Title: Quantum Field Theory for Cosmology - Achim Kempf - Lecture 4

Date: Jan 15, 2016 01:30 PM

URL: http://pirsa.org/16010004

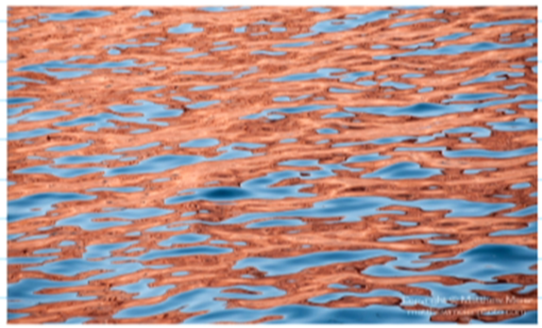
Abstract:

Note Title

From Heisenberg to Schrödinger picture

Water:

$\phi(x,t)$

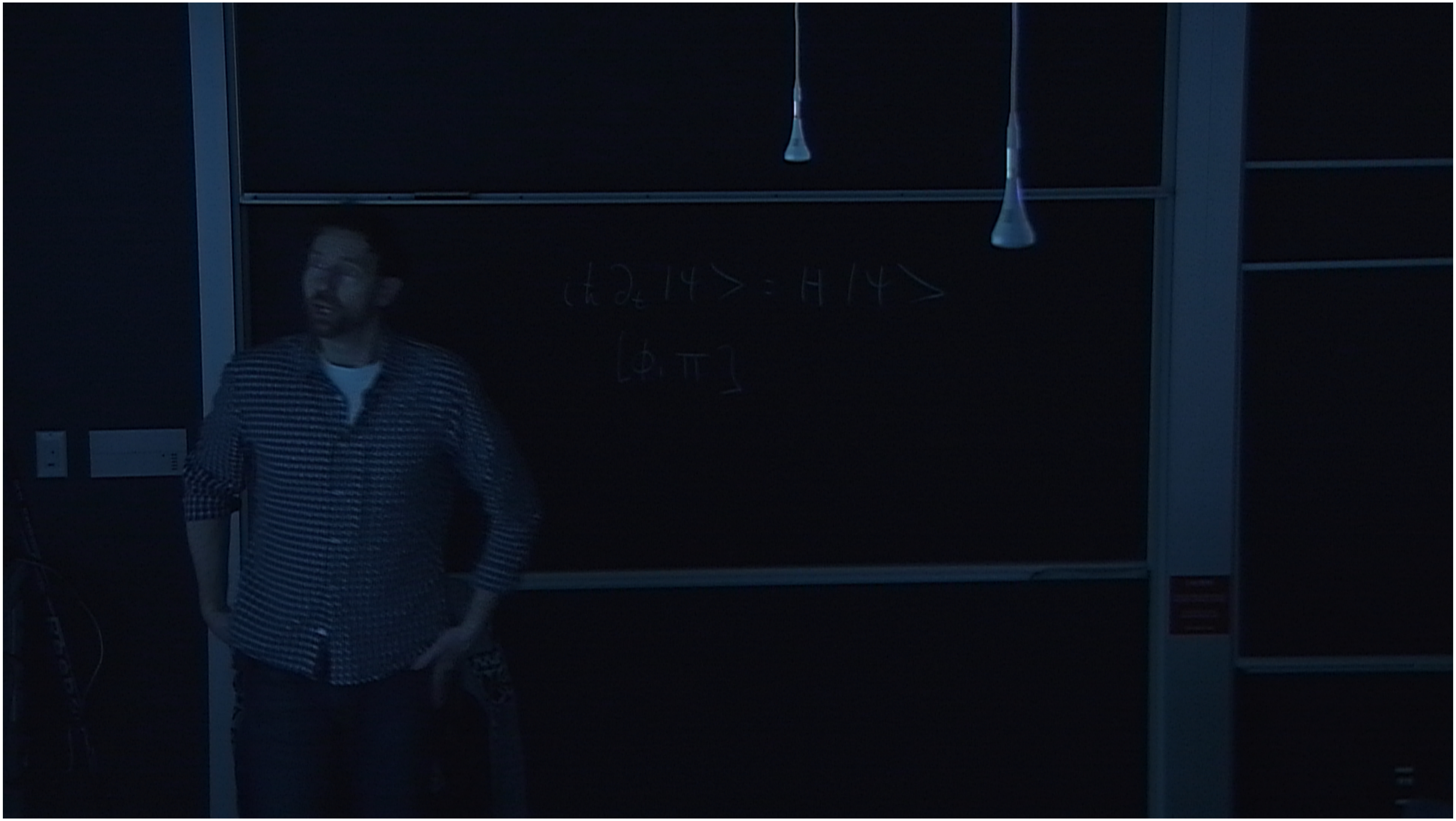


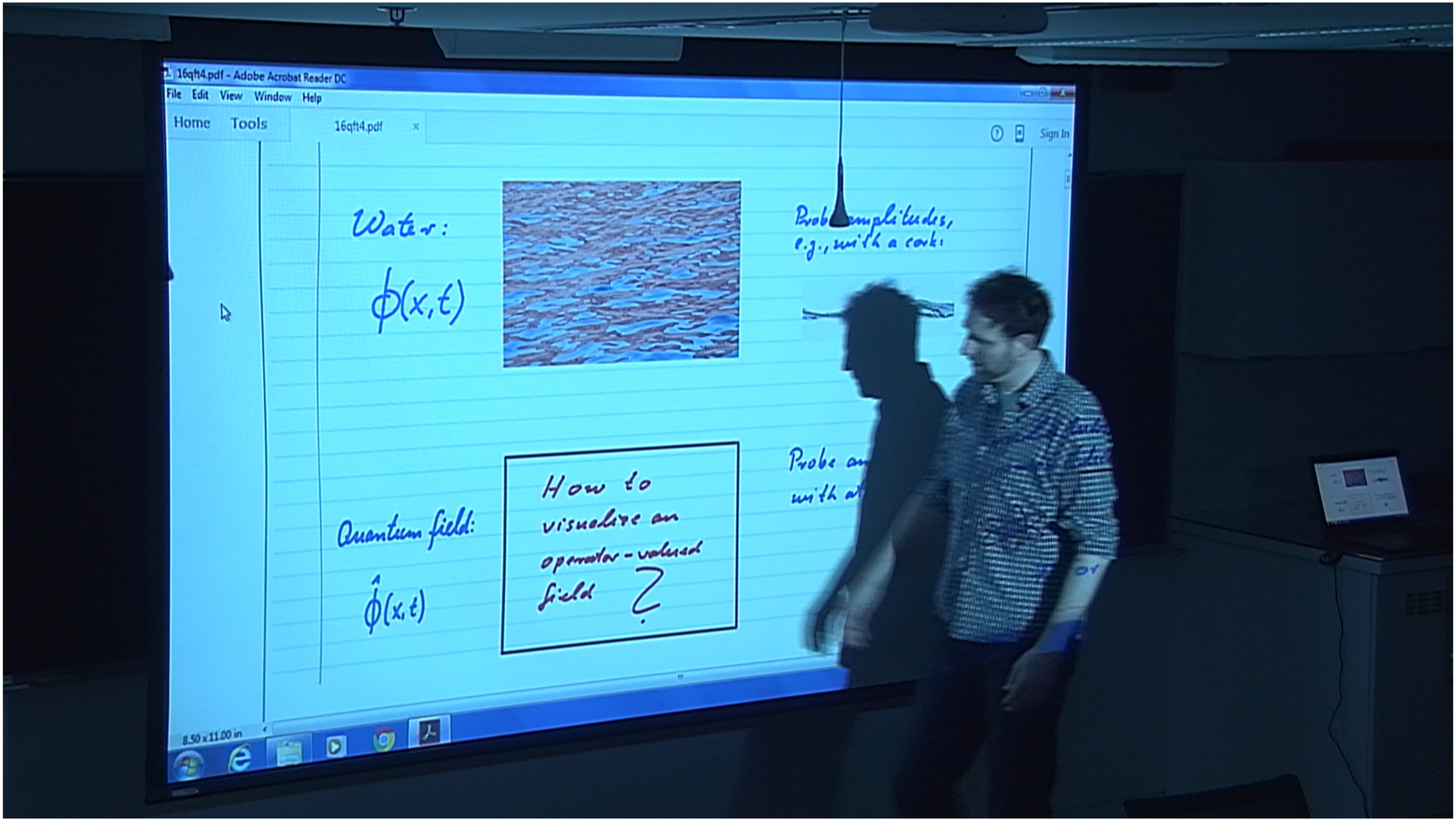
Probe amplitudes, e.g., with a cork:

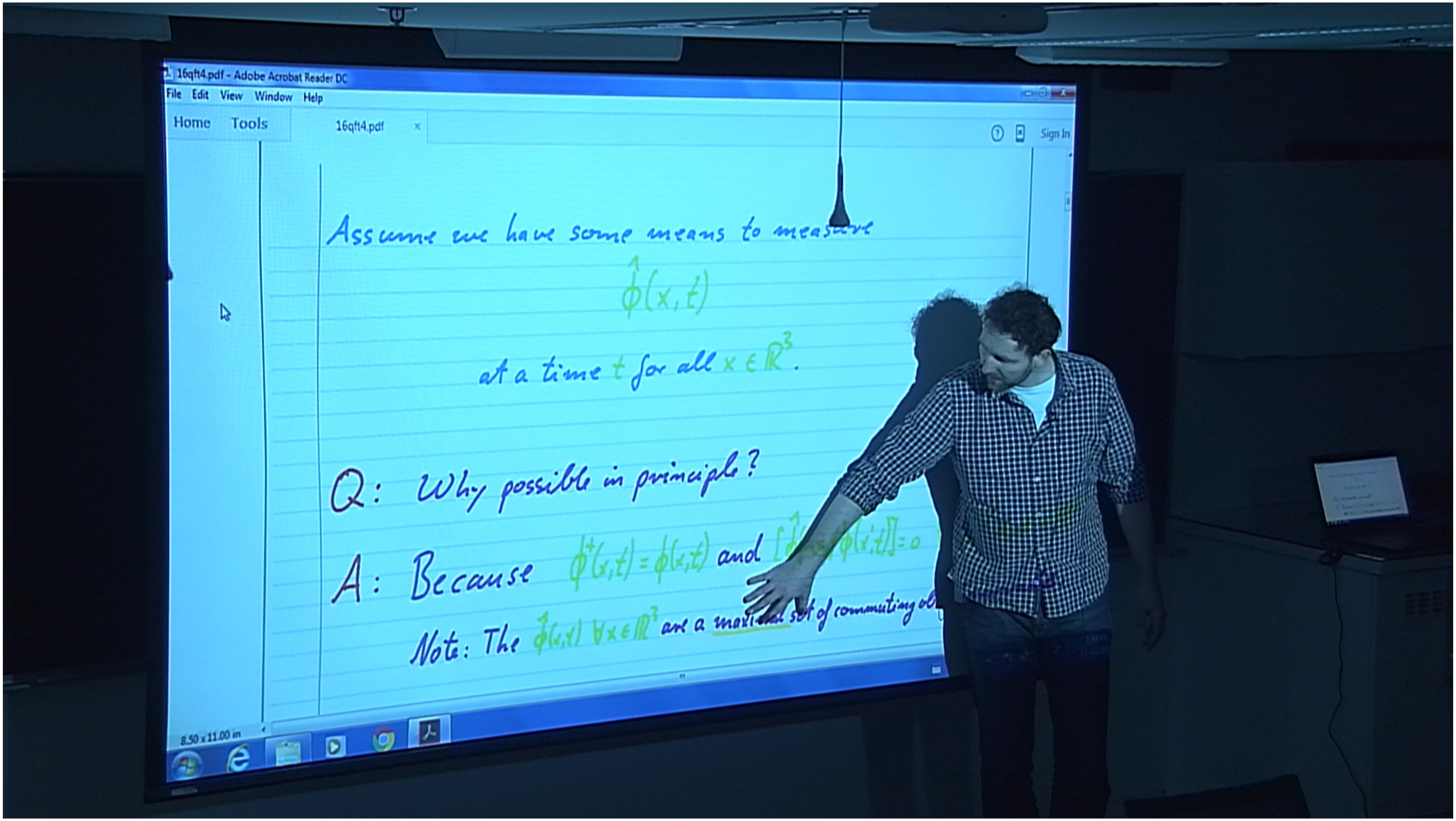
Quantum field:

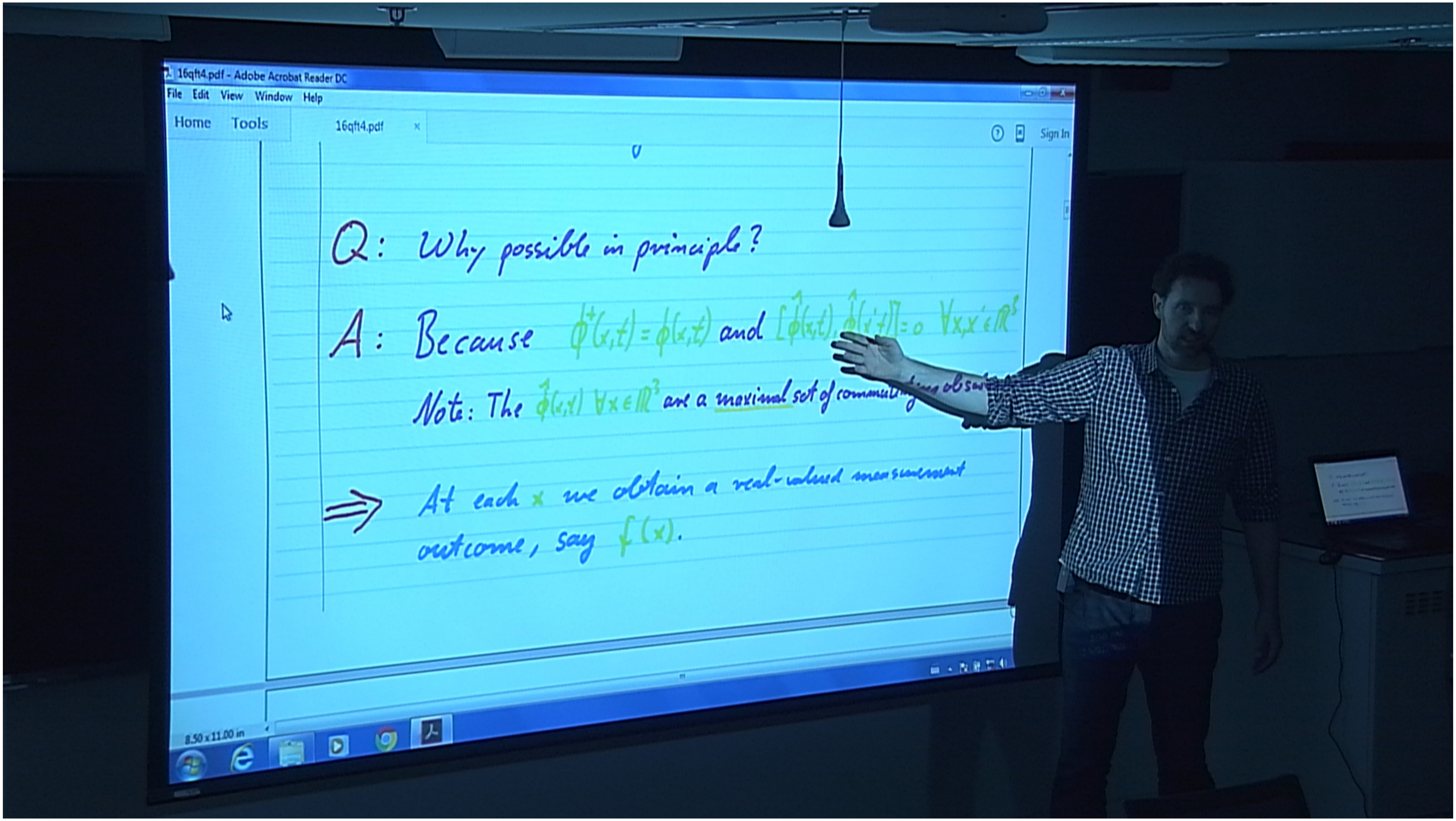
How to visualize an

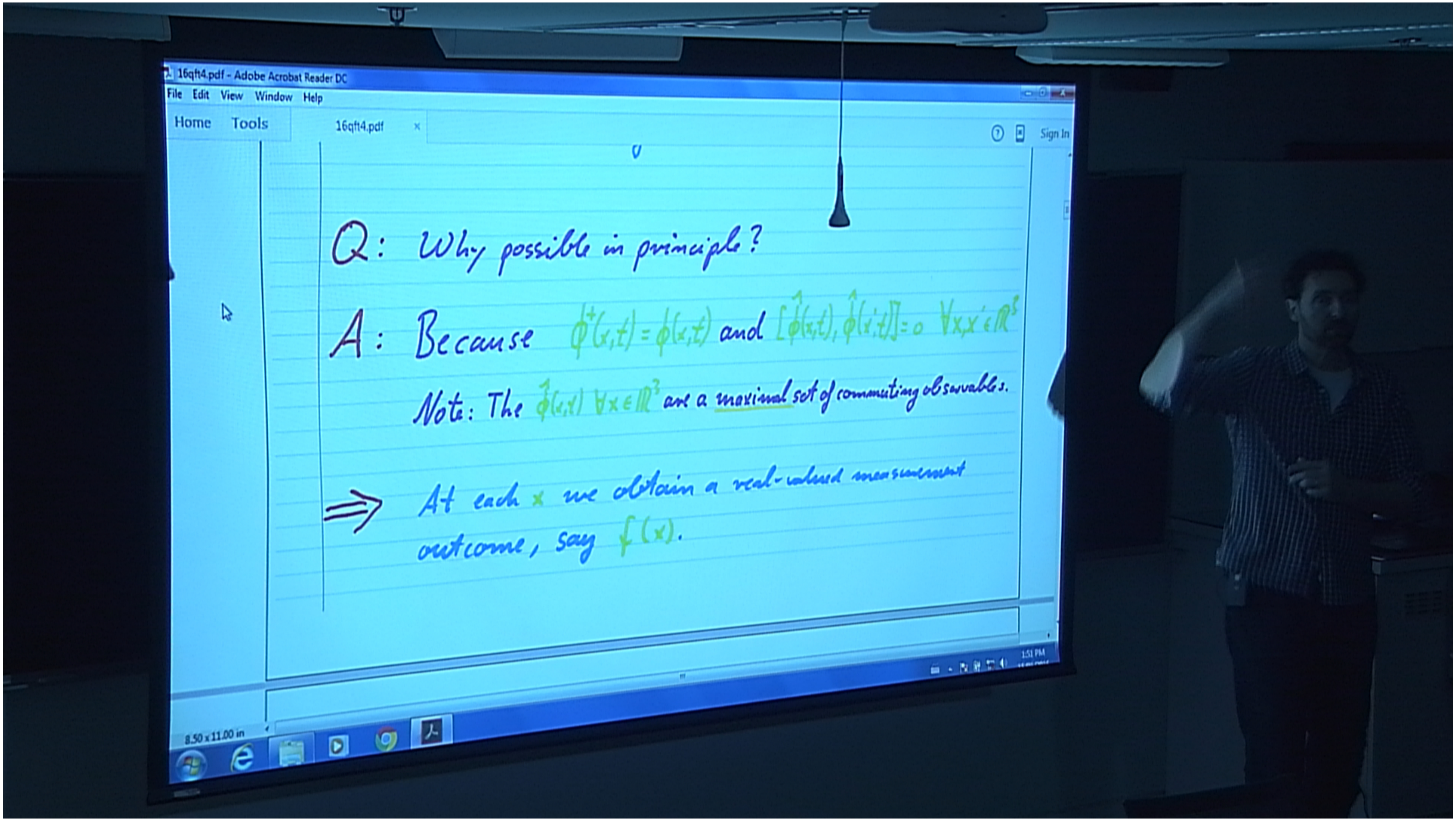
Probe amplitudes, e.g., with atoms (lecture 8):

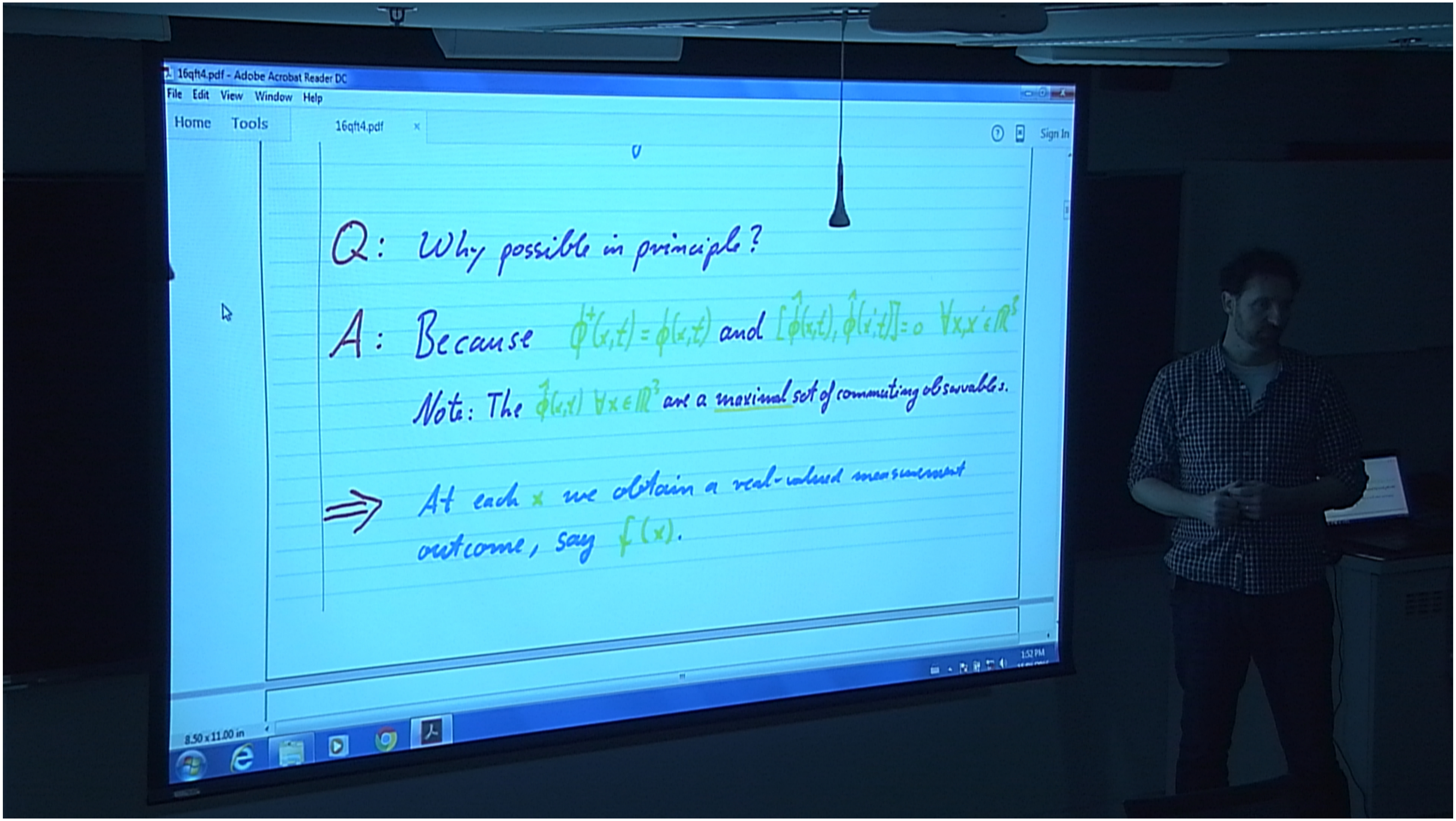

$i\hbar\partial_t|\psi\rangle = H|\psi\rangle$
$[\phi, \pi]$

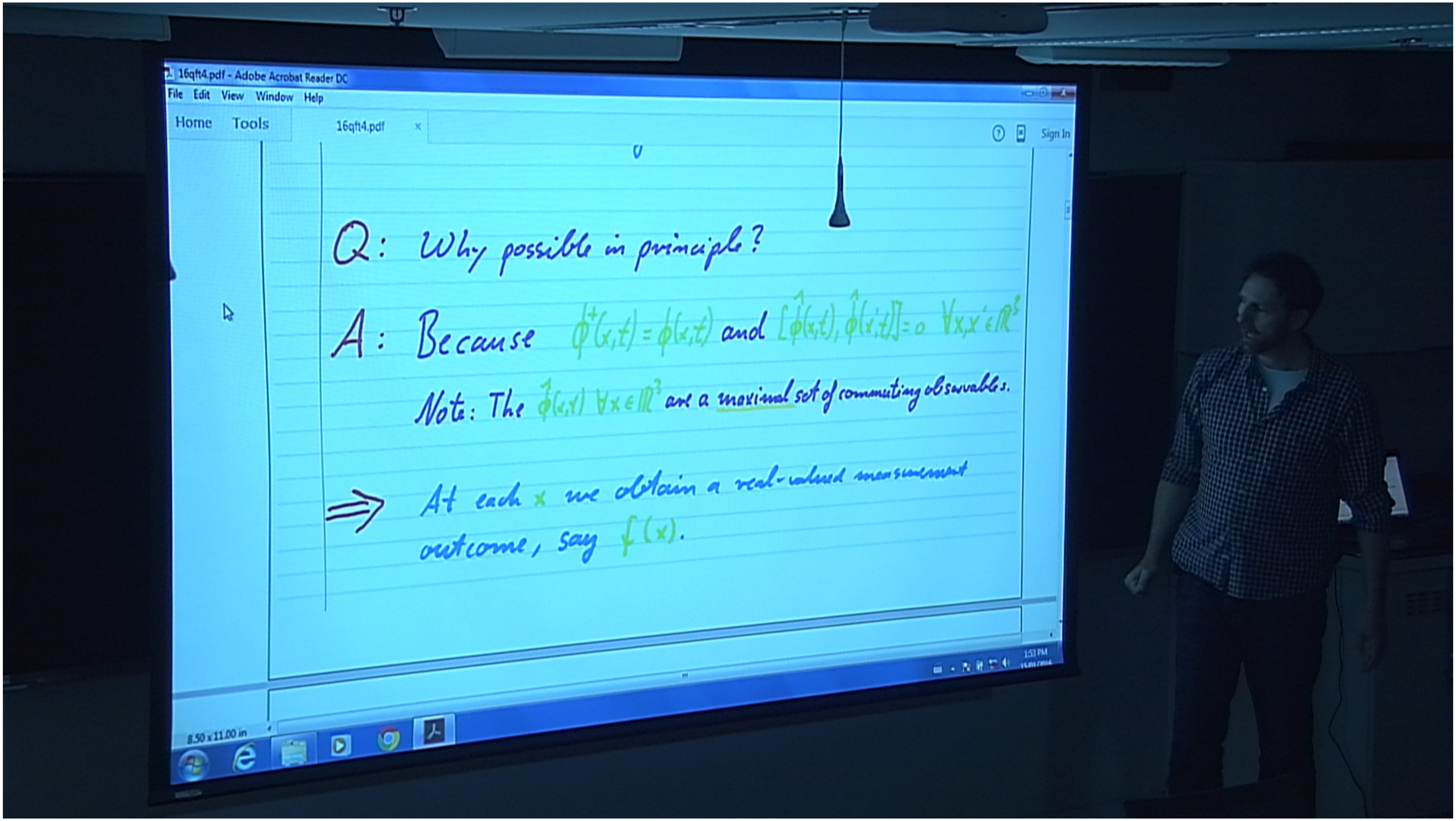

16qft4.pdf - Adobe Acrobat Reader DC
File Edit View Window Help
Home Tools
16qft4.pdf
Sign In
Water:
$\phi(x,t)$
e.g., with a cork:
Quantum field:
$\hat{\phi}(x,t)$
How to
visualize an
operator-valued
field ?
8.50 x 11.00 in

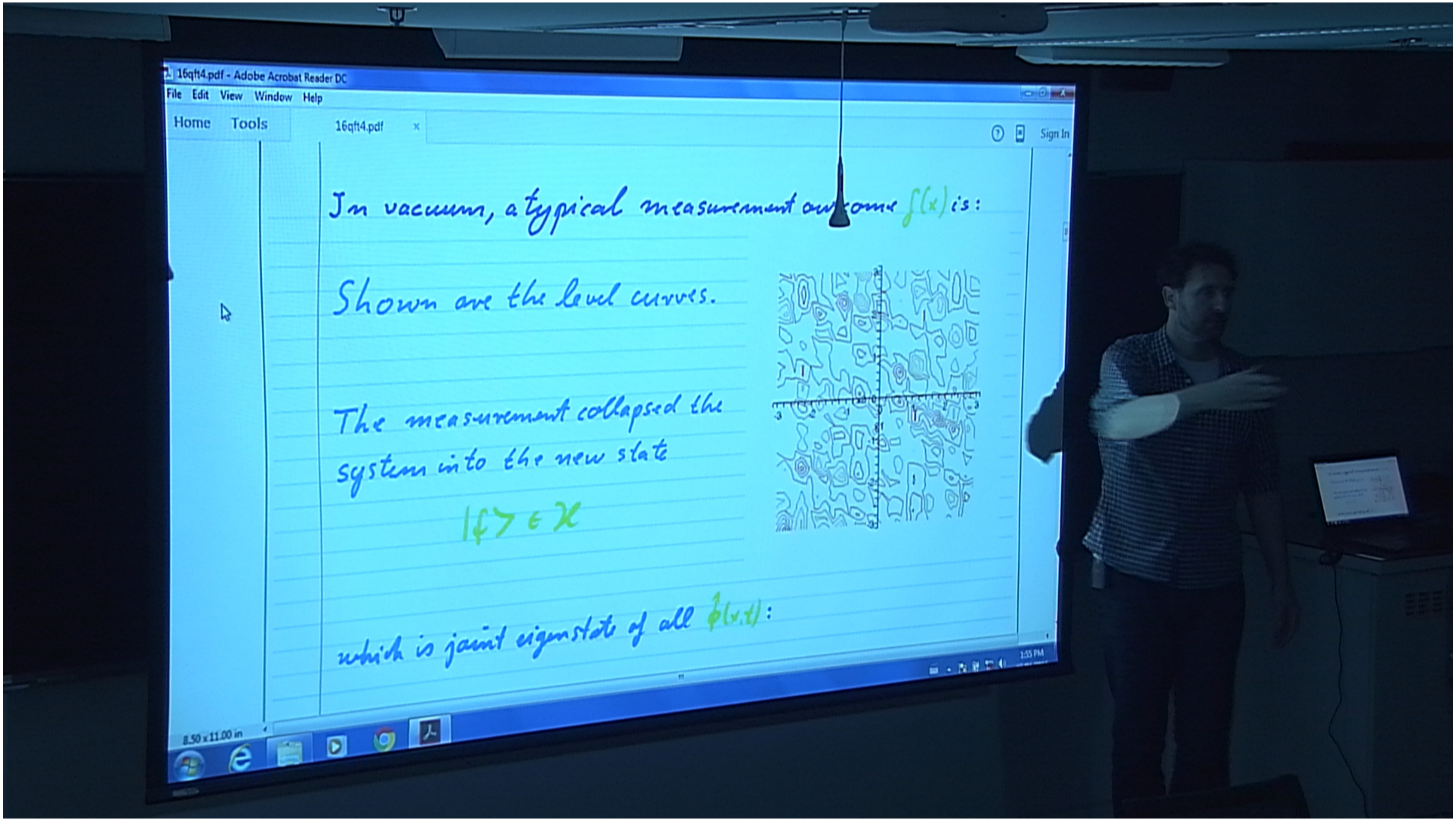

16qft4.pdf - Adobe Acrobat Reader DC
File Edit View Window Help
Home Tools
16qft4.pdf
Sign In
Assume we have some means to measure
$\hat{\phi}(x,t)$
at a time t for all $x \in \mathbb{R}^3$.
Q: Why possible in principle?
A: Because $\hat{\phi}^\dagger(x,t) = \hat{\phi}(x,t)$ and
Note: The $\hat{\phi}(x,t)$ $\forall x \in \mathbb{R}^3$ are a maximal set of commuting
8.50 x 11.00 in


16qft4.pdf - Adobe Acrobat Reader DC
File Edit View Window Help
Home Tools
16qft4.pdf
Sign In
Q: Why possible in principle?
A: Because $\hat{\phi}^{\dagger}(x,t) = \hat{\phi}(x,t)$ and $[\hat{\phi}(x,t), \hat{\phi}(x',t)] = 0 \ \forall x, x' \in \mathbb{R}^3$
Note: The $\hat{\phi}(x,t) \ \forall x \in \mathbb{R}^3$ are a maximal set of
$\Longrightarrow$ At each x we obtain a real-valued measurement outcome, say $f(x)$.
8.50 x 11.00 in


16qft4.pdf - Adobe Acrobat Reader DC
File Edit View Window Help
Home Tools
16qft4.pdf
Sign In
Q: Why possible in principle?
A: Because $\hat{\phi}^{\dagger}(x,t) = \hat{\phi}(x,t)$ and $[\hat{\phi}(x,t), \hat{\phi}(x',t)] = 0 \ \forall x, x' \in \mathbb{R}^3$
Note: The $\hat{\phi}(x,t) \ \forall x \in \mathbb{R}^3$ are a maximal set of commuting observables.
$\Longrightarrow$ At each x we obtain a real-valued measurement outcome, say $f(x)$.
8.50 x 11.00 in
1:51 PM

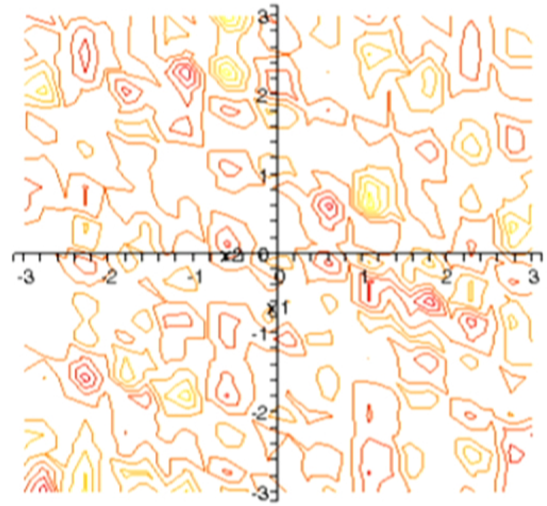

16qft4.pdf - Adobe Acrobat Reader DC
File Edit View Window Help
Home Tools
16qft4.pdf
Sign In
Q: Why possible in principle?
A: Because $\hat{\phi}^{\dagger}(x,t) = \hat{\phi}(x,t)$ and $[\hat{\phi}(x,t), \hat{\phi}(x',t)] = 0 \ \forall x, x' \in \mathbb{R}^3$
Note: The $\hat{\phi}(x,t) \ \forall x \in \mathbb{R}^3$ are a maximal set of commuting observables.
$\Rightarrow$ At each x we obtain a real-valued measurement outcome, say $f(x)$.
8.50 x 11.00 in
1:52 PM


