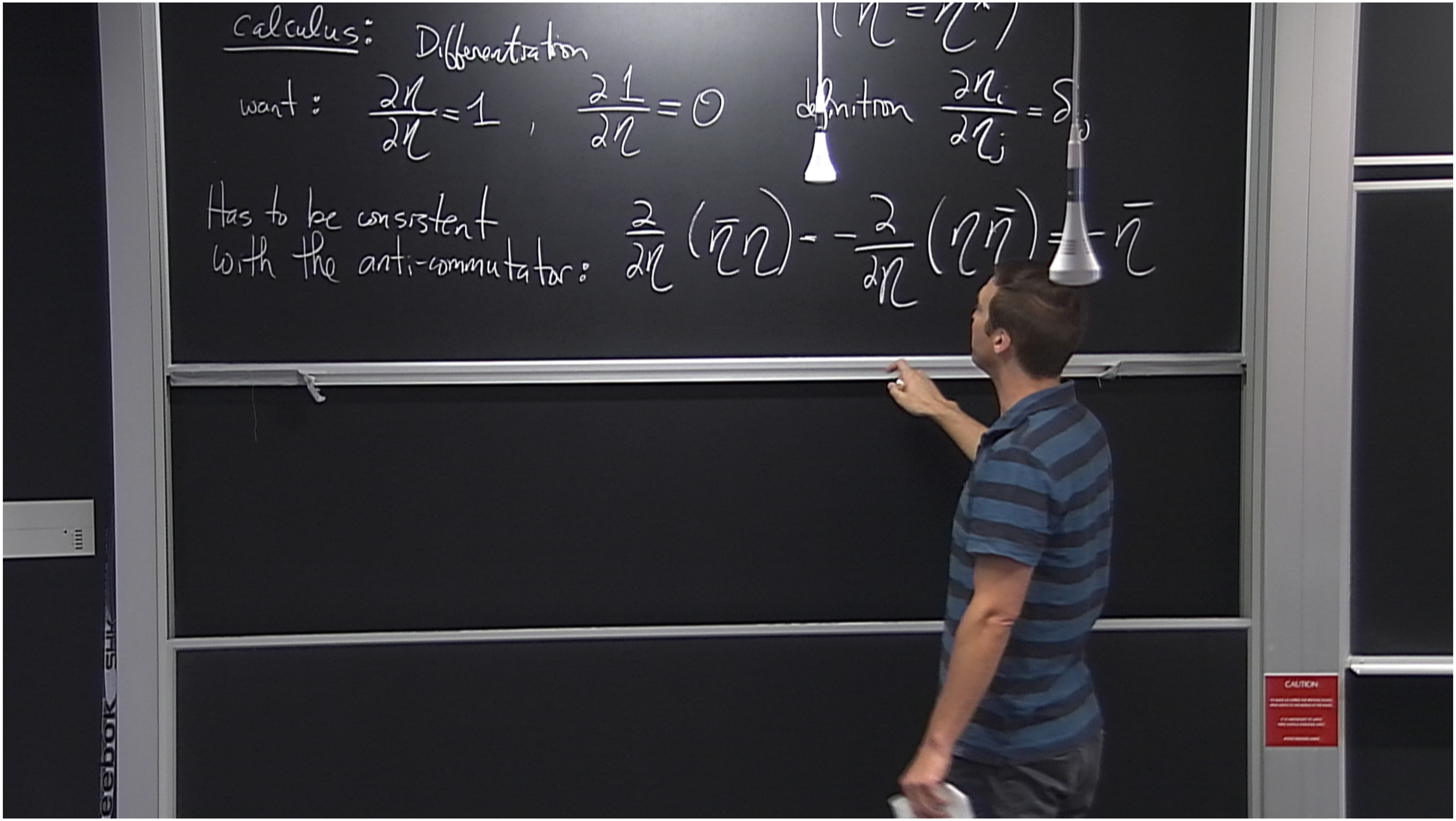
Title: PHYS 733 - Quantum Many-Body Physics (W2016) - Roger Melko - Lecture 7

Date: Jan 26, 2016 10:00 AM

URL: http://pirsa.org/16010037

Abstract:


Calculus: Differentiation
want: $\frac{\partial \eta}{\partial \eta} = 1$, $\frac{\partial 1}{\partial \eta} = 0$
definition $\frac{\partial \eta_i}{\partial \eta_j} = \delta_{ij}$
Has to be consistent with the anti-commutator: $\frac{\partial}{\partial \eta}(\bar{\eta}\eta) = -\frac{\partial}{\partial \eta}(\eta\bar{\eta}) = -\bar{\eta}$
CAUTION

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Then $\frac{\partial A}{\partial \eta} = c_1 - c_3\bar{\eta}$ $\qquad$ $\frac{\partial A}{\partial \bar{\eta}} = c_2 + c_3\eta$

$$\frac{\partial}{\partial \bar{\eta}}\frac{\partial}{\partial \eta}A = -c_3 \;, \quad \frac{\partial}{\partial \eta}\frac{\partial}{\partial \bar{\eta}}A = c_3 \Rightarrow \left\{\frac{\partial}{\partial \eta}, \frac{\partial}{\partial \bar{\eta}}\right\}$$

