Title: PSI 2015/2016 Gravitational Physics - Lecture 5

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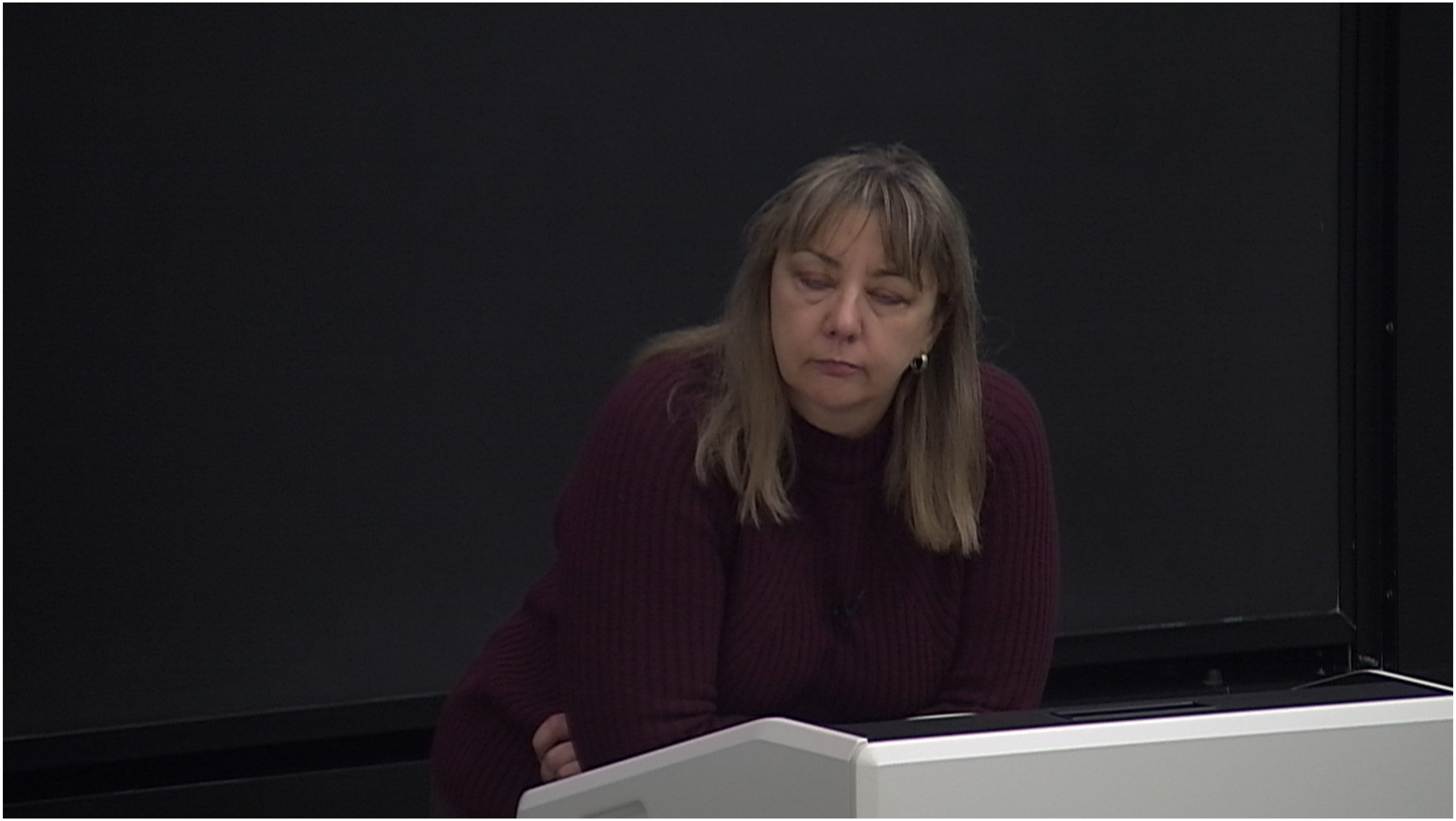
Abstract:

LECTURE 5 "BACK IN BLACK"

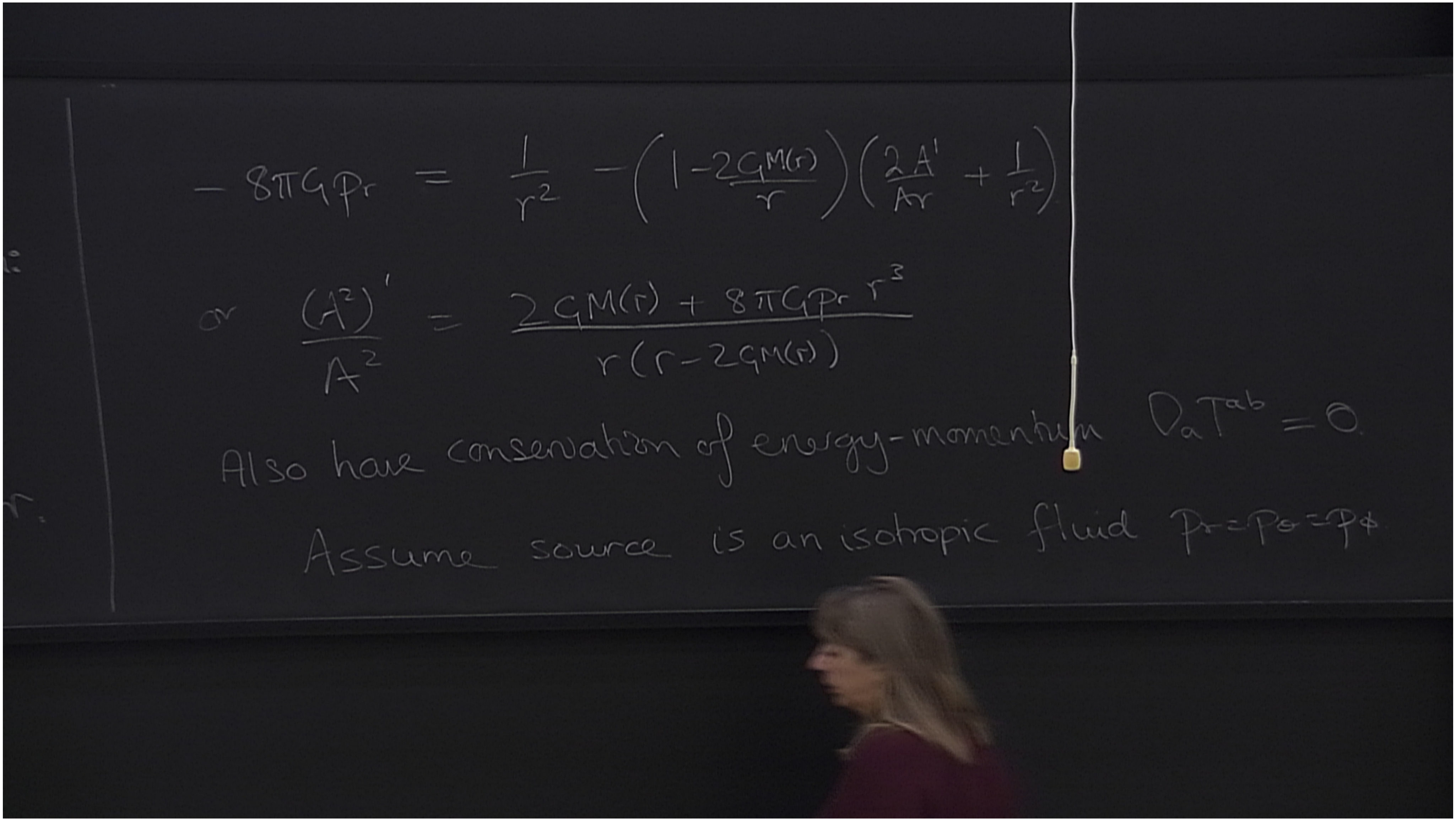


LECTURE 5 "BACK IN BLACK"

Take a closer look at sph. symm:

$$B^{-2} = 1 - \frac{2GM(r)}{r}$$

$M(r)$ intuitively is mass inside radius r.


