

Title: Cluster Structures on Higher Teichmuller Spaces for Classical Groups - Ian Le

Date: Jan 07, 2016 01:30 PM

URL: <http://pirsa.org/16010071>

Abstract:

Let S be a surface, G a semi-simple group of type B, C or D. I will explain why the moduli space of framed local systems $\mathcal{A}_{G,S}$ defined by Fock and Goncharov has the structure of a cluster variety, and fits inside a larger structure called a cluster ensemble. This was previously known only in type A. This gives a more direct proof of results of Fock and Goncharov for the symplectic and spin groups, and also allows one to quantize higher Teichmuller space in these cases. If time permits, I hope to talk about applications to counting tensor invariants of finite dimensional representations of these groups.

Cluster algebras on
Higher Teichmüller Space
for the classical groups

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 $A_n \quad B_n \quad C_n \quad D_n$

$Spin_{2n+1}$
 $S = \text{topological surface (w/)}$

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$\mathcal{A}_{G,S} = \text{moduli of } G\text{-local
systems on } S \text{ with
reduction to } UCG \text{ on
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monodromy is
unipotent
flat section of
associated G/U -bdl

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 Space or Hitchin component

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Cluster algebras on
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$$G = \begin{matrix} SL_{n+1} & SO_{2n+1} & Sp_{2n} & SO_{2n} \\ A_n & B_n & C_n & D_n \end{matrix}$$

$$S = \begin{matrix} Spin_{2n+1} & Spin_{2n} \\ \text{topological surface (w/d)} \end{matrix}$$

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systems on S with
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$A_{G,S}(R_{\geq 0})$ = higher Teichmüller
space or Hitchin component

S closed

$A_{G,S}(R) = \text{character variety}$
for $G(R)$
has many components,
one of them is $A_{G,S}(R_{\geq 0})$

$\mathcal{A}_{G,S}$ = higher Teichmüller
or Hitchin component

$\mathcal{A}_{G,S}$ = character variety
for $G(\mathbb{R})$
components,
then is $\mathcal{A}_{G,S}(\mathbb{R}_{>0})$

Two results about $\mathcal{A}_{G,S}$
(Fock-Goncharov)

- 1) $\mathcal{A}_{G,S}(\mathbb{R}_{>0})$ = Hitchin component
- 2) all representations
 $\pi_1 S \rightarrow G(\mathbb{R})$ in $\mathcal{A}_{G,S}(\mathbb{R}_{>0})$
are discrete faithful

$\mathcal{H}$ = higher Teichmüller
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Two results about $\mathcal{A}_{G,S}$
(Fock-Goncharov)

1) $\mathcal{A}_{G,S}(\mathbb{R}_{>0})$ = Hitchin component

2) all representations

$\pi_1 S \rightarrow G(\mathbb{R})$ in $\mathcal{A}_{G,S}(\mathbb{R}_{>0})$
are discrete & faithful

$$\pi_1 S \rightarrow SL_2 \mathbb{R} \xrightarrow{\uparrow \text{principal embedding}} G(\mathbb{R})$$

← Proofs: positive structure on $\mathcal{A}_{G,S}$
- positive atlas of
co-ordinates; transition
functions are subtraction
free

2 approaches to a positive
structure on $\mathcal{A}_{G,S}$

1) A_n case: write down the
cluster algebra structure

$$\pi_1 S \rightarrow SL_2 \mathbb{R} \hookrightarrow G(\mathbb{R})$$

↑
principal
embedding

2 approaches to a positive structure on $A_{g,S}$:

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2 approaches to a positive structure on $A_{G,S}$:

- 1) A_n case, write down the cluster algebra structure
- 2) other groups: Lusztig total positivity

← Proofs: positive structure on $A_{G,S}$

- positive atlas of co-ordinate: transition functions are subtraction free

Goal:

Give a cluster structure on $A_{G,S}$ for G a classical group

Why?

- 1) more economical
cluster charts $\subset$ positive charts
- e.g. on UCG has a positive structure
- generalized minors (A)
 - reduced words

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- generalized minors (A)
 - reduced words (X)
 - transposes
 - inverses

- ② more fundamental
 - spectral networks
 - snakes
 - mirror symmetry
- ③ cluster fns allow an explicit quantization $X_{G,S}$
- ④ applications counting tensor product invariants for G

Cluster algebra
Higher
for the

$$G = SL_{n+1},$$

$$A_n$$

$$S = \text{topological}$$

$(G/U)^n$
 $\parallel$
 $\text{Conf}_n G/U$
 finding & gluing

- Problems.
- ① Does it exist?
 - ② how do you know you have the right one?
 - ③ finding & computing w/ functions
 - ④ cluster algebra in MCG
- reduced words {
 folding {
 & reducing to SL_n equivariant

$A_{G,S}(R_{20})$
 Space

 S closed
 $A_{G,S}(R) =$
 to
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 one of them

ems.