16qft4.pdf - Adobe Acrobat Reader DC
File Edit View Window Help
Home Tools
16qft4.pdf
Sign In
Q: Why possible in principle?
A: Because $\hat{\phi}^{\dagger}(x,t) = \hat{\phi}(x,t)$ and $[\hat{\phi}(x,t), \hat{\phi}(x',t)] = 0 \ \forall x, x' \in \mathbb{R}^3$
Note: The $\hat{\phi}(x,t) \ \forall x \in \mathbb{R}^3$ are a maximal set of commuting observables.
$\Rightarrow$ At each x we obtain a real-valued measurement outcome, say $f(x)$.
8.50 x 11.00 in


16qft4.pdf - Adobe Acrobat Reader DC
File Edit View Window Help
Home Tools
16qft4.pdf
Sign In
In vacuum, a typical measurement outcome $f(x)$ is:
Shown are the level curves.
The measurement collapsed the
system into the new state
$|f\rangle \in \mathcal{H}$
which is joint eigenstate of all $\phi(x,t)$:
8.50 x 11.00 in

In vacuum, a typical measurement outcome $f(x)$ is:

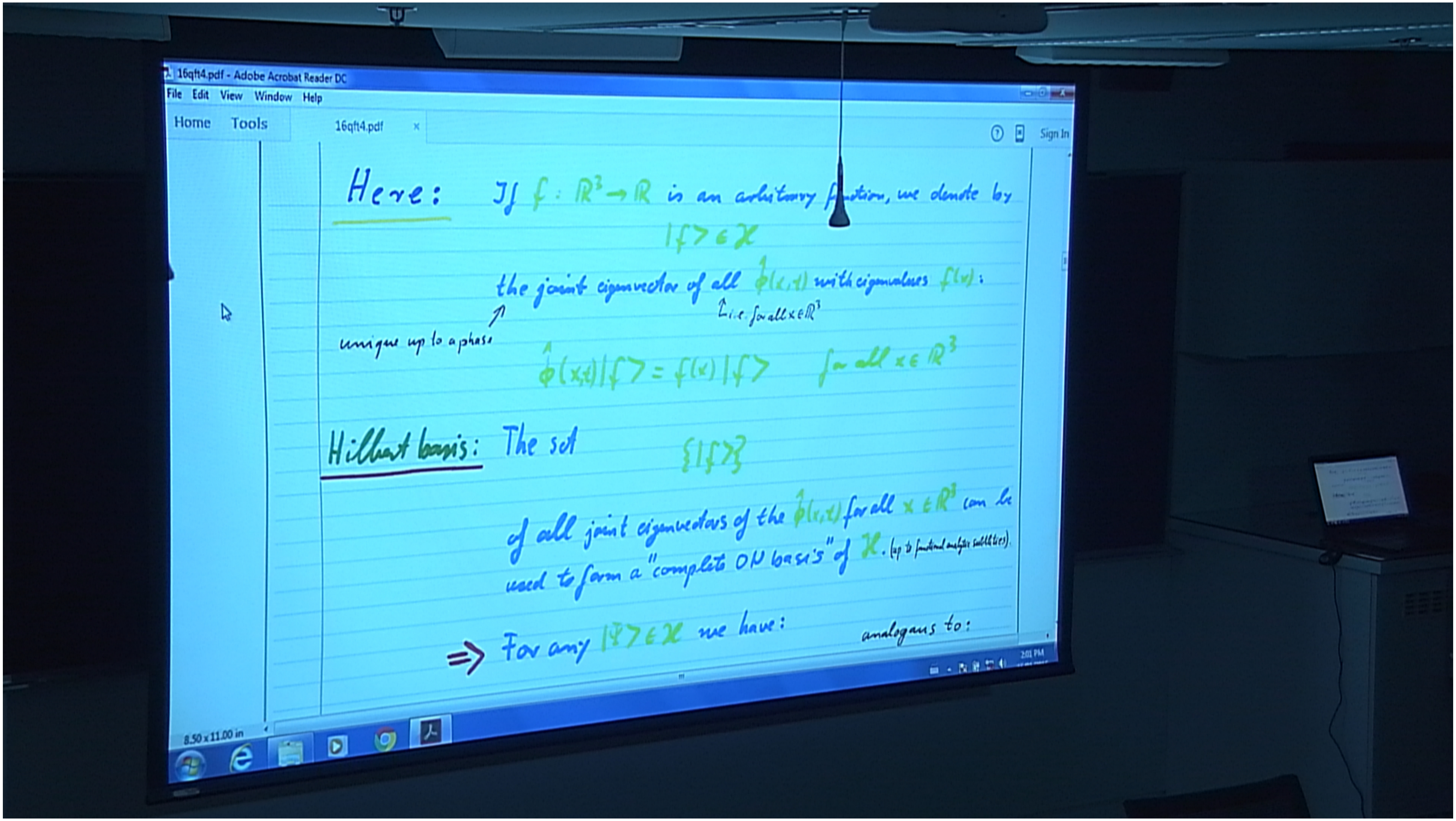
Shown are the level curves.

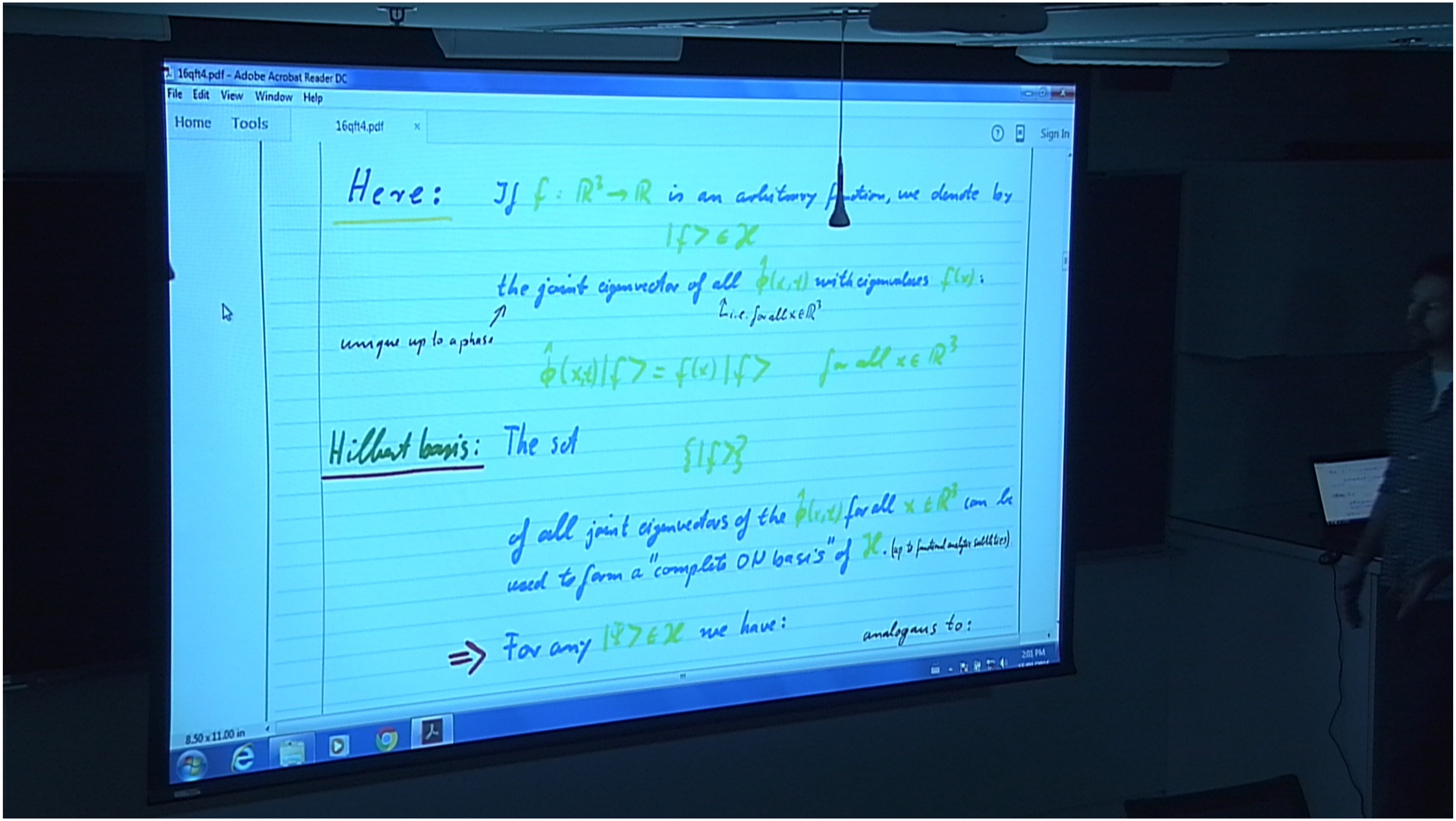


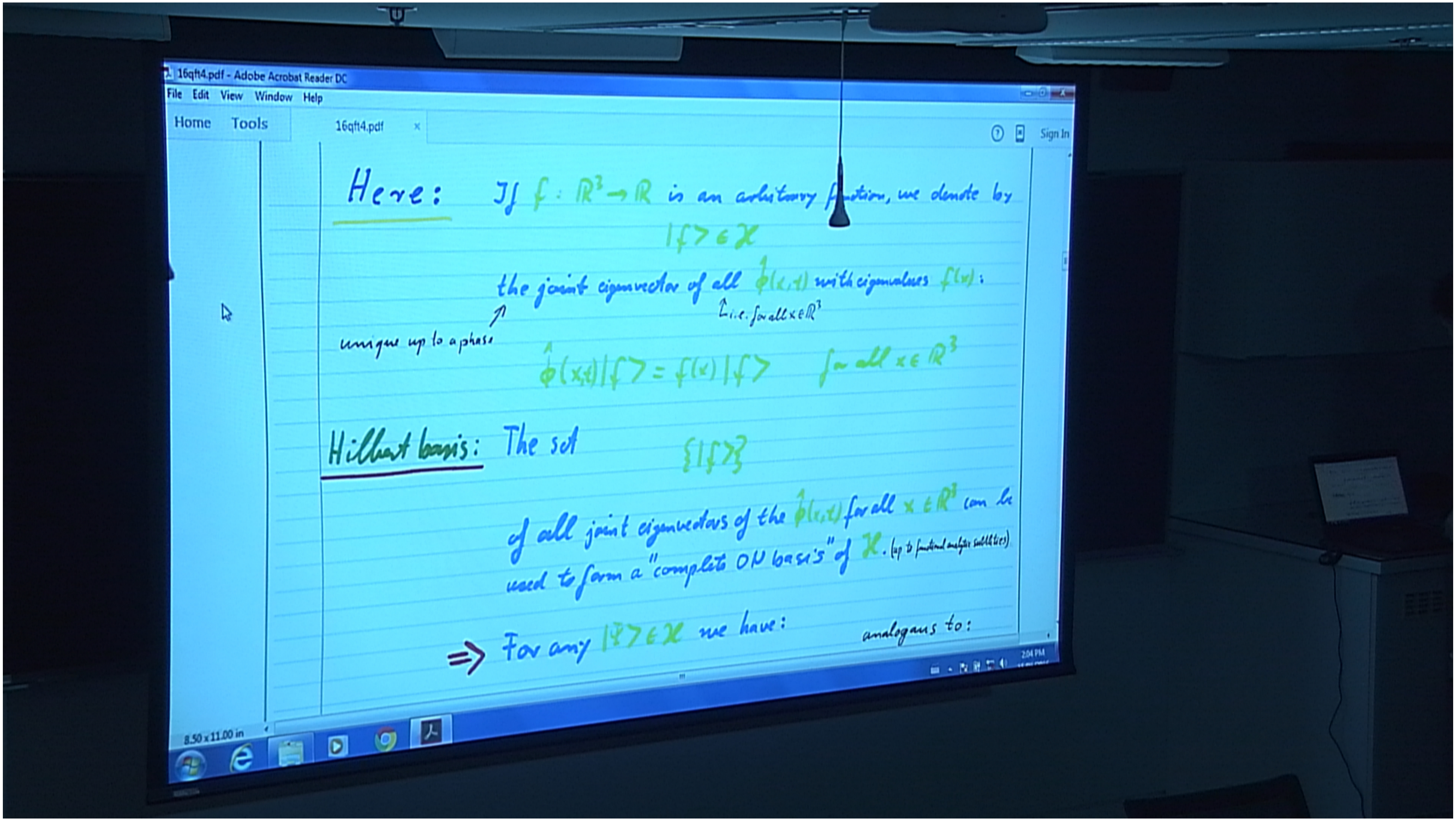
The measurement collapsed the system into the new state