$-8\pi G p_r = \frac{1}{r^2} - \left(1 - \frac{2GM(r)}{r}\right)\left(\frac{2A'}{Ar} + \frac{1}{r^2}\right)$
or $\frac{(A^2)'}{A^2} = \frac{2GM(r) + 8\pi G p_r r^3}{r(r-2GM(r))}$
Also have conservation of energy-momentum $D_a T^{ab} = 0$.
Assume source is an isotropic fluid $p_r = p_\theta = p_\phi$.

$$\nabla_{\hat{a}} T^{\hat{a}\hat{r}} = \partial_{\hat{r}}\, p_r + \Gamma_{\hat{a}}{}^{\hat{a}}{}_{\hat{r}}\, p_{\hat{r}} + \Gamma_{\hat{a}}{}^{\hat{r}}{}_{\hat{b}}\, T^{\hat{a}\hat{b}}$$

$$= \frac{p'}{B} + \left(\frac{A'}{AB} + \cancel{\frac{2}{rB}}\right) p + \frac{A'}{AB}\rho - \cancel{\frac{2p}{rB}}$$

$$= \frac{1}{B}\left(p' + \frac{A'}{A}(p+\rho)\right) = 0$$

$$\nabla_{\hat a} T^{\hat a \hat r} = \partial_{\hat r}\, p_r + \Gamma_{\hat a}{}^{\hat a}{}_{\hat r}\, p_r + \Gamma_{\hat a}{}^{\hat r}{}_{\hat b}\, T^{\hat a \hat b}$$

$$= \frac{p'}{B} + \left(\frac{A'}{AB} + \cancel{\frac{2}{rB}} \right) p + \frac{A'}{AB}\rho - \cancel{\frac{2p}{rB}}$$

$$= \frac{1}{B}\left(p' + \frac{A'}{A}(p+\rho) \right) = 0$$

System of eqns for a (fluid) star

$\hat{a}\,\hat{b}$

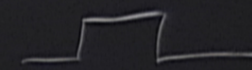
$\rho - \cancel{\frac{2\phi}{rB}}$

Model a star by a "top hat" distribution 

$$\rho = \begin{cases} \rho_0 & r < R \\ 0 & r \geq R \end{cases} \Rightarrow M(r) = \begin{cases} \frac{4}{3}\pi r^3 \rho_0 & r < R \\ \frac{4}{3}\pi R^3 \rho_0 = M & r \geq R \end{cases}$$

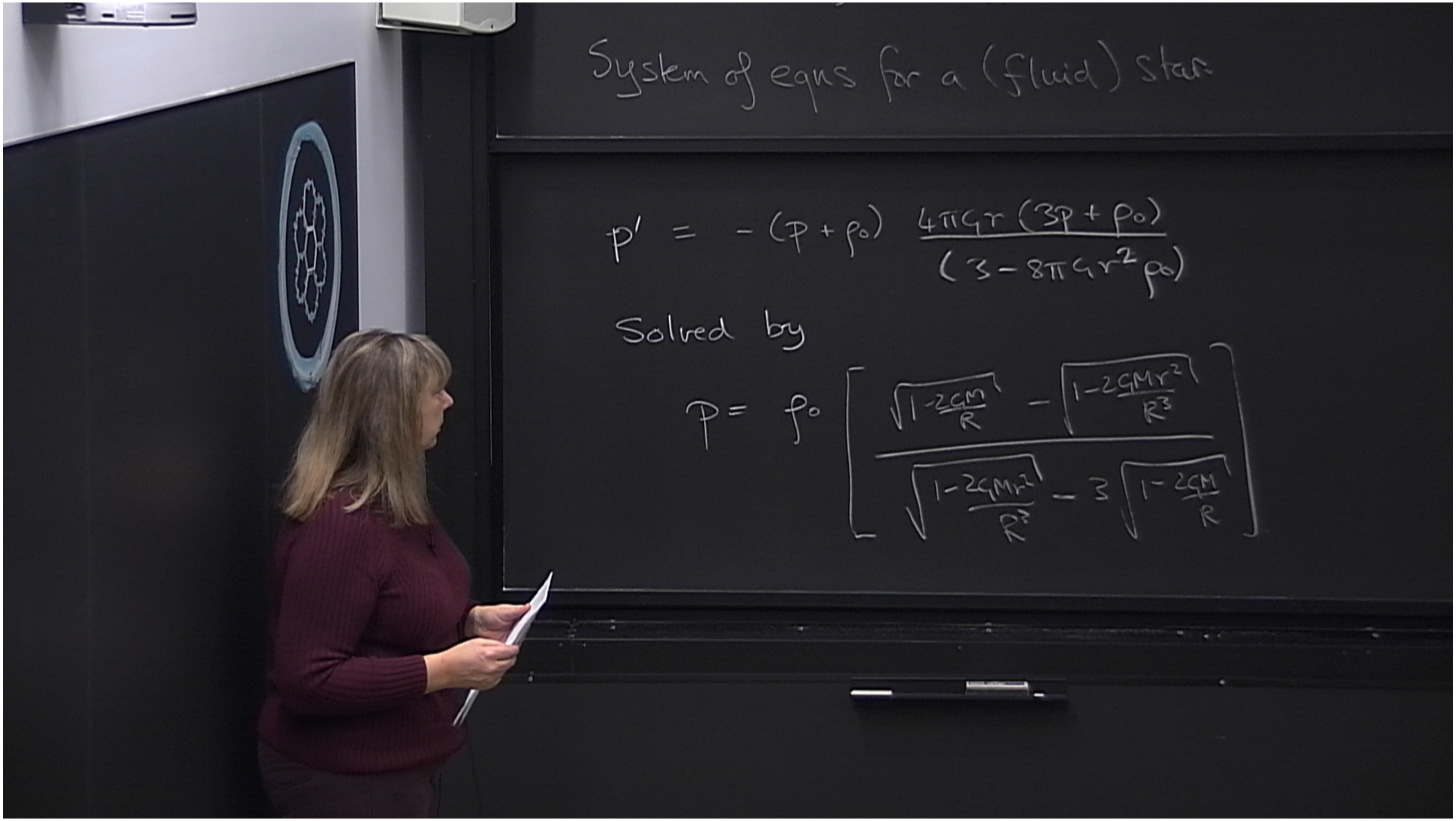
$T^{\hat a \hat b}$

$\rho - \frac{2\cancel{\rho}}{\cancel{r}B}$

Model a star by a "top hat" distribution 

$$\rho = \begin{cases} \rho_0 & r < R \\ 0 & r \geq R \end{cases} \Rightarrow M(r) = \begin{cases} \frac{4}{3}\pi r^3 \rho_0 & r < R \\ \frac{4}{3}\pi R^3 \rho_0 = M & r \geq R \end{cases}$$

Hence $$p' = -(p+\rho)\frac{[GM(r) + 4\pi G p r^3]}{r(r-2GM(r))}$$


System of eqns for a (fluid) star
$p' = -(p+\rho_0)\dfrac{4\pi G r(3p+\rho_0)}{(3-8\pi G r^2\rho_0)}$
Solved by
$p = \rho_0\left[\dfrac{\sqrt{1-\frac{2GM}{R}}-\sqrt{1-\frac{2GMr^2}{R^3}}}{\sqrt{1-\frac{2GMr^2}{R^3}}-3\sqrt{1-\frac{2GM}{R}}}\right]$

...s for a (fluid) star:

$$= -(p+\rho)\,\frac{4\pi G r^{\dots}}{3}\;\frac{(3p+\dots}{r(r-\dots}$$

$$\dots+\rho_0)\;\frac{4\pi G r\,(3p+\rho_0)}{(3-8\pi G r^2\rho_0)}$$

$$\dots\left[\frac{\sqrt{1-\frac{2GM}{R}}-\sqrt{1-\frac{2GMr^2}{R^3}}}{\sqrt{1-\frac{2GMr^2}{R^3}}-3\sqrt{1-\frac{2GM}{R}}}\right]$$

Satisfies $p \to 0$ at edge of star

Note $p(0) = \rho_0\,\dfrac{\sqrt{1-\frac{2GM}{r}}-1}{1-3\sqrt{1-\frac{2GM}{R}}}$

eqns for a (fluid) star

$$= -(p+\rho)\,\frac{4\pi G r^3}{3}\,\frac{(3p+\rho_0}{r(r-\frac{8\pi G}{3}}$$

$$(p+\rho_0)\,\frac{4\pi G r(3p+\rho_0)}{(3-8\pi G r^2\rho_0)}$$

ry

$$\rho_0\left[\frac{\sqrt{1-\frac{2GM}{R}}-\sqrt{1-\frac{2GMr^2}{R^3}}}{\sqrt{1-\frac{2GMr^2}{R^3}}-3\sqrt{1-\frac{2GM}{R}}}\right]$$

Satisfies $p \to 0$ at edge of star

Note $p(0) = \rho_0\,\frac{\sqrt{1-\frac{2GM}{R}}-1}{1-3\sqrt{1-\frac{2GM}{R}}}$

$$R = \frac{9GM}{4} \qquad p(0) \to \infty$$

at

$$\rho_0 \frac{\sqrt{1-\frac{2GM}{R}} - 1}{1 - 3\sqrt{1-\frac{2GM}{R}}}$$

$$p(0) \to \infty$$

i.e. a given mass cannot fit inside an arbitrarily small fluid star.

