Title: Quantum Clocks

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Abstract: <p>Time in quantum mechanics has duly received a lot of attention over the years. Perfect clocks which can turn on/off a particular interaction at a precise time that have been proposed only exist in infinite dimensions and have unphysical Hamiltonians (their spectrum is unbounded from below). It was this observation which led many to conclude that an operator for time cannot exist in quantum mechanics. Here, we prove rigorous results about the accuracy of finite dimensional clocks and show that they can well approximate their infinite dimensional counterparts under the right conditions.</p>

Quantum Clocks

Perimeter Institute, 19/01/2016

Presented by Mischa Woods[1,2] in collaboration with Ralph Silva[3], Michal Horodecki[4], Jonathan Oppenheim[1]

1: University College London, UK
2: QuTech Delft University of Technology, Netherlands
3: Université de Genève, Switzerland
4: University of Gdansk, Gdańsk

Outline

- Clocks and time in quantum Mechanics
  - Small Perez Clocks
  - Perfect clocks: Infinite dimensional but unphysical
- Our result: a large Perez Clock is "exponentially close to a Perfect Clock"
  - Our model
  - Our Theorem
  - Analogy between our clock and the perfect clock
  - Proof outline
- Applications
- Conclusions

Clocks and time in quantum Mechanics

- Small Perez Clocks

Asher Perez (1979): $H = \sum_{n=0}^{d-1} n\omega \, |n\rangle\langle n| \,,$

Angle basis: $|\theta_k\rangle = \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} e^{-i2\pi nk/d} \, |n\rangle \,,$

$$|n\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} e^{+i2\pi nk/d} \, |\theta_k\rangle \,.$$

[1]: *Measure of time by quantum clocks,* (1979)

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$d = 12$

At $t = \frac{2\pi}{\omega} \frac{12}{d}$: $\mathrm{e}^{-\mathrm{i}tH_c} \, |\theta_{12}\rangle = |\theta_{12}\rangle$

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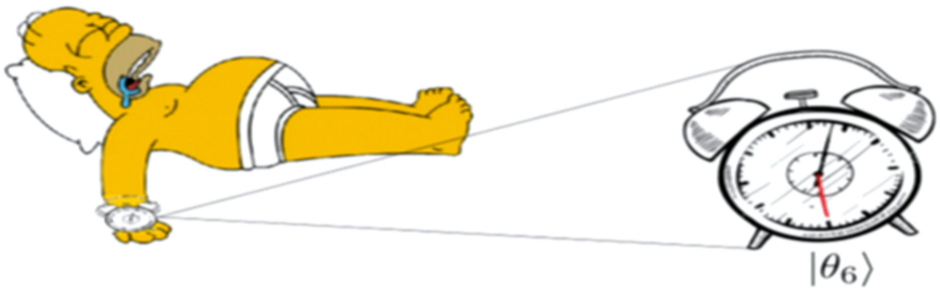
Clocks and time in quantum Mechanics

- Small Perez Clocks: they wake up in the morning but are fragile



$$H = \mathbb{1}_s + H_s^{\text{int}} \otimes |\theta_6\rangle\langle\theta_6| + H_c\,, \qquad \rho_s(0) \otimes |\theta_{12}\rangle$$