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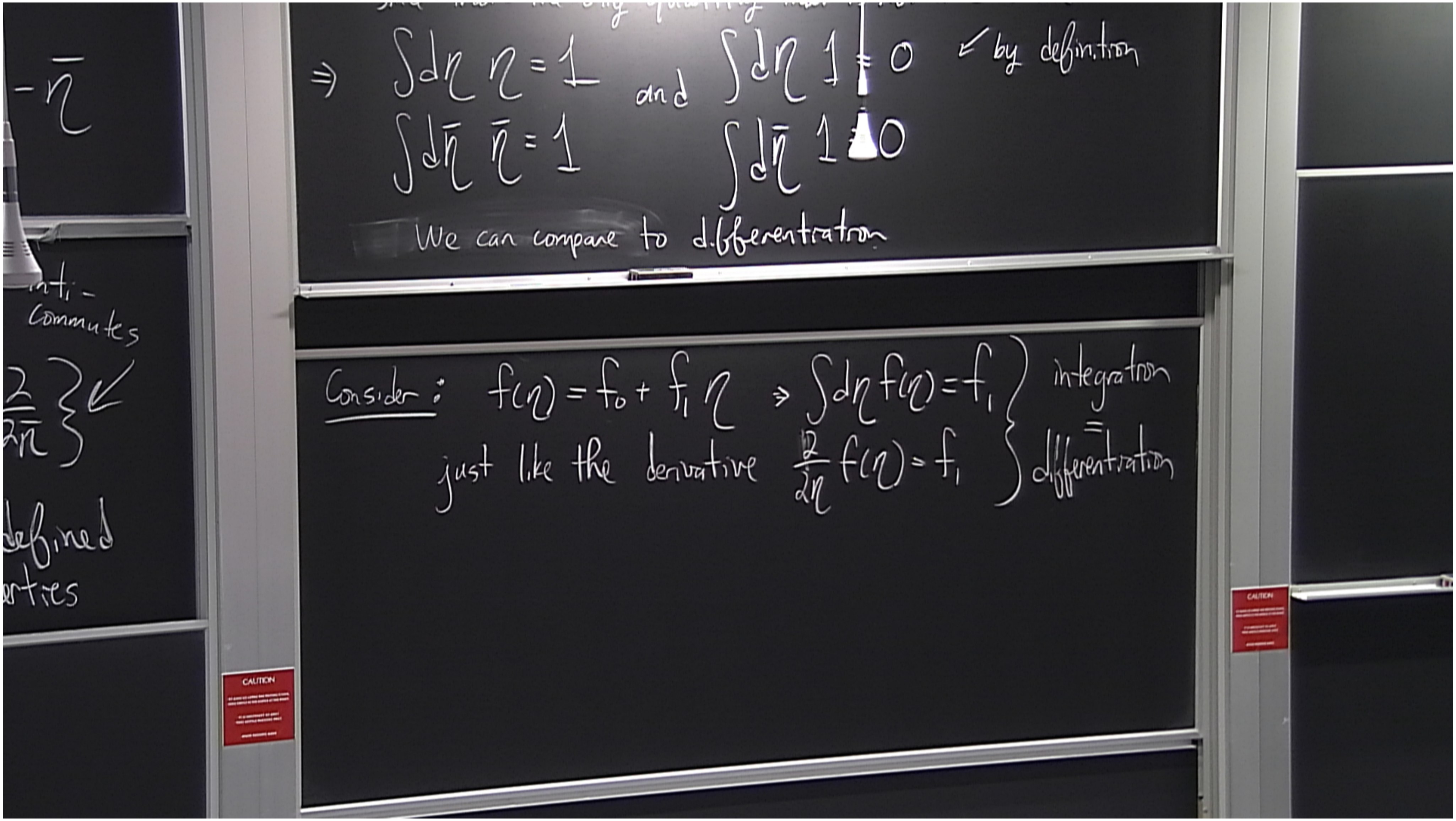
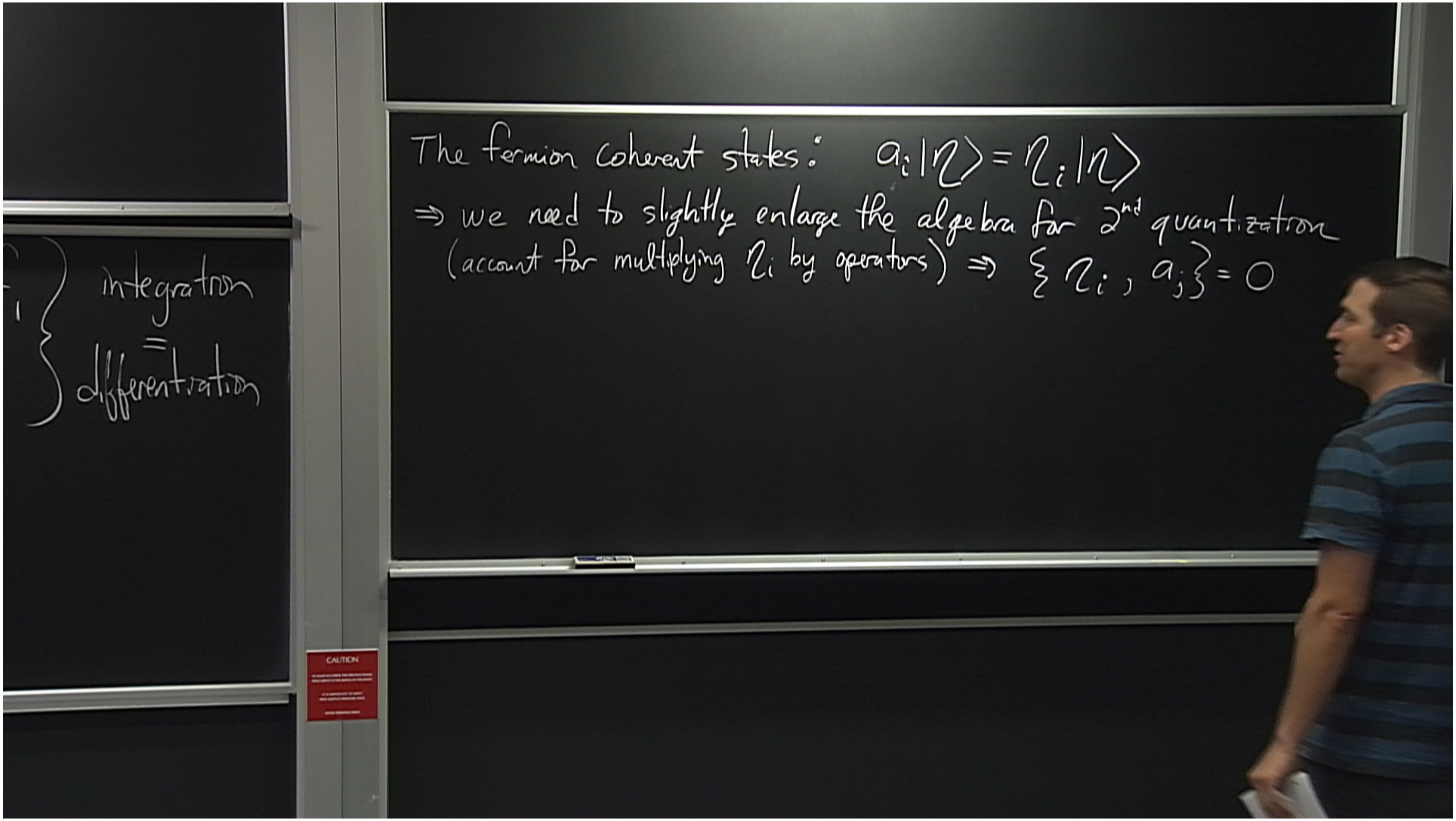
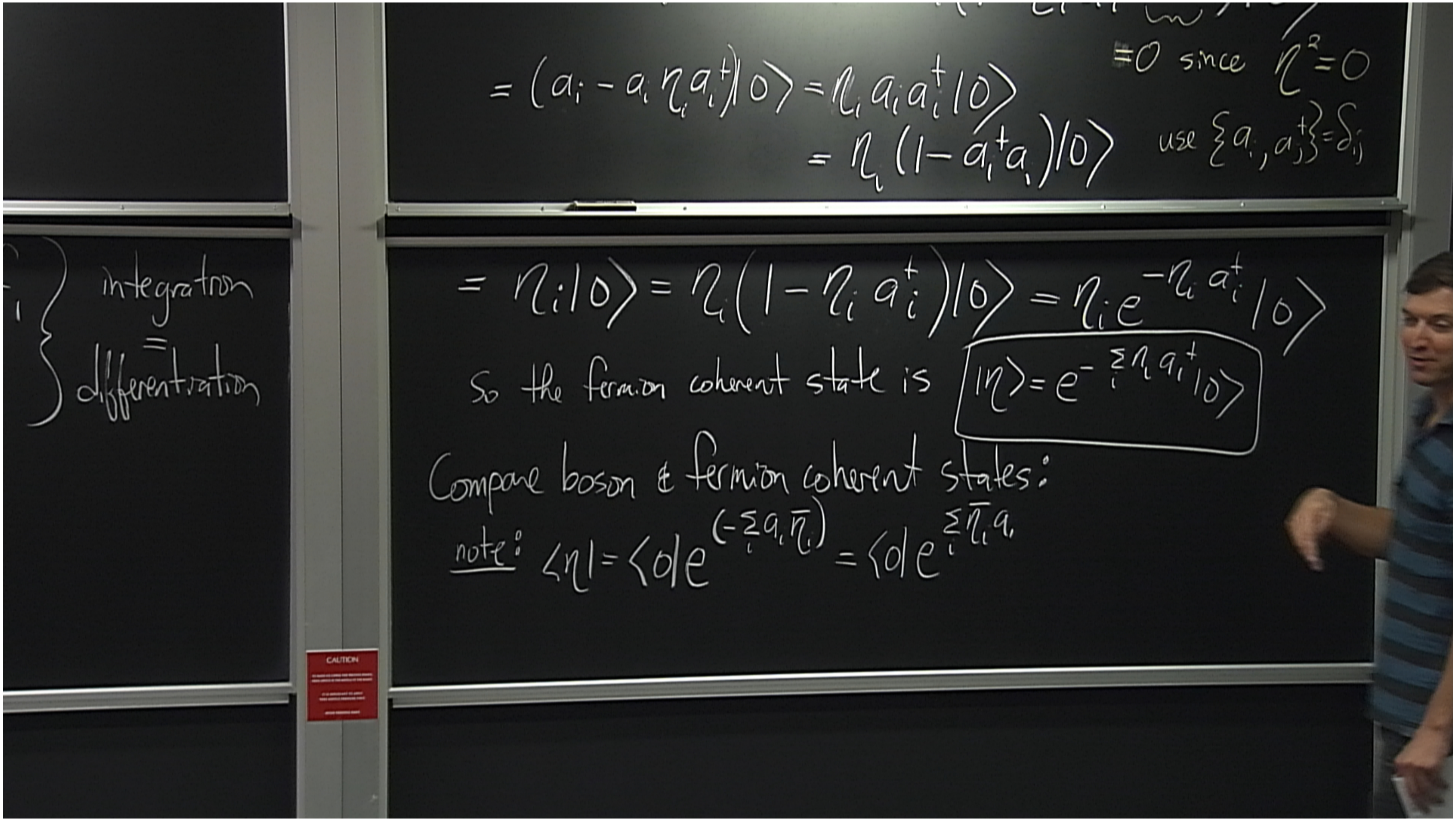
← anti-commutes

<u>Integration:</u> no Riemann sum analogy ⇒ definition is just defined to obey certan properties

e.g.