Simple example of Chandrasekhar Limit.

Penrose Diagrams

(also Brandon Carter)

Encode important causal information.

Penrose Diagrams

(also Brandon Carter)

- Encode important causal information.

Penrose Diagrams

(also Brandon Carter)

Encode important causal information.

Schw:
$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2 d\Omega_{II}^2$$

Tortoise coord

$$r^* = \int \frac{dr}{1-\frac{2GM}{r}} = r + 2GM \ln\left(\frac{r-2GM}{2GM}\right)$$

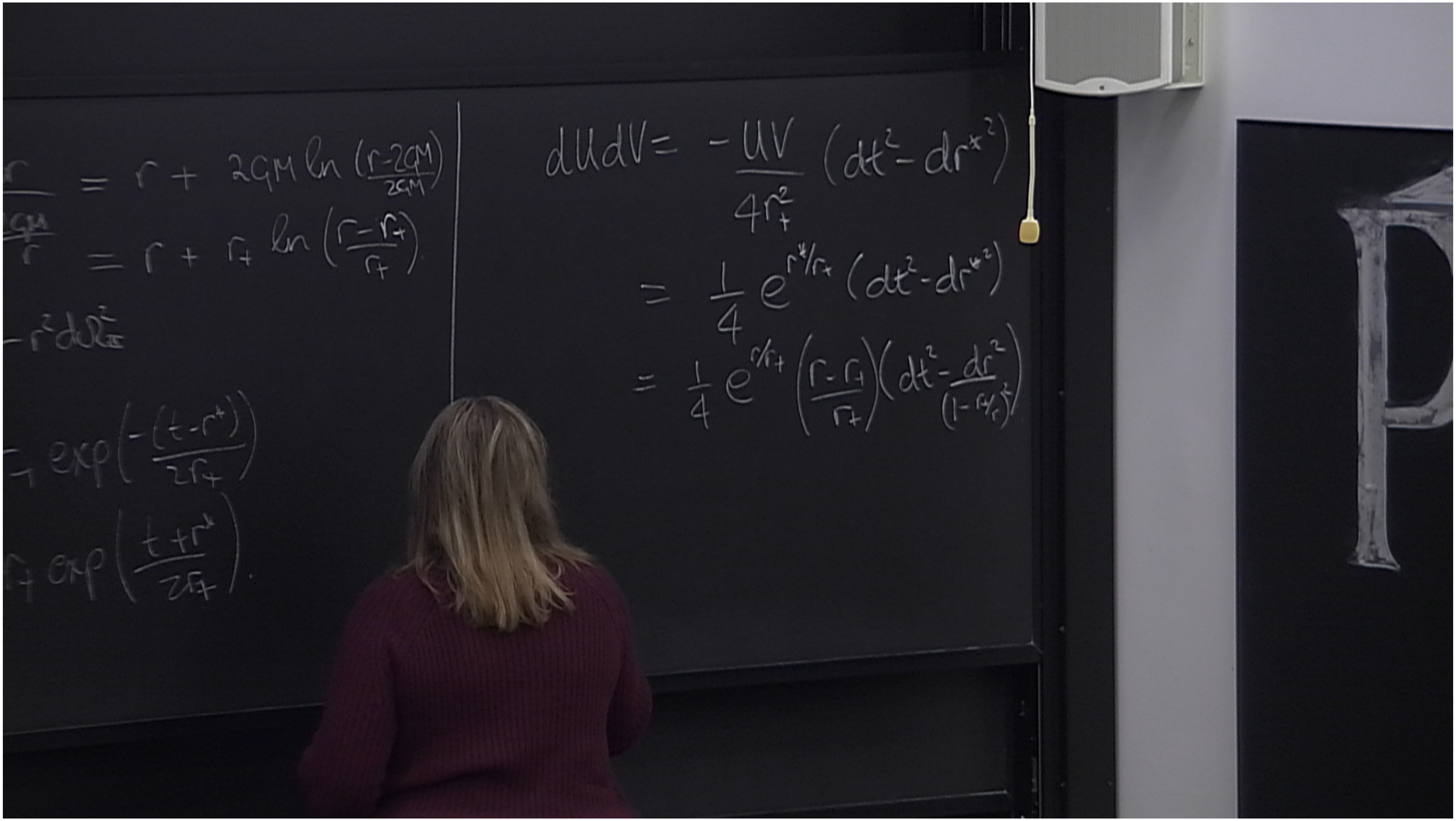
$$= r + r_+ \ln\left(\frac{r-r_+}{r_+}\right)$$

$$ds^2 \to V_s\left(dt^2 - dr^{*2}\right) - r^2 d\Omega^2_{II}$$

KRUSKALS:

$$U = -r_+ \exp\left(-\frac{(t-r^*)}{2r_+}\right)$$

$$V = r_+ \exp\left(\frac{t+r^*}{2r_+}\right).$$


$= r + 2GM \ln\left(\frac{r-2GM}{2GM}\right)$
$= r + r_+ \ln\left(\frac{r-r_+}{r_+}\right)$
$dUdV = -\frac{UV}{4r_+^2}(dt^2 - dr^{*2})$
$= \frac{1}{4} e^{r^*/r_+}(dt^2 - dr^{*2})$
$= \frac{1}{4} e^{r/r_+}\left(\frac{r-r_+}{r_+}\right)\left(dt^2 - \frac{dr^2}{(1-r_+/r)^2}\right)$

