Title: PSI 2015/2016 Gravitational Physics - Lecture 2

Date: Jan 05, 2016 09:00 AM

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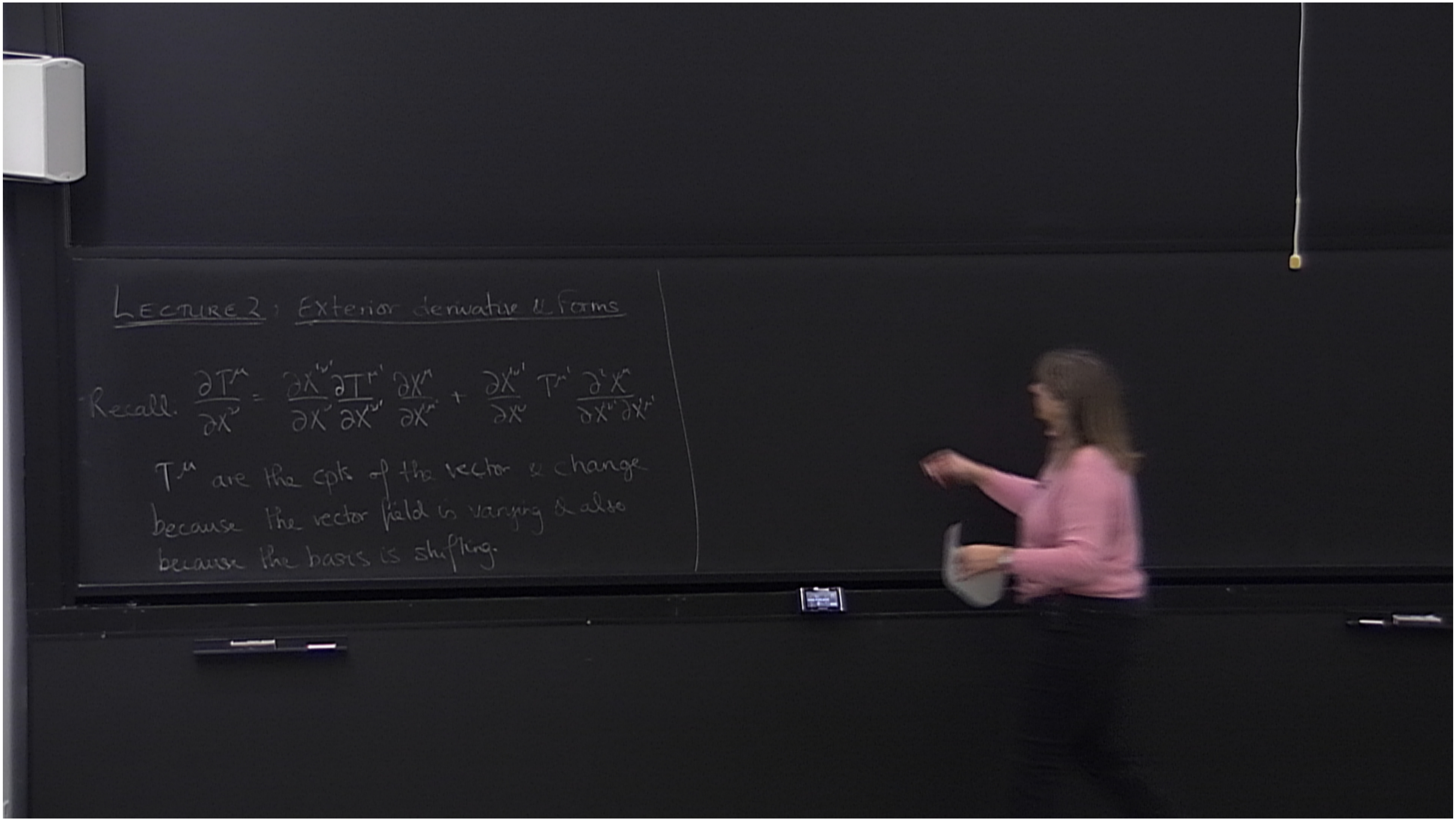
Abstract:

LECTURE 2: Exterior derivative & forms

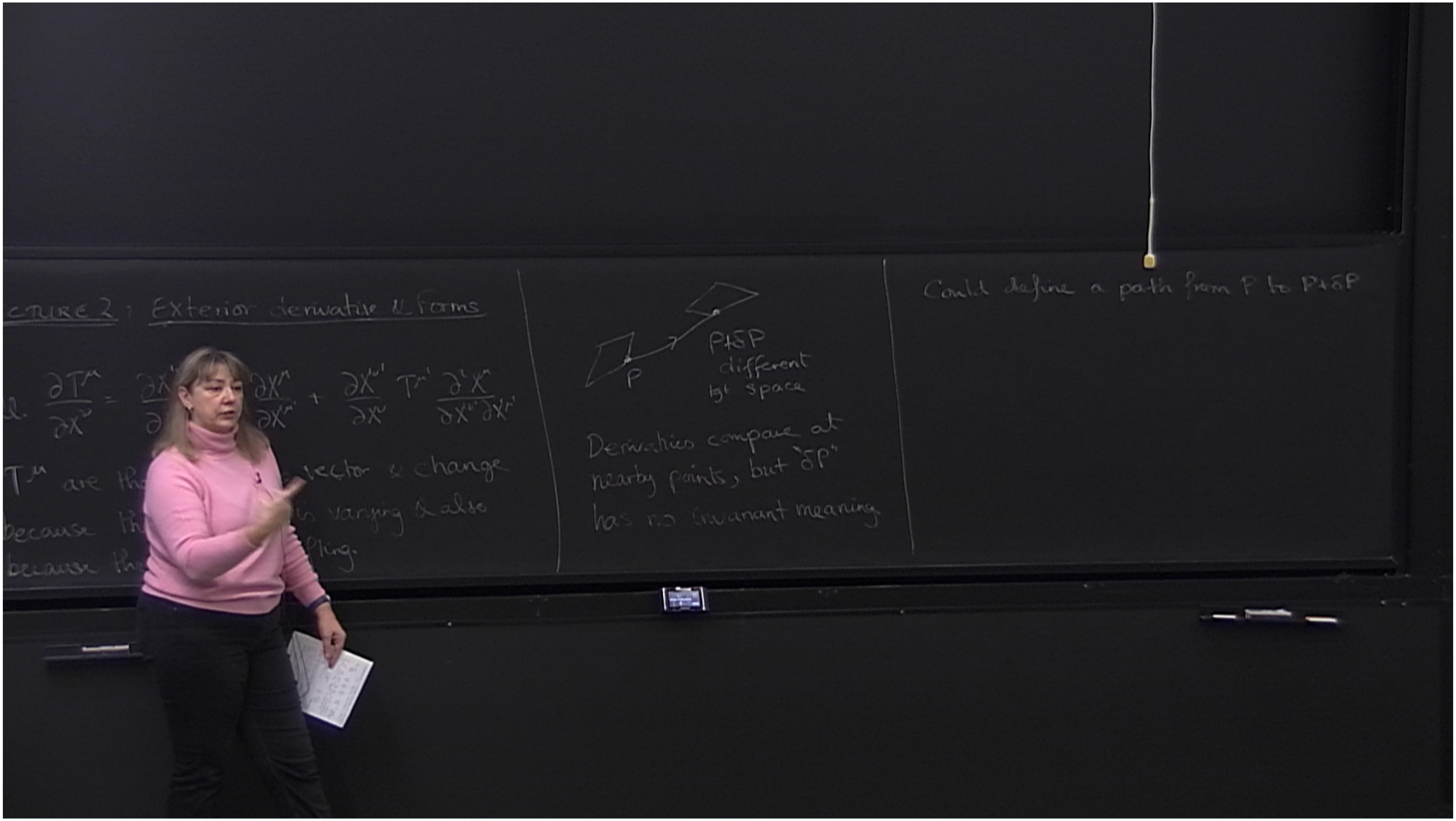
$$\frac{\partial T^\mu}{\partial x^\nu} = \frac{\partial x'^{\nu'}}{\partial x^\nu} T^{\mu'} \frac{\partial x^\mu}{\partial x'^{\mu'}} + \frac{\partial x'^{\nu'}}{\partial x^\nu} T^{\mu'} \frac{\partial^2 x^\mu}{\partial x'^{\nu'} \partial x'^{\mu'}}$$



LECTURE 2: Exterior derivative & forms

Recall. $$\frac{\partial T^{\mu}}{\partial x^{\nu}} = \frac{\partial x^{\nu'}}{\partial x^{\nu}} \frac{\partial T^{\mu'}}{\partial x^{\nu'}} \frac{\partial x^{\mu}}{\partial x^{\mu'}} + \frac{\partial x^{\nu'}}{\partial x^{\nu}} T^{\mu'} \frac{\partial^2 x^{\mu}}{\partial x^{\nu'} \partial x^{\mu'}}$$

T^{μ} are the cpts of the vector & change because the vector field is varying & also because the basis is shifting.



CTURE 2: Exterior derivative & forms

$\frac{\partial T^\mu}{\partial X^\nu} = \frac{\partial X^{\nu'}}{\partial} \ldots \frac{\partial X^\mu}{\partial X^{\mu'}} + \frac{\partial X^{\nu'}}{\partial X^\nu} T^{\mu'} \frac{\partial^2 X^\mu}{\partial X^{\nu'} \partial X^{\mu'}}$

