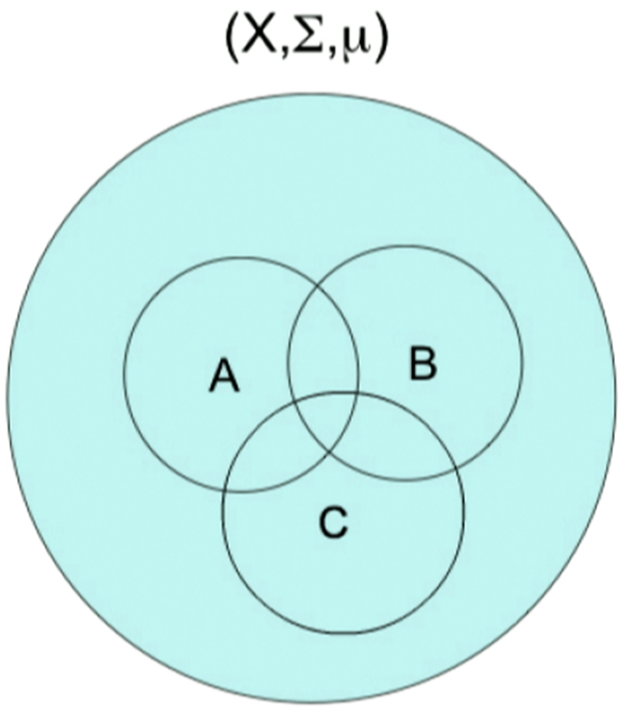
Weak Classicality If $Q \subseteq S$ contains only mutually orthogonal operators spanning $\mathcal{H}$, then $\Sigma_Q \subseteq \Sigma$ and $(X, \Sigma_Q, \mu_{|\Sigma_Q})$ is a classical probability space, where Σ_Q is the smallest algebra for X containing $\{A_i : P_i \in Q\}$.

No Finite Null Cover There is no finite collection $\{B_1, \ldots, B_m\} \subseteq \Sigma$ such that $\mu(B_i) = 0$ for all $i \in \{1, \ldots, m\}$ and $\bigcup_{i=1}^{m} B_i = X$.

Constraints

Weak Classicality If $Q \subseteq S$ contains only mutually orthogonal operators spanning $\mathcal{H}$, then $\Sigma_Q \subseteq \Sigma$ and $(X, \Sigma_Q, \mu_{|\Sigma_Q})$ is a classical probability space, where Σ_Q is the smallest algebra for X containing $\{A_i : P_i \in Q\}$.

No Finite Null Cover There is no finite collection $\{B_1, \ldots, B_m\} \subseteq \Sigma$ such that $\mu(B_i) = 0$ for all $i \in \{1, \ldots, m\}$ and $\bigcup_{i=1}^{m} B_i = X$.


(X,Σ,μ)
A
B
C

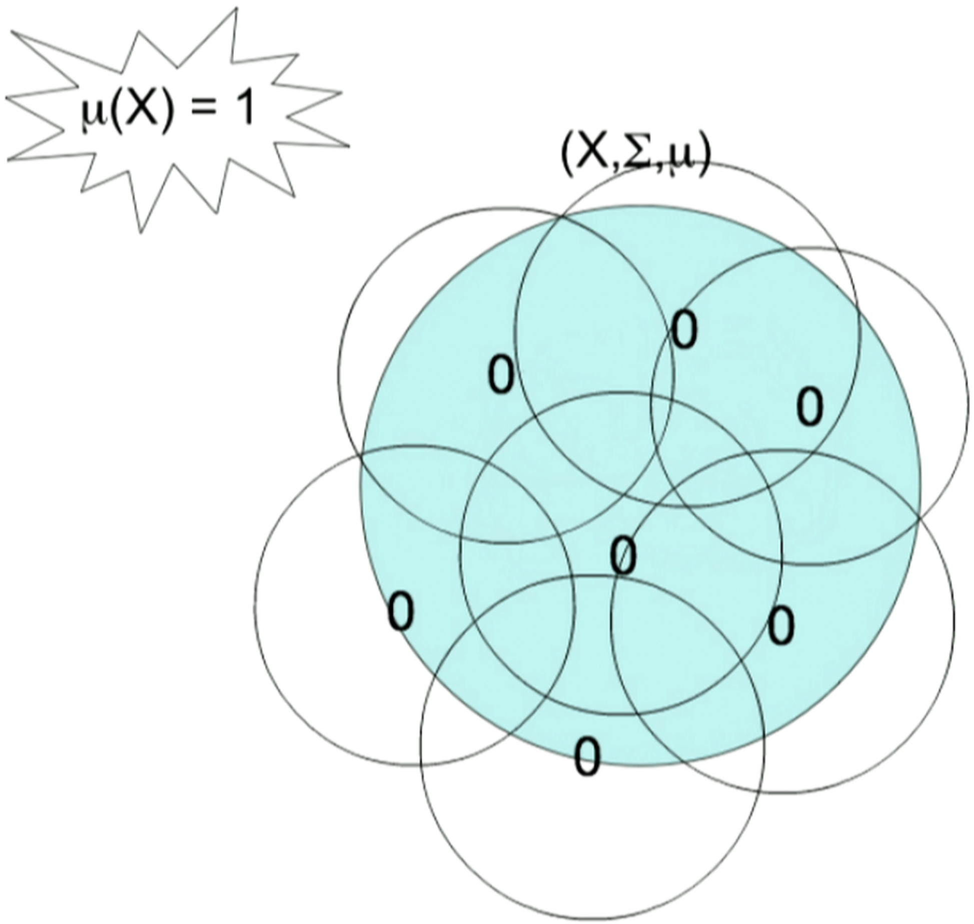
A probability space representation satisfies Weak Classicality just in case each experiment can be described on its own in classical terms.