$$GM \ln\left(\frac{r-2GM}{2GM}\right)$$

$$\ln\left(\frac{r-r_+}{r_+}\right)$$

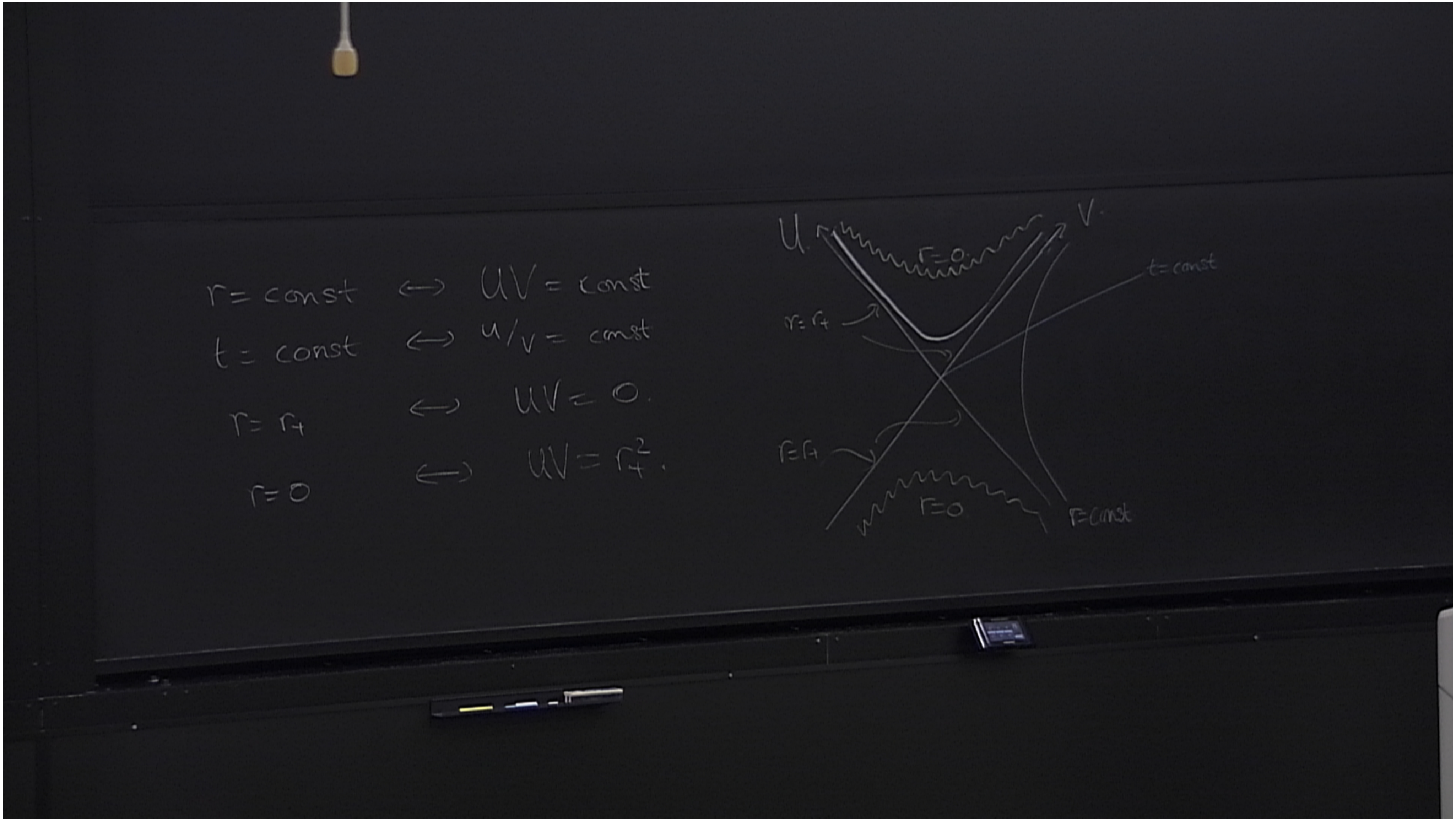
$$dU dV = -\frac{UV}{4r_+^2}\left(dt^2 - dr^{*2}\right)$$

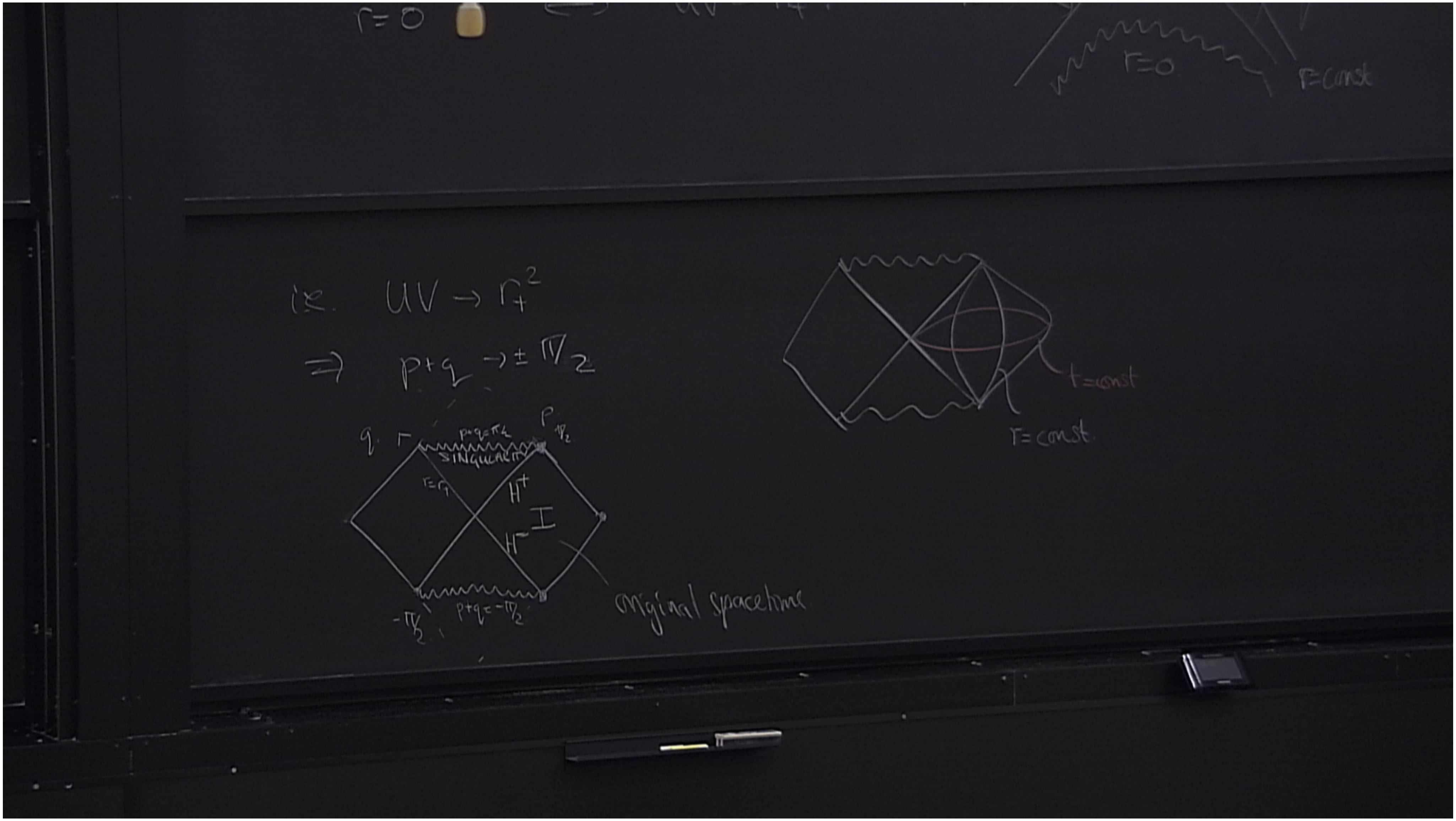
$$= \frac{1}{4} e^{r^*/r_+}\left(dt^2 - dr^{*2}\right)$$

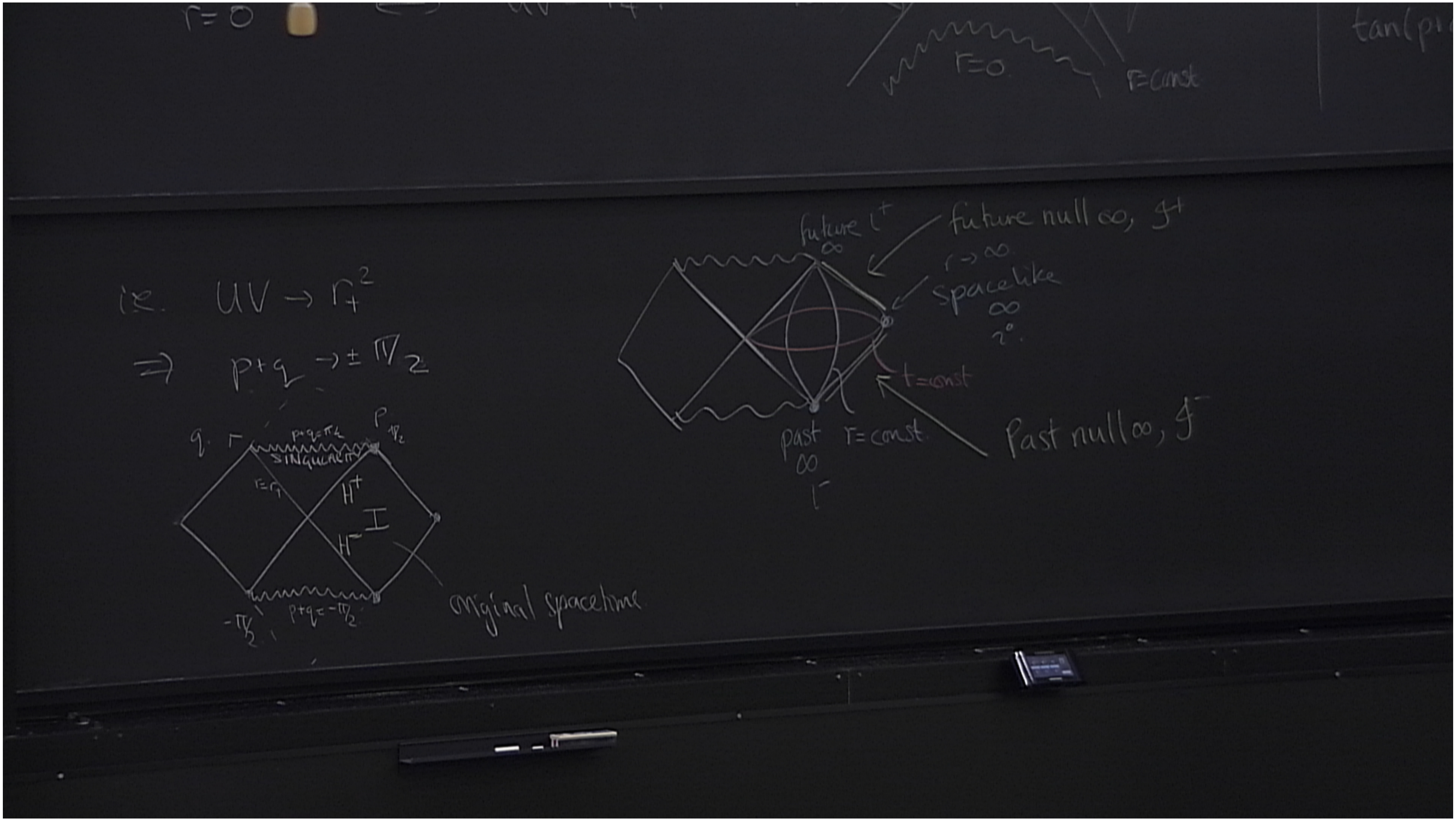
$$= \frac{1}{4} e^{r/r_+}\left(\frac{r-r_+}{r_+}\right)\left(dt^2 - \frac{dr^2}{(1-r_+/r)^2}\right)$$