Clocks and time in quantum Mechanics

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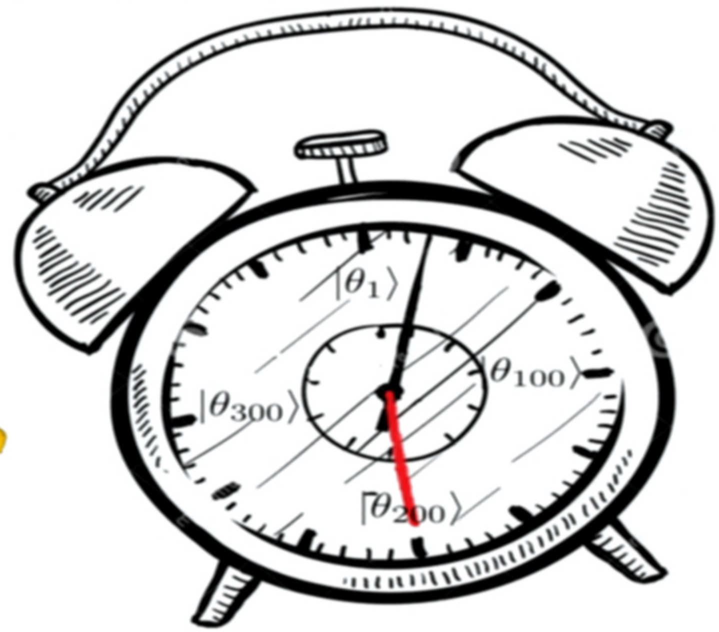
Clocks and time in quantum Mechanics

- Solution: make clock very Big!

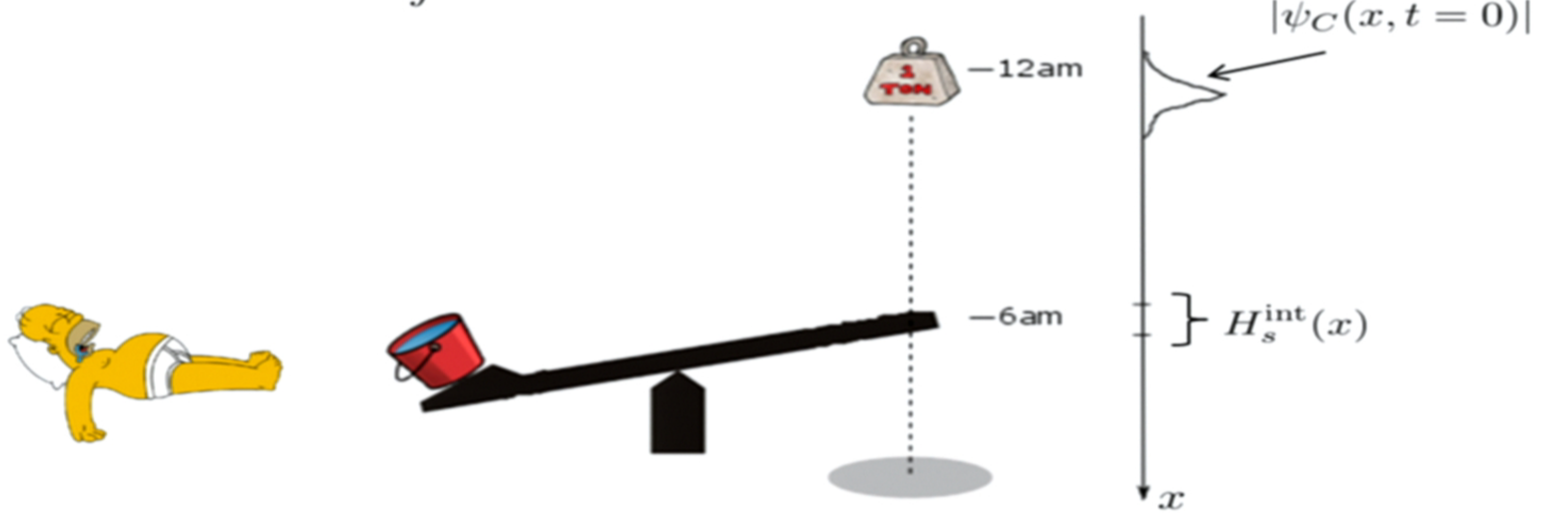
$$d >> 12$$



Clocks and time in quantum Mechanics

- Perfect clocks: Infinite dimensional but unphysical [1]

$$H = \mathbb{1}_s + \int dx H_s^{\text{int}}(x) \otimes |x\rangle\langle x| + P$$