The interpretation of No Finite Null Cover is trickier.

Constraints

Weak Classicality If $Q \subseteq S$ contains only mutually orthogonal operators spanning $\mathcal{H}$, then $\Sigma_Q \subseteq \Sigma$ and $(X, \Sigma_Q, \mu_{|\Sigma_Q})$ is a classical probability space, where Σ_Q is the smallest algebra for X containing $\{A_i : P_i \in Q\}$.

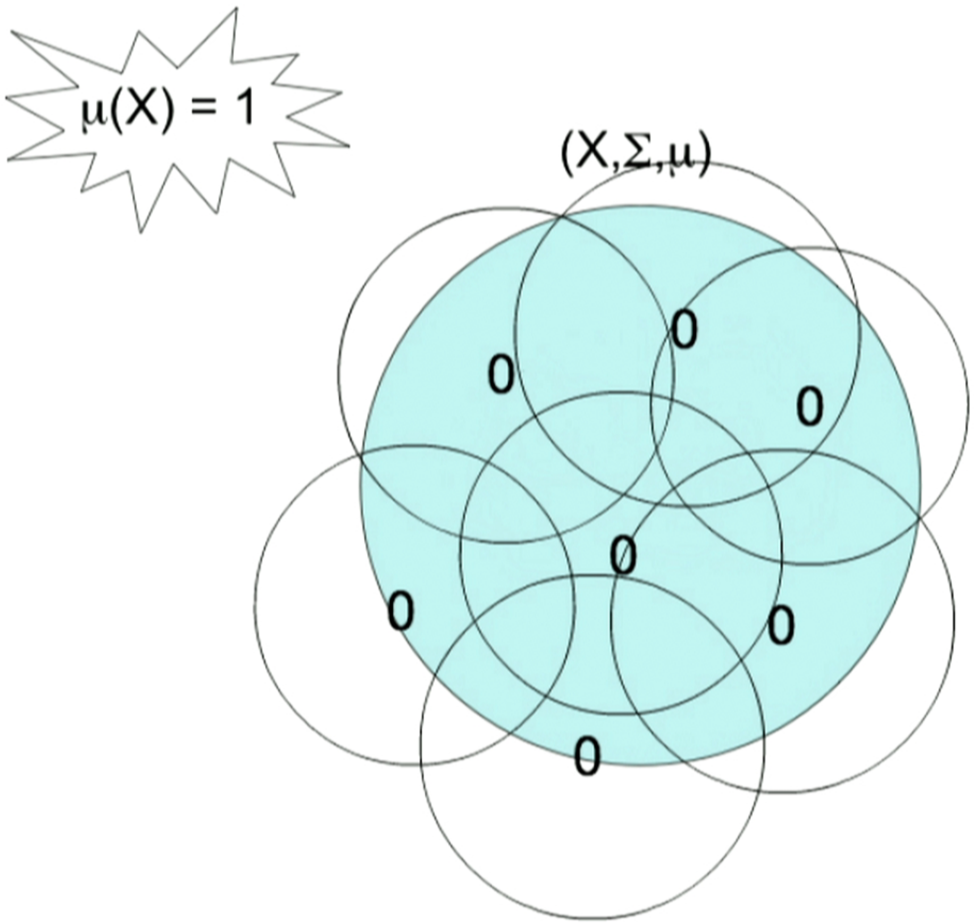
No Finite Null Cover There is no finite collection $\{B_1, \ldots, B_m\} \subseteq \Sigma$ such that $\mu(B_i) = 0$ for all $i \in \{1, \ldots, m\}$ and $\bigcup_{i=1}^{m} B_i = X$.


μ(X) = 1
(X,Σ,μ)
0
0
0
0
0
0
0

A probability space with a Finite Null Cover contains a certain kind of pathology or irrationality.

If an agent sets her degrees of belief by the probabilities of a space with a Finite Null Cover, then she is subject to a *Dutch Book*.

A probability space with a Finite Null Cover contains a certain kind of pathology or irrationality.


μ(X) = 1
(X,Σ,μ)
0
0
0
0
0
0
0
0

If an agent sets her degrees of belief by the probabilities of a space with a Finite Null Cover, then she is subject to a *Dutch Book*.

A Dutch Book is a series of bets for which the agent is guaranteed to lose money.

If a weak noncontextual probability space contains a Finite Null Cover, then it contains a Dutch Book.

A Dutch Book is a series of bets for which the agent is guaranteed to lose money.

The interpretation of a Finite Null Cover as a pathology does not depend on a subjective belief interpretation of probability.

If we try to give a frequency interpretation of a Finite Null Cover, then we also find an internal inconsistency.

The interpretation of a Finite Null Cover as a pathology does not depend on a subjective belief interpretation of probability.

If we try to give a frequency interpretation of a Finite Null Cover, then we also find an internal inconsistency.

I think Weak Classicality and No Finite Null Cover are plausible constraints on a weak noncontextual probability space...

...at least if want to understand that space as providing a determinate hidden variable theory for quantum mechanics.

Conclusions

Negative Results

Anyone who wants to use alternative probability theories faces a dilemma.

- either give up one of the constraints,
- or else fail to represent all quantum mechanical experiments.

Positive Results

The many different "no-go" theorems can be organized in a hierarchy.

Positive Results

Kochen-Specker theorem $\Rightarrow$ Weak Probability Theorem $\Rightarrow$ Fine/Pitowsky Theorem $\Rightarrow$ Bell's Theorem