⇒ ∫dη η = 1 and ∫dη 1 = 0 ↙ by definition
∫dη̄ η̄ = 1 ∫dη̄ 1 = 0
We can compare to differentiation
Consider: f(η) = f₀ + f₁ η ⇒ ∫dη f(η) = f₁
just like the derivative ∂/∂η f(η) = f₁
integration = differentiation
anti-commutes
defined
erties


integration
=
differentiation
The fermion coherent states: $a_i|\eta\rangle = \eta_i|\eta\rangle$
⇒ we need to slightly enlarge the algebra for 2nd quantization
(account for multiplying η_i by operators) ⇒ $\{\eta_i, a_j\} = 0$


$=0$ since $\eta^2=0$
$= (a_i - a_i \eta_i a_i^\dagger)|0\rangle = \eta_i a_i a_i^\dagger |0\rangle$
$= \eta_i (1 - a_i^\dagger a_i)|0\rangle$
use $\{a_i, a_j^\dagger\} = \delta_{ij}$
integration = differentiation
$= \eta_i |0\rangle = \eta_i (1 - \eta_i a_i^\dagger)|0\rangle = \eta_i e^{-\eta_i a_i^\dagger}|0\rangle$
So the fermion coherent state is $|\eta\rangle = e^{-\sum_i \eta_i a_i^\dagger}|0\rangle$
Compare boson & fermion coherent states:
note: $\langle\eta| = \langle 0| e^{(-\sum_i a_i \bar{\eta}_i)} = \langle 0| e^{\sum_i \bar{\eta}_i a_i}$

$$= (a_i - a_i \eta_i a_i^\dagger)|0\rangle = \eta_i a_i a_i^\dagger |0\rangle$$

$=0$ since $\eta^2 = 0$

$$= \eta_i (1 - a_i^\dagger a_i)|0\rangle$$

use $\{a_i, a_j^\dagger\} = \delta_{ij}$

integration = differentiation

$$= \eta_i |0\rangle = \eta_i (1 - \eta_i a_i^\dagger)|0\rangle = \eta_i e^{-\eta_i a_i^\dagger}|0\rangle$$

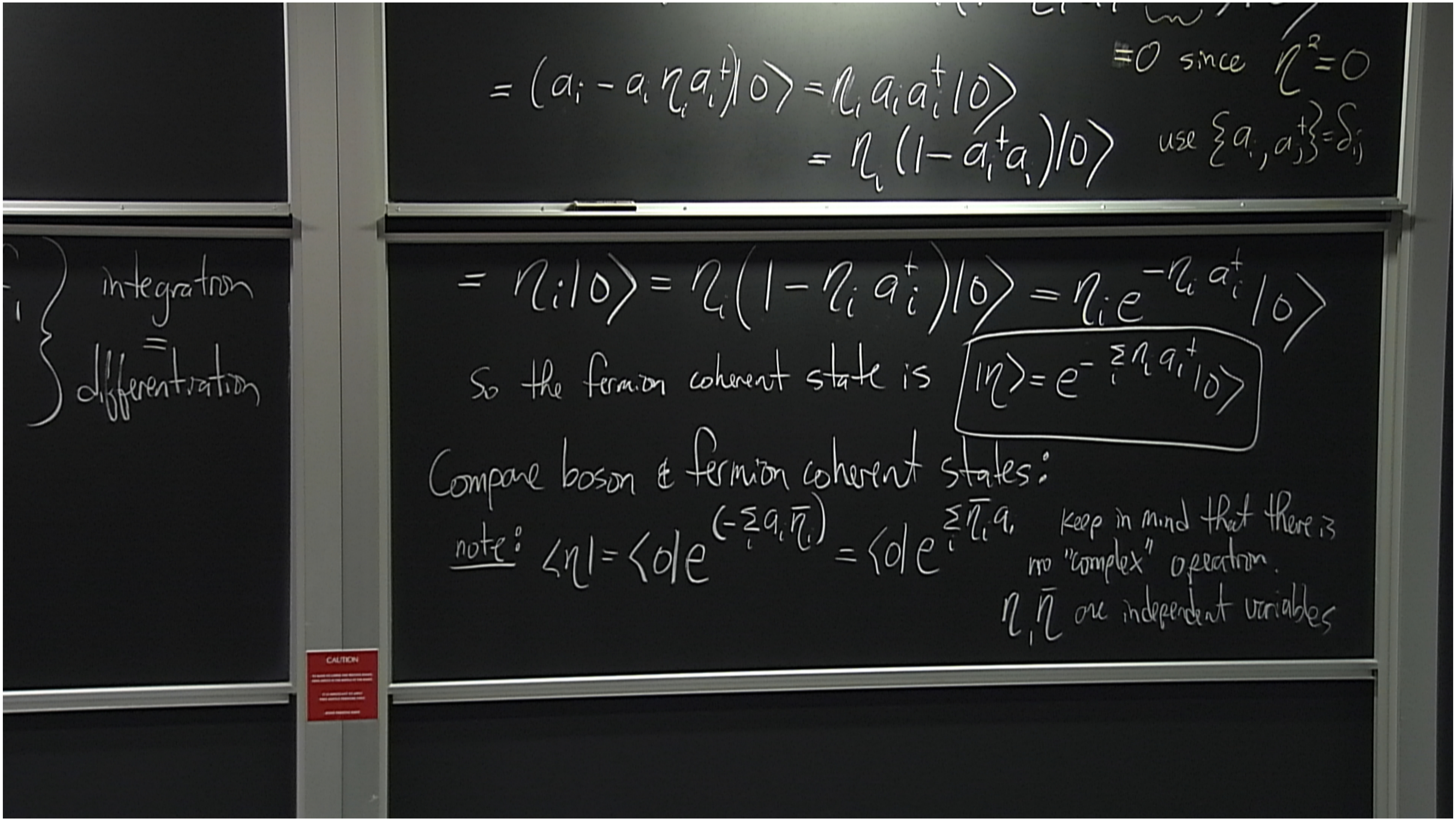
So the fermion coherent state is $|\eta\rangle = e^{-\sum_i \eta_i a_i^\dagger}|0\rangle$