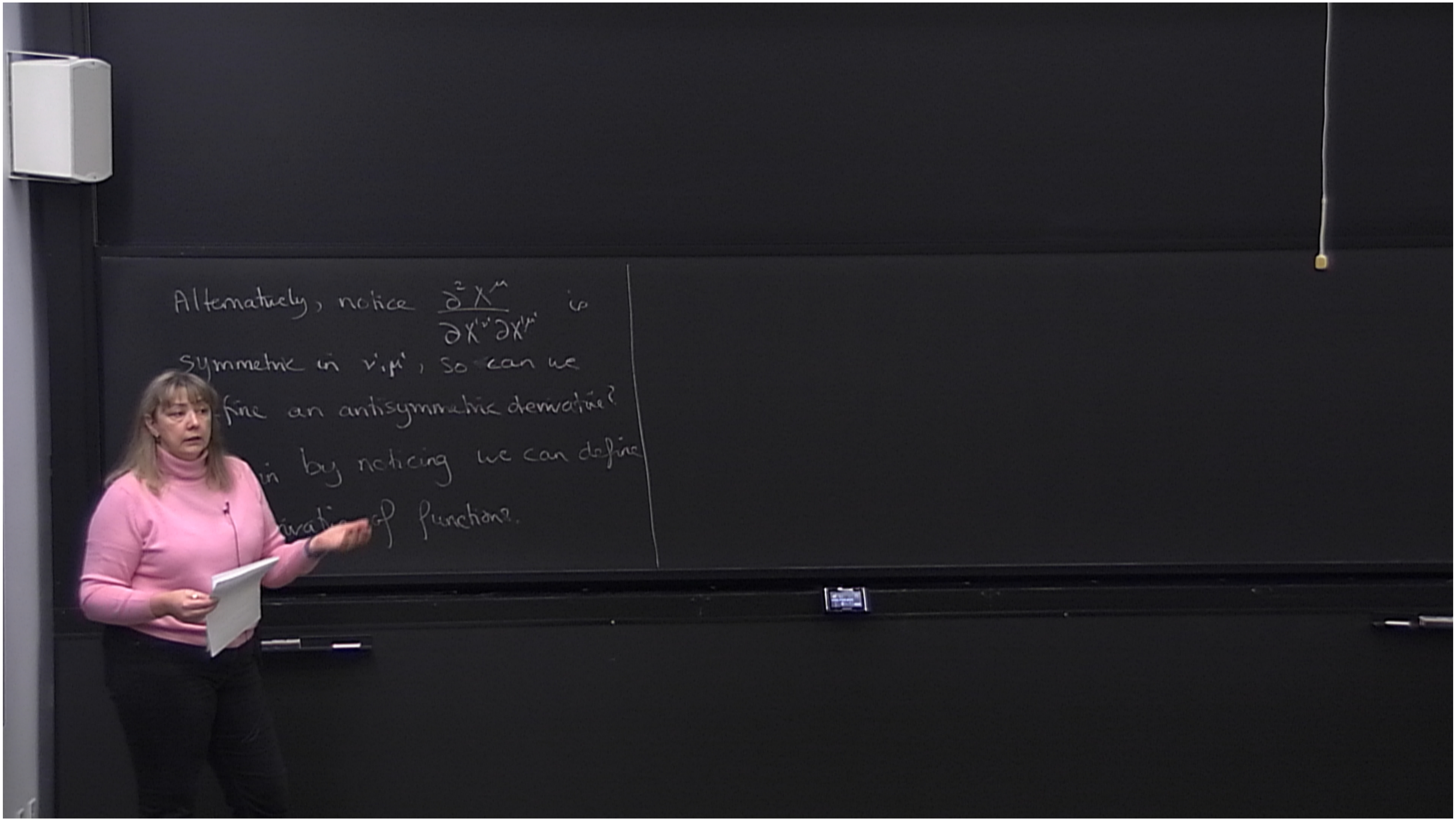
T^μ are th … vector & change

because th … is varying & also

because th … pling.

Derivatives compare at nearby points, but "δP" has no covariant meaning.

Could define a path from P to P+δP



Alternatively, notice $\frac{\partial^2 X^\mu}{\partial X'^{\nu'} \partial X'^{\mu'}}$ is

symmetric in ν', μ', so can we

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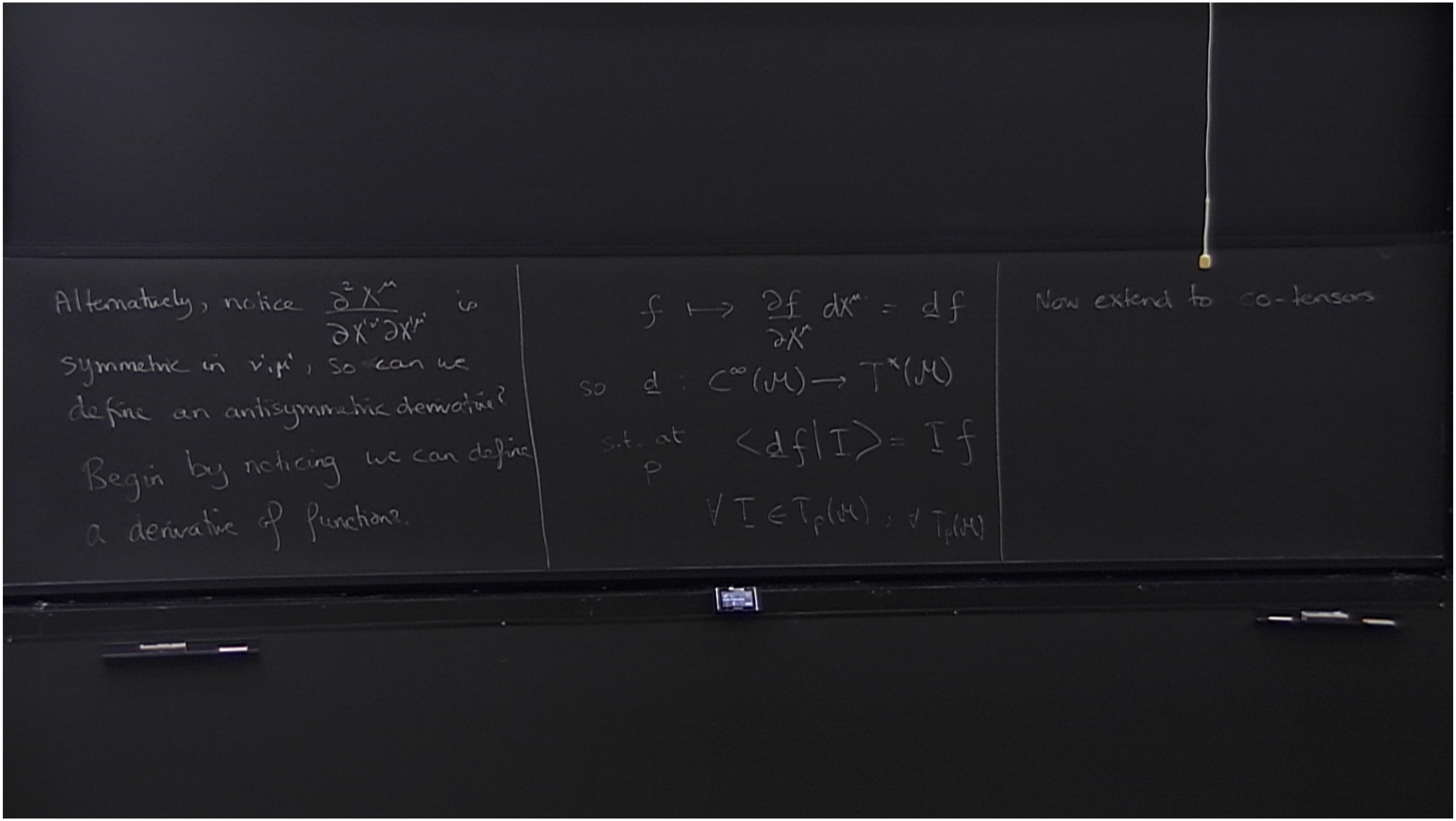


Alternatively, notice $\frac{\partial^2 X^\mu}{\partial X'^{\nu'} \partial X'^{\mu'}}$ is symmetric in ν', μ', so can we define an antisymmetric derivative?

Begin by noticing we can define a derivative of functions.

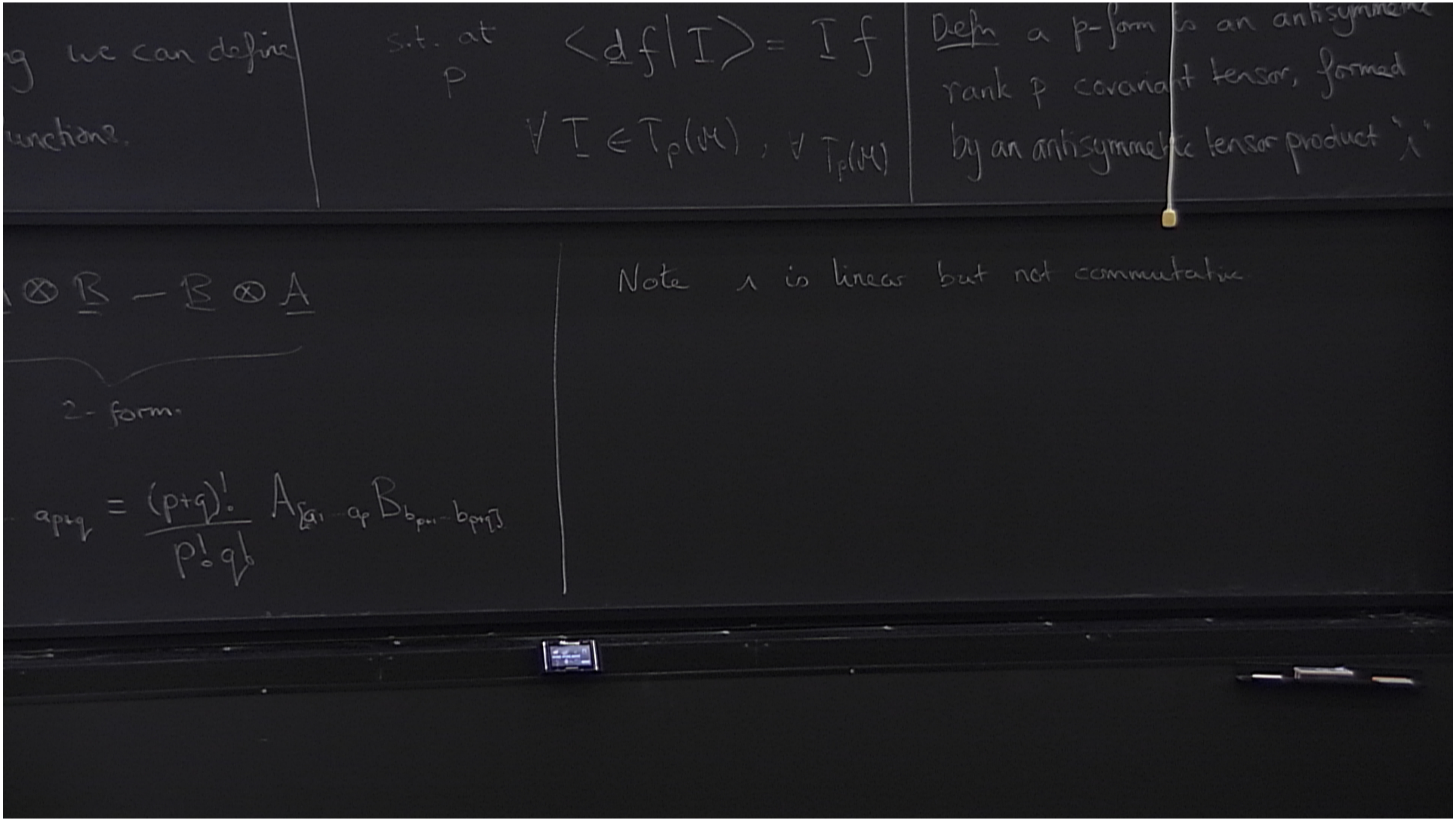
$$f \longmapsto \frac{\partial f}{\partial X^\mu} dX^\mu = \underline{d} f$$

so $\underline{d} : C^\infty(\mathcal{M}) \longrightarrow T^*(\mathcal{M})$

s.t. at p $\langle \underline{d} f | \underline{I} \rangle = \underline{I} f$

$\forall\, \underline{I} \in T_p(\mathcal{M})$; $\forall\, T_p(\mathcal{M})$

Now extend to co-tensors



g we can define

unctions.