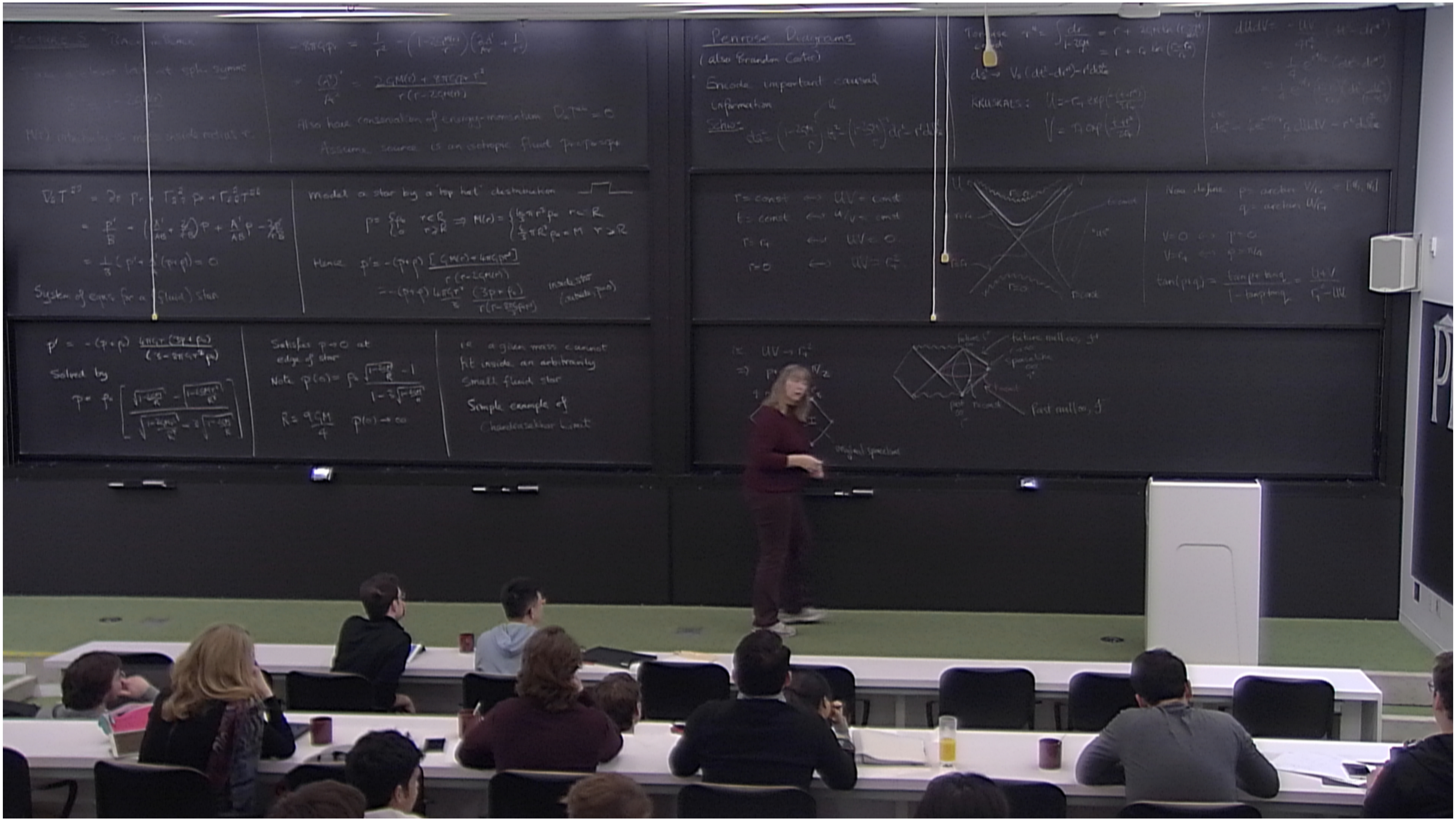
i.e.

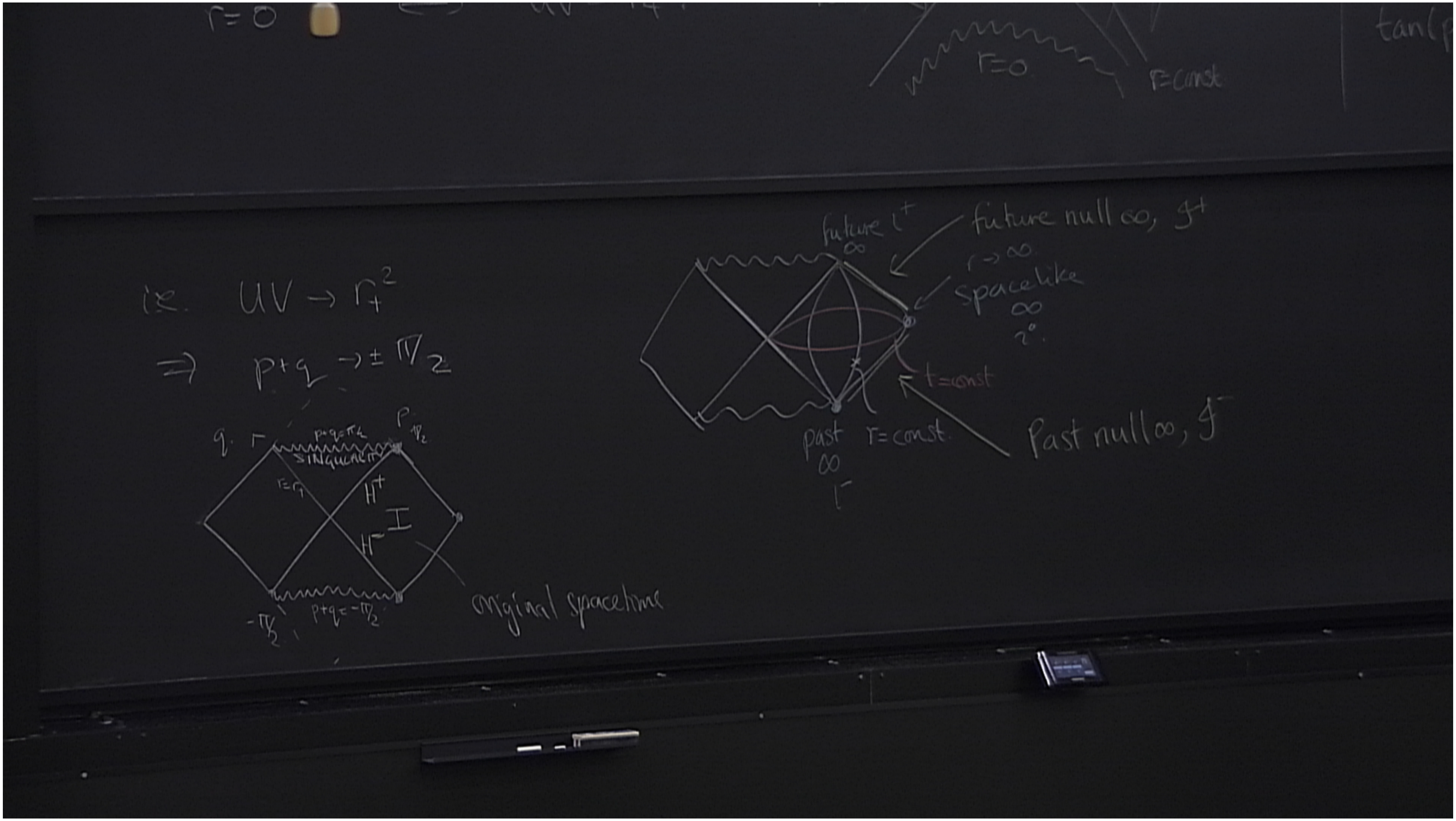
$$ds^2 = 4e^{-r/r_+}\frac{r_+}{r}\, dU dV - r^2 d\Omega_{\mathrm{II}}^2$$