Does it exist?

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cluster algebra $\subset G$

SL_n

Cluster algebras

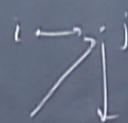
Seed: I - indexst parameterizing
functions

$I_0 \subset I$ frozen variables

b_{ij} - skew symmetrizable
matrix $I \times I$

$d_i, i \in I$ rational #s
that skew symmetrize B .

encode with a quiver



$$b_{ij} = 1$$

$$b_{ji} = -1$$

white vertices & black vertices

Two

(F

1) A

2) all

π_i

are

ems.

Does it exist?

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Cluster algebras

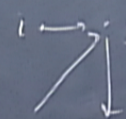
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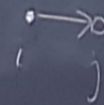
$$b_{ij} = 1$$

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white vertices & black vertices

$$d = \frac{1}{2}$$

$$d = 1$$



$$\begin{array}{cc} i & j \\ i & 0 \quad 1 \\ j & -2 \quad 0 \end{array}$$

Two
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$A_i \quad i \in I$
 $\uparrow$
 functions
 Given $\{A_i\}_{i \in I}$ and $k \in I$

$$A_k \rightarrow A'_k$$

$$A_k A_{k'} = \prod_{b_{ki} > 0} A_i^{b_{ki}} + \prod_{b_{ki} < 0} A_i^{-b_{ki}}$$

$$\pi_1 S \rightarrow SL_2 \mathbb{R} \hookrightarrow G(\mathbb{R})$$

$\uparrow$
principal embedding

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Give a

$A_{G,S}$

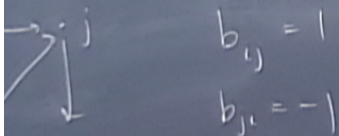
Why?

1) more

ch

e.g.

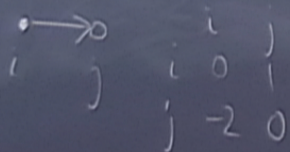
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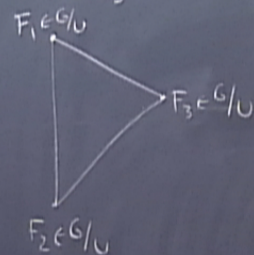
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Under mutation, B-matrix changes