[1]: A. Malabarba et el, ArXiv:1412.1338

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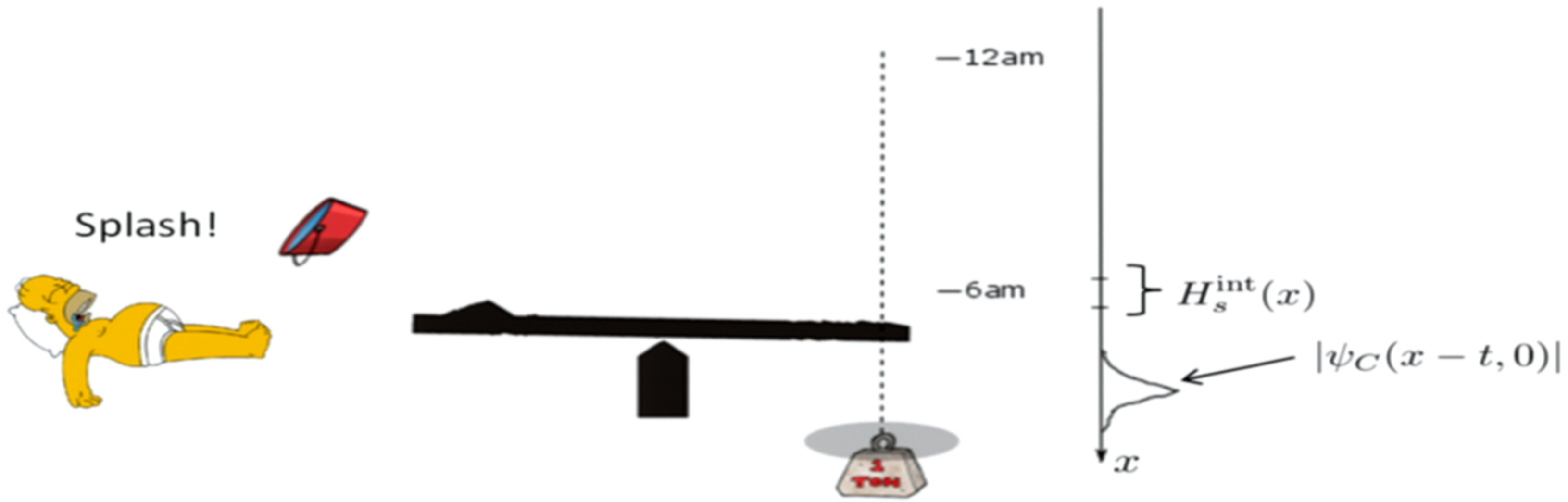


Clocks and time in quantum Mechanics

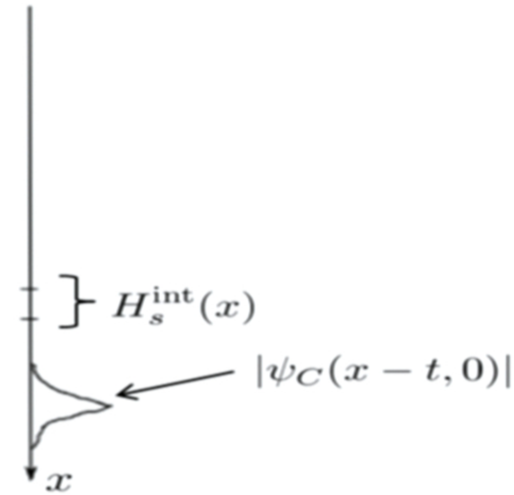
- Perfect clock and time-energy uncertainty relation

$$H = \mathbb{1}_s + \int dx H_s^{\text{int}}(x) \otimes |x\rangle\langle x| + P$$