s.t. at p $\langle \underline{d} f | \underline{I} \rangle = \underline{I} f$

$\forall \underline{I} \in T_p(M)$, $\forall T_p(M)$

Defn a p-form is an antisymme

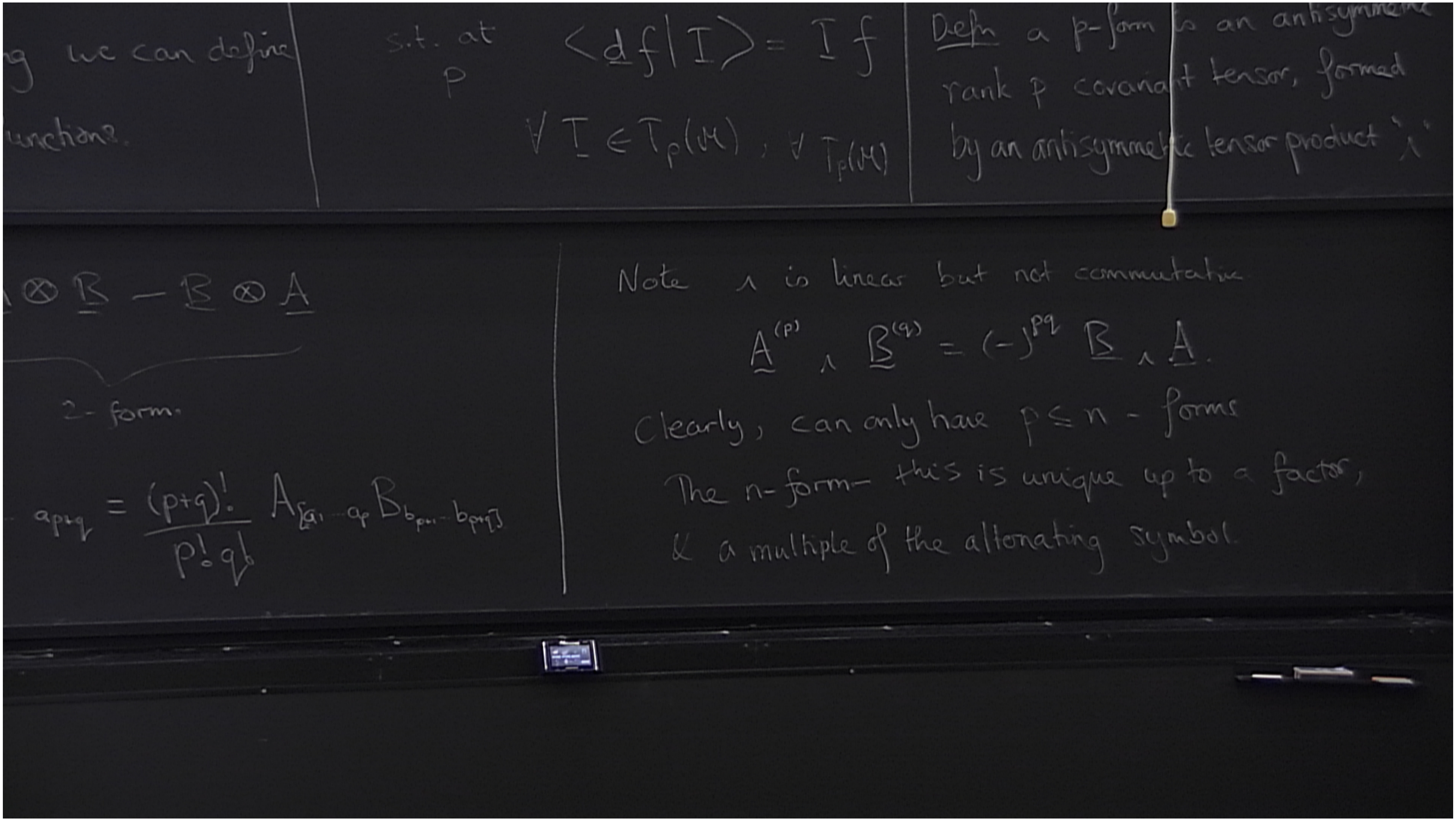
rank p covariant tensor, formed

by an antisymmetric tensor product '$\wedge$'

$$\underbrace{\otimes \underline{B} - \underline{B} \otimes \underline{A}}_{\text{2-form.}}$$

$$\ldots a_{p+q} = \frac{(p+q)!}{p!q!} A_{[a_1 \ldots a_p} B_{b_{p+1} \ldots b_{p+q}]}$$

Note $\wedge$ is linear but not commutative.



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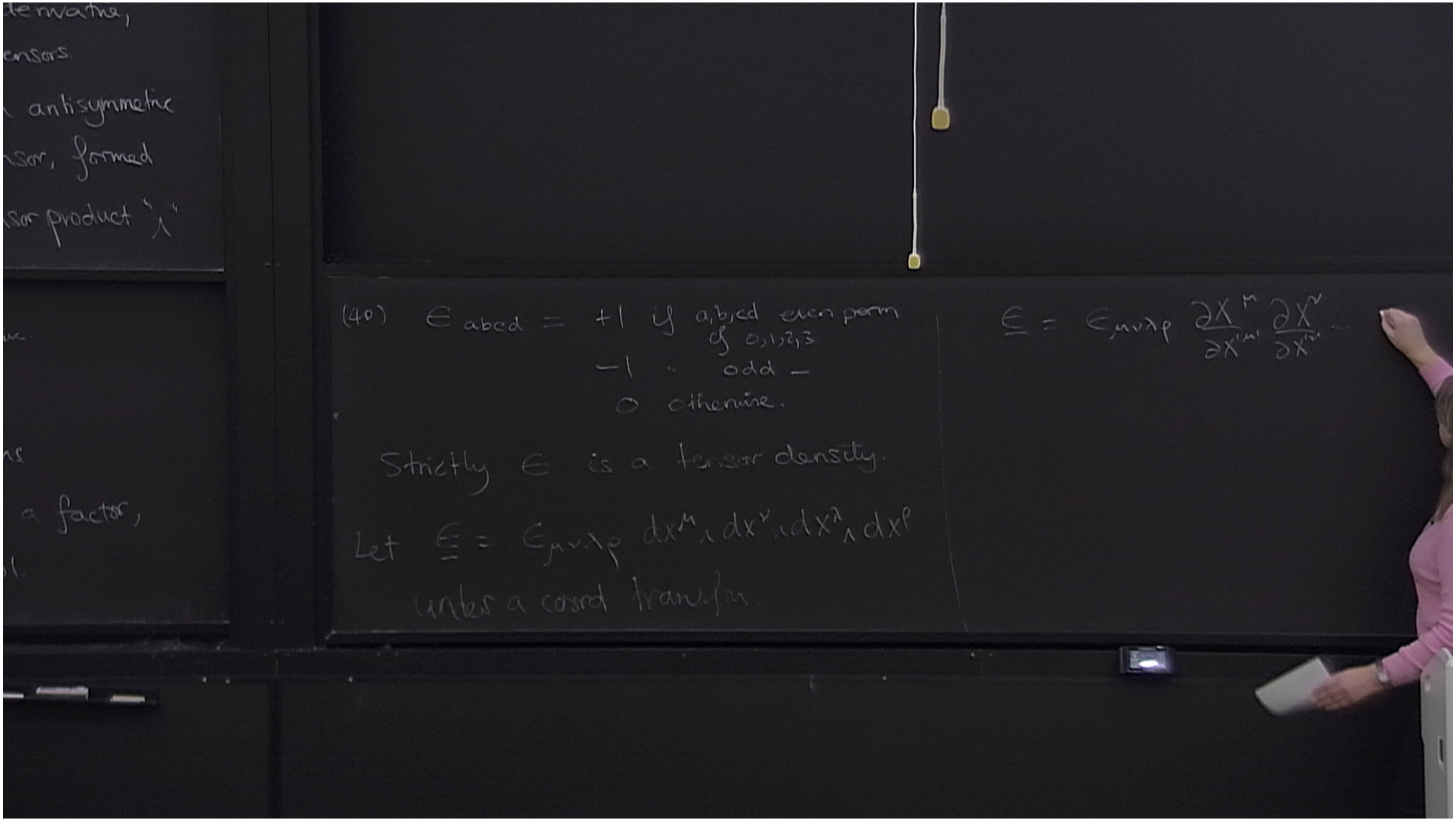
$$\underline{A}^{(p)} \wedge \underline{B}^{(q)} = (-)^{pq}\, \underline{B} \wedge \underline{A}.$$

Clearly, can only have $p \leq n$ - forms

The n-form — this is unique up to a factor, & a multiple of the alternating symbol.



derivative,

ensors

antisymmetric

nsor, formed

nsor product "∧"

a factor,

(40) $\epsilon_{abcd} = +1$ if a,b,c,d even perm of 0,1,2,3

-1 " odd –

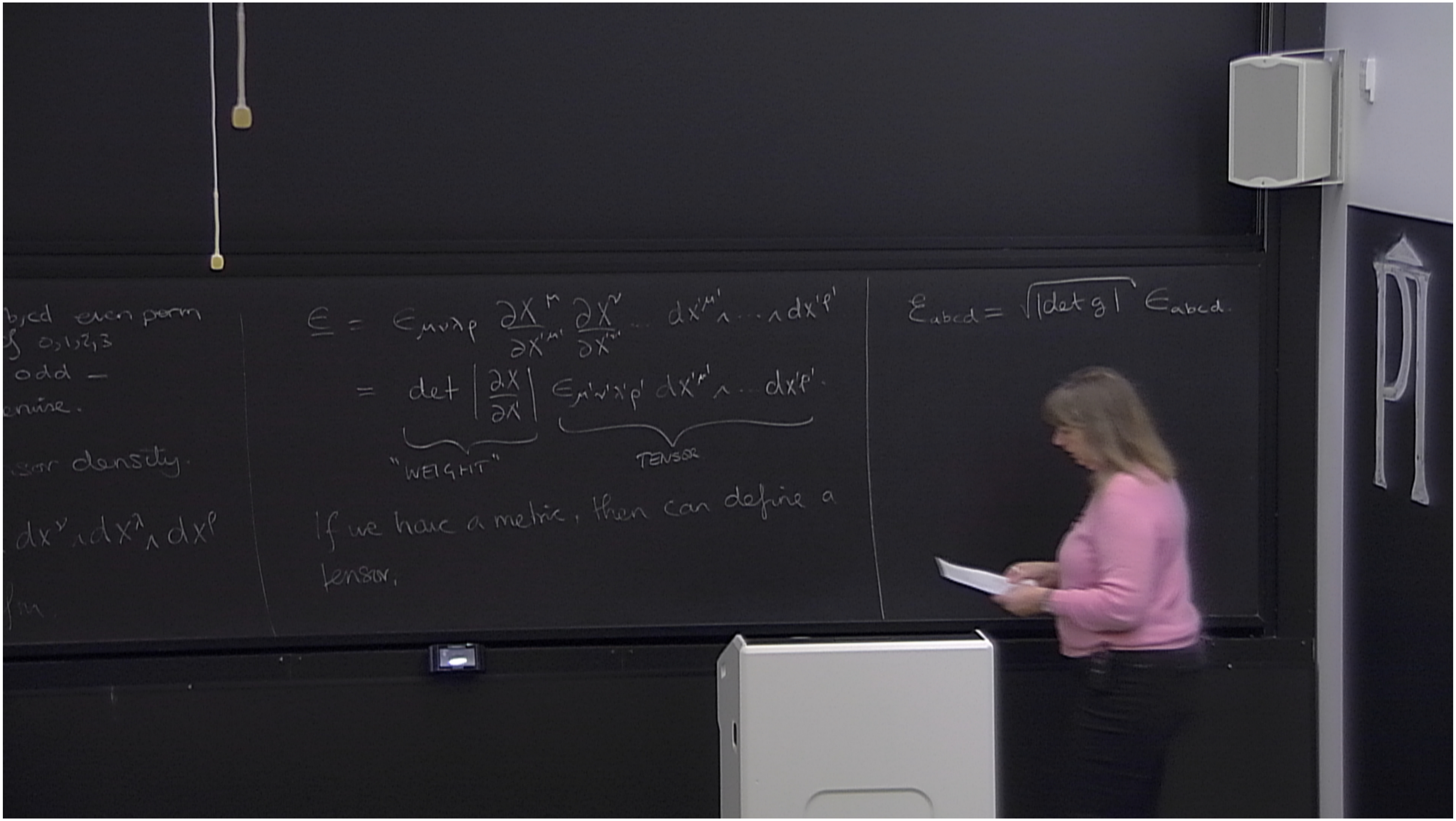
0 otherwise.