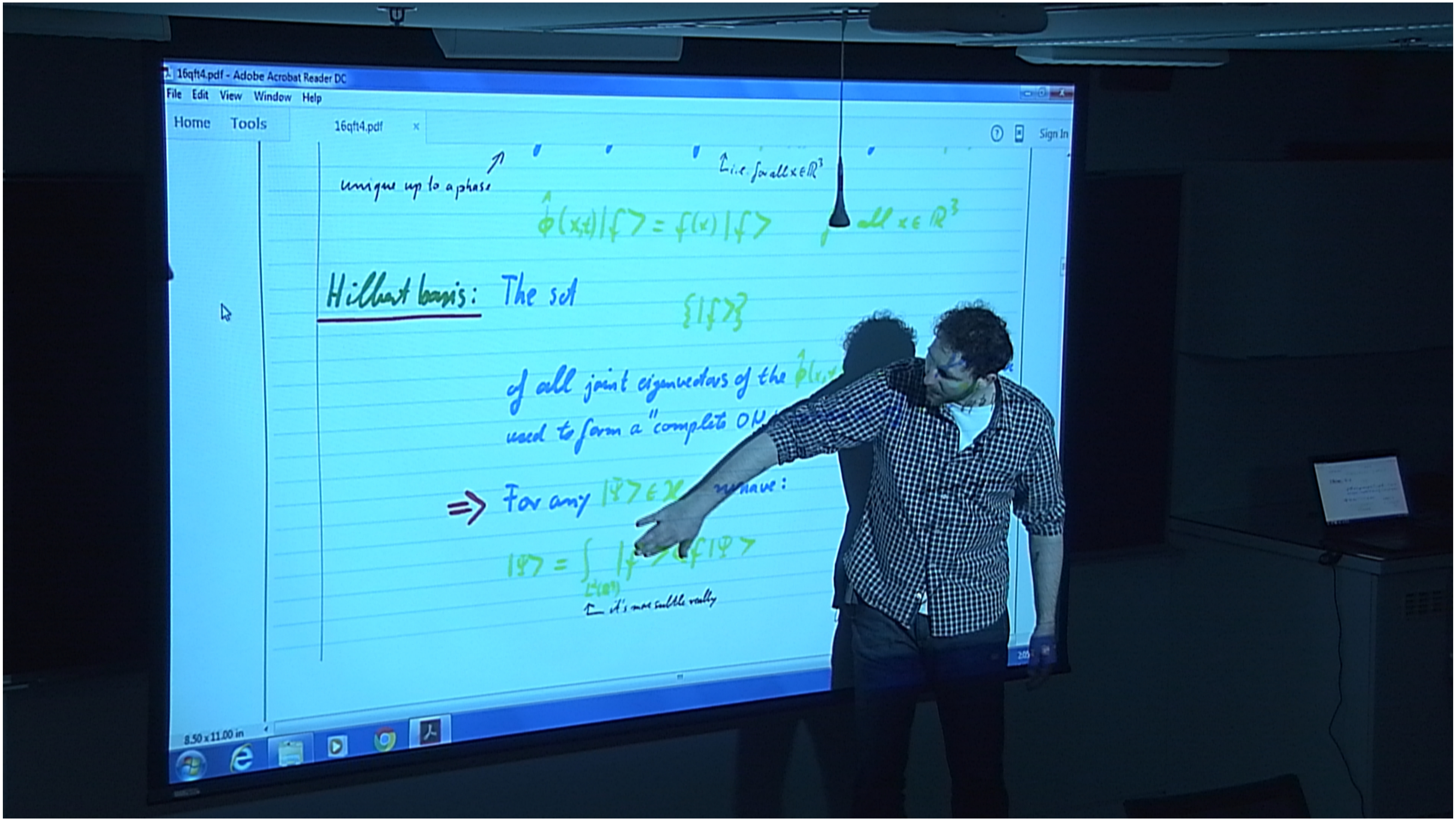
$$|f\rangle \in \mathcal{H}$$

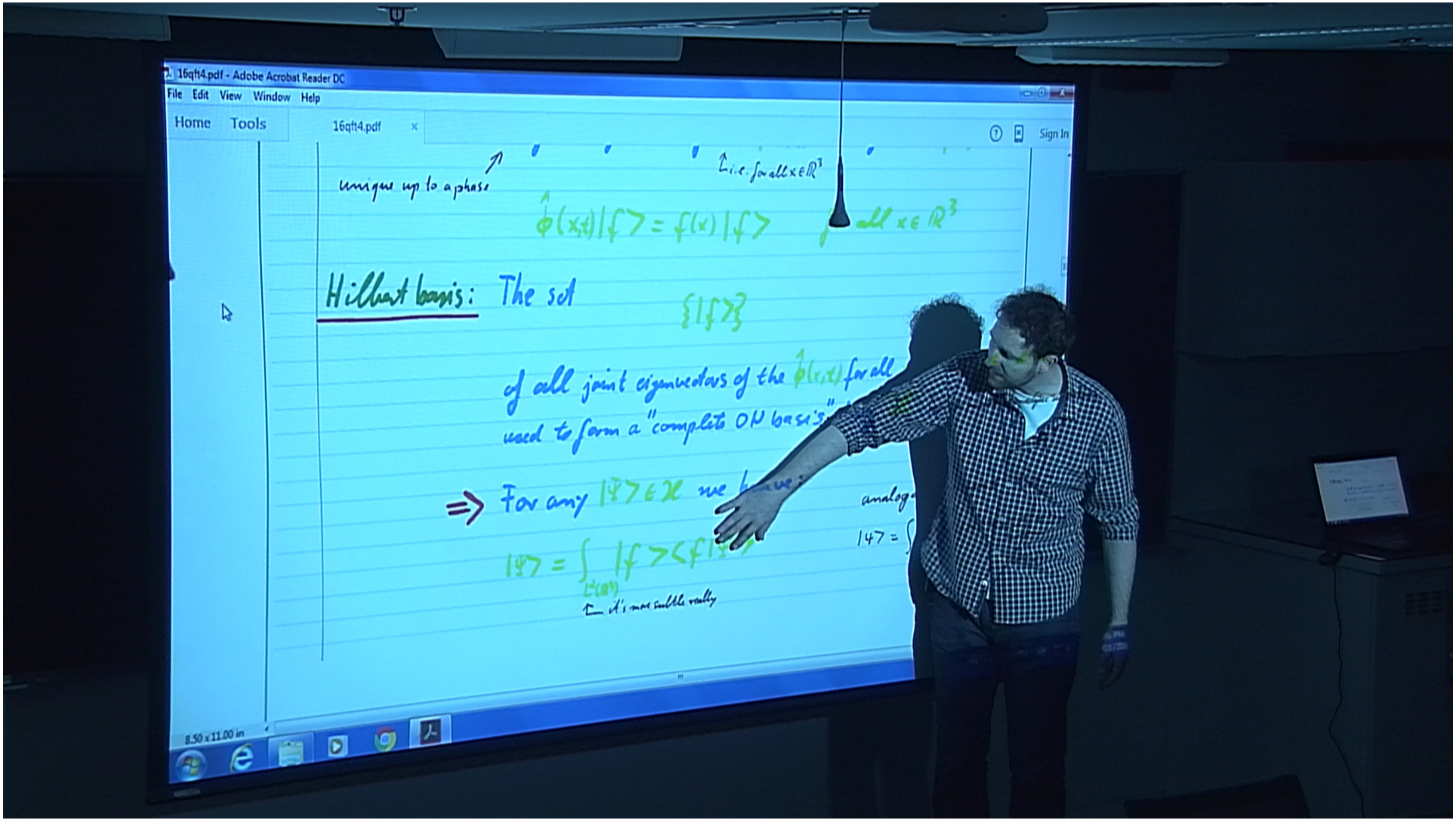
which is joint eigenstate of all $\hat{\phi}(x,t)$:

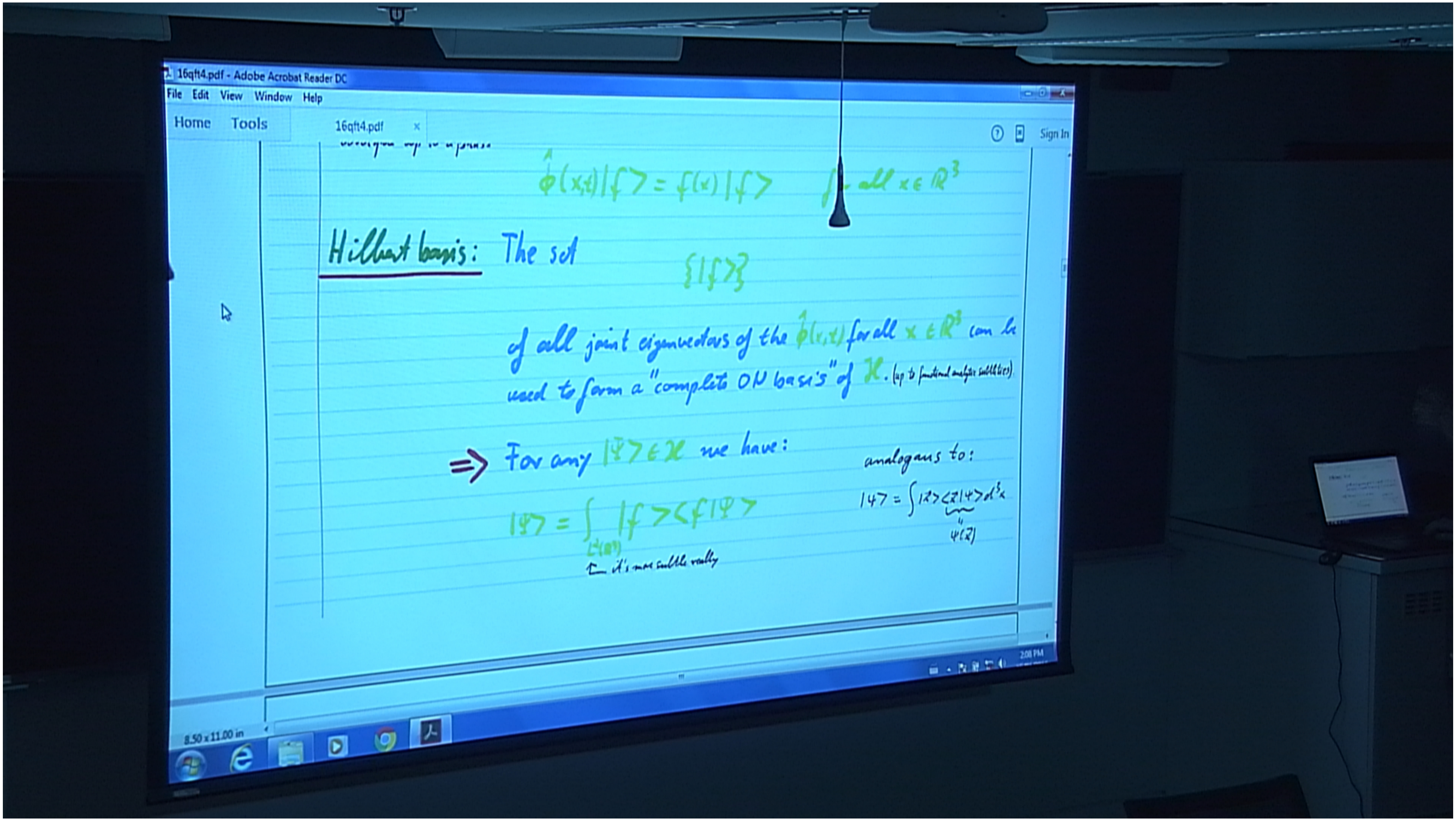

16qft4.pdf - Adobe Acrobat Reader DC
File Edit View Window Help
Home Tools
16qft4.pdf
Sign In
Here: If $f : \mathbb{R}^3 \to \mathbb{R}$ is an arbitrary function, we denote by
$|f\rangle \in \mathcal{H}$
the joint eigenvector of all $\hat{\phi}(x,t)$ with eigenvalues $f(x)$:
i.e. for all $x \in \mathbb{R}^3$
unique up to a phase
$\hat{\phi}(x,t)|f\rangle = f(x)|f\rangle$ for all $x \in \mathbb{R}^3$
Hilbert basis: The set
$\{|f\rangle\}$
of all joint eigenvectors of the $\hat{\phi}(x,t)$ for all $x \in \mathbb{R}^3$ can be
used to form a "complete ON basis" of $\mathcal{H}$. (up to functional analytic subtleties).
$\Rightarrow$ For any $|\Psi\rangle \in \mathcal{H}$ we have:
analogous to:
8.50 x 11.00 in
2:01 PM

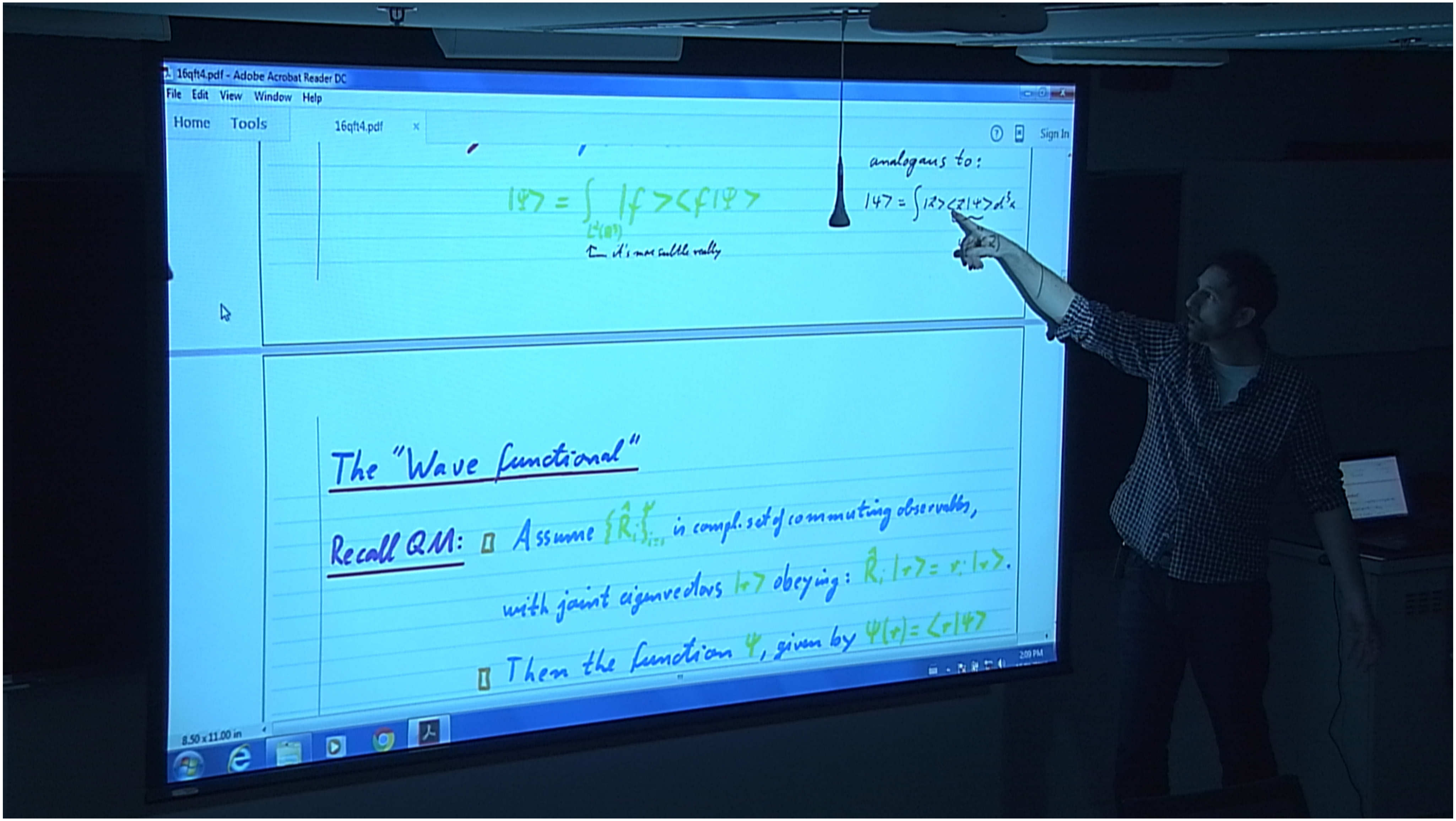

16qft4.pdf - Adobe Acrobat Reader DC
File Edit View Window Help
Home Tools
16qft4.pdf
Sign In
Here: If f : ℝ³ → ℝ is an arbitrary function, we denote by
|f⟩ ∈ ℋ
the joint eigenvector of all φ̂(x,t) with eigenvalues f(x):
i.e. for all x ∈ ℝ³
unique up to a phase
φ̂(x,t)|f⟩ = f(x)|f⟩ for all x ∈ ℝ³
Hilbert basis: The set
{|f⟩}
of all joint eigenvectors of the φ̂(x,t) for all x ∈ ℝ³ can be used to form a "complete ON basis" of ℋ. (up to functional analytic subtleties)
⇒ For any |Ψ⟩ ∈ ℋ we have:
analogous to:
8.50 x 11.00 in

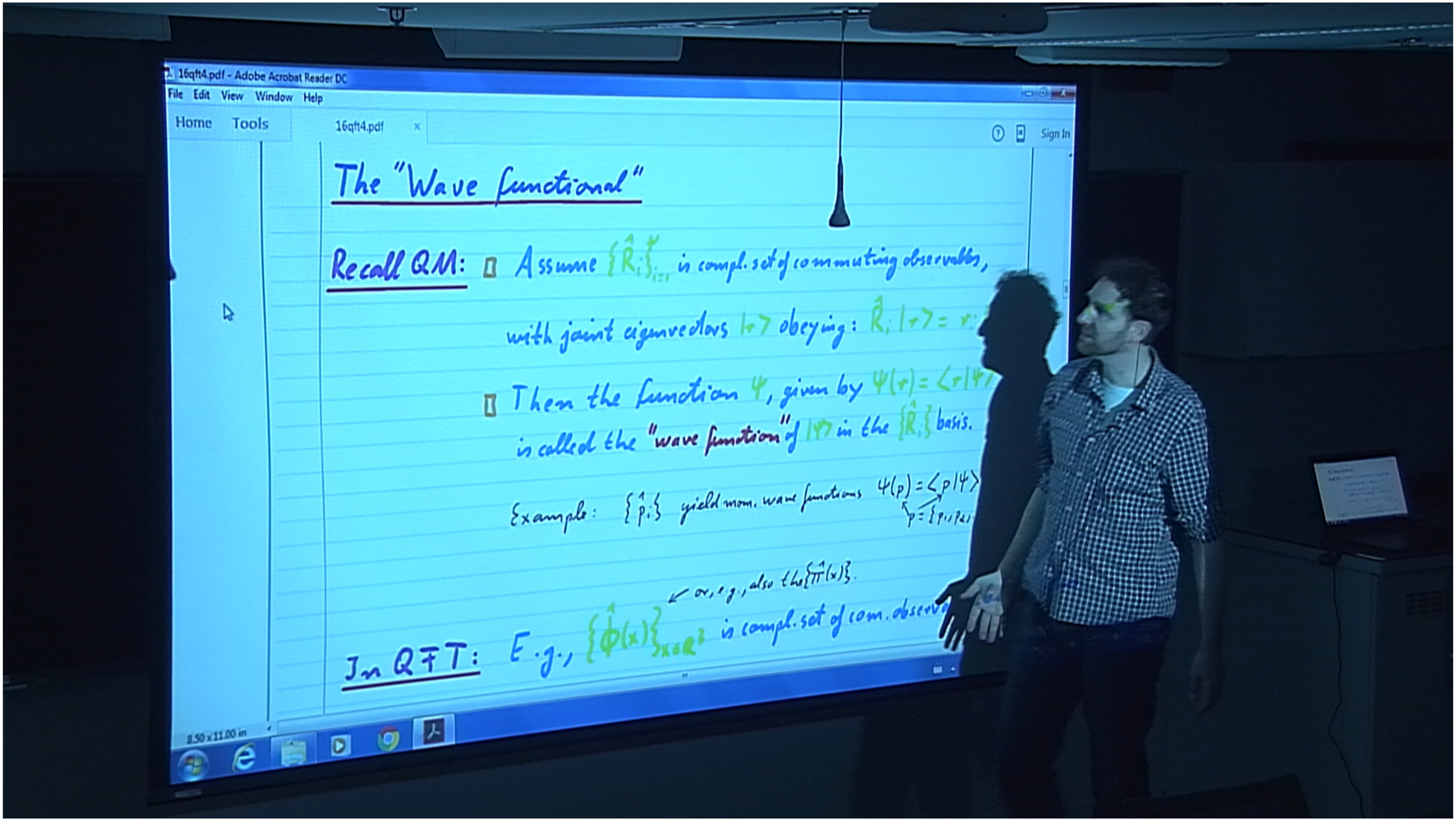

16qft4.pdf - Adobe Acrobat Reader DC
File Edit View Window Help
Home Tools
16qft4.pdf
Sign In
Here: If f : ℝ³ → ℝ is an arbitrary function, we denote by
|f⟩ ∈ ℋ
the joint eigenvector of all φ̂(x,t) with eigenvalues f(x):
i.e. for all x ∈ ℝ³
unique up to a phase
φ̂(x,t)|f⟩ = f(x)|f⟩ for all x ∈ ℝ³
Hilbert basis: The set
{|f⟩}
of all joint eigenvectors of the φ̂(x,t) for all x ∈ ℝ³ can be used to form a "complete ON basis" of ℋ. (up to functional analytic subtleties)
⇒ For any |Ψ⟩ ∈ ℋ we have:
analogous to:
8.50 x 11.00 in
2:04 PM