$$\frac{\Delta \mathcal{T}}{|<\dot{\mathcal{T}}>|}\Delta H \geq \frac{1}{2}$$

$$1 = \dot{\mathcal{T}} = -\mathrm{i}[\mathcal{T}, H]$$

$$H = P, \quad \mathcal{T} = x$$



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Our result

- Our model

Ideal case: $U = \sum_n \mathrm{e}^{-\mathrm{i}\Omega_n} |\phi_n\rangle\langle\phi_n|_S$ Such that: $[U, H_s] = 0$

$$\rho_S^{\mathrm{ideal}}(t) = \begin{cases} \mathrm{e}^{-\mathrm{i}tH_s}\rho(0)\mathrm{e}^{\mathrm{i}tH_s} = \sum_{n,m}\rho_{n,m}(t)\,|\phi_n\rangle\langle\phi_m|_S & \text{if } t < t^* \\ \sum_{n,m}\rho_{n,m}(t)\mathrm{e}^{-\mathrm{i}(\Omega_m-\Omega_n)}\,|\phi_n\rangle\langle\phi_m|_S & \text{if } t \geq t^* \end{cases}$$

What we have:

$$H = H_C \otimes \mathbb{1}_S + H_{\mathrm{int}} + \mathbb{1}_C \otimes H_S = H_C \otimes \mathbb{1}_S + \left(\sum_n \Omega_n\,|\phi_n\rangle\langle\phi_n|_S\right) V_C + \mathbb{1}_C \otimes H_S$$

$$\rho_{SC}(0) = \rho_S(0) \otimes \rho_C(0); \qquad \rho_C(0) = |\psi\rangle\langle\psi|_C$$

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Our result

- Result

$$V_c = \sum_{k=1}^{d} \omega \tilde{V}(\frac{2\pi}{d}k)\, |\theta_k\rangle\langle\theta_k| \,, \qquad |\psi\rangle_C := |(k_0, 0)\rangle$$

$$|\psi(k_0,\Delta)\rangle_C = \left(\frac{1}{d}\right)^{1/4} \sum_{k=\lfloor k_0 \rfloor+1-d/2}^{\lfloor k_0 \rfloor+d/2} \mathrm{e}^{-\frac{\pi}{d}(k-k_0)^2} \mathrm{e}^{\mathrm{i}\pi(k-k_0)} \mathrm{e}^{-\mathrm{i}\frac{2\pi}{d}\int_{k-\Delta}^{k} dx \tilde{V}\left(\frac{2\pi x}{d}\right)} |\theta_k\rangle$$

Analytic extension: $\tilde{V}(x) = \begin{cases} 1) & \text{smooth} \\ 2) & \text{periodic with period } d \\ 3) & \int_0^{2\pi} dx \tilde{V}(x) = \Omega; \quad 0 \le \Omega < 2\pi \end{cases}$

Theorem: $\mathrm{e}^{-\mathrm{i}t(H_C+V_C)}\, |\psi(k_0,0)\rangle = |\psi(k_0 + \frac{d}{T}t, \frac{d}{T}t\rangle + |\epsilon\rangle \,, \quad \forall t, k_0 \in \mathbb{R}, \; \frac{d}{2} \ge 1$

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With: $\|\,|\epsilon\rangle\,\|_2 \le c_0\, t\, \mathrm{e}^{-\frac{\pi}{4}\frac{d}{(b+1)^2}}$