Strictly ϵ is a tensor density.

Let $\underline{\epsilon} = \epsilon_{\mu\nu\lambda\rho}\, dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\rho$

under a coord transfm.

$$\underline{\epsilon} = \epsilon_{\mu\nu\lambda\rho} \frac{\partial x^\mu}{\partial x'^{\mu'}} \frac{\partial x^\nu}{\partial x'^{\nu'}} \cdots$$



b,cd even perm

f 0,1,2,3

odd −

enwise.

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$dx^\nu \wedge dx^\lambda \wedge dx^\rho$

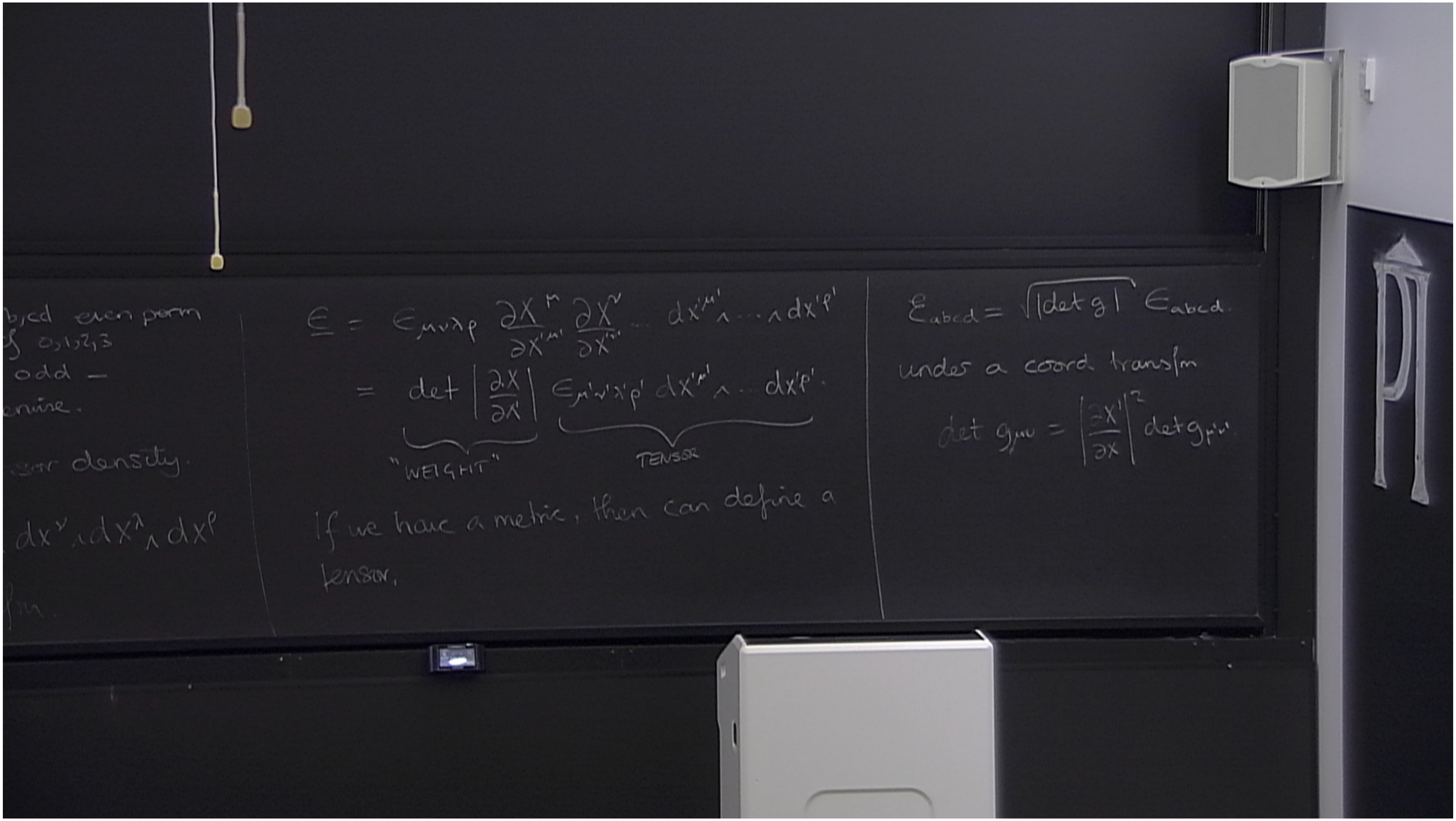
fm.

$$\underline{\epsilon} = \epsilon_{\mu\nu\lambda\rho} \frac{\partial x^\mu}{\partial x'^{\mu'}} \frac{\partial x^\nu}{\partial x'^{\nu'}} \cdots dx'^{\mu'} \wedge \cdots \wedge dx'^{\rho'}$$

$$= \underbrace{\det\left|\frac{\partial x}{\partial x'}\right|}_{\text{"WEIGHT"}} \underbrace{\epsilon_{\mu'\nu'\lambda'\rho'}\, dx'^{\mu'} \wedge \cdots dx'^{\rho'}}_{\text{TENSOR}}.$$

If we have a metric, then can define a tensor.

$$\varepsilon_{abcd} = \sqrt{|\det g|}\; \epsilon_{abcd}.$$



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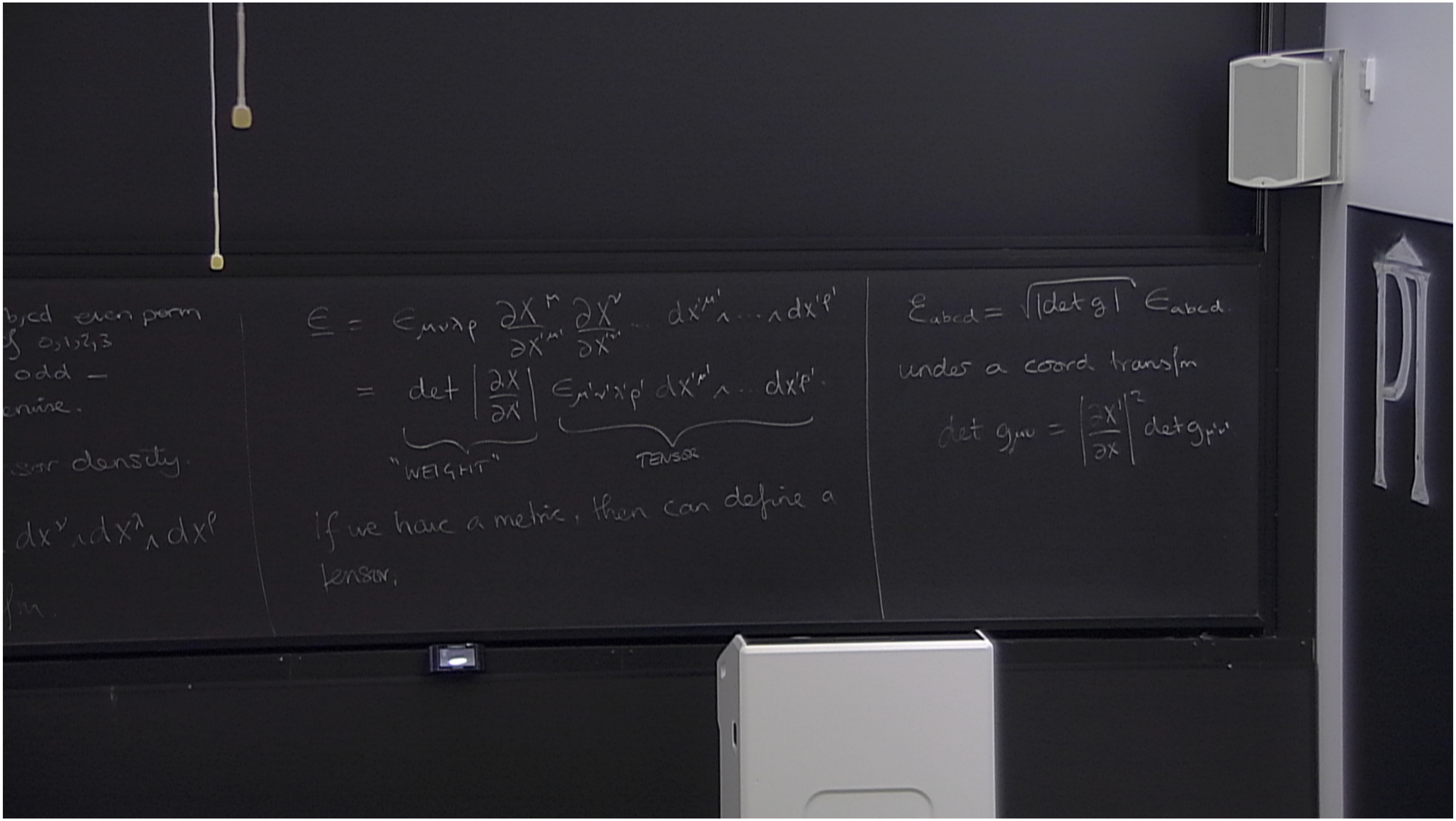
under a coord transfm

$$\det g_{\mu\nu} = \left| \frac{\partial x'}{\partial x} \right|^{2} \det g_{\mu'\nu'}$$



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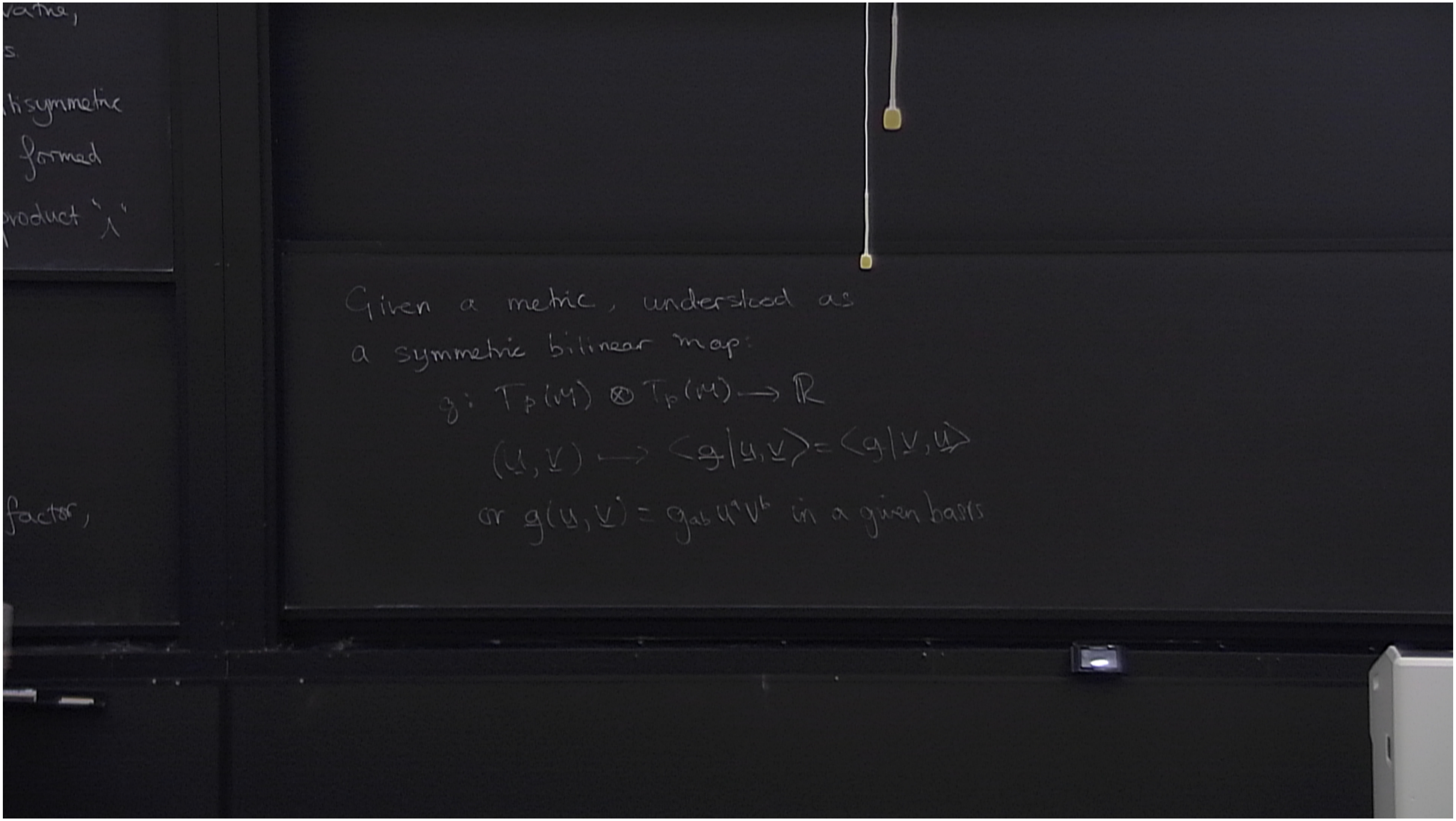


vative,

s.