16qft4.pdf - Adobe Acrobat Reader DC
File Edit View Window Help
Home Tools
16qft4.pdf
Sign In
unique up to a phase
i.e. for all x ∈ ℝ³
φ̂(x,t)|f> = f(x)|f>
all x ∈ ℝ³
Hilbert basis: The set
{|f>}
of all joint eigenvectors of the φ̂(x,t
used to form a "complete ON
⟹ For any |Ψ> ∈ ℋ
we have:
|Ψ> = ∫ |f><f|Ψ>
L²(ℝ³)
it's more subtle really
8.50 x 11.00 in

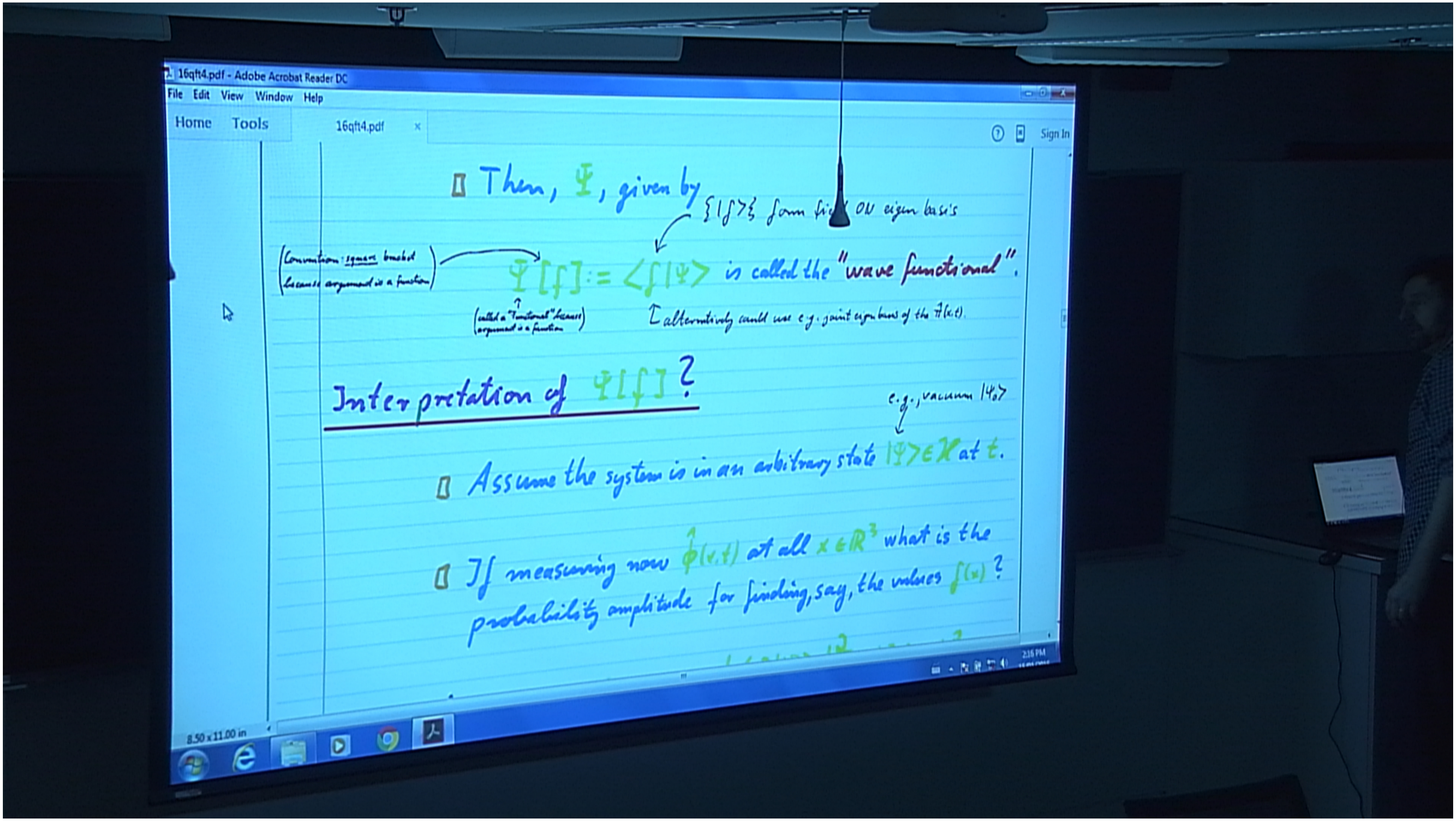

16qft4.pdf - Adobe Acrobat Reader DC
File Edit View Window Help
Home Tools
16qft4.pdf
Sign In
unique up to a phase
i.e. for all x ∈ ℝ³
Hilbert basis: The set
of all joint eigenvectors of the
used to form a "complete ON basis"
For any
it's more subtle really
8.50 x 11.00 in

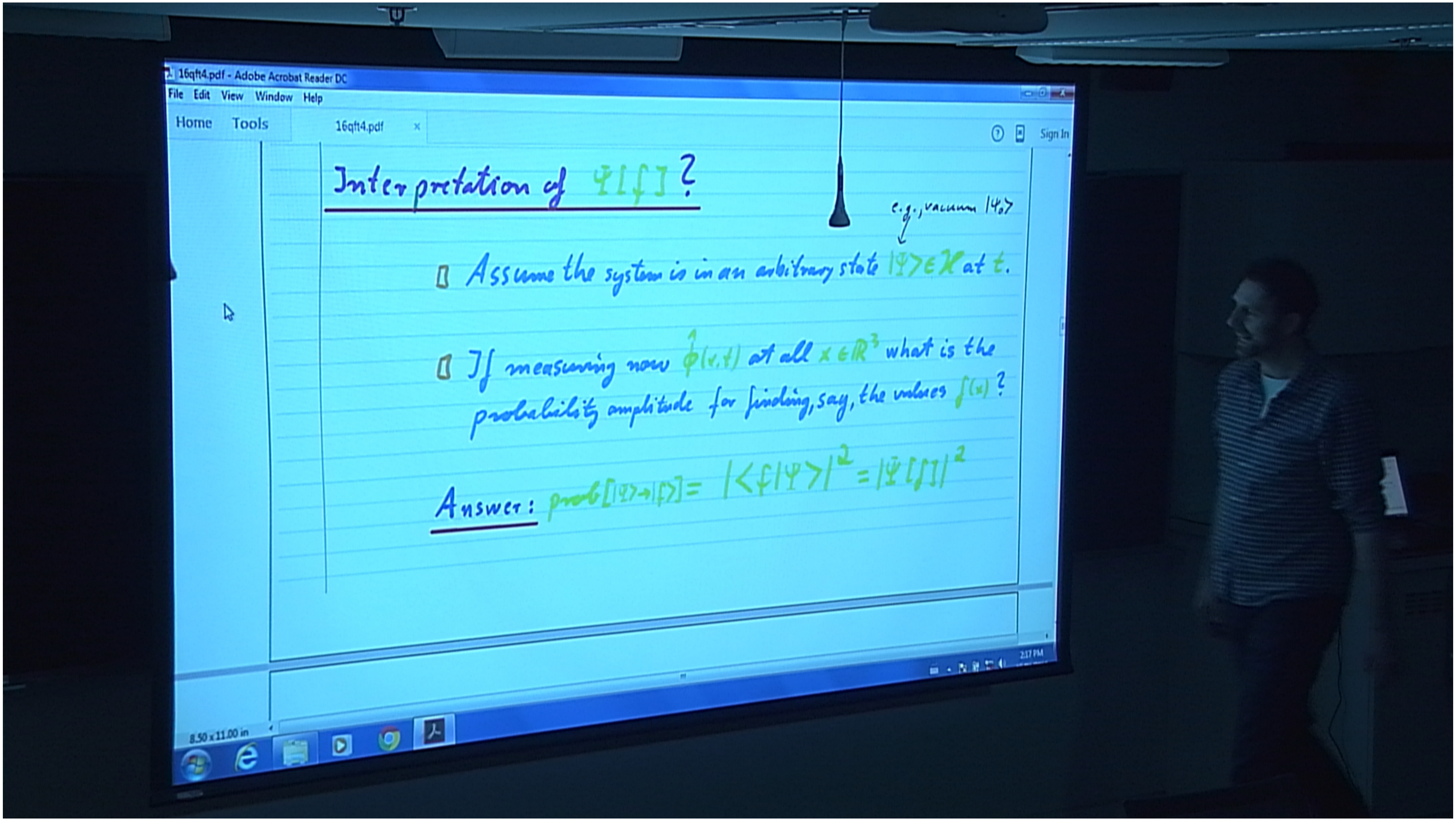

16qft4.pdf - Adobe Acrobat Reader DC
File Edit View Window Help
Home Tools
16qft4.pdf
Sign In
φ̂(x,t)|f> = f(x)|f> for all x ∈ ℝ³
Hilbert basis: The set
{|f>}
of all joint eigenvectors of the φ̂(x,t) for all x ∈ ℝ³ can be
used to form a "complete ON basis" of ℋ. (up to functional analytic subtleties)
⇒ For any |Ψ> ∈ ℋ we have:
|Ψ> = ∫ |f><f|Ψ>
L²(ℝ³)
it's more subtle really
analogous to:
|ψ> = ∫ |x><x|ψ> d³x
ψ(x)
8.50 x 11.00 in
2:08 PM

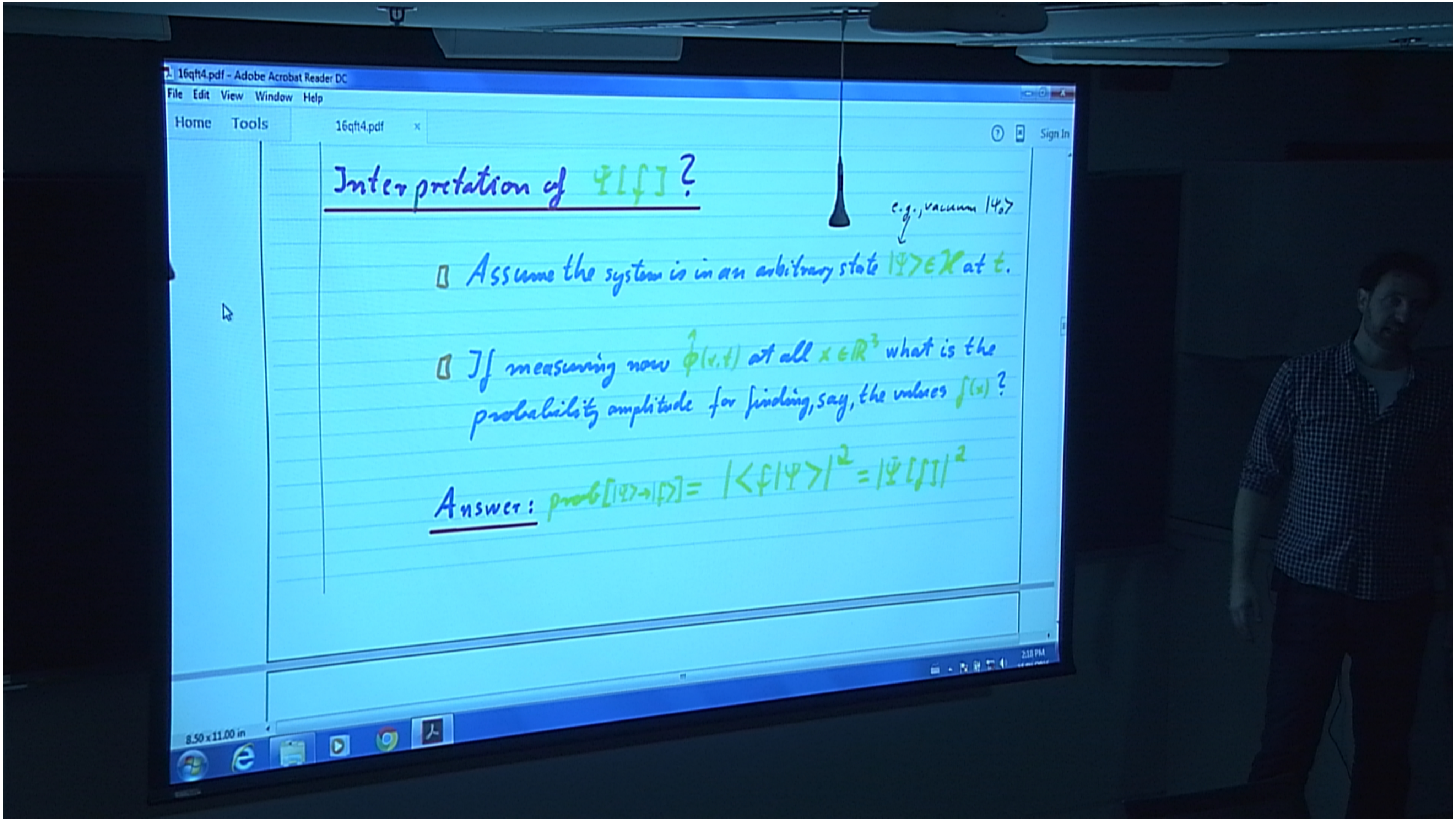

16qft4.pdf - Adobe Acrobat Reader DC
File Edit View Window Help
Home Tools
16qft4.pdf
Sign In
analogous to:
|Ψ> = ∫ |f><f|Ψ>
it's more subtle really
The "Wave functional"
Recall QM:
Assume {R̂ᵢ} is compl. set of commuting observables,
with joint eigenvectors |r> obeying: R̂ᵢ|r> = rᵢ|r>.
Then the function Ψ, given by Ψ(r) = <r|Ψ>
8.50 x 11.00 in