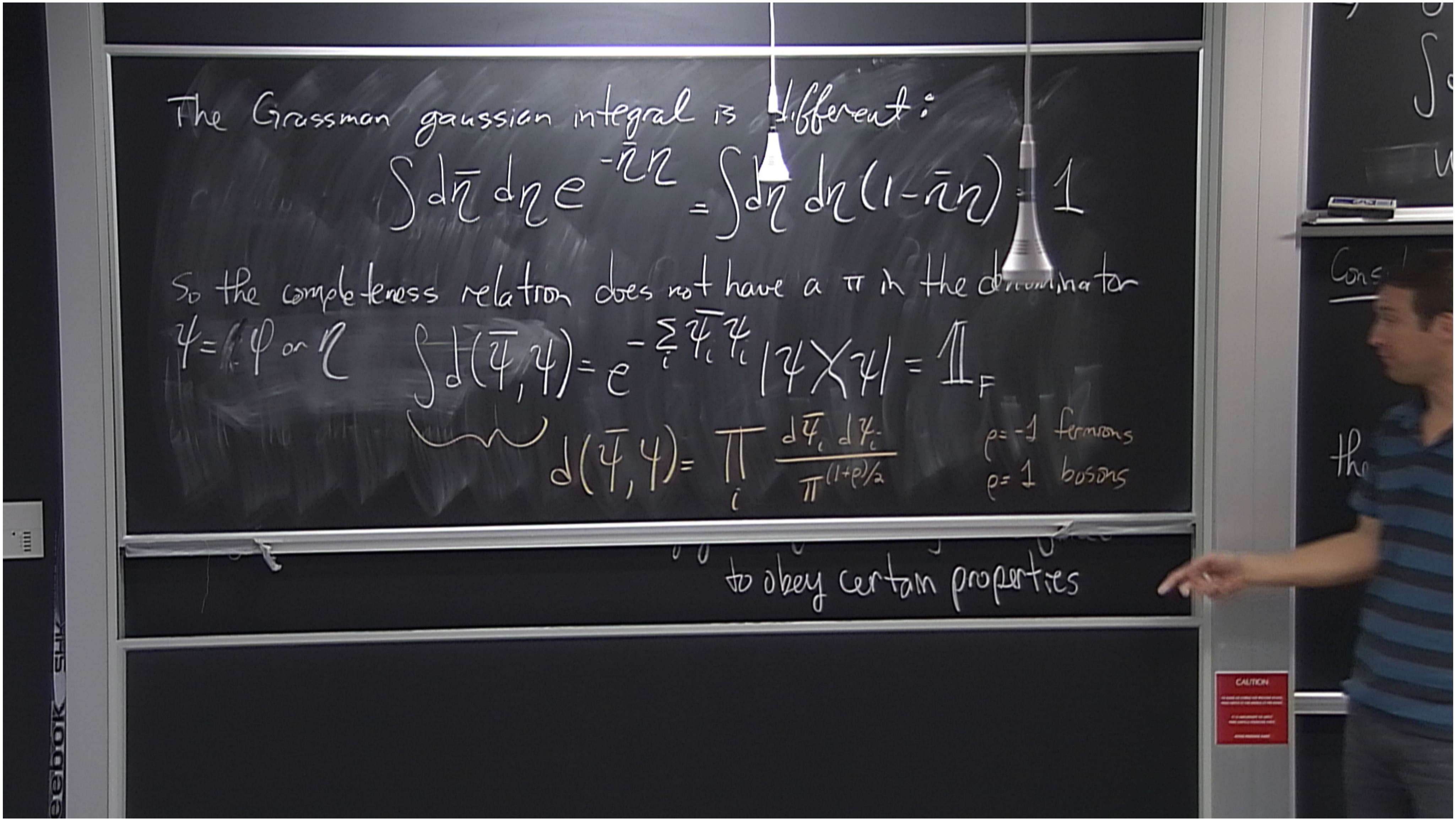
Compare boson & fermion coherent states:

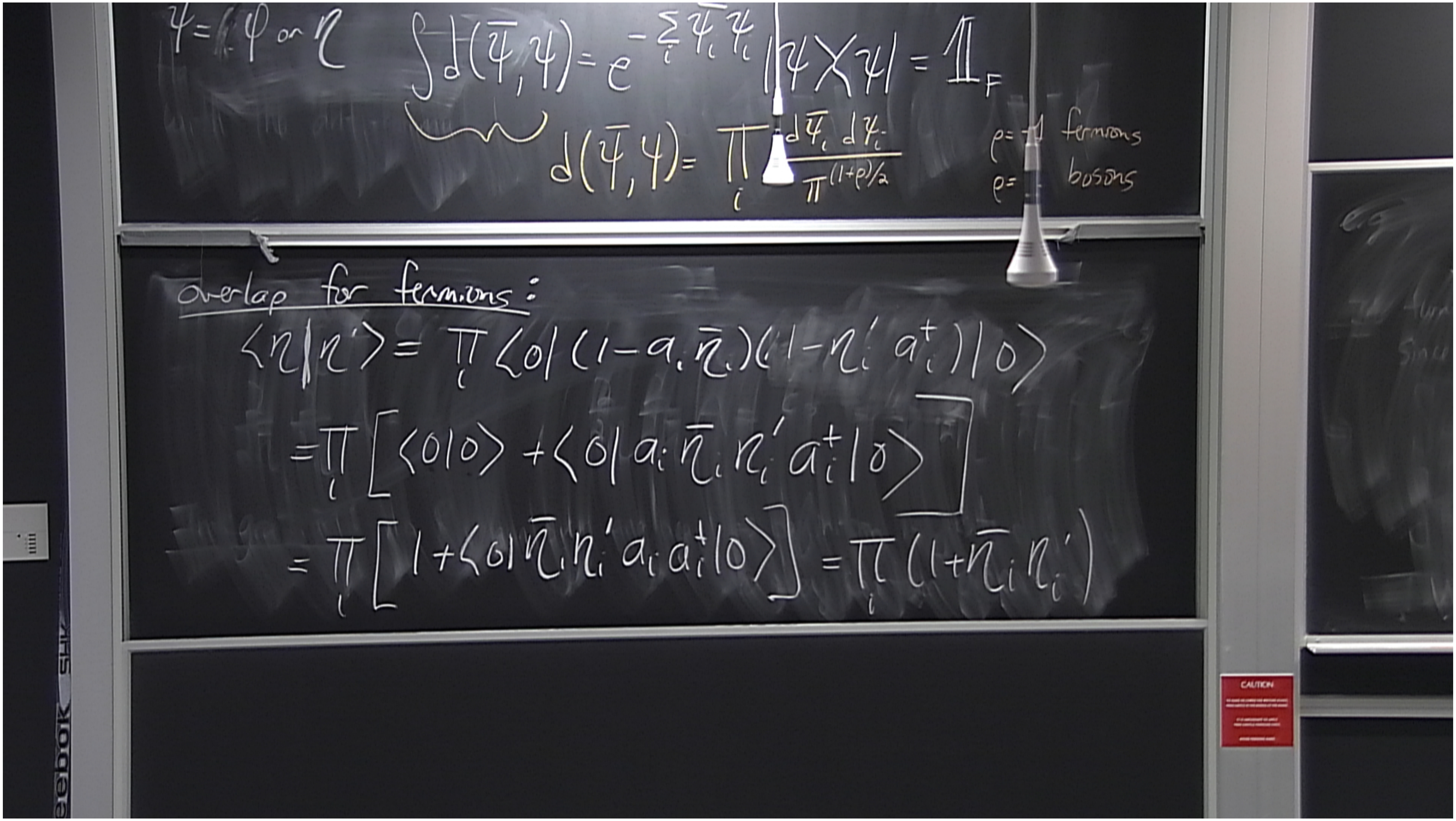
note: $\langle \eta| = \langle 0| e^{(-\sum_i a_i \bar{\eta}_i)} = \langle 0| e^{\sum_i \bar{\eta}_i a_i}$

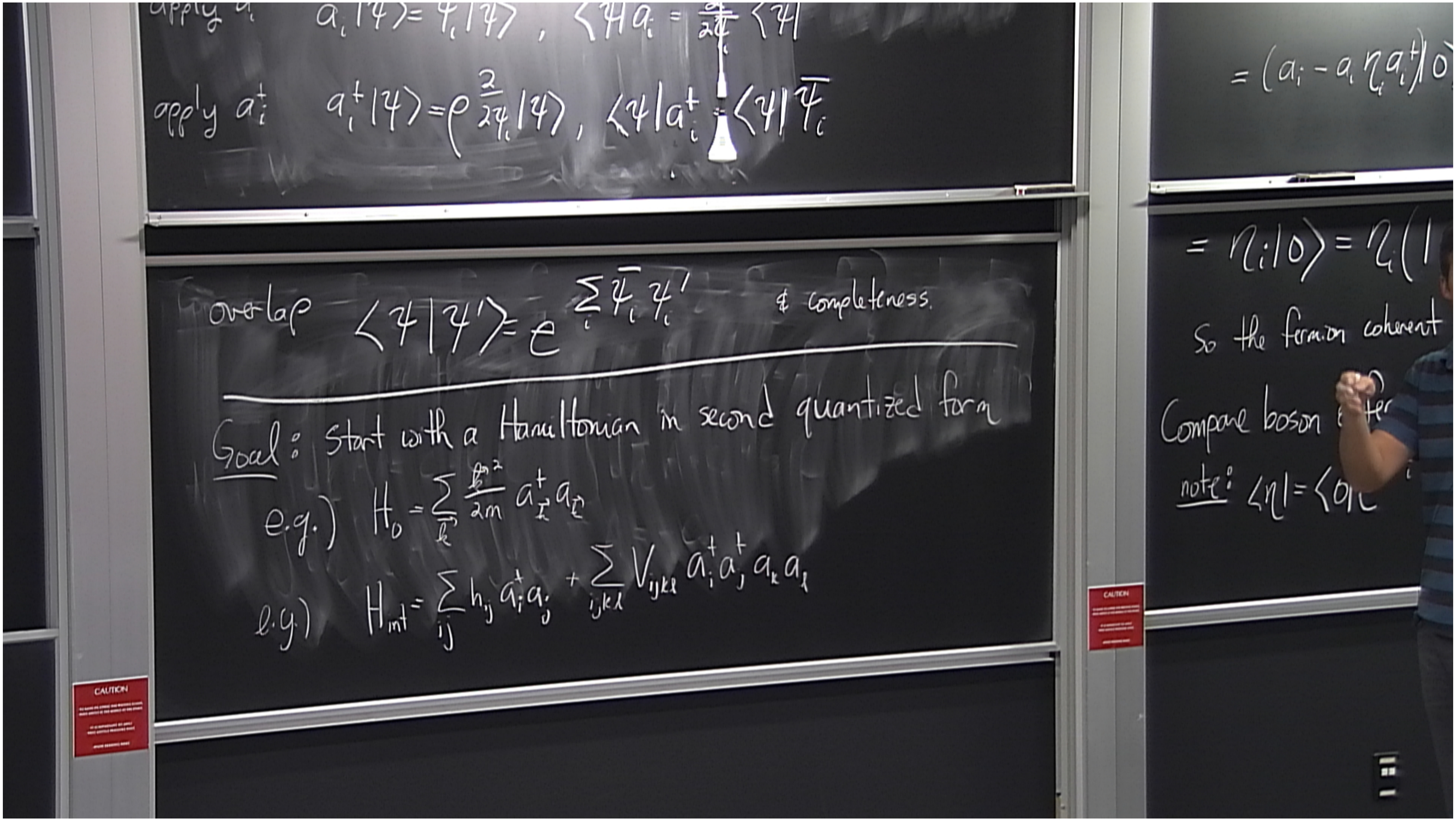
Keep in mind that there is no "complex" operation.