tisymmetric

formed

product "$\wedge$"

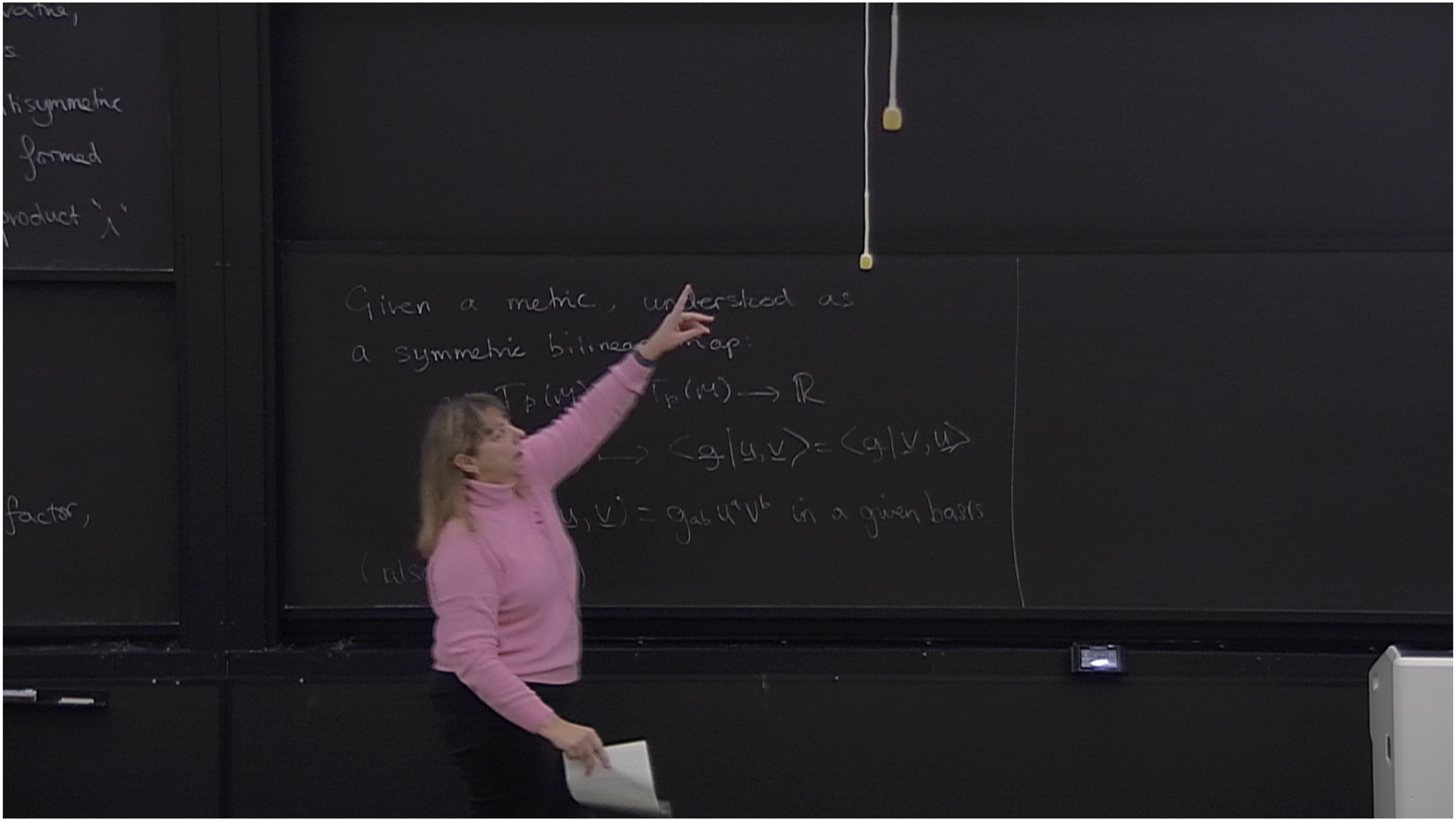
factor,

Given a metric, understood as a symmetric bilinear map:

$$g : T_p(M) \otimes T_p(M) \longrightarrow \mathbb{R}$$

$$(\underline{u}, \underline{v}) \longmapsto \langle \underline{g} | \underline{u}, \underline{v} \rangle = \langle g | \underline{v}, \underline{u} \rangle$$

or $\underline{g}(\underline{u}, \underline{v}) = g_{ab} u^a V^b$ in a given basis



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Given a metric, understood as

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$T_p(M) \quad T_p(M) \longrightarrow \mathbb{R}$

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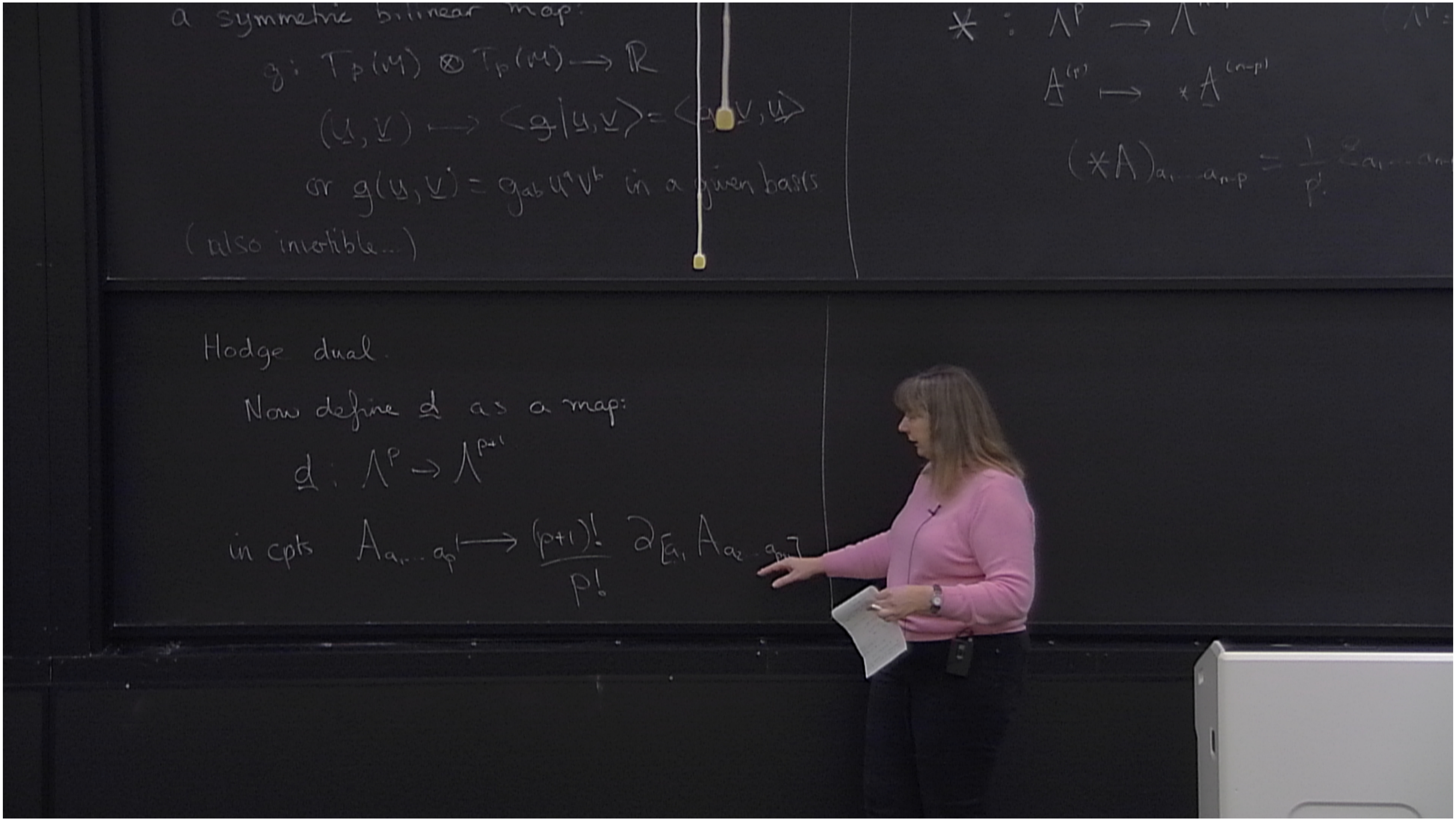


a symmetric bilinear map:

$g : T_p(M) \otimes T_p(M) \longrightarrow \mathbb{R}$

$(\underline{u}, \underline{v}) \longmapsto \langle \underline{g} | \underline{u}, \underline{v} \rangle = \langle \underline{g} | \underline{v}, \underline{u} \rangle$

or $\underline{g}(\underline{u}, \underline{v}) = g_{ab} u^a v^b$ in a given basis

(also invertible...)