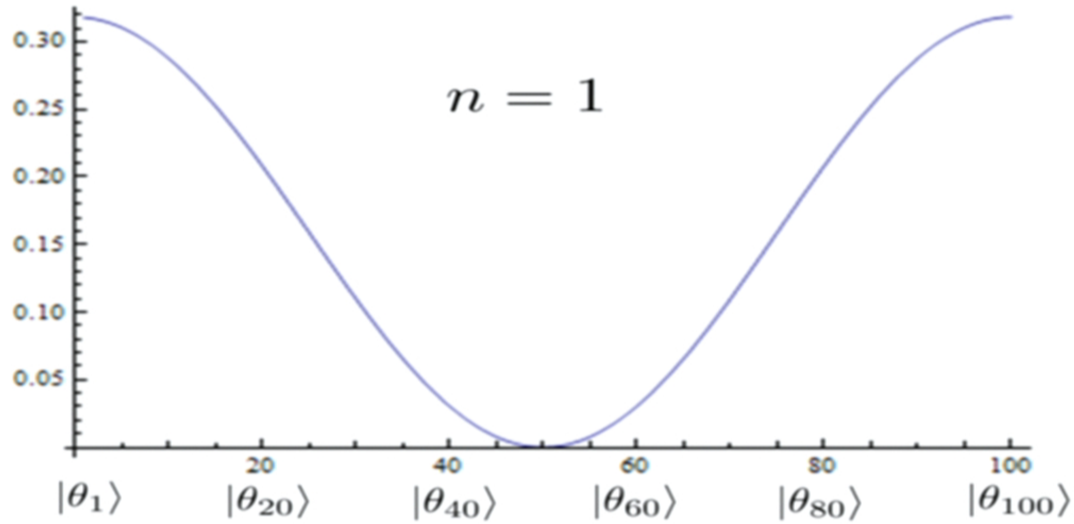
$$b := \frac{1}{\min\{b_L, b_R\}} \sup_{k \in \mathbb{N}^+} \left(\frac{1}{2\pi}\right)^{\frac{k-1}{k}} \left(2 \max_{x\in[0,2\pi]} \left|\tilde{V}^{(k-1)}(x)\right|\right)^{1/k}$$

$$\min\{b_L, b_R\} \sim \ln(d)$$

Our result

- Explicit example: Cosine potential $\tilde{V}\left(\frac{2\pi}{d}x\right) = A\cos^{2n}\left(\frac{\pi x}{d}\right)$

$$b \leq \frac{e^2}{\sqrt{\pi}}\frac{n\sqrt{n}}{\min\{b_L, b_R\}}$$



Our result

- Analogy with "Perfect Clock" in our model: $H = H_c + V_c = -\mathrm{i}\frac{\partial}{\partial x} + V(x)$

Solution: $\psi(x,t) = \psi(x-t,0)\,\mathrm{e}^{-\mathrm{i}\int_{x-t}^{x} V(y)dy}$

Our Model:

$$\underbrace{\langle\theta_k|\,\mathrm{e}^{-\mathrm{i}t(H_C+V_C)}\,|\psi(k_0,0)\rangle}_{\psi(k,t)} = \langle\theta_k|\psi(k_0+\frac{d}{T}t,\frac{d}{T}t\rangle + \langle\theta_k|\epsilon\rangle$$

$$\stackrel{\epsilon}{\approx} \underbrace{\left(\frac{1}{d}\right)^{1/4}\mathrm{e}^{-\frac{\pi}{d}(k-k_0-\frac{d}{T}t)^2}\mathrm{e}^{\mathrm{i}\pi(k-k_0-\frac{d}{T}t)}}_{\psi(k-\frac{d}{T}t,0)}\mathrm{e}^{-\mathrm{i}\frac{2\pi}{d}\int_{k-\frac{d}{T}t}^{k}\tilde{V}(\frac{2\pi}{d}y)dy}$$

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Our result

- Proof overview (intuition)

$|\psi\rangle_c =$

- 1st: apply $\mathrm{e}^{-\mathrm{i}\delta H_c} |\psi\rangle_c =$ D.F.T. + Phase Shift + I.D.F.T. $= |\psi\rangle_c'$
- 2nd: apply $\mathrm{e}^{-\mathrm{i}\delta V_c} |\psi\rangle_c' =$ Phase Shift
- 3rd: apply Lie Product Formula $\lim_{m\to\infty} \left(\mathrm{e}^{-\mathrm{i}tV_c/m}\mathrm{e}^{-\mathrm{i}tH_C/m}\right)^m = \mathrm{e}^{-\mathrm{i}t(V_C+H_c)}$