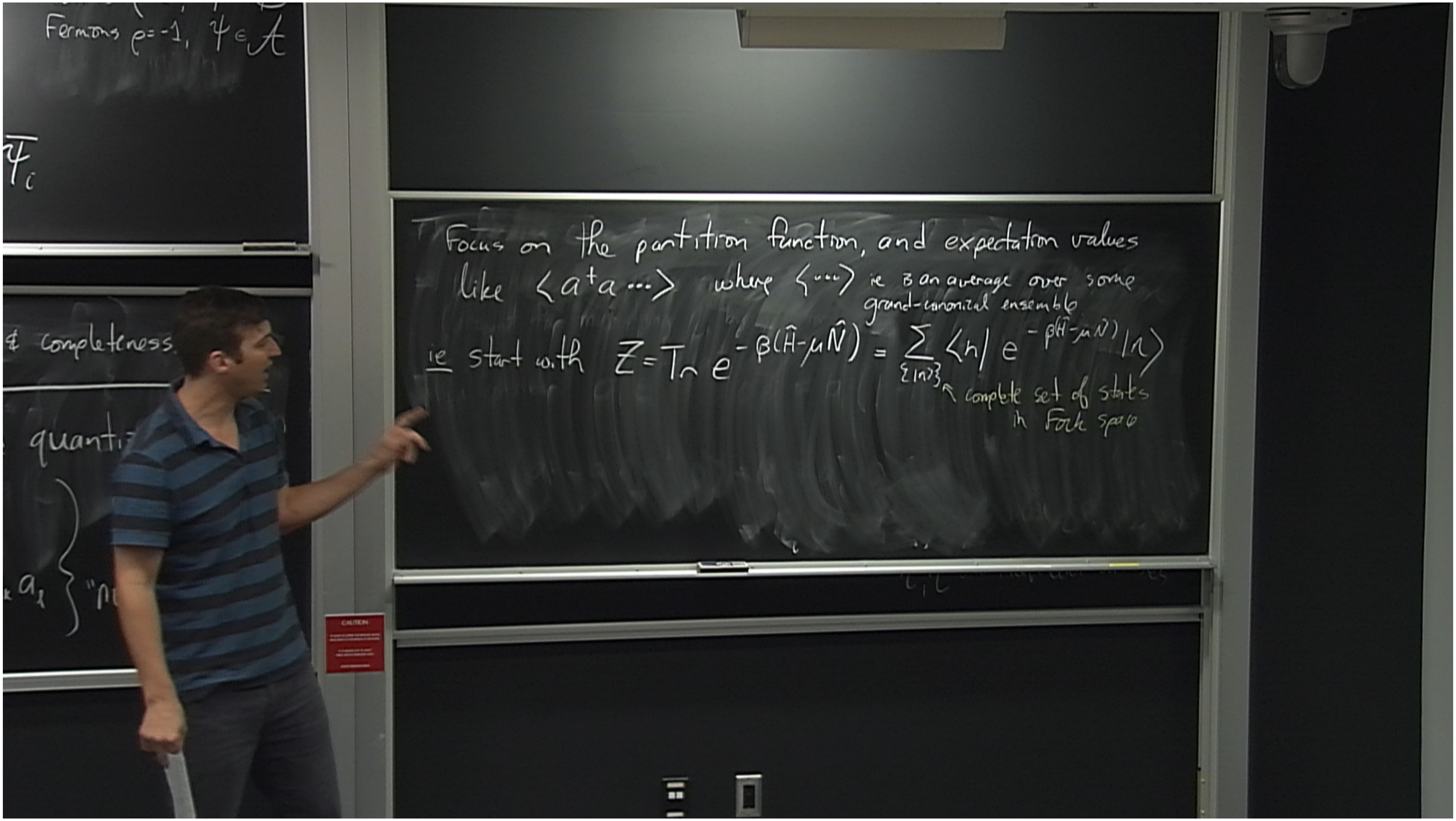
$\eta, \bar{\eta}$ are independent variables