$* : \Lambda^p \longrightarrow \Lambda^{n-p}$

$\underline{A}^{(p)} \longmapsto * \underline{A}^{(n-p)}$

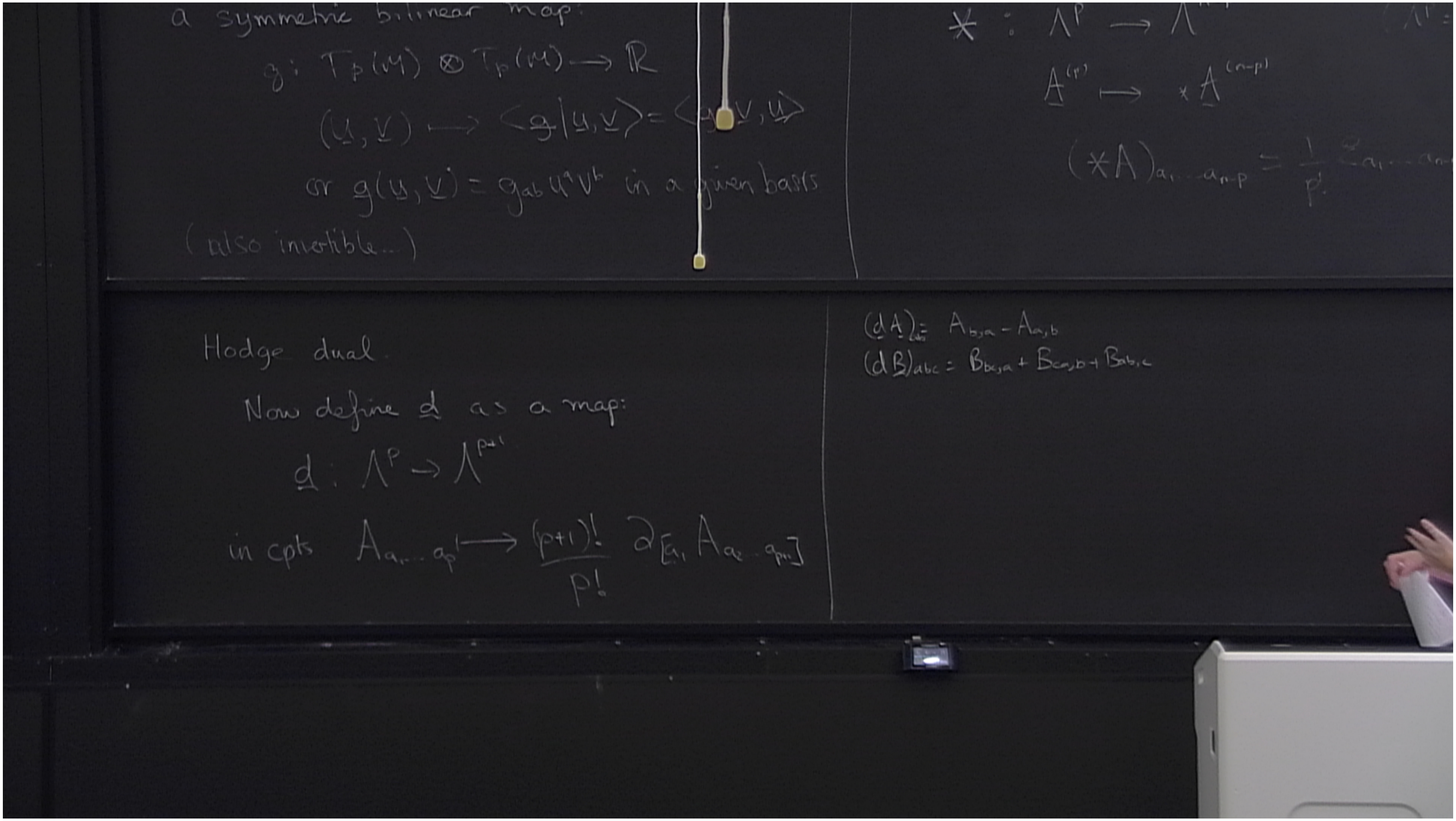
$(*A)_{a_1 \cdots a_{n-p}} = \frac{1}{p!} \epsilon_{a_1 \cdots a_{n-}}$

Hodge dual.

Now define $\underline{d}$ as a map:

$\underline{d} : \Lambda^p \rightarrow \Lambda^{p+1}$

in cpts $A_{a_1 \cdots a_p} \longmapsto \frac{(p+1)!}{p!} \partial_{[a_1} A_{a_2 \cdots a_{p+1}]}$



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$(\Lambda^p =$

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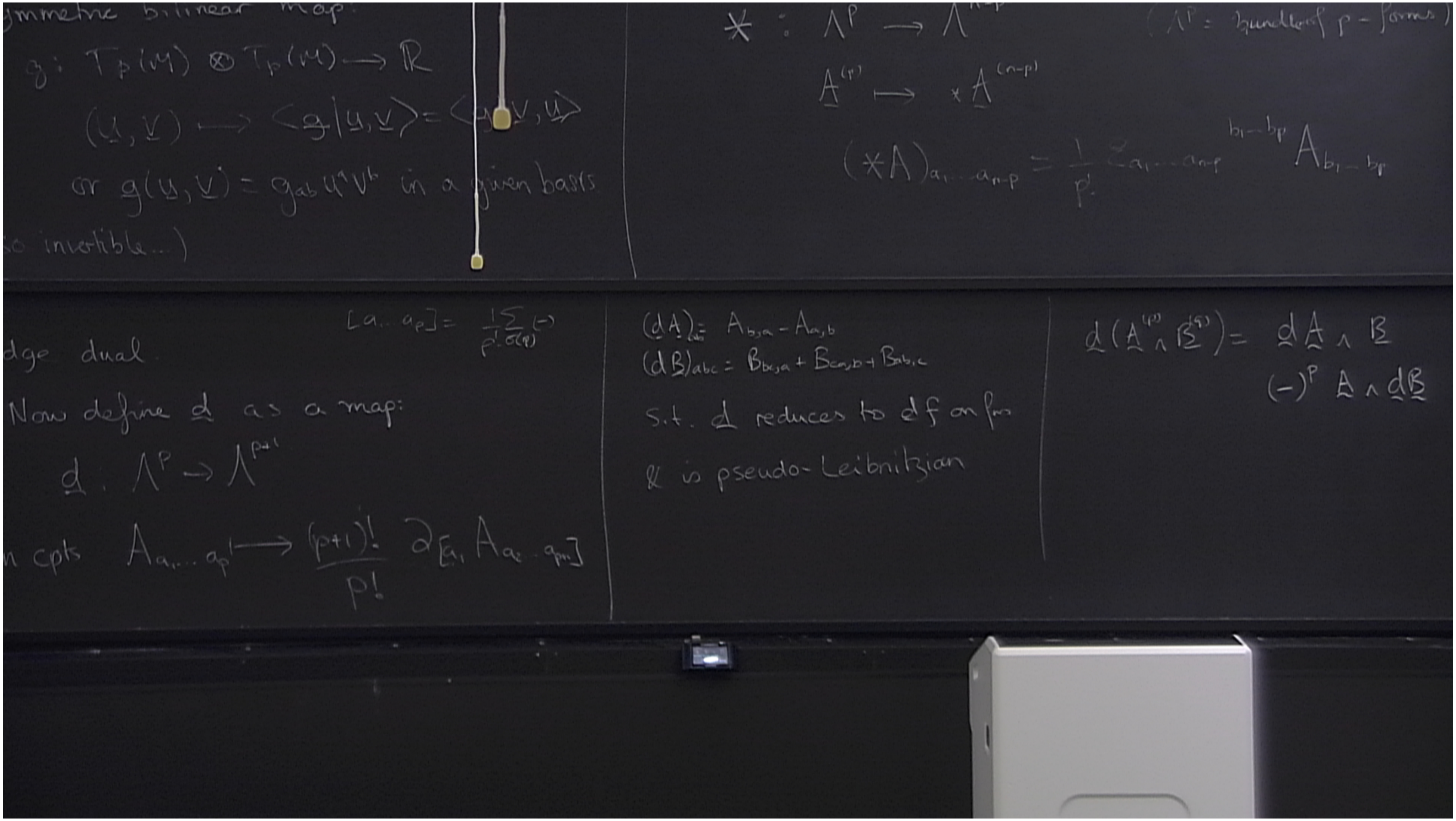
$$(\underline{d}\underline{A})_{ab} = A_{b,a} - A_{a,b}$$

$$(\underline{d}\underline{B})_{abc} = B_{bc,a} + B_{ca,b} + B_{ab,c}$$



ymmetric bilinear map:

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or $\underline{g}(\underline{u}, \underline{v}) = g_{ab} u^a v^b$ in a given basis

so invertible...)

$* : \Lambda^p \longrightarrow \Lambda^{n-p}$ (Λ^p = bundle of p-forms)

$\underline{A}^{(p)} \longmapsto * \underline{A}^{(n-p)}$

$$(*A)_{a_1 \dots a_{n-p}} = \frac{1}{p!} \varepsilon_{a_1 \dots a_{n-p}}{}^{b_1 \dots b_p} A_{b_1 \dots b_p}$$

$[a_1 \dots a_p] = \frac{1}{p!} \sum_{\sigma(p)} (-)$

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$(\underline{d}\underline{A})_{ab} = A_{b,a} - A_{a,b}$

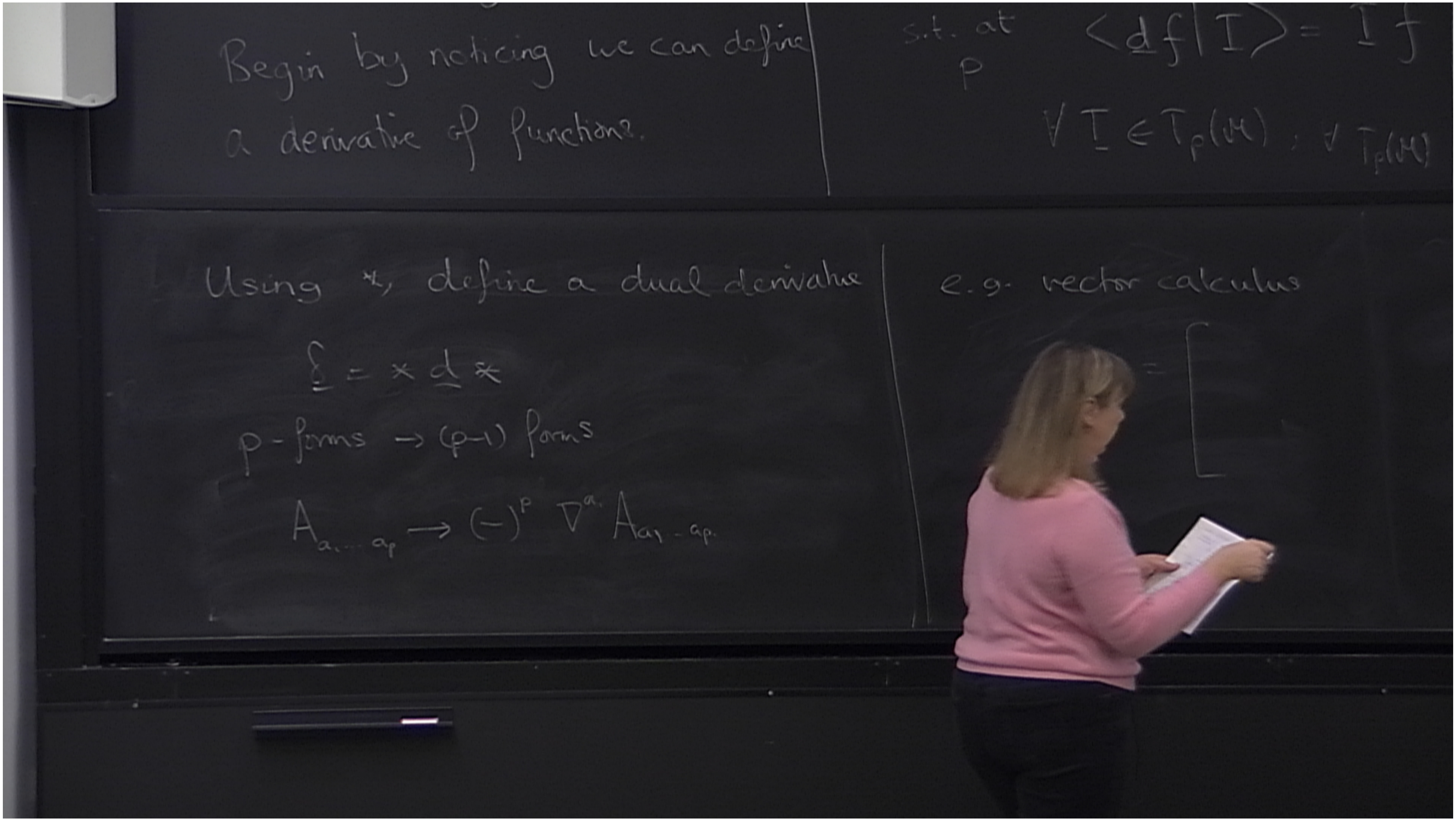
$(\underline{d}\underline{B})_{abc} = B_{bc,a} + B_{ca,b} + B_{ab,c}$

s.t. $\underline{d}$ reduces to df on fns

& is pseudo-Leibnitzian

$\underline{d}(\underline{A}^{(p)} \wedge \underline{B}^{(q)}) = \underline{d}\underline{A} \wedge \underline{B}$