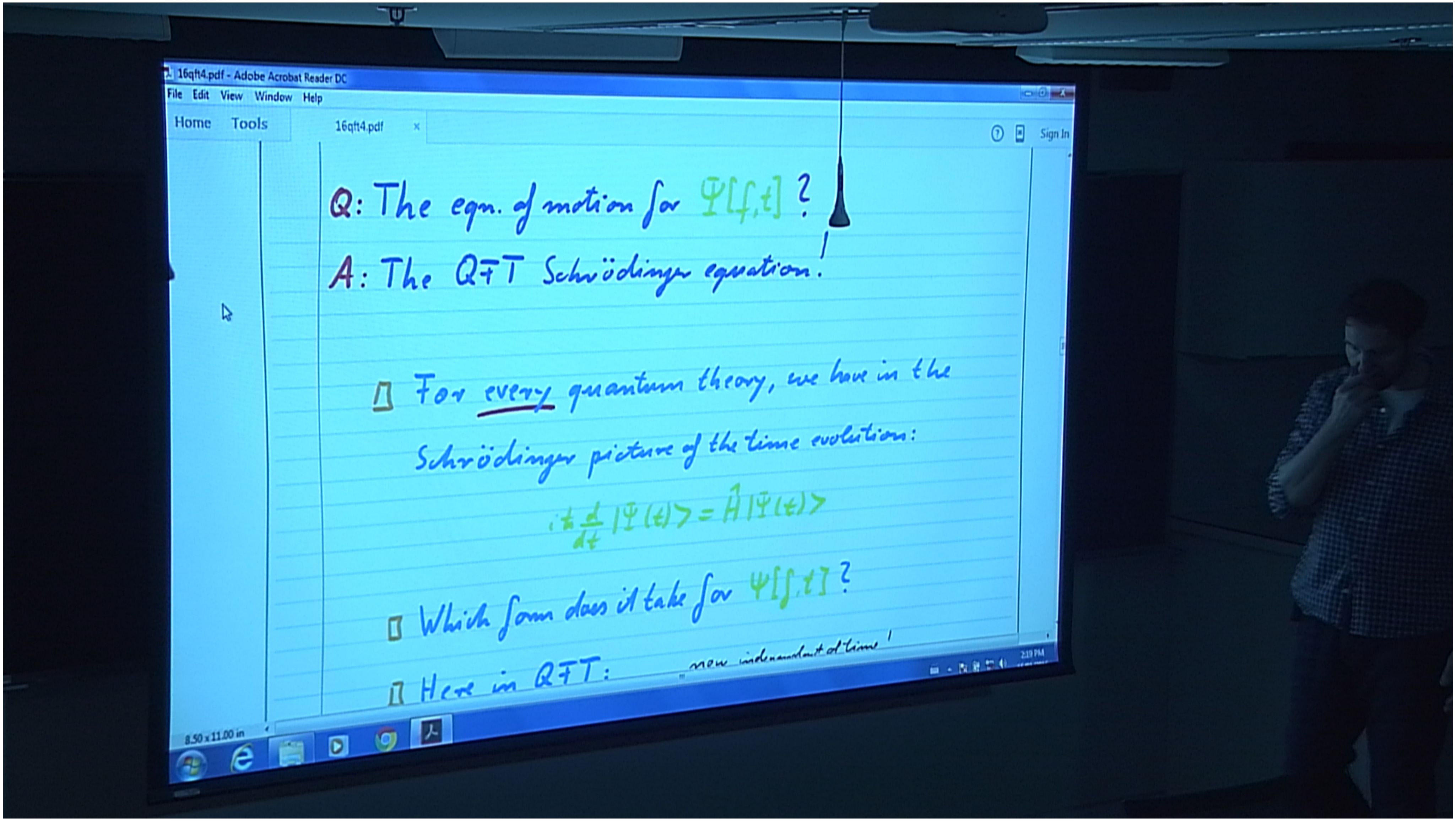


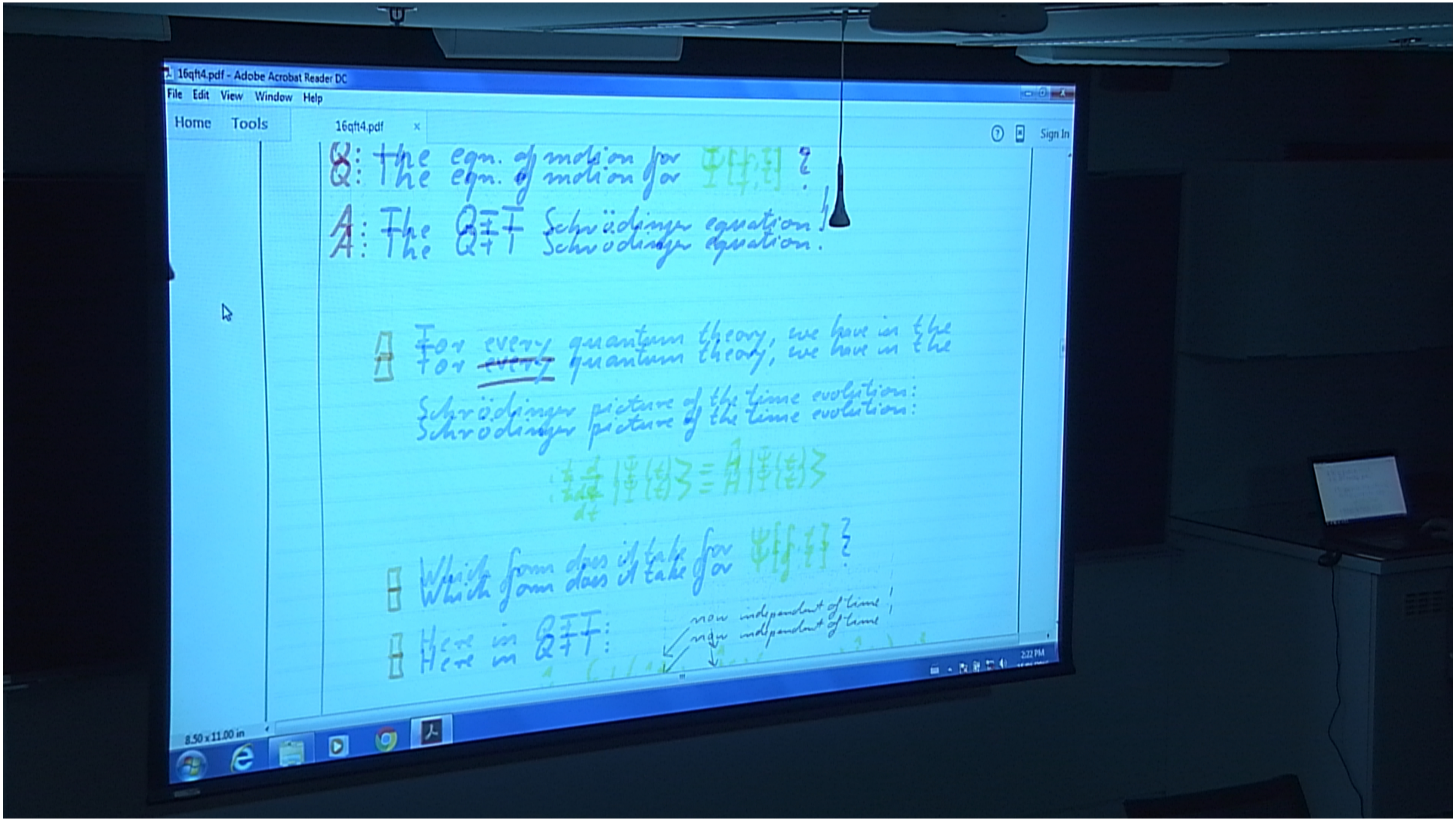
16qft4.pdf - Adobe Acrobat Reader DC
File Edit View Window Help
Home Tools
16qft4.pdf
Sign In
The "Wave functional"
Recall QM: Assume $\{\hat{R}_i\}$ is compl. set of commuting observables,
with joint eigenvectors $|r\rangle$ obeying:
Then the function ψ, given by $\psi(r) = \langle r|\psi\rangle$
is called the "wave function" of $|\psi\rangle$ in the $\{\hat{R}_i\}$ basis.
Example: $\{\hat{p}_i\}$ yield mom. wave functions $\psi(p) = \langle p|\psi\rangle$
or, e.g., also the $\{\hat{\pi}(x)\}$.
In QFT: E.g., $\{\hat{\phi}(x)\}_{x\in\mathbb{R}^3}$ is compl. set of com. observ
8.50 x 11.00 in

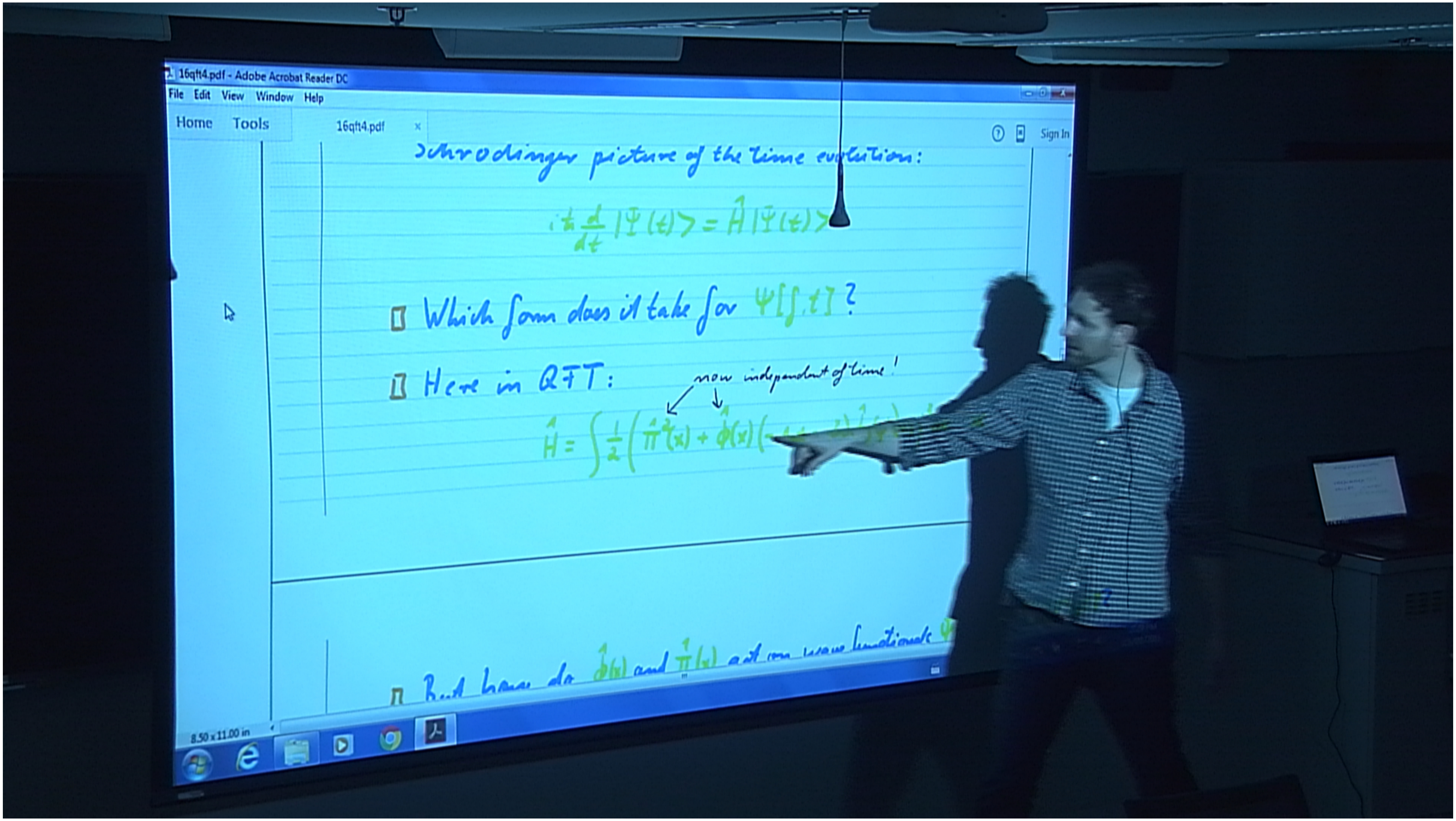

16qft4.pdf - Adobe Acrobat Reader DC
File Edit View Window Help
Home Tools
16qft4.pdf
Sign In
Then, Ψ, given by
{|f>} form field ON eigen basis
(Convention: square bracket because argument is a function)
Ψ[f] := <f|Ψ> is called the "wave functional".
(called a "functional" because argument is a function)
Alternatively could use e.g. joint eigenbasis of the π(x,t).
Interpretation of Ψ[f] ?
e.g., vacuum |Ψ0>
Assume the system is in an arbitrary state |Ψ>∈H at t.
If measuring now φ̂(x,t) at all x∈R³ what is the probability amplitude for finding, say, the values f(x) ?
8.50 x 11.00 in

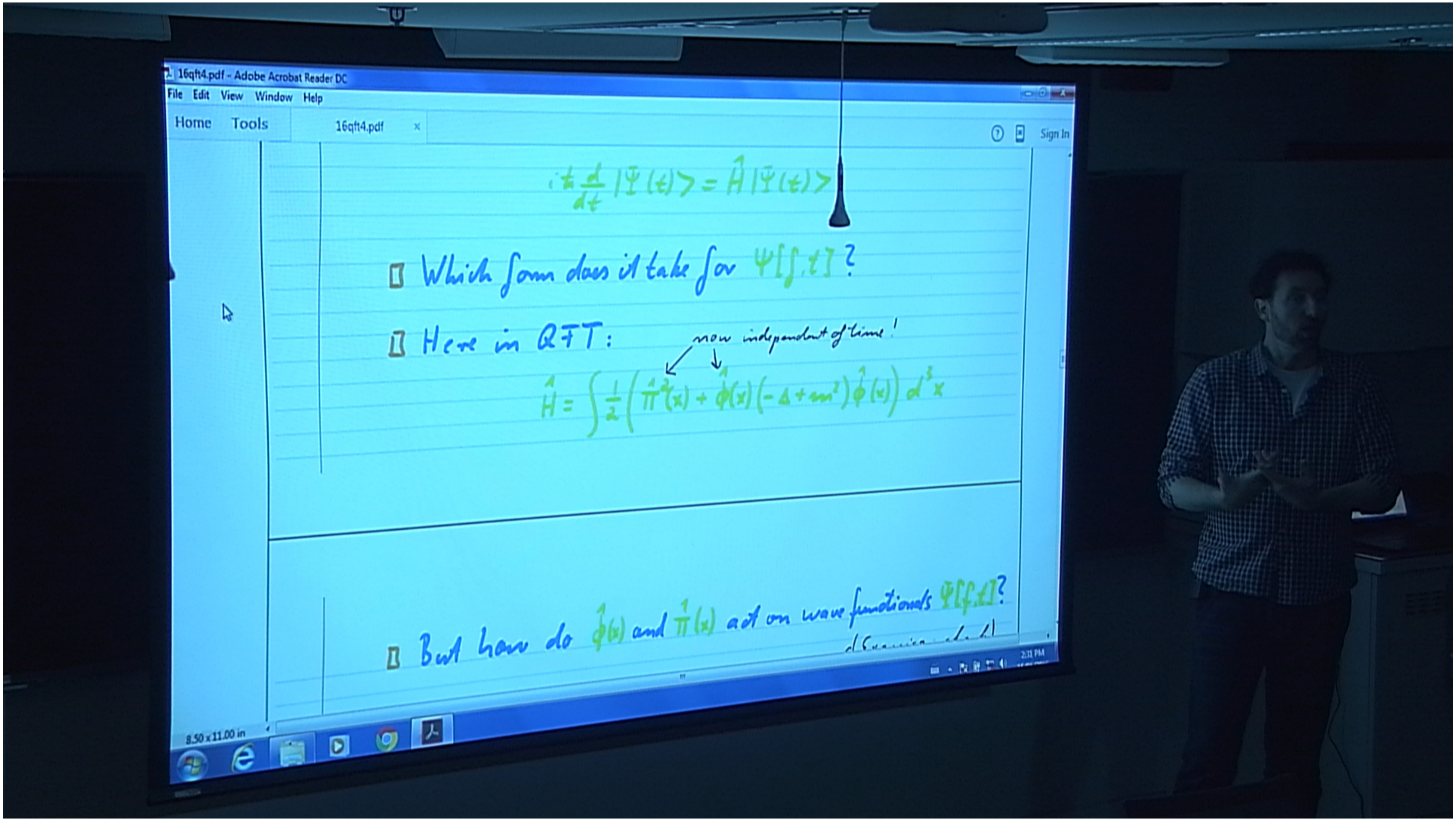

16qft4.pdf - Adobe Acrobat Reader DC
File Edit View Window Help
Home Tools
16qft4.pdf
Sign In
Interpretation of Ψ[f]?
e.g., vacuum |Ψ₀>
Assume the system is in an arbitrary state |Ψ> ∈ H at t.
If measuring now φ̂(x,t) at all x ∈ R³ what is the probability amplitude for finding, say, the values f(x)?
Answer: prob[|Ψ>→|f>] = |<f|Ψ>|² = |Ψ[f]|²
8.50 x 11.00 in
2:17 PM

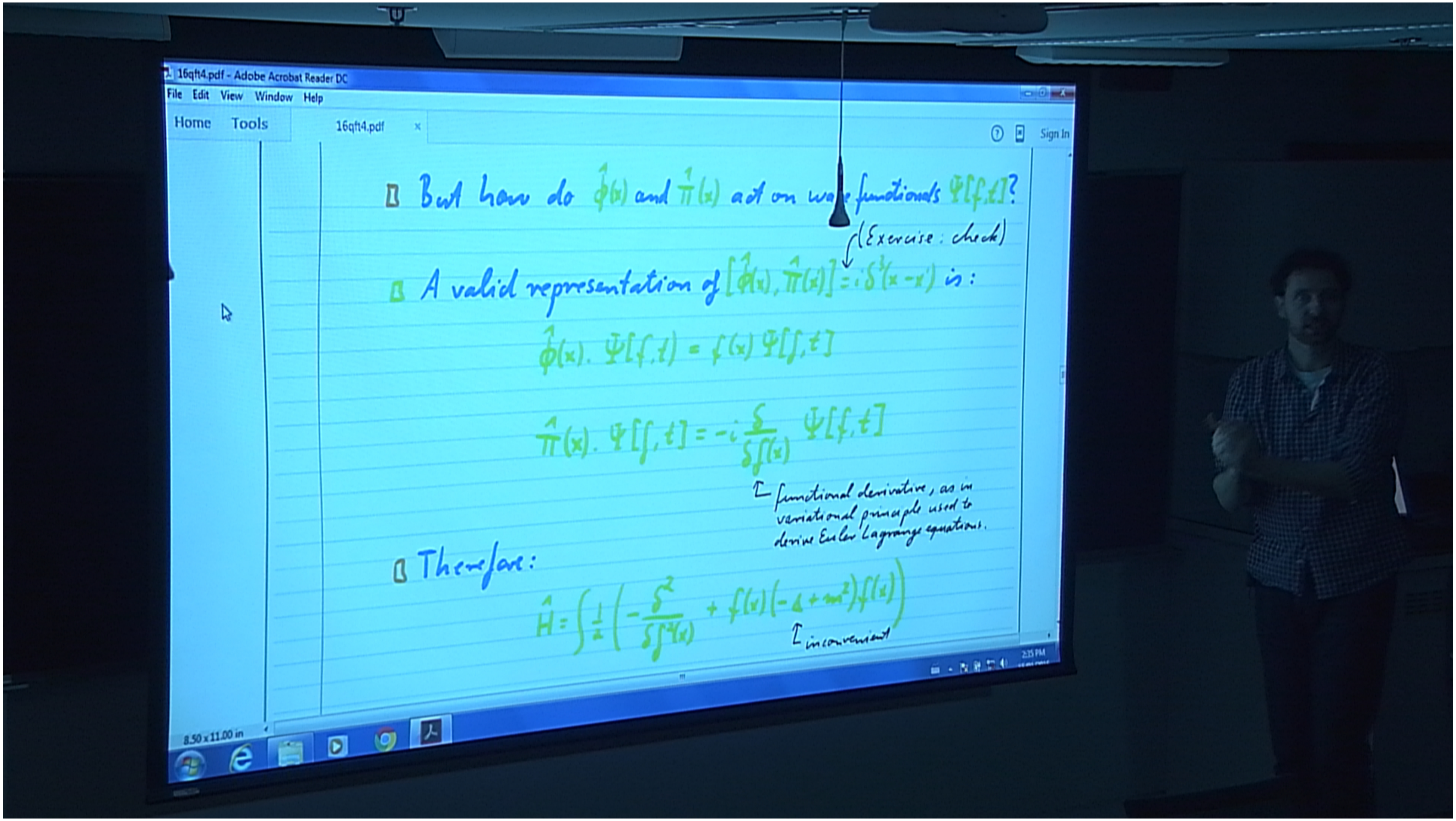

16qft4.pdf - Adobe Acrobat Reader DC
File Edit View Window Help
Home Tools
16qft4.pdf
Sign In
Interpretation of $\Psi[f]$?
e.g., vacuum $|\Psi_0\rangle$
Assume the system is in an arbitrary state $|\Psi\rangle \in \mathcal{H}$ at t.
If measuring now $\hat{\phi}(x,t)$ at all $x \in \mathbb{R}^3$ what is the probability amplitude for finding, say, the values $f(x)$?
Answer: $\text{prob}[|\Psi\rangle \to |f\rangle] = |\langle f|\Psi\rangle|^2 = |\Psi[f]|^2$
8.50 x 11.00 in
2:18 PM