Cluster structure on
 $\text{Conf}_3 G/U$ for $G = \text{Sp}_8$



$$F_1 = u_1 u_2 u_3 u_4 u_5 u_6 u_7 u_8$$

where these spaces are isotropic

$$\Lambda^k \cong \Lambda^{8-k}$$

$$F_1 = u_1, u_2, u_3, \dots, u_7$$

$$F_2 = v_1, \dots, v_7$$

$$F_3 = w_1, \dots, w_7$$

Goal

Give a

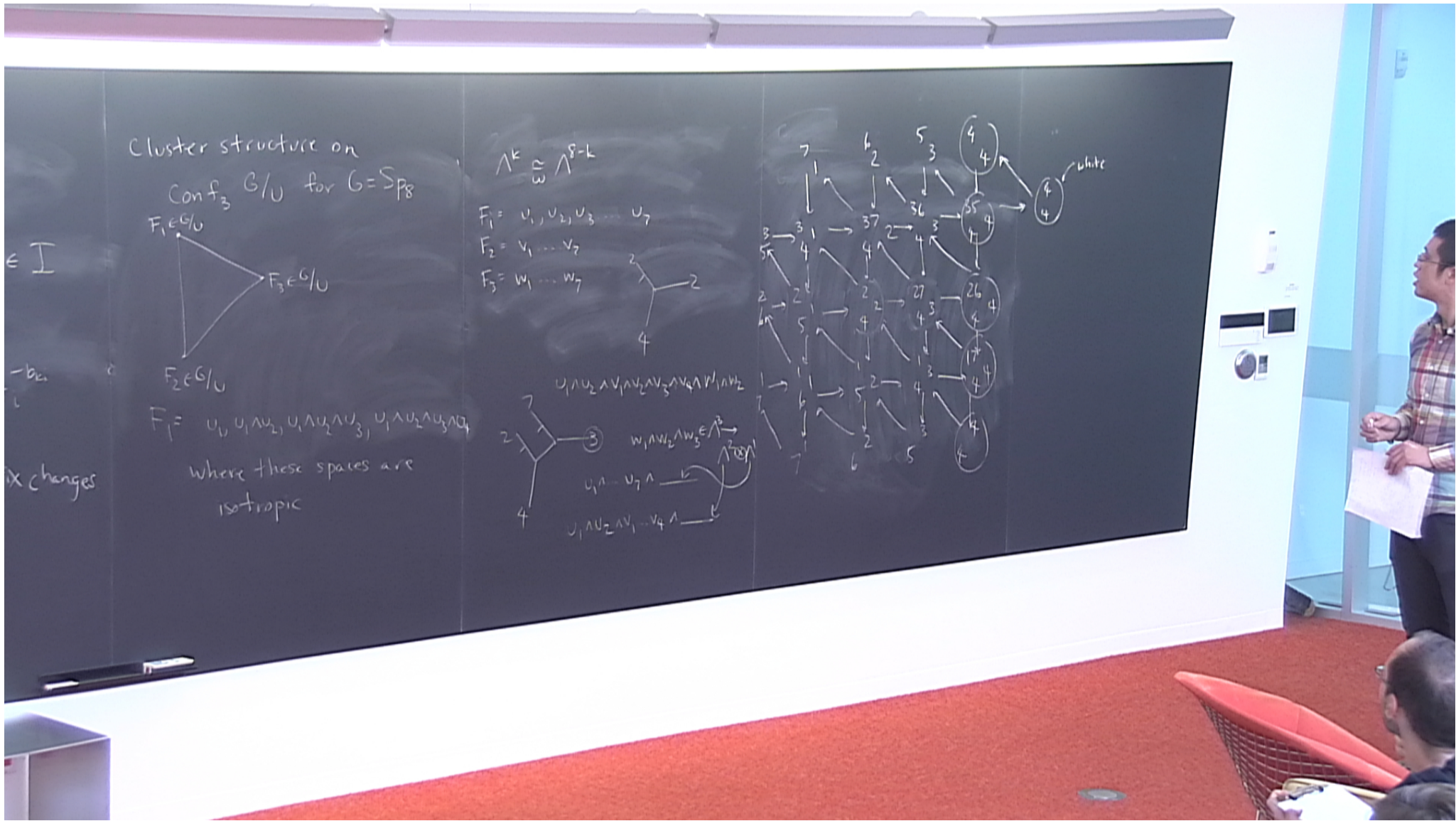
A_G

Why?

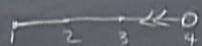
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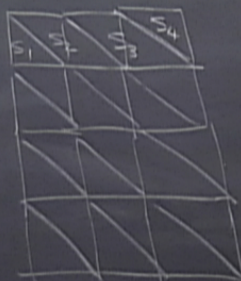
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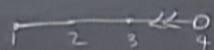
Sp_8



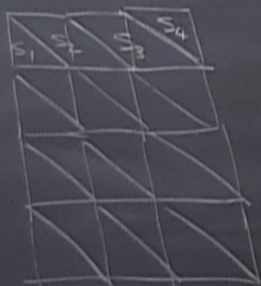
$$w_0 \in W \quad w_0 = (s_1 s_2 s_3 s_4)^4$$



Sp_8



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$$Sp_{2n} \subset SL_{2n}$$

$$Sp_{2n} = (SL_{2n})^\sigma$$

Dynkin auto
of A_{2n-1}

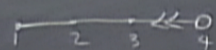


$$s_3 s_5$$

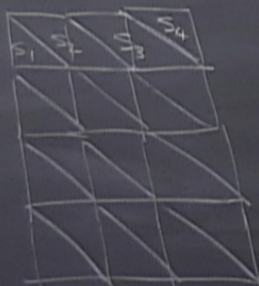
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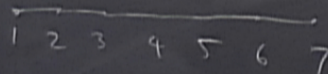
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$$Sp_{2n} \subset SL_{2n}$$

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Dynkin auto
of A_{2n-1}



$$(s_1 s_7 s_2 s_6 s_3 s_5 s_4)^4 = w_0 \in W$$