- Analyze setup in D.F.T. → F.T. limit ($d \to \infty$) $\langle x|\psi\rangle_c = \mathrm{e}^{-ax^2+bix}$

$$\mathrm{FT}\left(\mathrm{e}^{-ax^2+bix}\right)[p] \sim \mathrm{e}^{-(b+p)^2/4a}$$

$$\mathrm{PhaseShift}\left(\mathrm{e}^{-(a+p)^2/4a}\right)[p] \sim \mathrm{e}^{-(a+p)^2/4a}\mathrm{e}^{-\mathrm{i}\delta p}$$

$$\mathrm{IFT}\left(\mathrm{e}^{-(a+p)^2/4a}\mathrm{e}^{-\mathrm{i}\delta p}\right)[x] \sim \mathrm{e}^{-a(x-\delta)^2+b\mathrm{i}(x-\delta)}$$

$$\mathrm{PhaseShift}\left(\mathrm{e}^{-a(x-\delta)^2+b\mathrm{i}(x-\delta)}\right)[x] \sim \mathrm{e}^{-a(x-\delta)^2+b\mathrm{i}(x-\delta)}\mathrm{e}^{-\mathrm{i}\delta\Omega} = \langle x-\delta|\psi\rangle_c\,\mathrm{e}^{-\mathrm{i}\delta\Omega}$$

Our result

- Proof overview (More detailed)

Lemma 1: infinitesimal evolution under Clock Hamiltonian

$$e^{-i\frac{2\pi}{\omega d}\delta H_c} \sum_{k=\lfloor k_0 \rfloor+1-d/2}^{\lfloor k_0 \rfloor+d/2} \tilde{\psi}(k_0,\Delta,k) = \sum_{k=\lfloor k_0 \rfloor+1-d/2}^{\lfloor k_0 \rfloor+d/2} \tilde{\psi}(k_0,\Delta,k-\delta)\,|\theta_k\rangle + |\epsilon\rangle$$

With $\|\,|\epsilon\rangle\,\|_2 = c\,\delta\, e^{-\frac{d}{(1+b)^2}}$

Essential Proof tools:

1) Poisson Summation formulas (forward and backward types)
2) Integration by parts ~ d times
3) Generalized Leibniz Rule
4) Rodriguez formulas for Hermite Polynomials, and orthogonality conditions of Hermite Polynomials
6) Faà di Bruno's formula
7) Bell Polynomials, Bell Numbers, and analytic upper bounds to Bell numbers
8) Sterling’s formula
9) Fundamental theorem of Calculus

Our result

- Proof overview (More detailed)

Lemma 2: infinitesimal evolution under the interaction potential

$$e^{-i\frac{2\pi}{\omega d}\delta V_c}|\Psi\rangle = \sum_k \langle\theta_k|\Psi\rangle\, e^{-i\frac{2\pi}{d}\int_{k-\delta}^{k}\tilde{V}(y)dy}|\theta_k\rangle + |\epsilon\rangle$$

With $\| \, |\epsilon\rangle \, \|_2 = \mathcal{O}\left(\delta^2\right)$

Essential Proof tools:

1) Taylor's remainder theorem

Conclusions

- To the best of our knowledge, 1st quantitative study of the accuracy of finite dimensional quantum mechanical clocks
- Clocks studies behave like angular momentum versions of the "Perfect" linear momentum clocks
- Clock error:
    1. grows linearly in time
    2. decays exponentially in clock dimension
    3. decay rate governed by how steep the potential is

A few potential Applications

- T-E uncertainty relation for finite dimensional clocks
- Foundational issues in quantum mechanics: The alternative "Tick game" [1]
- Implantation of unitaries in quantum computation
- Implementability of energy preserving unitaries in Quantum Thermodynamics

[1] ArXiv: 1506.01373 (S. Rankovic, et al)

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