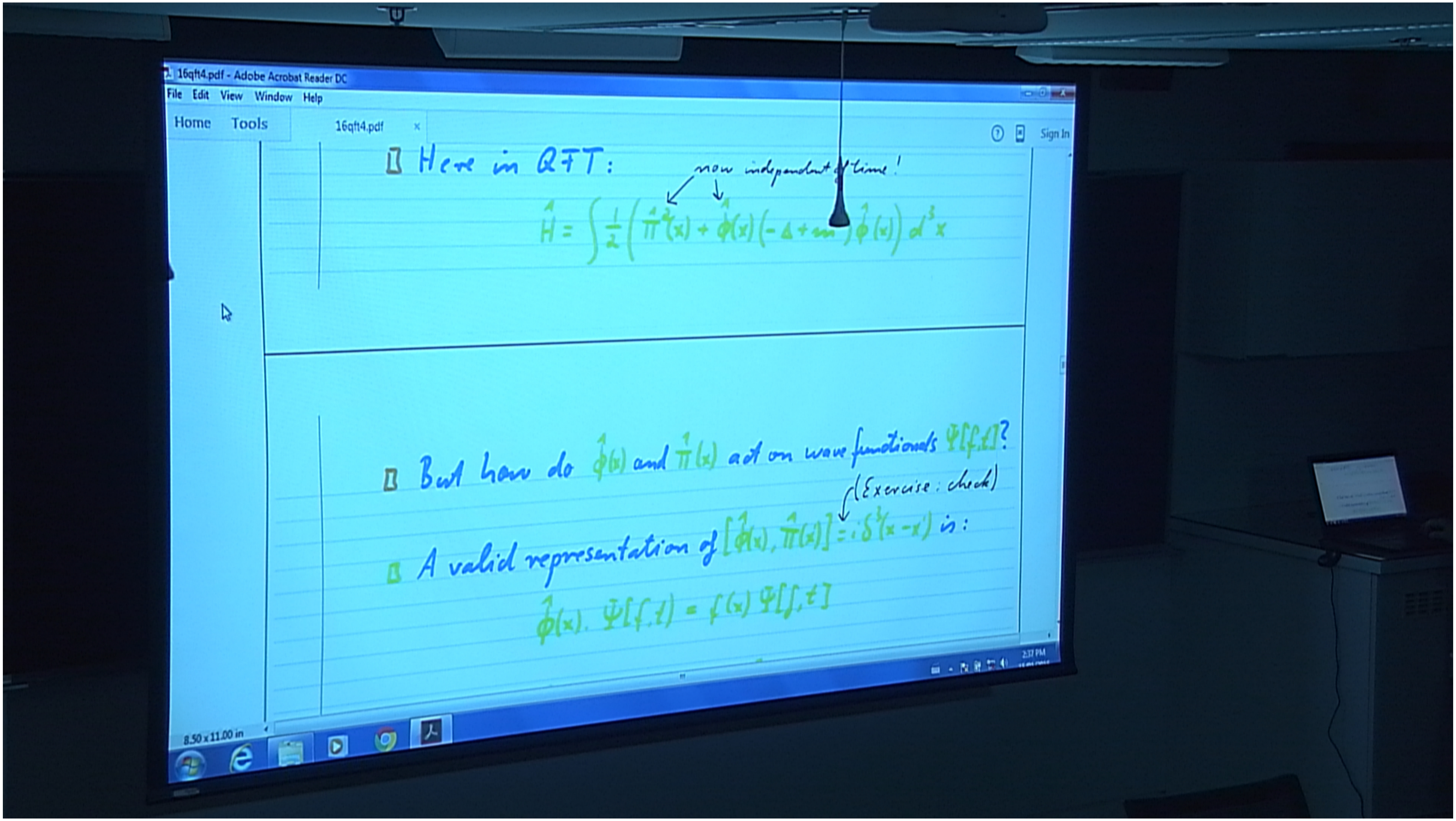

16qft4.pdf - Adobe Acrobat Reader DC
File Edit View Window Help
Home Tools 16qft4.pdf
Sign In
Q: The eqn. of motion for Ψ[f,t] ?
A: The QFT Schrödinger equation!
For every quantum theory, we have in the
Schrödinger picture of the time evolution:
iħ d/dt |Ψ(t)> = Ĥ|Ψ(t)>
Which form does it take for Ψ[f,t] ?
Here in QFT:
8.50 x 11.00 in

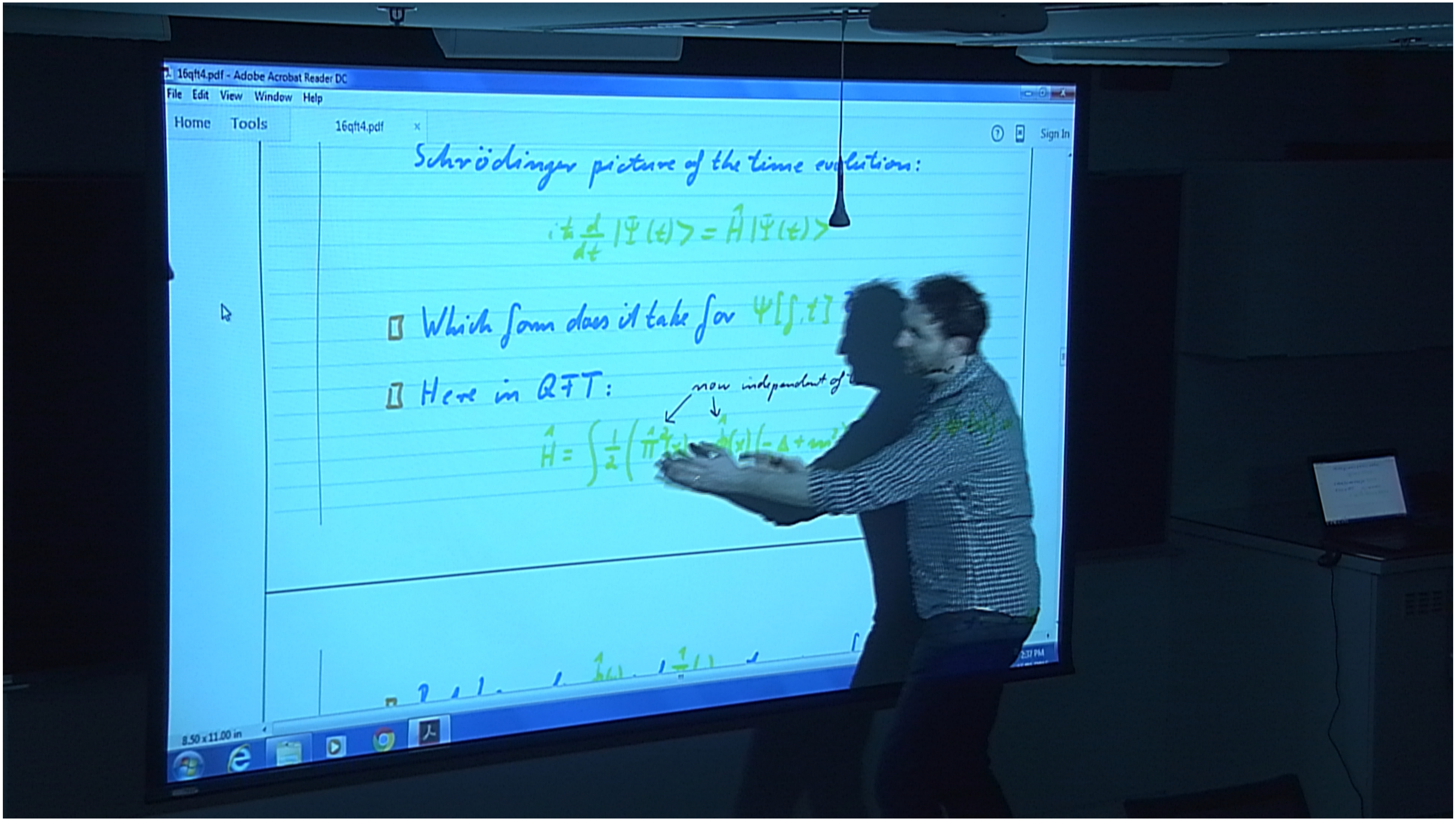

16qft4.pdf - Adobe Acrobat Reader DC
File Edit View Window Help
Home Tools
16qft4.pdf
Sign In
Q: The eqn. of motion for Ψ[φ,t] ?
A: The QFT Schrödinger equation!
For every quantum theory, we have in the Schrödinger picture of the time evolution:
i ħ d/dt |Ψ(t)> = Ĥ |Ψ(t)>
Which form does it take for Ψ[φ,t] ?
Here in QFT:
now independent of time
8.50 x 11.00 in

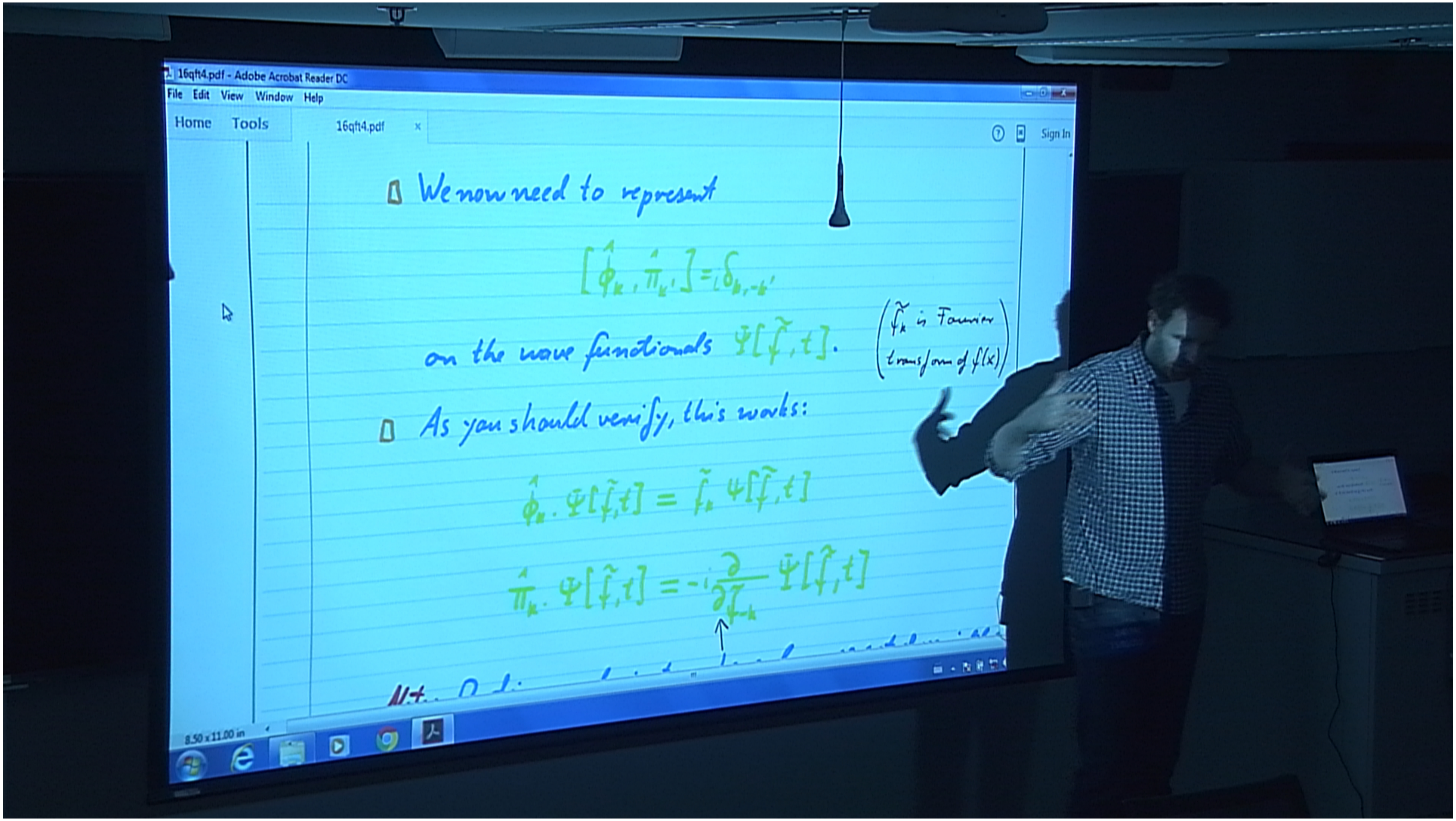

16qft4.pdf - Adobe Acrobat Reader DC
File Edit View Window Help
Home Tools
16qft4.pdf
Sign In
Schrodinger picture of the time evolution:
Which form does it take for Ψ[φ,t]?
Here in QFT:
now independent of time!
8.50 x 11.00 in

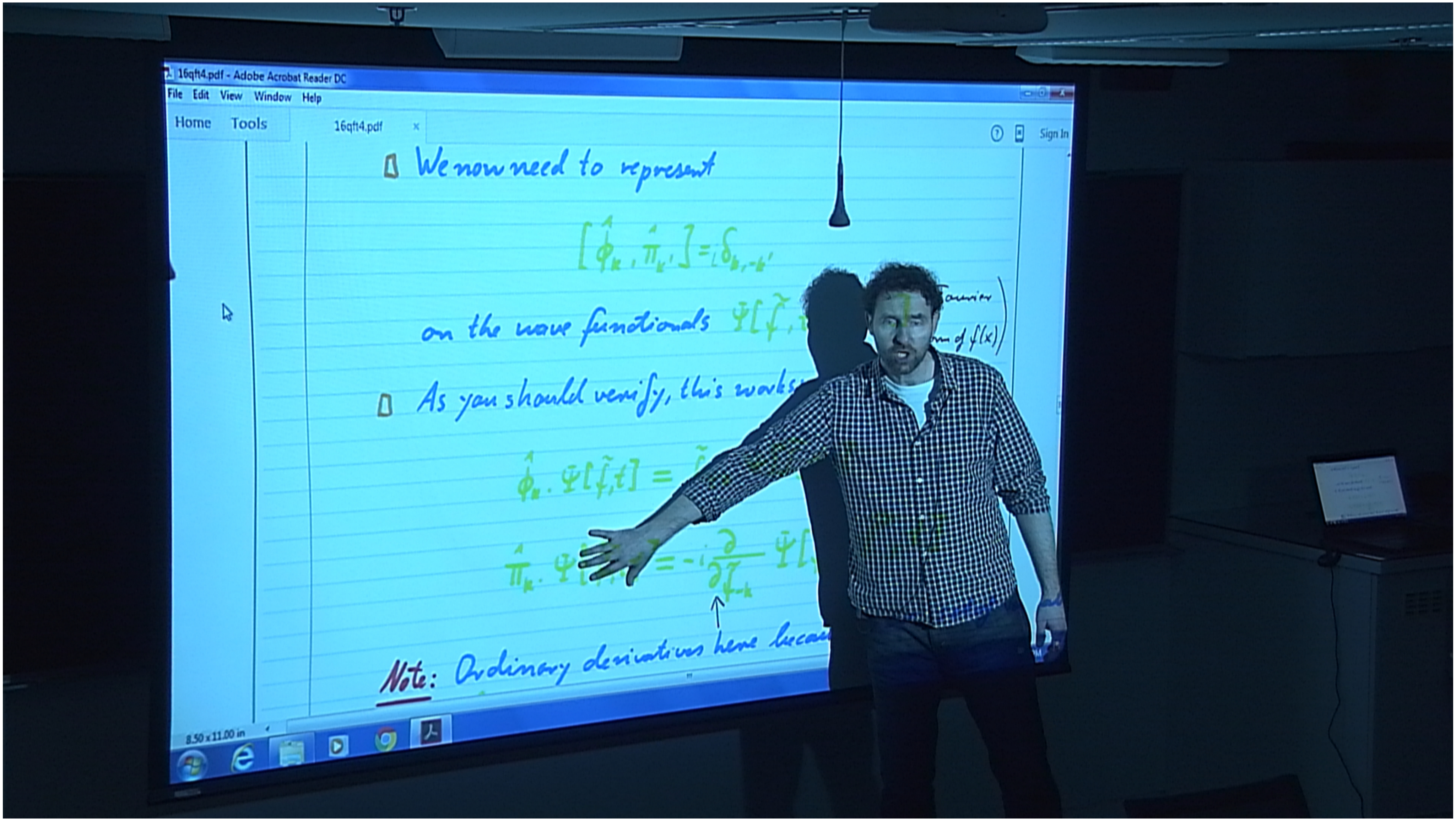

16qft4.pdf - Adobe Acrobat Reader DC
File Edit View Window Help
Home Tools
16qft4.pdf
Sign In
Which form does it take for Ψ[φ,t] ?
Here in QFT:
now independent of time!
But how do φ̂(x) and π̂(x) act on wave functionals Ψ[φ,t]?
8.50 x 11.00 in