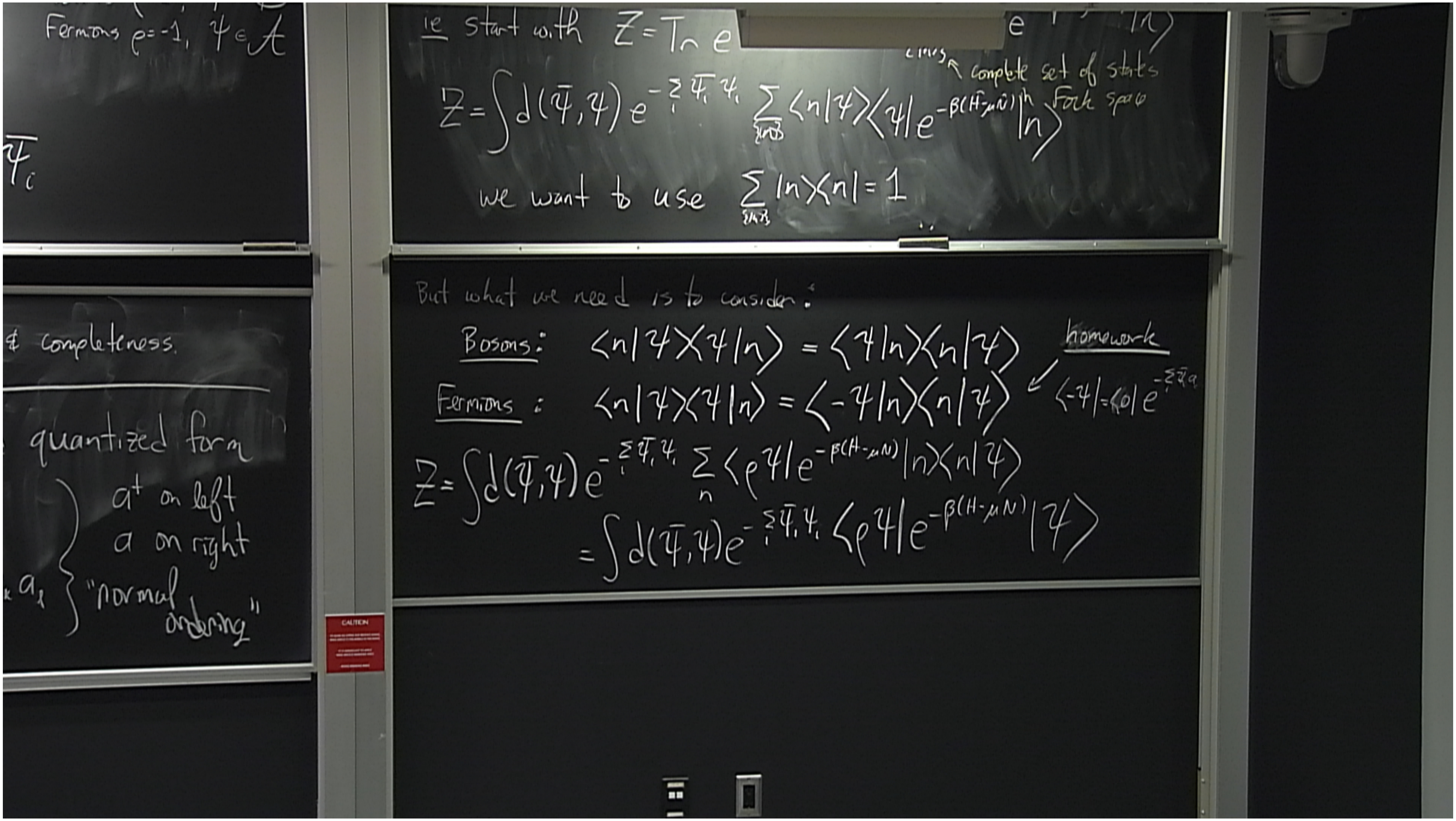

= (a_i − a_i η_i a_i^†)|0⟩ = η_i a_i a_i^†|0⟩
=0 since η² = 0
= η_i (1 − a_i^† a_i)|0⟩
use {a_i, a_j^†} = δ_ij
integration
=
differentiation
= η_i|0⟩ = η_i(1 − η_i a_i^†)|0⟩ = η_i e^(−η_i a_i^†)|0⟩
So the fermion coherent state is |η⟩ = e^(−Σ_i η_i a_i^†)|0⟩
Compare boson & fermion coherent states:
note: ⟨η| = ⟨0|e^(−Σ_i a_i η̄_i) = ⟨0|e^(Σ_i η̄_i a_i)
Keep in mind that there is no "complex" operation.
η, η̄ are independent variables
CAUTION

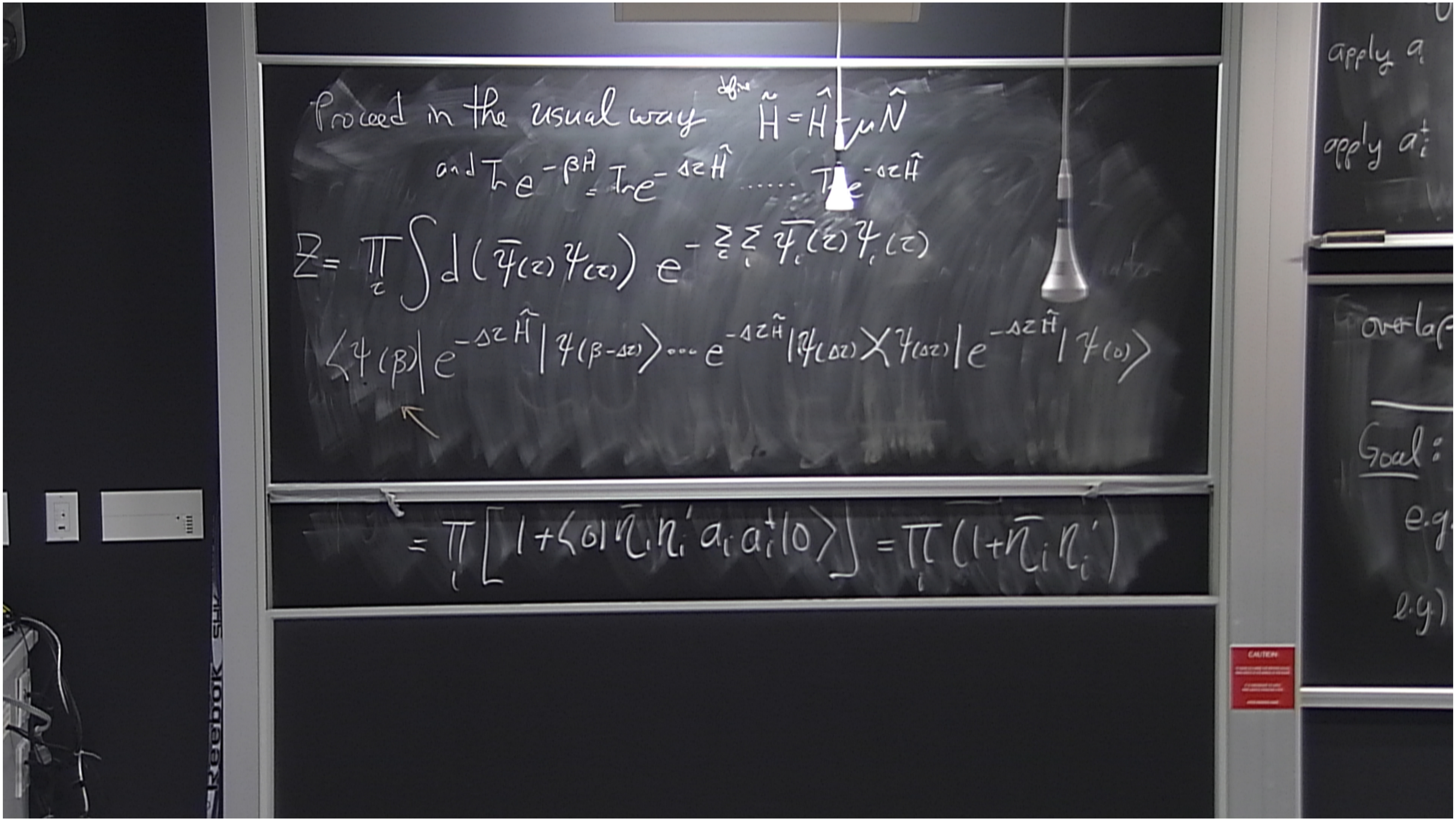

The Grassman gaussian integral is different:
$\int d\bar{\eta}\, d\eta\, e^{-\bar{\eta}\eta} = \int d\bar{\eta}\, d\eta\, (1-\bar{\eta}\eta) = 1$
So the completeness relation does not have a π in the denominator
$\psi = \varphi$ or η
$\int d(\bar{\psi},\psi)\, e^{-\sum_i \bar{\psi}_i \psi_i} |\psi\rangle\langle\psi| = \mathbb{1}_F$
$d(\bar{\psi},\psi) = \prod_i \frac{d\bar{\psi}_i\, d\psi_i}{\pi^{(1+\rho)/2}}$
ρ = -1 fermions
ρ = 1 bosons
to obey certain properties