$(-)^p \underline{A} \wedge \underline{d}\underline{B}$



Begin by noticing we can define a derivative of functions.

s.t. at p $\langle \underline{d}f | \underline{I} \rangle = \underline{I} f$

$\forall \underline{I} \in T_p(\mathcal{M})$; $\forall T_p(\mathcal{M})$

Using $*$, define a dual derivative

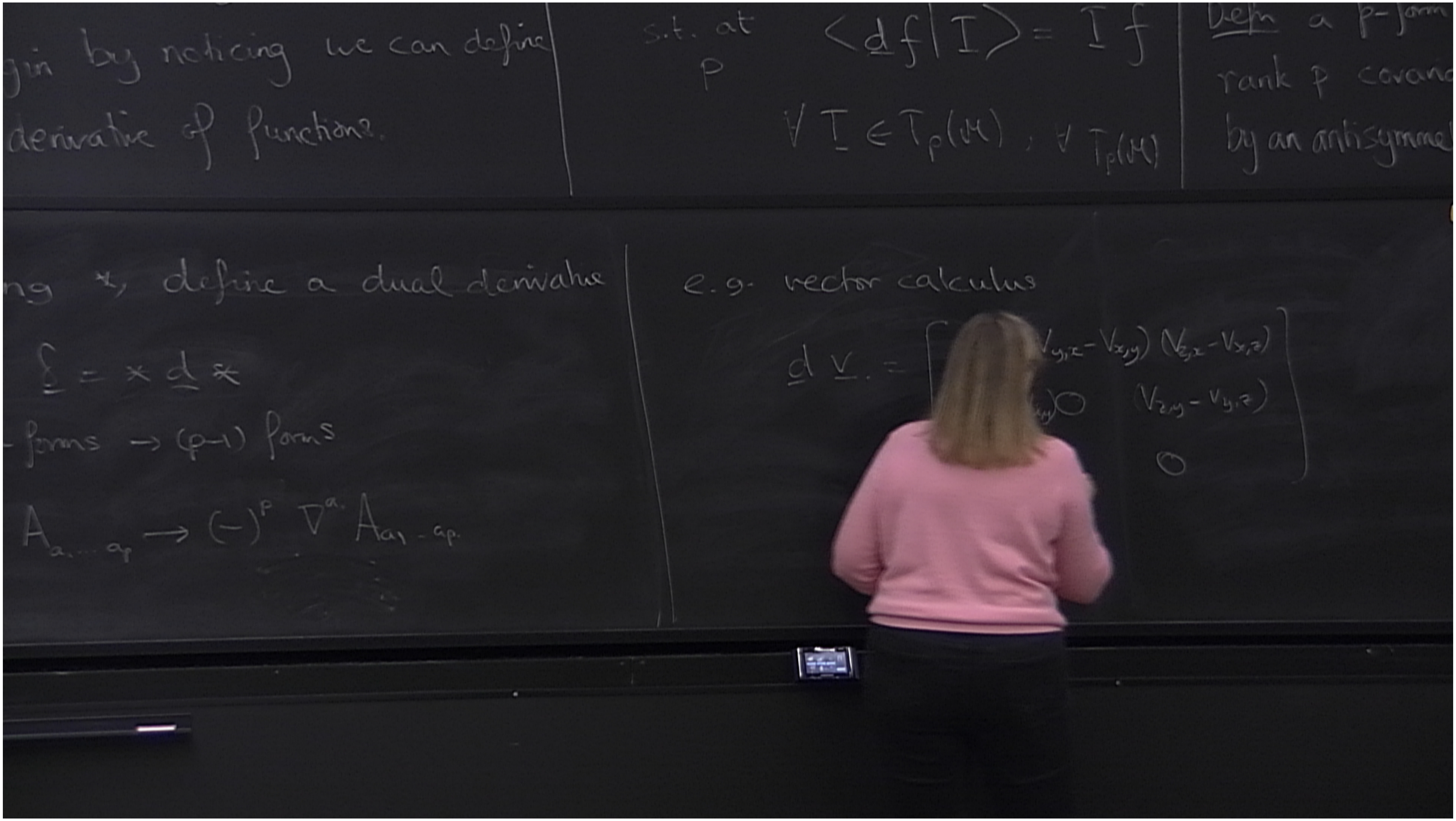
$$\underline{\delta} = * \underline{d} *$$

p-forms → (p−1) forms

$$A_{a_1 \dots a_p} \to (-)^p \nabla^{a_1} A_{a_1 \dots a_p}$$

e.g. vector calculus

$= \Big[$



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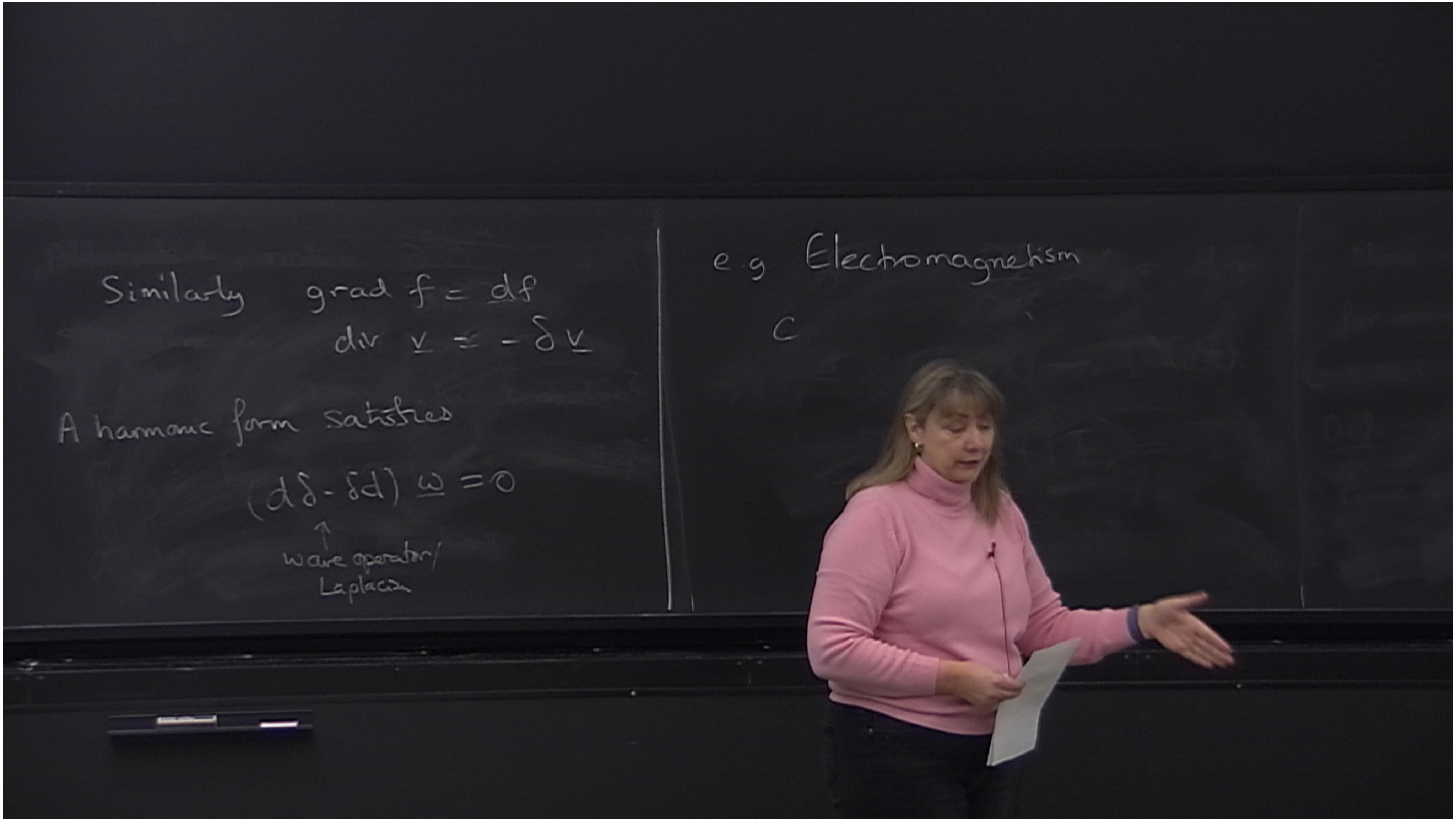
-forms $\rightarrow$ (p-1) forms

$A_{a_1 \dots a_p} \longrightarrow (-)^p \nabla^{a_1} A_{a_1 \dots a_p}$

e.g. vector calculus

$\underline{d}\, \underline{V}. = \Big[\quad (V_{y,x} - V_{x,y}) \ (V_{z,x} - V_{x,z}) \quad \bigcirc \quad (V_{z,y} - V_{y,z}) \quad \bigcirc \Big]$



