Title: Quantum Field Theory for Cosmology - Achim Kempf - Lecture 7

Date: Jan 25, 2016 01:30 PM

URL: http://pirsa.org/16010007

Abstract:

* Recall $\hat{H}(t) = \omega\left(a^+(t)\,a(t) + \frac{1}{2}\right) - \frac{1}{\sqrt{2\omega}}\left(a^+(t) + a(t)\right) J(t)$

$$= \begin{cases} \omega\left(a^+_{in}\, a_{in} + \frac{1}{2}\right) & \text{for } t < 0 \\ \text{something} & \text{for } 0 \leq t \leq T \\ \omega\left(a^+_{out}\, a_{out} + \frac{1}{2}\right) & \text{for } T < t \end{cases}$$

Here, $a_{in} := a(0)$, $a_{out} := a(T)$ and $a_{out} = a_{in} + J_0$

with: $J_0 := 1 \frac{i}{\sqrt{2\omega}} \int_0^T J(t')\, e^{i\omega t'}\, dt'$

* For $t<0$, we diagonalized the Hamiltonian

$\hat{H}(t) = \omega\left(a^{+}_{in}\, a_{in} + \frac{1}{2}\right) = \hat{H}$ ...

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$$\hat{H}(t) = \omega\left(a^+_{in}a_{in} + \frac{1}{2}\right) = \hat{H}_{t<0} = \text{const.}$$

by using $[a_{in}, a^+_{in}] = 1$ to construct its eigenbasis:

$$\hat{H}_{t<0}|n_{in}\rangle = E^{(in)}_n |n_{in}\rangle$$

Namely:

$$E^{(in)}_n = \omega\left(n + \frac{1}{2}\right) \quad , n = 0, 1, 2, 3 \ldots$$

$$|n_{in}\rangle := \frac{1}{\sqrt{n!}}(a^+_{in})^n |0_{in}\rangle$$

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Namely:

$$E_n^{(in)} = \omega\left(n + \frac{1}{2}\right) \quad , n = 0, 1, 2, 3 \ldots$$

$$|n_{in}\rangle := \frac{1}{\sqrt{n!}} (a_{in}^{+})^{n} |0_{in}\rangle$$

Note: The set $\{|n_{in}\rangle\}$ is a Hilbert basis of the Hilbert space $\mathcal{H}$.

* By $t > T$, the Hamiltonian has become a different operator:

$$\hat{H}(t) = \omega\left(a_{out}^{+} a_{out} + \frac{1}{2}\right) = \hat{H}_{t>T} = \text{const.}$$

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$$a_{out} |0_{out}\rangle = 0$$

* We define the set of vectors $\{|n_{out}\rangle\}$:

$$|n_{out}\rangle := \frac{1}{\sqrt{n!}} (a^+_{out})^n |0_{out}\rangle$$

The operators $\hat{H}_{t<0}$ and $\hat{H}_{t>T}$ are different and have different eigenvectors: $|n_{in}\rangle$ and $|n_{out}\rangle$. Why are the eigenvalues the same? They both describe a free oscillator of frequency ω.

* Proposition:

$$\hat{H}_{t>T} |n_{out}\rangle = E_n^{(out)} |n_{out}\rangle \quad \text{with} \quad E_n^{(out)} = \omega\left(n + \frac{1}{2}\right) = E_n^{(in)}$$

* Proposition:

The set $\{|n_{out}\rangle\}$ is a ON Hilbert basis of the Hilbert space $\mathcal{H}$.