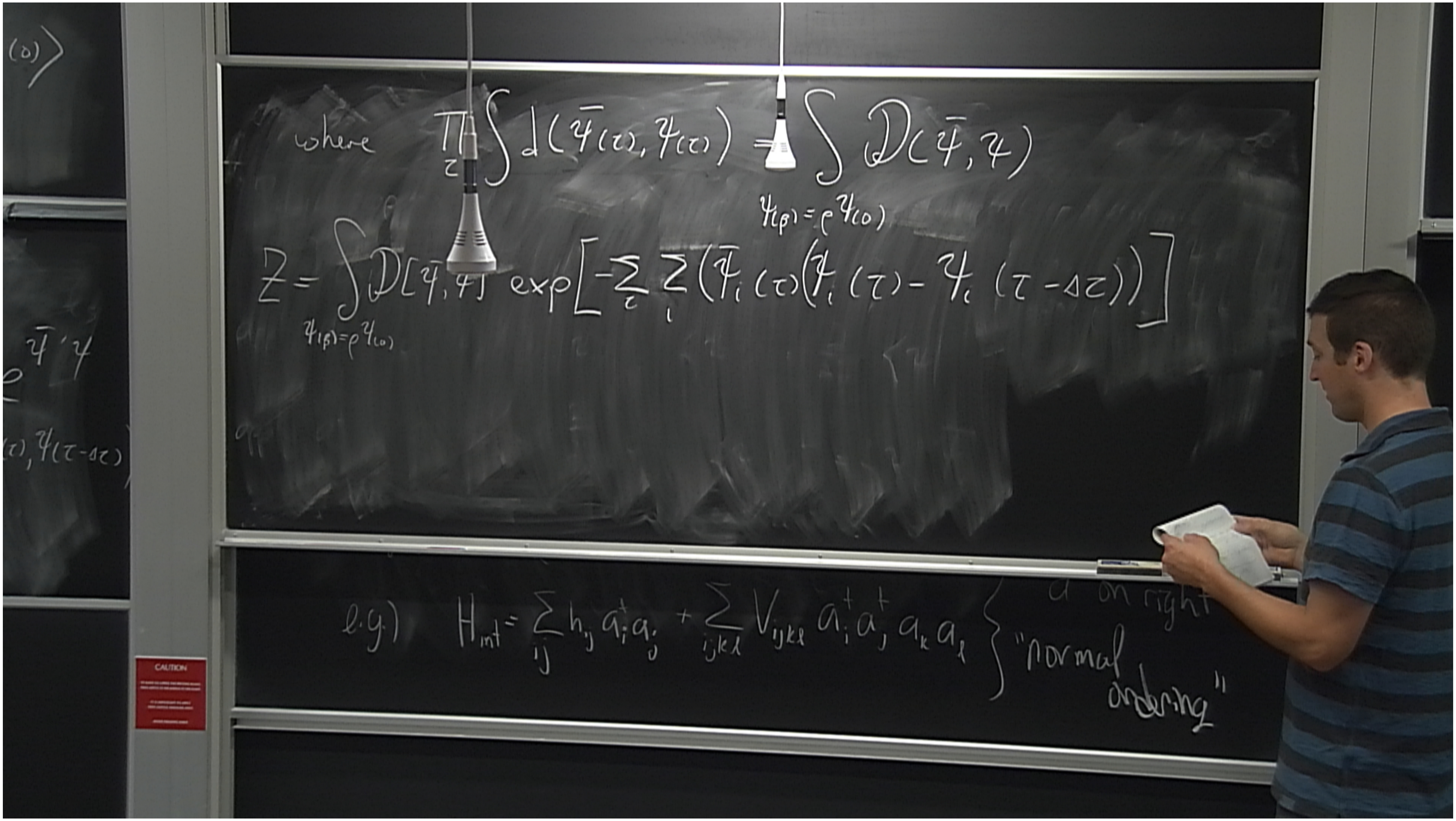

ψ = φ or η
∫d(ψ̄,ψ) = e^{-Σ_i ψ̄_i ψ_i} |ψ⟩⟨ψ| = 1_F
d(ψ̄,ψ) = ∏_i dψ̄_i dψ_i / π^{(1+ρ)/2}
ρ=-1 fermions
ρ= bosons
overlap for fermions:
⟨η|η'⟩ = ∏_i ⟨0|(1−a_i η̄_i)(1−η'_i a_i^†)|0⟩
= ∏_i [⟨0|0⟩ + ⟨0|a_i η̄_i η'_i a_i^†|0⟩]
= ∏_i [1+⟨0|η̄_i η'_i a_i a_i^†|0⟩] = ∏_i (1+η̄_i η'_i)

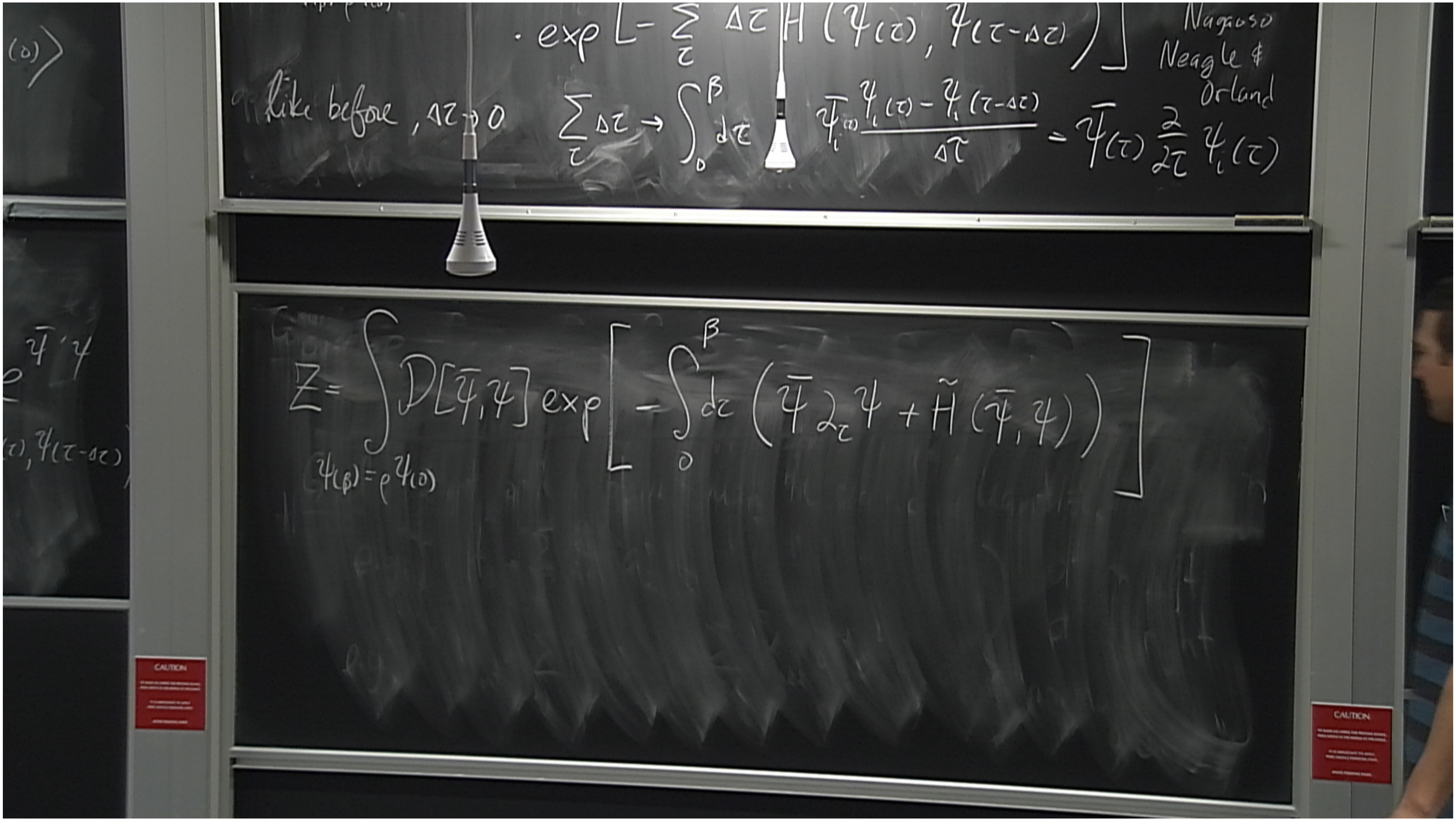

apply a_i $a_i|\psi\rangle = \psi_i|\psi\rangle$, $\langle\psi|a_i = \frac{\partial}{\partial\bar{\psi}_i}\langle\psi|$
apply $a_i^\dagger$ $a_i^\dagger|\psi\rangle = \frac{\partial}{\partial\psi_i}|\psi\rangle$, $\langle\psi|a_i^\dagger = \langle\psi|\bar{\psi}_i$
overlap $\langle\psi|\psi'\rangle = e^{\sum_i \bar{\psi}_i \psi_i'}$ & completeness.
Goal: Start with a Hamiltonian in second quantized form
e.g.) $H_0 = \sum_k \frac{k^2}{2m} a_k^\dagger a_k$
e.g.) $H_{int} = \sum_{ij} h_{ij} a_i^\dagger a_j + \sum_{ijkl} V_{ijkl} a_i^\dagger a_j^\dagger a_k a_l$
$= (a_i - a_i \eta_i a_i^\dagger)|0\rangle$
$= \eta_i|0\rangle = \eta_i(1$
So the fermion coherent
Compare boson
note: $\langle\eta| = \langle 0|$
CAUTION