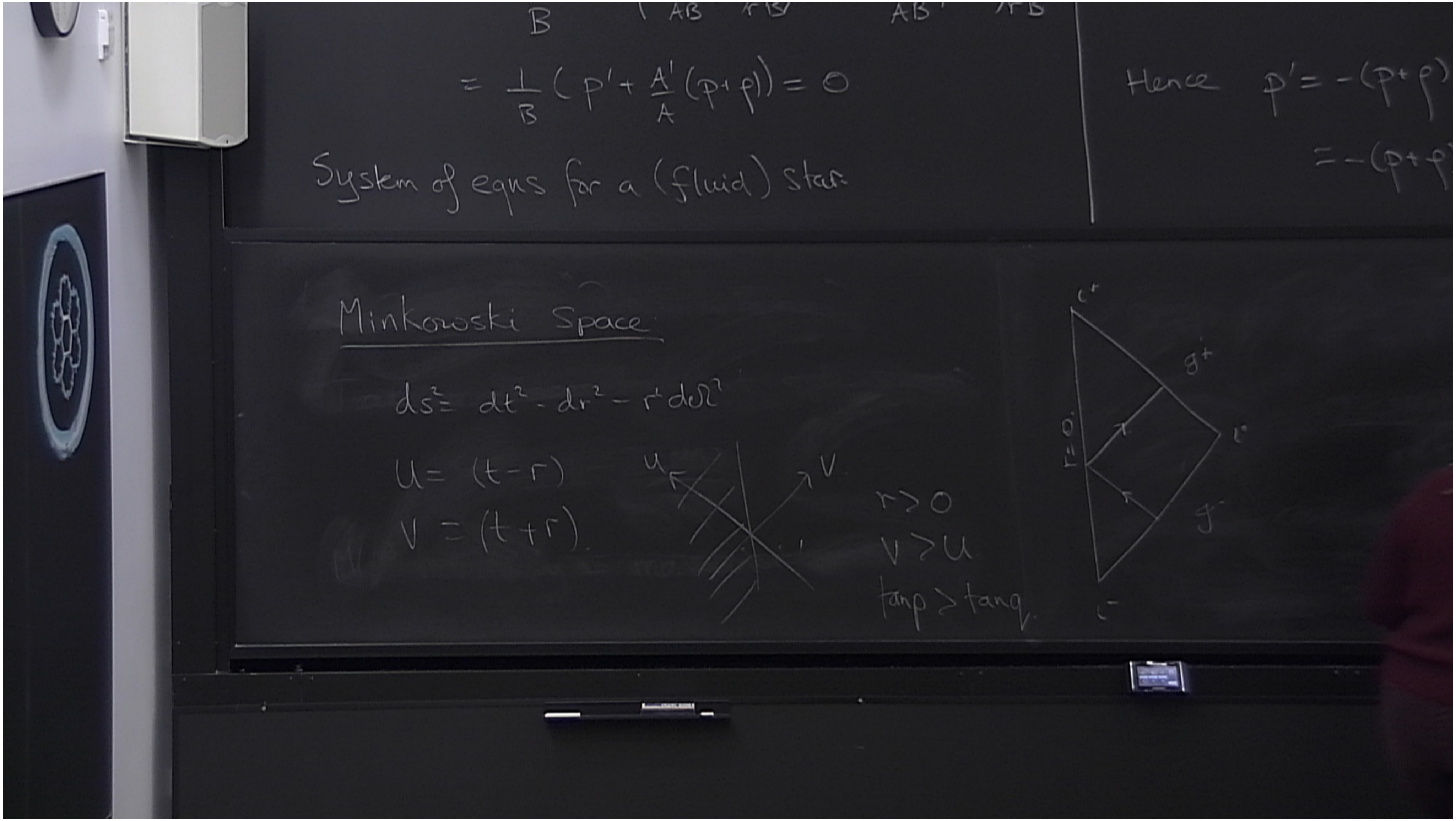

$r = \text{const} \longleftrightarrow UV = \text{const}$
$t = \text{const} \longleftrightarrow U/V = \text{const}$
$r = r_+ \longleftrightarrow UV = 0.$
$r = 0 \longleftrightarrow UV = r_+^2.$
U.
V.
r=0
t=const
r=r+
r=r+
r=0
r=const


r=0
r=0
r=const
i.e. UV → r_+^2
⇒ p+q → ±π/2
q
r
p
π/2
p+q=π/2
SINGULARITY
r=r_+
H+
H−
I
original spacetime
−π/2
p+q=−π/2
t=const
r=const


r=0
r=0
r=const
i.e. UV → r+²
⇒ p+q → ±π/2
future i+
∞
future null ∞, J+
r → ∞
spacelike
∞
i°
t=const
past
∞
i-
r=const.
Past null ∞, J-
q
r
p
π/2
p+q=π/2
SINGULARITY
r=r+
H+
H-
I
original spacetime
p+q=-π/2
-π/2


Penrose Diagrams
(also Brandon Carter)
Encode important causal information
Model a star by a 'top hat' distribution
Simple example of Chandrasekhar Limit


i.e. UV → r+²
⇒ p+q → ± π/2
SINGULARITY
H+
H−
I
original spacetime
future i+ ∞
future null ∞, J+
r → ∞
spacelike ∞ i°
t=const
r=const
past ∞ i−
Past null ∞, J−


= 1/B (p' + A'/A (p+ρ)) = 0
System of eqns for a (fluid) star:
Hence p' = -(p+ρ)
= -(p+ρ
Minkowski Space
ds² = dt² - dr² - r² dΩ²
u = (t - r)
v = (t + r)
u
v
r > 0
v > u
tan p > tan q
i+
𝒥+
i0
r = 0
𝒥-
i-

$$r\,(r-2GM(r))$$

star:

$$= -(p+\rho)\,\frac{4\pi G r^3}{3}\,\frac{(3p+\rho_0)}{r\left(r-\frac{8\pi G\rho_0 r^3}{3}\right)}$$

inside star
(outside, $p=0$)

<u>de Sitter</u>

$$A^2 = B^{-2} = 1 - \frac{2GM}{r} - \frac{r^2}{\ell^2} \qquad \Lambda = \frac{3}{\ell^2}$$

$g_{tt} \to 0$ not just at $r \approx 2GM$ ← black hole

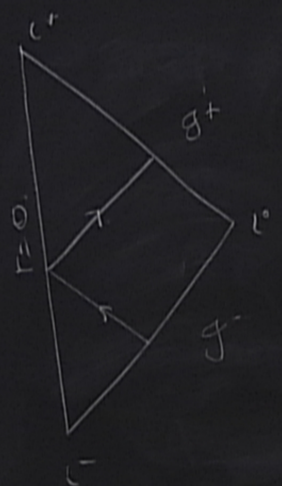
also at $r \approx \ell$ ← cosmological event h