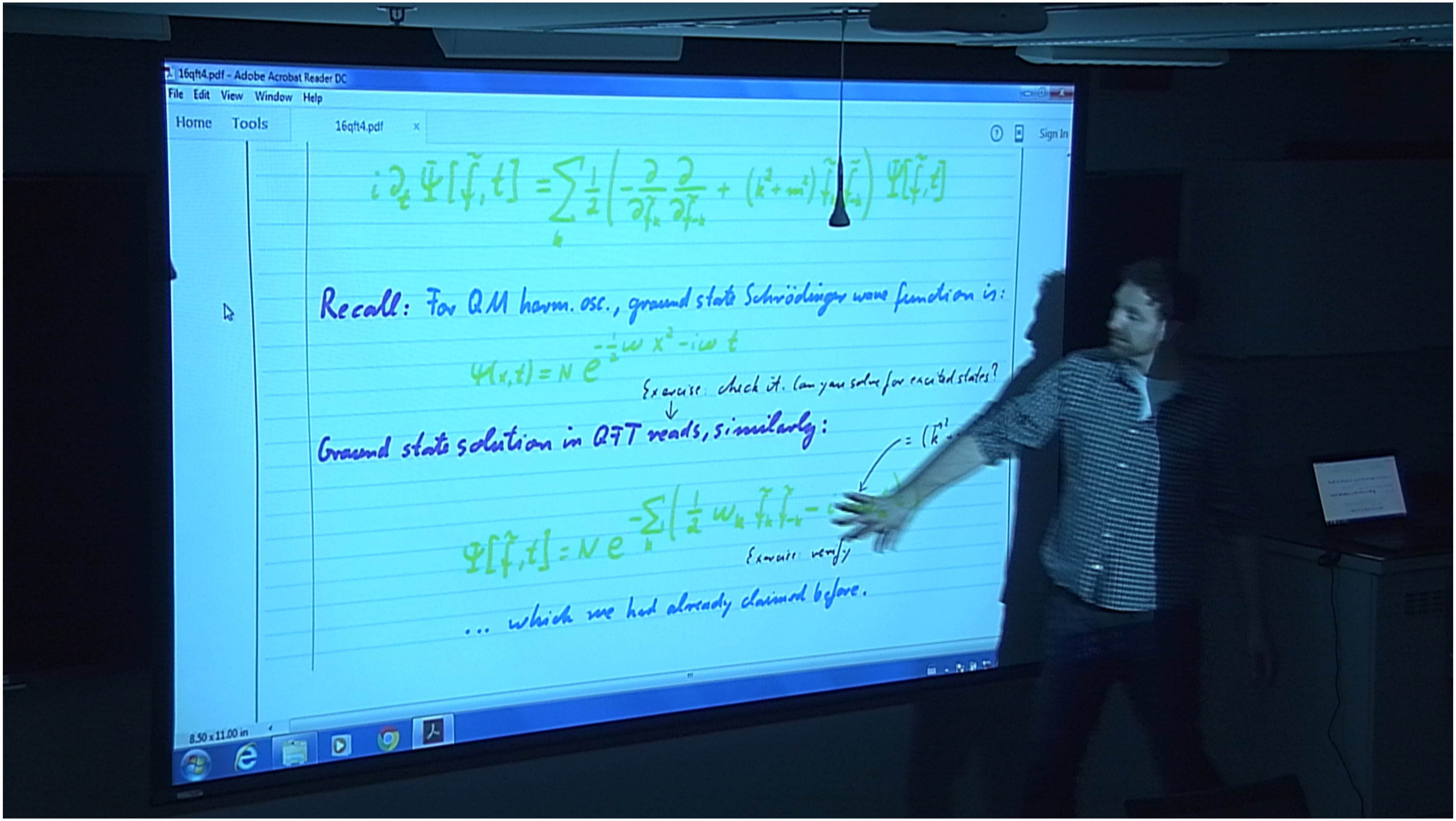

16qft4.pdf - Adobe Acrobat Reader DC
File Edit View Window Help
Home Tools
16qft4.pdf
Sign In
But how do $\hat{\phi}(x)$ and $\hat{\pi}(x)$ act on wave functionals $\Psi[f,t]$?
(Exercise: check)
A valid representation of $[\hat{\phi}(x), \hat{\pi}(x')] = i\delta^3(x-x')$ is:
$\hat{\phi}(x).\Psi[f,t] = f(x)\Psi[f,t]$
$\hat{\pi}(x).\Psi[f,t] = -i\frac{\delta}{\delta f(x)}\Psi[f,t]$
functional derivative, as in variational principle used to derive Euler Lagrange equations.
Therefore:
$\hat{H} = \int \frac{1}{2}\left(-\frac{\delta^2}{\delta f^2(x)} + f(x)(-\Delta + m^2)f(x)\right)$
inconvenient
8.50 x 11.00 in
2:35 PM

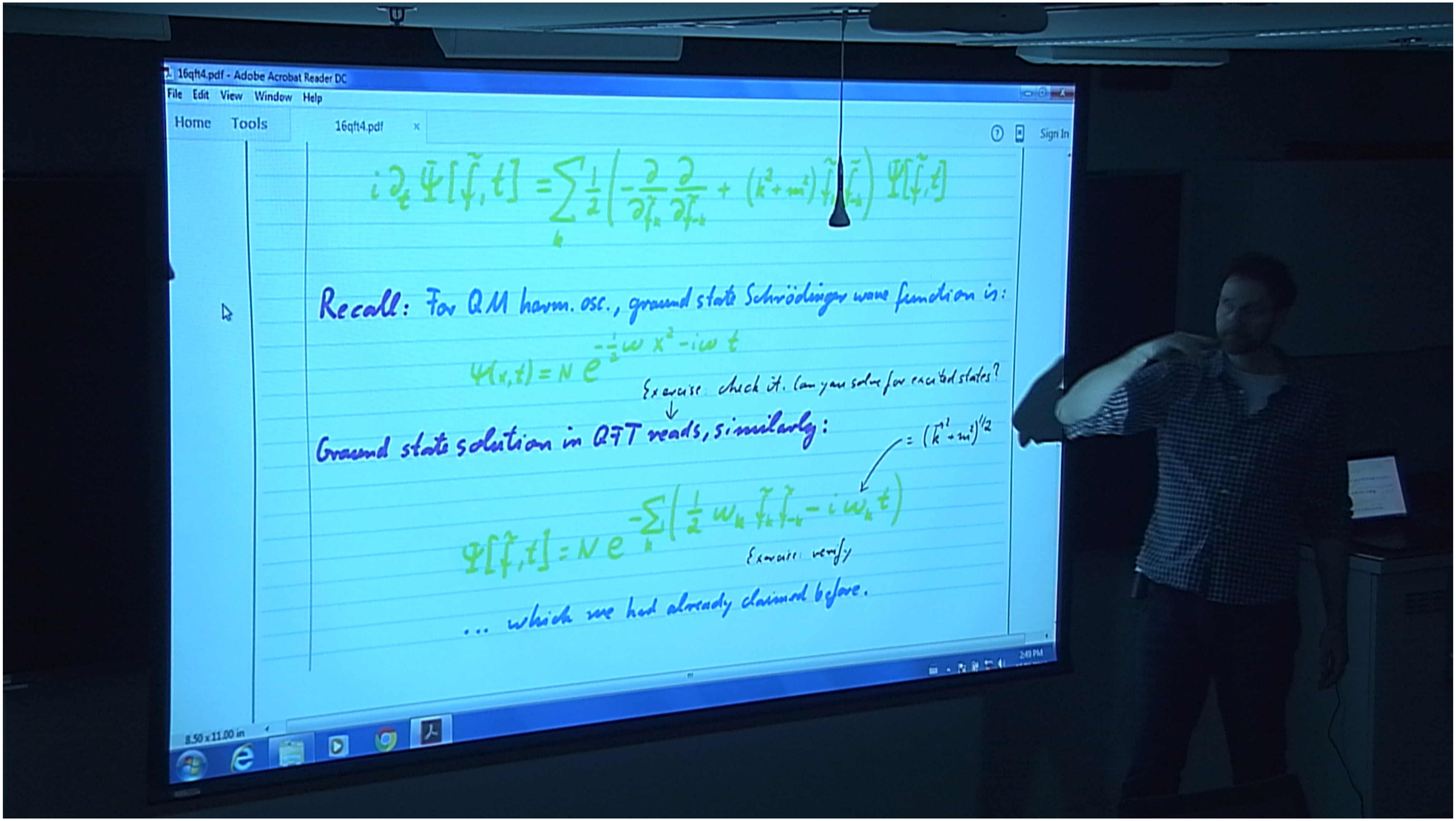

16qft4.pdf - Adobe Acrobat Reader DC
File Edit View Window Help
Home Tools
16qft4.pdf
Sign In
Here in QFT:
now independent of time!
$\hat{H} = \int \frac{1}{2}\left(\hat{\pi}^2(x) + \hat{\phi}(x)(-\Delta + m^2)\hat{\phi}(x)\right) d^3x$
But how do $\hat{\phi}(x)$ and $\hat{\pi}(x)$ act on wave functionals $\Psi[f,t]$?
(Exercise: check)
A valid representation of $[\hat{\phi}(x), \hat{\pi}(x')] = i\delta^3(x-x')$ is:
$\hat{\phi}(x).\ \Psi[f,t] = f(x)\,\Psi[f,t]$
8.50 x 11.00 in
2:37 PM

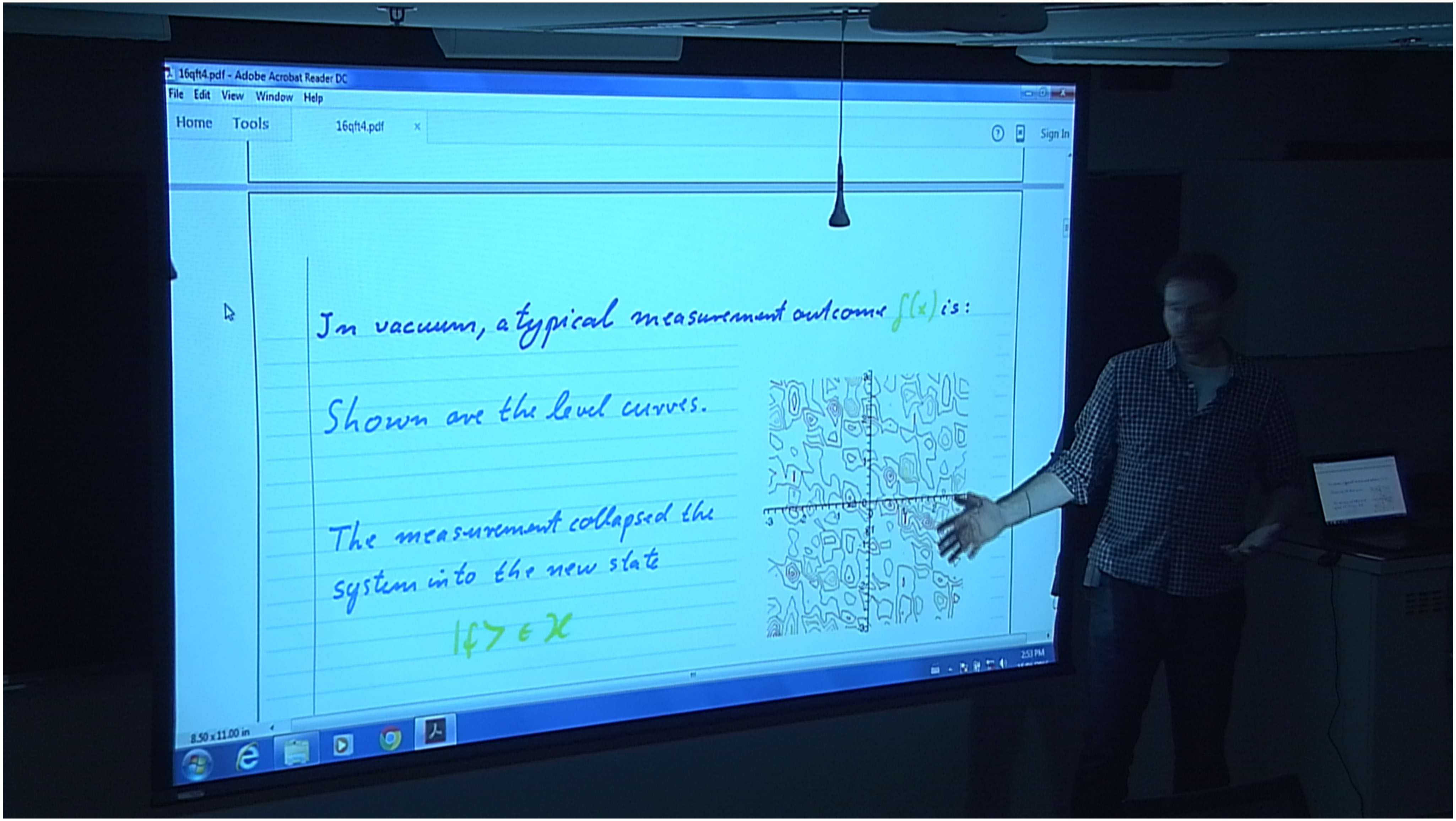

16qft4.pdf - Adobe Acrobat Reader DC
File Edit View Window Help
Home Tools
16qft4.pdf
Sign In
Schrödinger picture of the time evolution:
$i\hbar \frac{d}{dt}|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle$
Which form does it take for
Here in QFT:
now independent of
8.50 x 11.00 in


16qft4.pdf - Adobe Acrobat Reader DC
File Edit View Window Help
Home Tools
16qft4.pdf
Sign In
We now need to represent
$[\hat{\phi}_k, \hat{\pi}_{k'}] = i\delta_{k,-k'}$
on the wave functionals $\Psi[\tilde{f}, t]$.
($\tilde{f}_k$ is Fourier transform of $f(x)$)
As you should verify, this works:
$\hat{\phi}_k . \Psi[\tilde{f},t] = \tilde{f}_k \Psi[\tilde{f},t]$
$\hat{\pi}_k . \Psi[\tilde{f},t] = -i\frac{\partial}{\partial \tilde{f}_{-k}} \Psi[\tilde{f},t]$
8.50 x 11.00 in


16qft4.pdf - Adobe Acrobat Reader DC
File Edit View Window Help
Home Tools
16qft4.pdf
Sign In
We now need to represent
$[\hat{\phi}_k, \hat{\pi}_{k'}] = i\delta_{k,-k'}$
on the wave functionals
As you should verify, this works
Note: Ordinary derivatives here
8.50 x 11.00 in


16qft4.pdf - Adobe Acrobat Reader DC
File Edit View Window Help
Home Tools
16qft4.pdf
Sign In
Recall: For QM harm. osc., ground state Schrödinger wave function is:
$\Psi(x,t) = N e^{-\frac{1}{2}\omega x^2 - i\omega t}$
Exercise: check it. Can you solve for excited states?
Ground state solution in QFT reads, similarly:
Exercise: verify
... which we had already claimed before.
8.50 x 11.00 in


16qft4.pdf - Adobe Acrobat Reader DC
File Edit View Window Help
Home Tools
16qft4.pdf
Sign In
$i\partial_t \Psi[\tilde{\varphi}, t] = \sum_k \frac{1}{2}\left(-\frac{\partial}{\partial\tilde{\varphi}_k}\frac{\partial}{\partial\tilde{\varphi}_{-k}} + (k^2+m^2)\tilde{\varphi}_k\tilde{\varphi}_{-k}\right)\Psi[\tilde{\varphi},t]$
Recall: For QM harm. osc., ground state Schrödinger wave function is:
$\Psi(x,t) = N e^{-\frac{1}{2}\omega x^2 - i\omega t}$
Exercise: check it. Can you solve for excited states?
Ground state solution in QFT reads, similarly:
$= (k^2+m^2)^{1/2}$
$\Psi[\tilde{\varphi},t] = N e^{-\sum_k\left(\frac{1}{2}\omega_k\tilde{\varphi}_k\tilde{\varphi}_{-k} - i\omega_k t\right)}$
Exercise: verify
... which we had already claimed before.
8.50 x 11.00 in
2:49 PM


16qft4.pdf - Adobe Acrobat Reader DC
File Edit View Window Help
Home Tools
16qft4.pdf
Sign In
In vacuum, a typical measurement outcome f(x) is:
Shown are the level curves.
The measurement collapsed the
system into the new state
|f⟩ ∈ H
8.50 x 11.00 in
2:53 PM