Similarly $\text{grad}\, f = \underline{df}$

$\text{div}\, \underline{v} = -\delta \underline{v}$

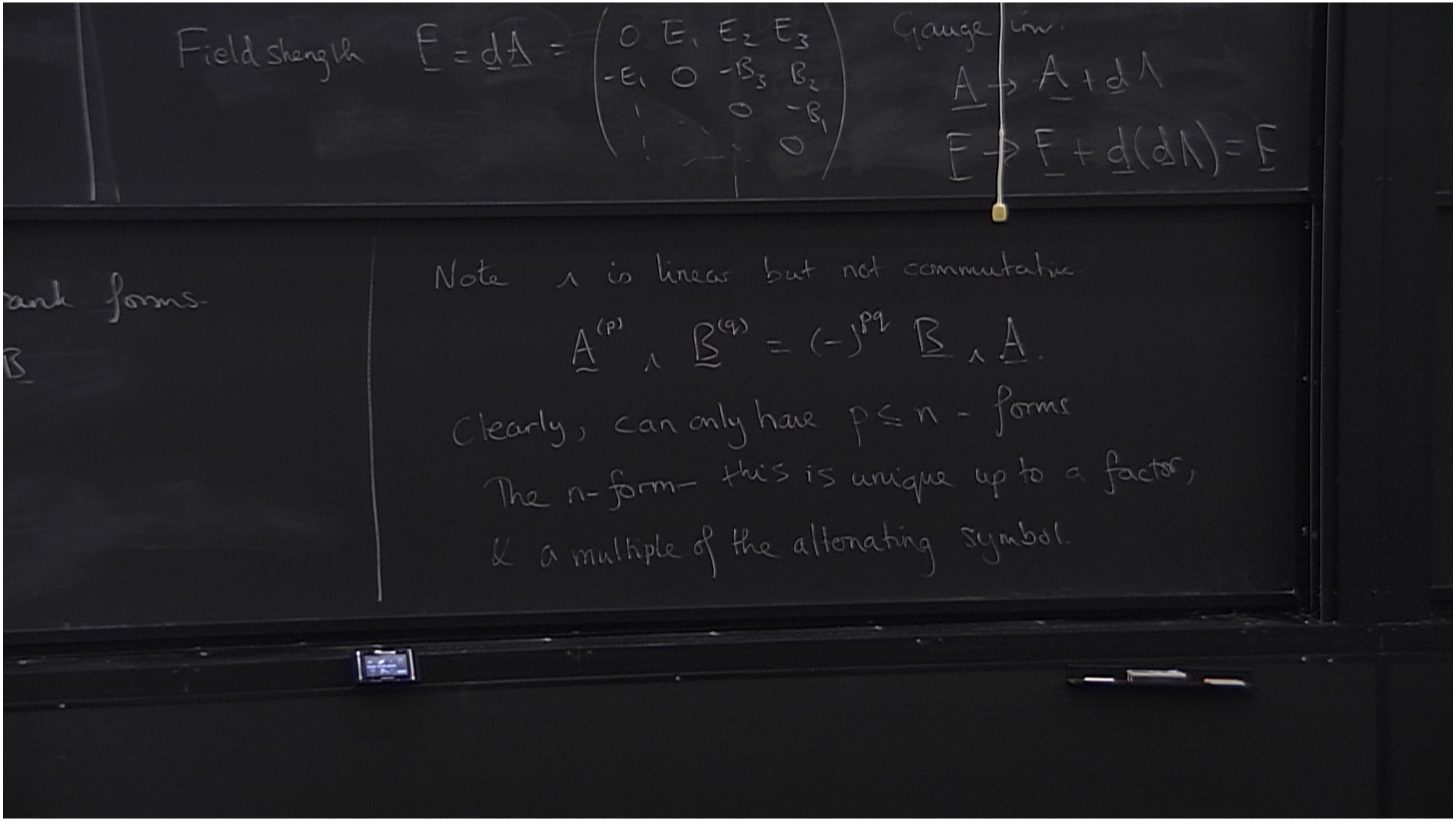
A harmonic form satisfies

$$(d\delta - \delta d)\, \underline{\omega} = 0$$

↑ wave operator/ Laplacian

e.g. Electromagnetism

c



Field strength $\underset{\sim}{F} = \underset{\sim}{d}\underset{\sim}{A} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ \vdots & & 0 & -B_1 \\ \vdots & & & 0 \end{pmatrix}$

Gauge inv.

$\underline{A} \to \underline{A} + \underline{d}\Lambda$

$\underline{F} \to \underline{F} + \underline{d}(\underline{d}\Lambda) = \underline{F}$

ank forms

$\underline{B}$

Note $\wedge$ is linear but not commutative

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The n-form — this is unique up to a factor,

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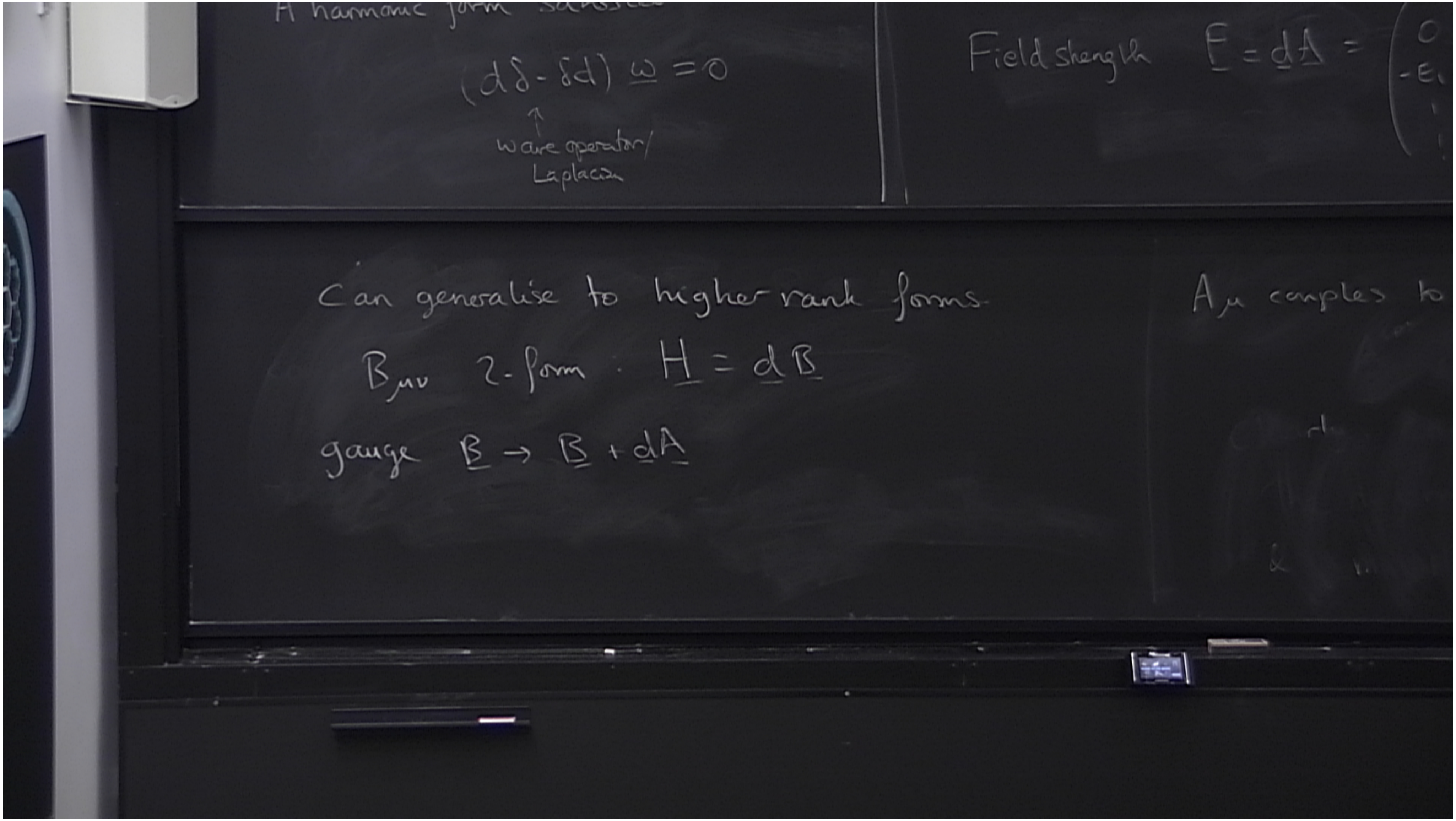


A harmonic form

$$(d\delta - \delta d)\,\underline{\omega} = 0$$

↑ wave operator/ Laplacian

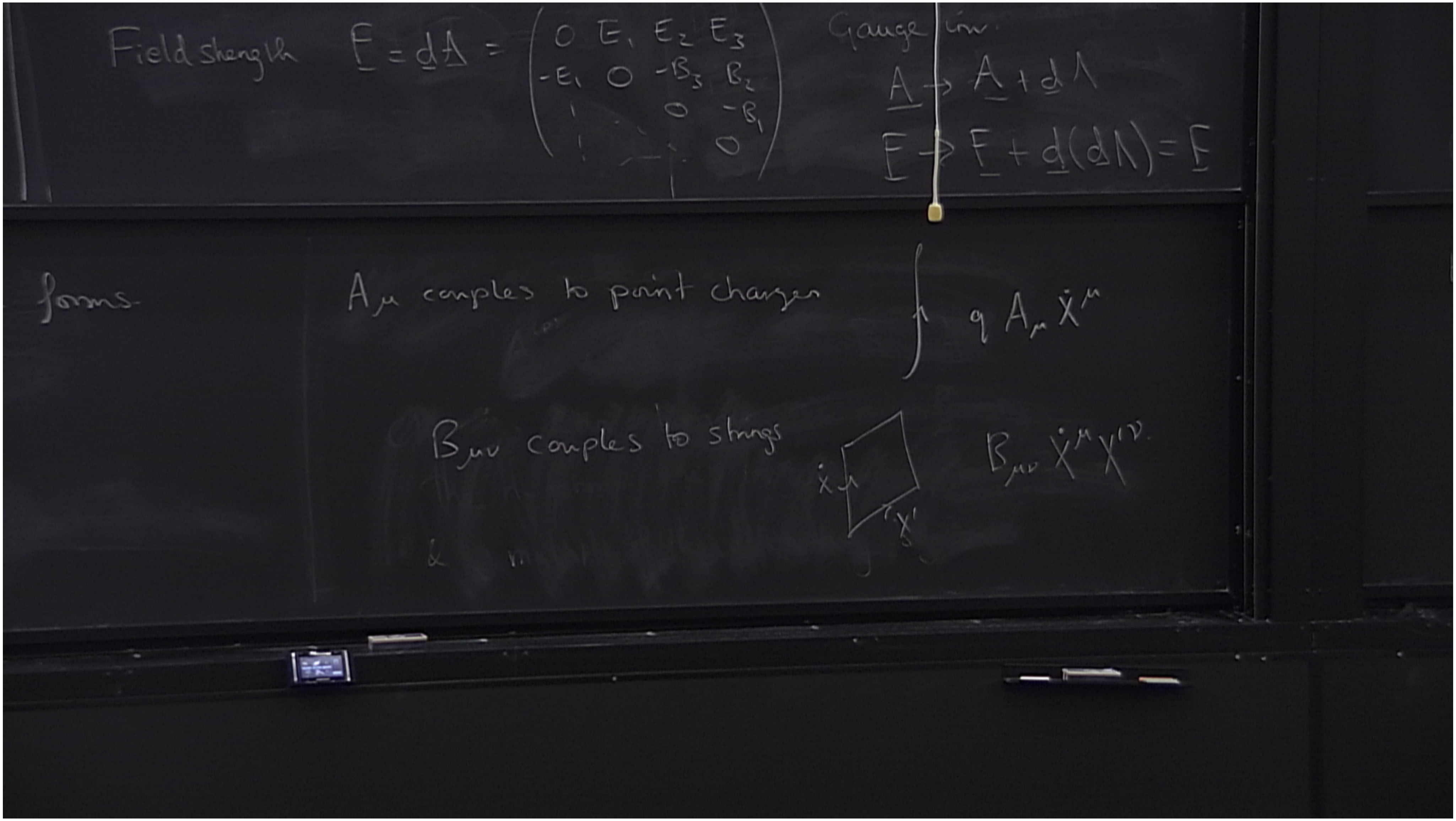
Field strength $\underline{F} = \underline{d}\,\underline{A} = \begin{pmatrix} 0 \\ -E \end{pmatrix}$

Can generalise to higher rank forms.

$B_{\mu\nu}$ 2-form : $\underline{H} = \underline{d}\,\underline{B}$

gauge $\underline{B} \to \underline{B} + \underline{d}\underline{A}$

A_μ couples to



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Gauge inv.

$\underline{A} \to \underline{A} + \underline{d}\Lambda$

$\underline{F} \to \underline{F} + \underline{d}(\underline{d}\Lambda) = \underline{F}$

forms.

A_μ couples to point charges $\int q A_\mu \dot{X}^\mu$

$B_{\mu\nu}$ couples to strings $B_{\mu\nu} \dot{X}^\mu X'^\nu$.