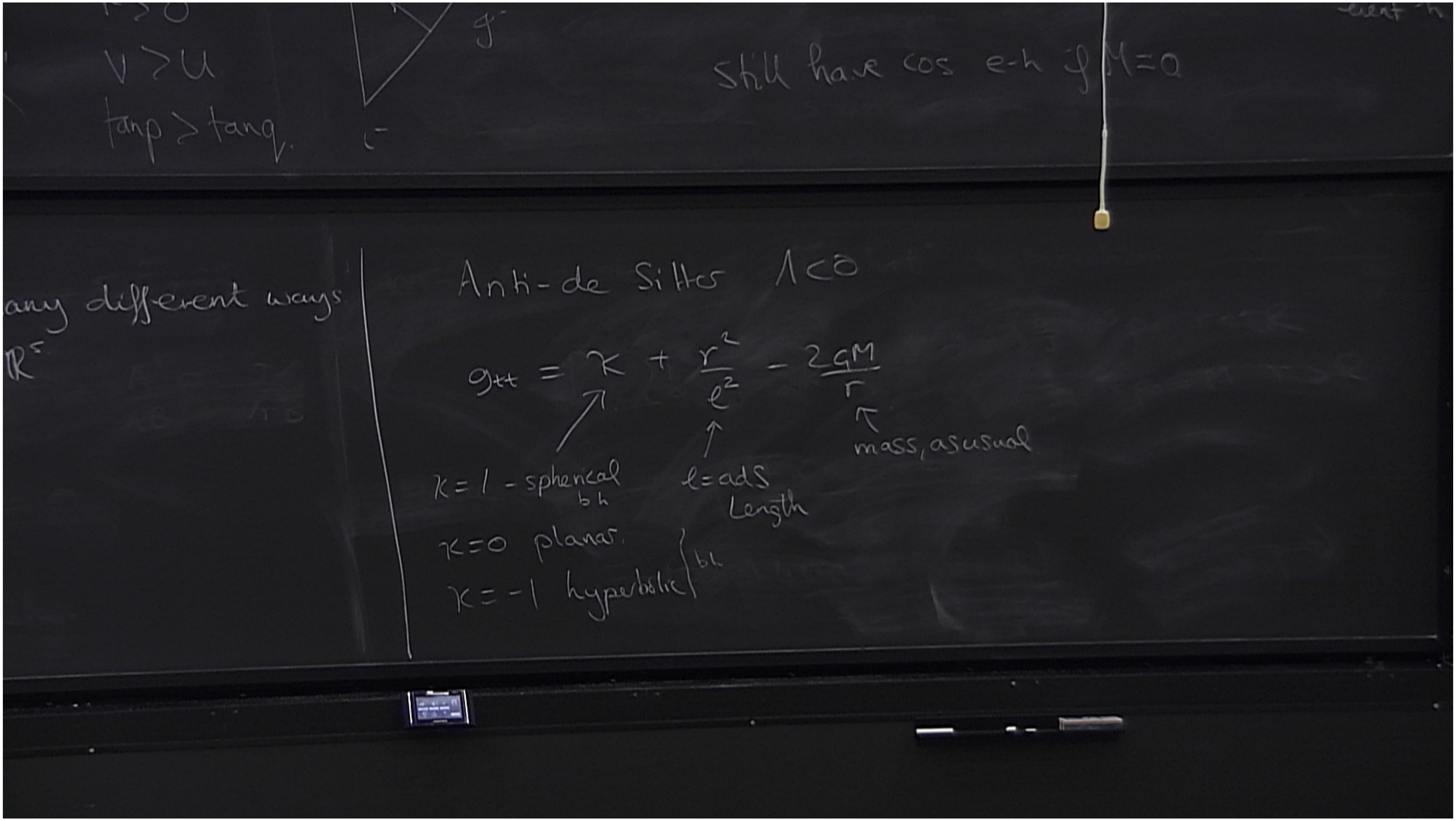
V

$r > 0$

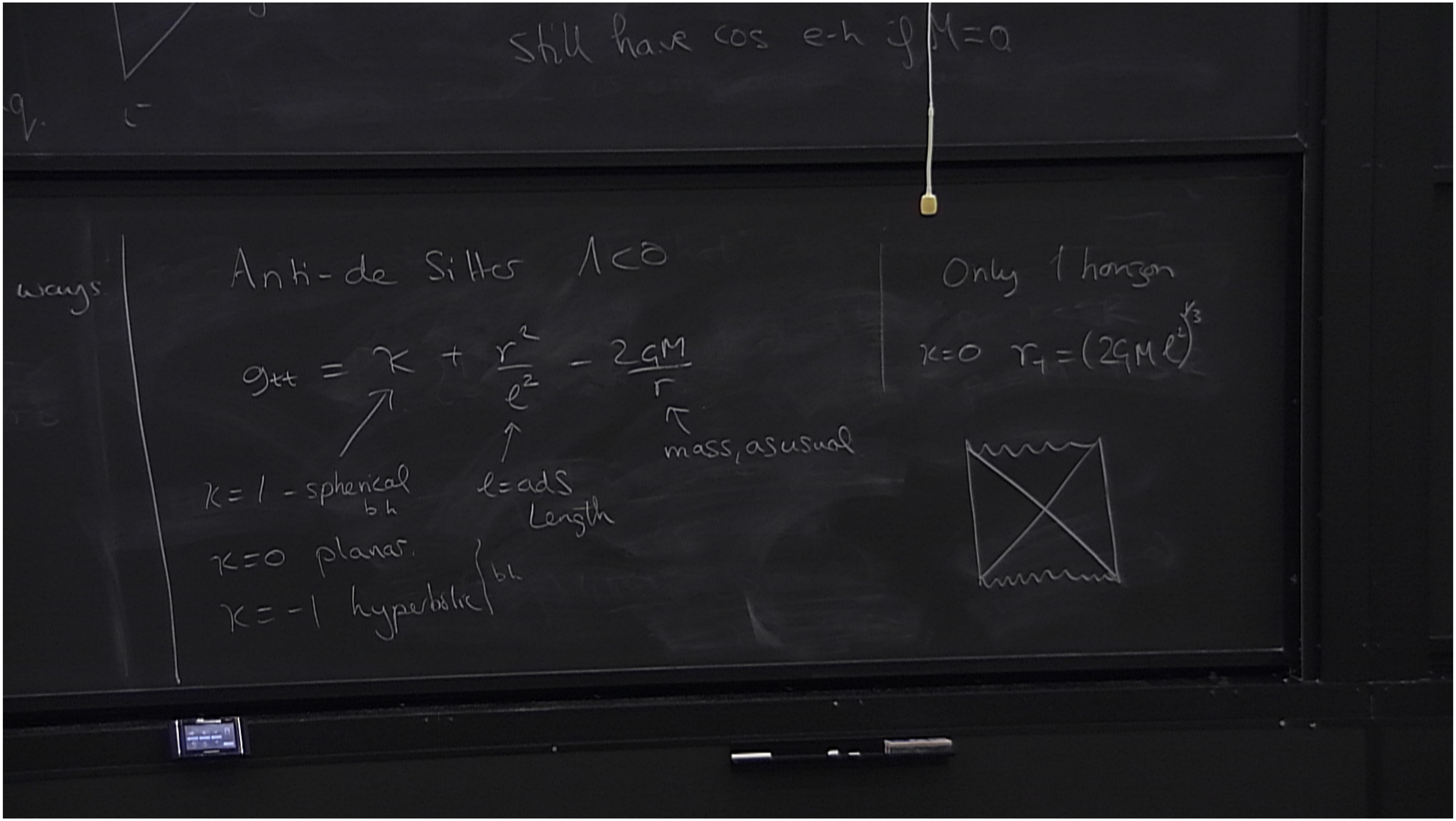
$V > U$

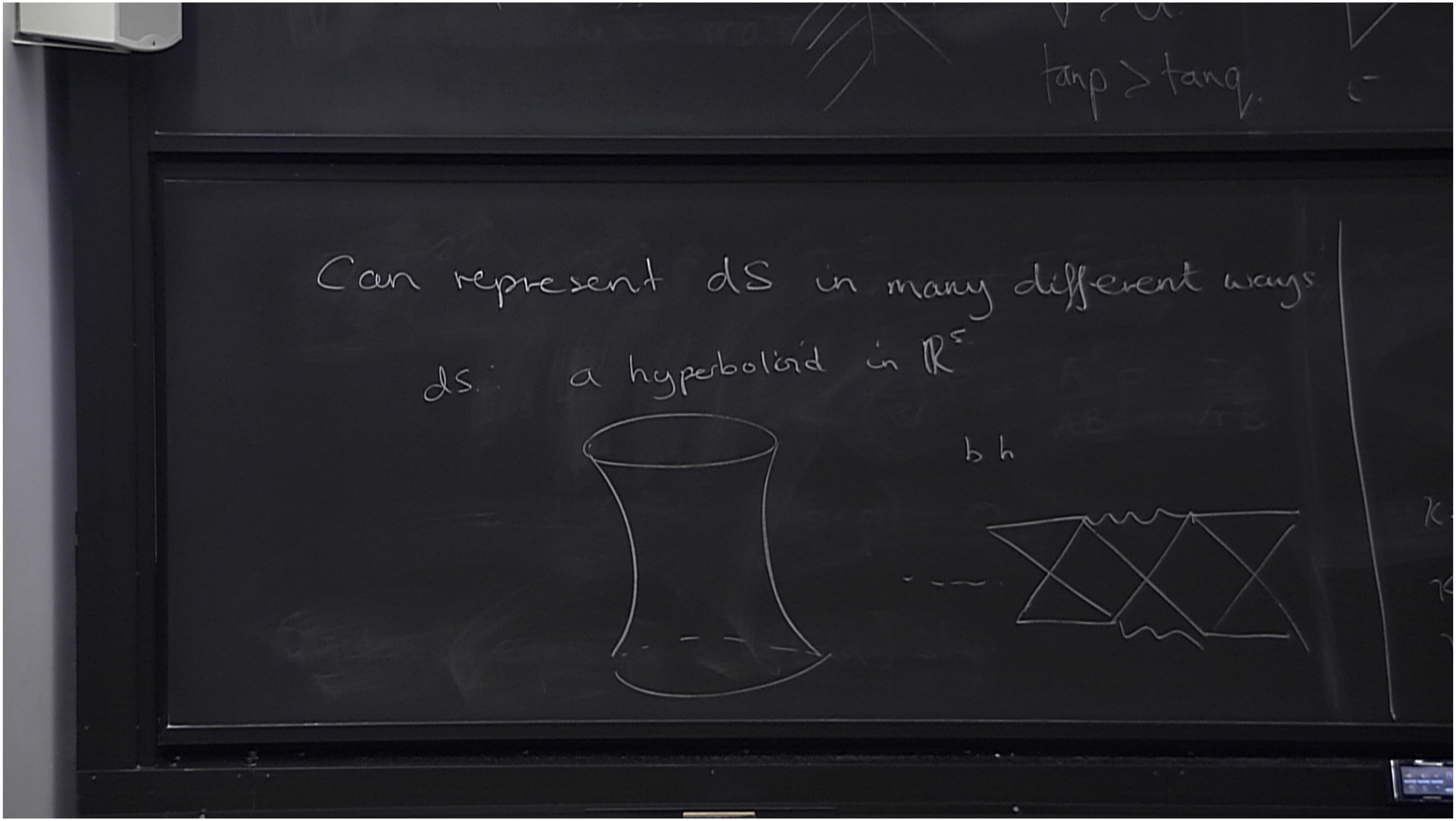
$\tan p > \tan q$

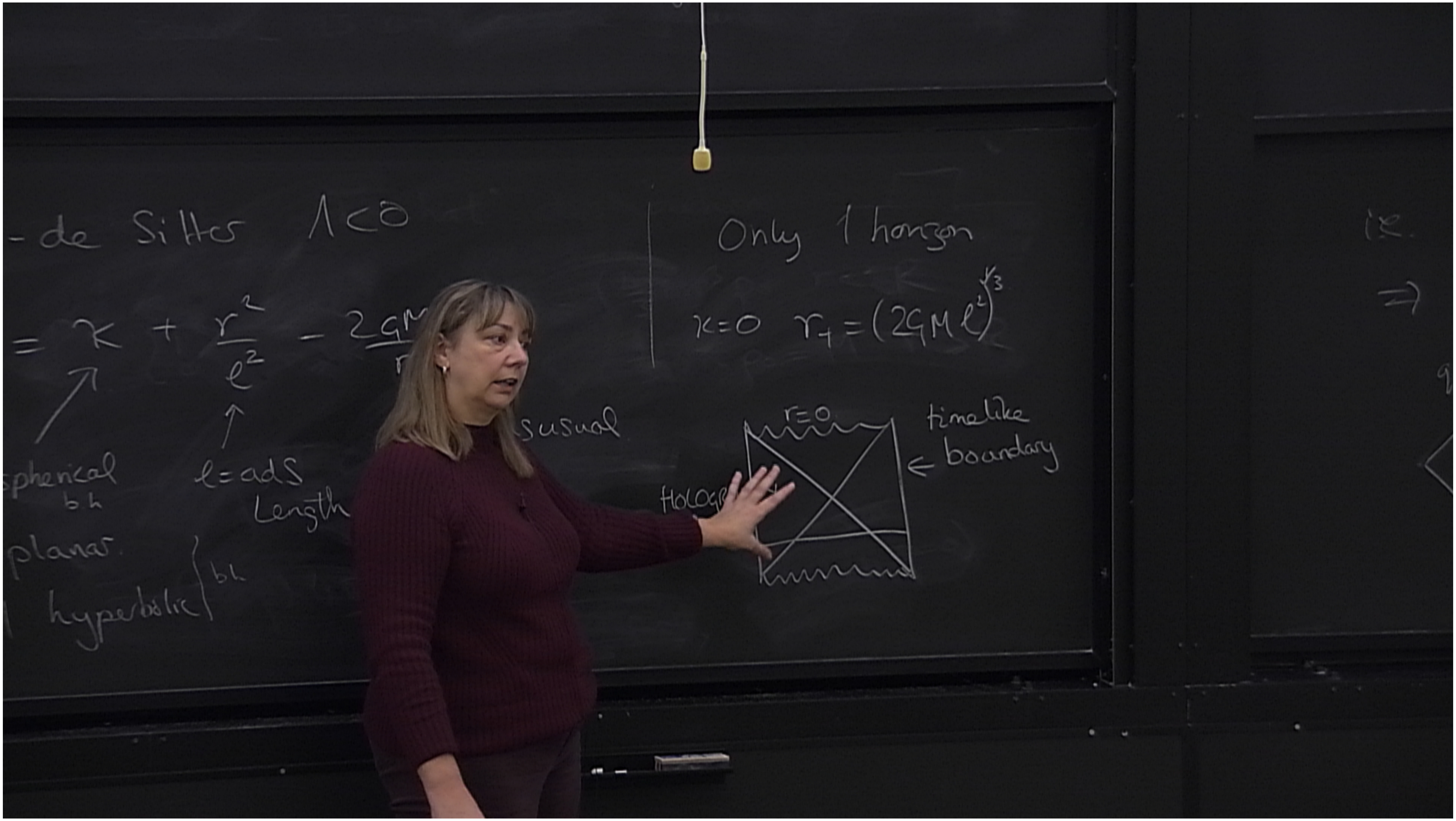




V > U
tanp > tanq.
Still have cos e-h if M=0
any different ways
ℝ5
Anti-de Sitter Λ<0
$g_{tt} = \kappa + \frac{r^2}{\ell^2} - \frac{2GM}{r}$
κ=1 - spherical bh
ℓ=adS Length
mass, as usual
κ=0 planar
κ=-1 hyperbolic
bh


Still have cos e-h if M=0
ways
Anti-de Sitter $\Lambda<0$
$g_{tt} = \kappa + \frac{r^2}{\ell^2} - \frac{2GM}{r}$
$\kappa=1$ - spherical bh
ℓ=adS Length
mass, as usual
$\kappa=0$ planar
$\kappa=-1$ hyperbolic bh
Only 1 horizon
$\kappa=0$ $r_+=(2GM\ell^2)^{1/3}$


tanp > tanq.
Can represent dS in many different ways
dS: a hyperboloid in R^5
b h


de Sitter Λ<0
Only 1 horizon
k=0 r+ = (2GMl²)^1/3
r=0
timelike
← boundary
spherical
bh
l = adS
Length
planar
hyperbolic
bh
usual
i.e.