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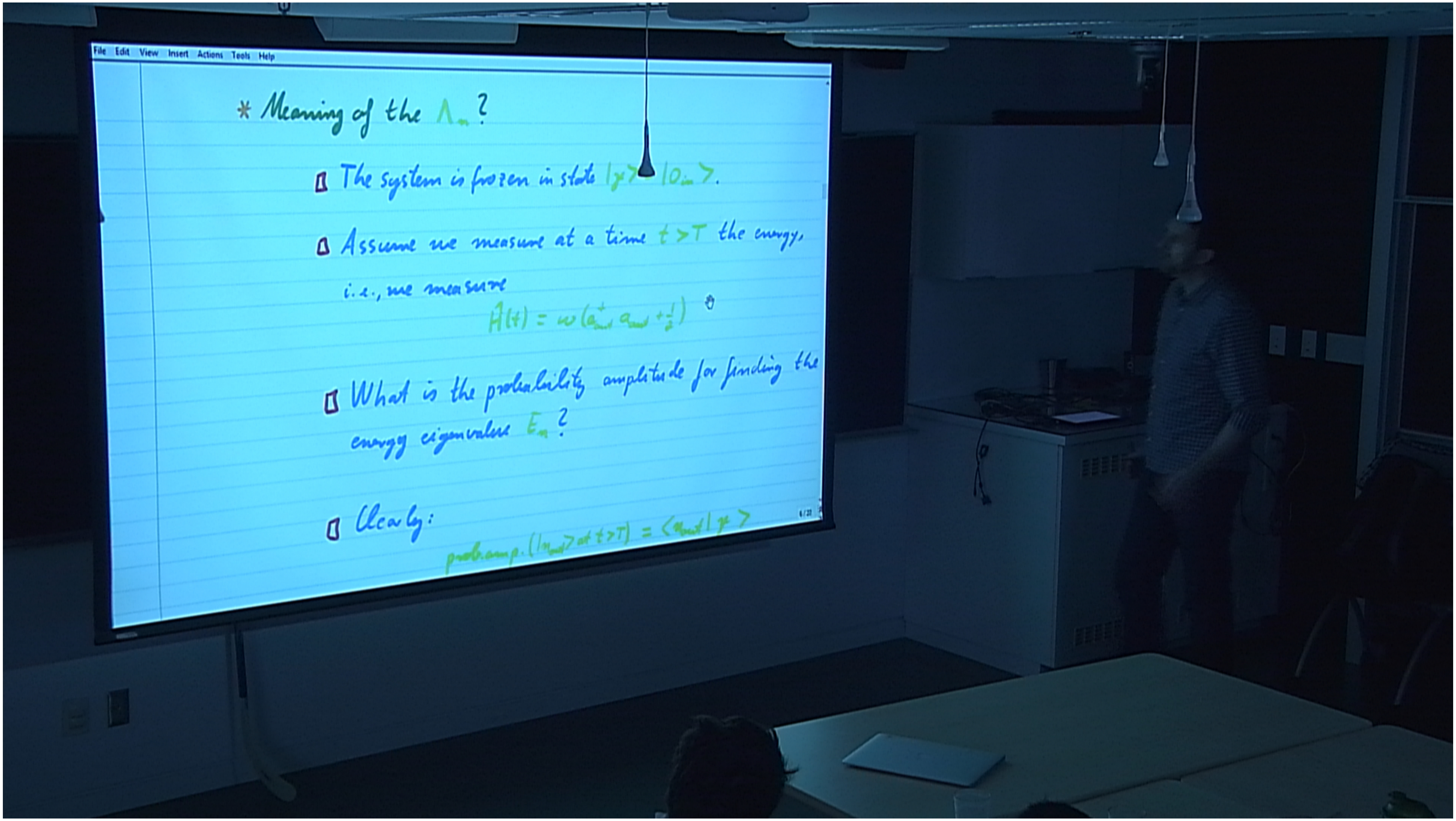
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* Meaning of the Λm ?
The system is frozen in state |ψ> = |0in>.
Assume we measure at a time t > T the energy,
i.e., we measure
H(t) = ω(a+out aout + 1/2)
What is the probability amplitude for finding the energy eigenvalue Em ?
Clearly:
prob.amp.(|nout> at t>T) = <nout| ψ>

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Clearly:

$$\text{prob.amp.}(|n_{out}\rangle \text{ at } t>T) = \langle n_{out} | \psi \rangle$$

i.e.:

$$\text{prob.}(|n_{out}\rangle \text{ at } t>T) = |\langle n_{out} | \psi \rangle|^2$$

Calculate:

$$\langle n_{out} | \psi \rangle = \langle n_{out} | 0_{in} \rangle$$

$$= \langle n_{out} | \sum_m \Lambda_m | m_{out} \rangle$$

$$= \Lambda_n$$

$\Rightarrow$ If the oscillator started in its ground state, then

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Calculation of Λ_n:

Proposition: $\Lambda_n = e^{-\frac{1}{2}|J_0|^2} \frac{1}{\sqrt{n!}} J_0^n$

Proof: The claim is that $|0_{in}\rangle = \sum_n e^{-\frac{1}{2}|J_0|^2} \frac{1}{\sqrt{n!}} J_0^n \, |n_{out}\rangle$.