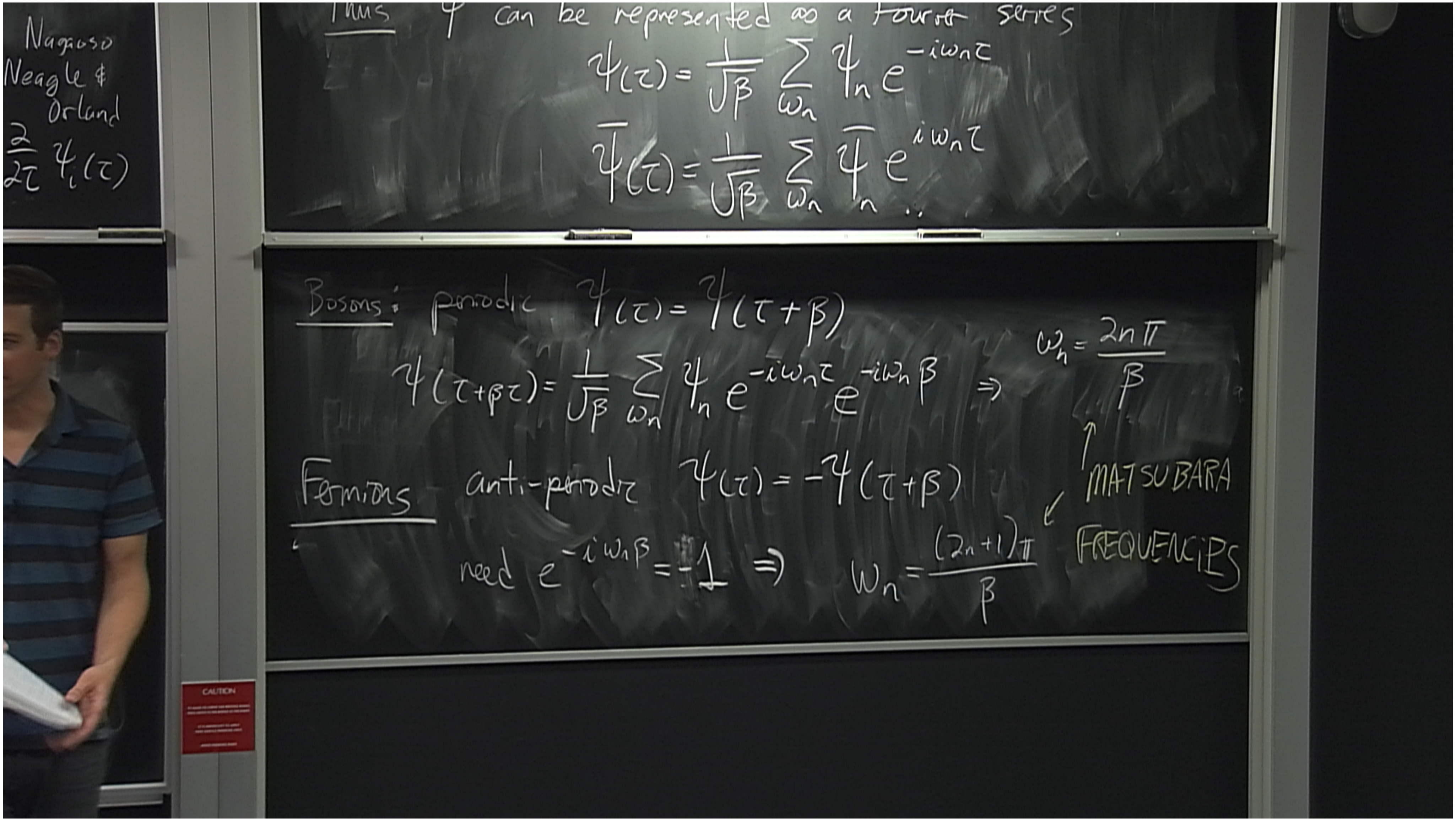

Fermions ρ=-1, ψ ∈ 𝒜
& completeness
Focus on the partition function, and expectation values
like ⟨a†a ...⟩ where ⟨...⟩ is an average over some grand-canonical ensemble
ie start with Z = Tr e^{-β(Ĥ-μN̂)} = Σ_{{|n⟩}} ⟨n| e^{-β(Ĥ-μN̂)} |n⟩
complete set of states in Fock space


Fermions $\rho = -1$, $\psi \in \mathcal{A}$
ie start with $Z = \mathrm{Tr}\, e$
complete set of states
Fock space
$Z = \int d(\bar\psi,\psi)\, e^{-\sum_i \bar\psi_i \psi_i} \sum_{\{n\}} \langle n|\psi\rangle\langle\psi| e^{-\beta(\hat H - \mu\hat N)} |n\rangle$
we want to use $\sum_{\{n\}} |n\rangle\langle n| = 1$
But what we need is to consider:
Bosons: $\langle n|\psi\rangle\langle\psi|n\rangle = \langle\psi|n\rangle\langle n|\psi\rangle$
homework
Fermions: $\langle n|\psi\rangle\langle\psi|n\rangle = \langle -\psi|n\rangle\langle n|\psi\rangle$
$\langle -\psi| = \langle 0| e^{-\sum_i \bar\psi_i a_i}$
$Z = \int d(\bar\psi,\psi)\, e^{-\sum_i \bar\psi_i \psi_i} \sum_n \langle \rho\psi| e^{-\beta(H-\mu N)} |n\rangle\langle n|\psi\rangle$
$= \int d(\bar\psi,\psi)\, e^{-\sum_i \bar\psi_i \psi_i} \langle \rho\psi| e^{-\beta(H-\mu N)} |\psi\rangle$
& completeness.
quantized form
$a^\dagger$ on left
a on right
"normal ordering"


Proceed in the usual way define $\tilde{H} = \hat{H} - \mu \hat{N}$
and $\mathrm{Tr}\, e^{-\beta \tilde{H}} = \mathrm{Tr}\, e^{-\Delta\tau \tilde{H}} \cdots \cdots \mathrm{Tr}\, e^{-\Delta\tau \tilde{H}}$
$Z = \prod_\tau \int d(\bar{\psi}(\tau)\psi(\tau))\, e^{-\sum_\tau \sum_i \bar{\psi}_i(\tau)\psi_i(\tau)}$
$\langle \psi(\beta)| e^{-\Delta\tau \tilde{H}} |\psi(\beta-\Delta\tau)\rangle \cdots e^{-\Delta\tau \tilde{H}} |\psi(\Delta\tau)\rangle\langle \psi(\Delta\tau)| e^{-\Delta\tau \tilde{H}} |\psi(0)\rangle$
$= \prod_i \left[1 + \langle 0| \bar{\eta}_i \eta_i' a_i a_i^\dagger |0\rangle \right] = \prod_i (1 + \bar{\eta}_i \eta_i')$
apply a_i
apply $a_i^\dagger$
overlap
Goal:
e.g.
e.g.)
CAUTION


where $\prod_\tau \int d(\bar{\psi}(\tau), \psi(\tau)) = \int \mathcal{D}(\bar{\psi}, \psi)$
$\psi(\beta) = e\,\psi(0)$
$Z = \int_{\psi(\beta)=e\psi(0)} \mathcal{D}[\bar{\psi}, \psi] \exp\left[-\sum_\tau \sum_i \left(\bar{\psi}_i(\tau)\left(\psi_i(\tau) - \psi_i(\tau - \Delta\tau)\right)\right)\right]$
e.g.) $H_{int} = \sum_{ij} h_{ij} a_i^\dagger a_j + \sum_{ijkl} V_{ijkl} a_i^\dagger a_j^\dagger a_k a_l$
a on right
"normal ordering"


· exp [− Σ_τ Δτ H (ψ̄(τ), ψ(τ−Δτ))]
Nagaosa
Neagle &
Orland
like before, Δτ → 0
Σ_τ Δτ → ∫_0^β dτ
ψ̄_i(τ) (ψ_i(τ) − ψ_i(τ−Δτ)) / Δτ = ψ̄(τ) ∂/∂τ ψ_i(τ)
Z = ∫ D[ψ̄, ψ] exp [−∫_0^β dτ (ψ̄ ∂_τ ψ + H̃(ψ̄, ψ))]
ψ(β) = ρ ψ(0)


Nagaoso
Neagle &
Orland
Thus ψ can be represented as a Fourier series
Bosons: periodic
Fermions anti-periodic
need
MATSUBARA
FREQUENCIES
CAUTION