We need to check that indeed: $a_{in}|0_{in}\rangle = 0$

Using $a_{out} = a_{in} + J_0$, we need to check: $(a_{out} - J_0)\,|0_{in}\rangle = 0$

Indeed:

$$(a_{out} - J_0) \sum_n e^{-\frac{1}{2}|J_0|^2} \frac{1}{\sqrt{n!}} J_0^n \, |n_{out}\rangle$$

$\frac{1}{n!}$

$\parallel$

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$$= e^{-\frac{1}{2}|J_0|^2}\left(a_{out} - J_0\right) e^{J_0 a^{+}_{out}} |0_{out}\rangle$$

$$= e^{-\frac{1}{2}|J_0|^2}\Big(\underbrace{a_{out}\, e^{J_0 a^{+}_{out}}} - J_0\, e^{J_0 a^{+}_{out}}\Big)|0_{out}\rangle$$

$\parallel$ using $AB = [A,B] + BA$

$$= e^{-\frac{1}{2}|J_0|^2}\Big(\overbrace{\underbrace{[a_{out}, e^{J_0 a^{+}_{out}}]} + e^{J_0 a^{+}}\, a_{out}} - J_0\, e^{J_0 a^{+}_{out}}\Big)|0_{out}\rangle$$

$\parallel$

$$\overset{(*)}{=} e^{-\frac{1}{2}|J_0|^2}\Big(\overbrace{(J_0 - J_0)}\, e^{J_0 a^{+}_{out}} + e^{J_0 a^{+}_{out}}\, a_{out}\Big)|0_{out}\rangle = 0 \quad ✓$$

Note: In the last step, (*), we used that: $[a_{out}, e^{J_0 a^{+}_{out}}] = J_0\, e^{J_0 a^{+}_{out}}$.

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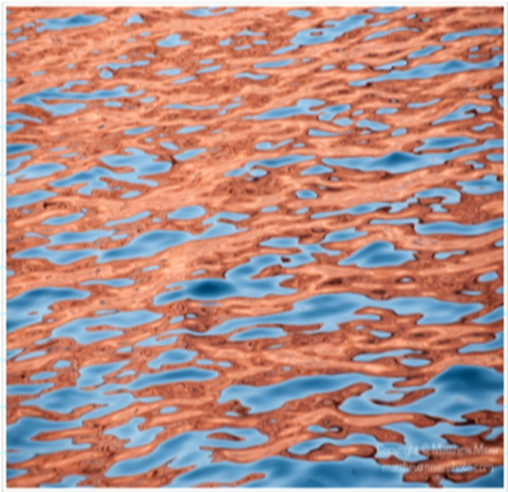
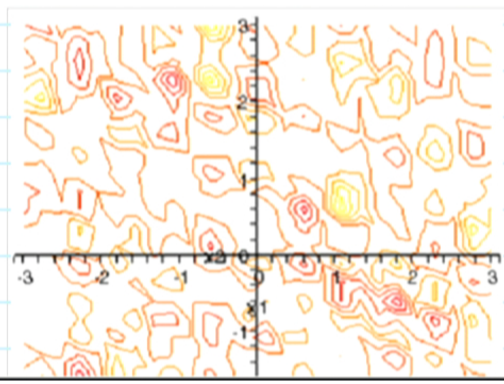
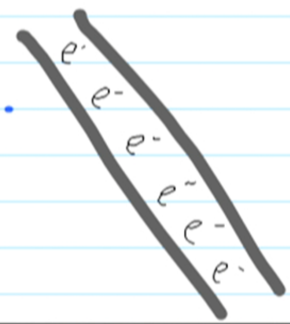
Drive mode oscillators in QFT:



Making waves...



Making EM waves...



e^+

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Recall:

$$\hat{H}(t) = \frac{1}{2}\int_{\mathbb{R}^3} \hat{\pi}^2(x,t) - \hat{\phi}(x,t)(\Delta - m^2)\hat{\phi}(x,t) + J(x,t)\hat{\phi}(x,t)\, d^3x$$

Example interpretation:

* $\hat{\phi}(x,t)$ may be viewed as a slightly simplified version of the quantum electromagnetic field.

* $J(x,t)$ may be viewed as a simplified version of a given classical electric charge and current density function.

Example:

A (Klein-Gordon) charge traveling a path $\vec{v}^i(t)$:

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$$[\hat{\phi}(x,t), \hat{\pi}(x',t)] = i\,\delta^3(x-x')$$

* Fourier transformed,

$$\hat{\phi}_k(t) := \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \hat{\phi}(x,t)\, e^{-ikx}\, d^3x$$

we need to solve:

$$\ddot{\hat{\phi}}_k(t) + \underbrace{(\overbrace{k^2}^{=\sum_{i=1}^{3} k_i^2} + m^2)}_{=:\,\omega_k^2}\, \hat{\phi}_k(t) = 0 \qquad (EoM)$$

Definition: ω_k^2

$$[\hat{\phi}_k(t), \hat{\pi}_{k'}(t)] = i\,\delta^3(k+k') \qquad (CCRs)$$

* Recall: $\hat{\phi}^+(x,t) = \hat{\phi}(x,t)$ means $\hat{\phi}_k^+(t) = \hat{\phi}_{-k}(t)$.

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