

Title: Quantum Field Theory for Cosmology - Achim Kempf - Lecture 2

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URL: <http://pirsa.org/16010002>

Abstract:

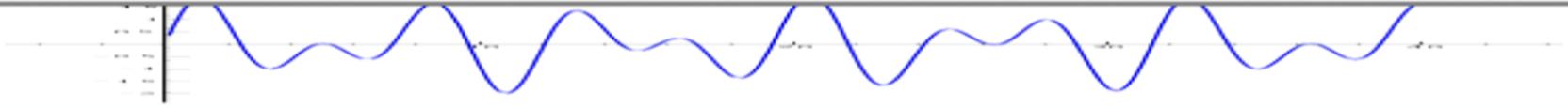


# A taste of quantum fields

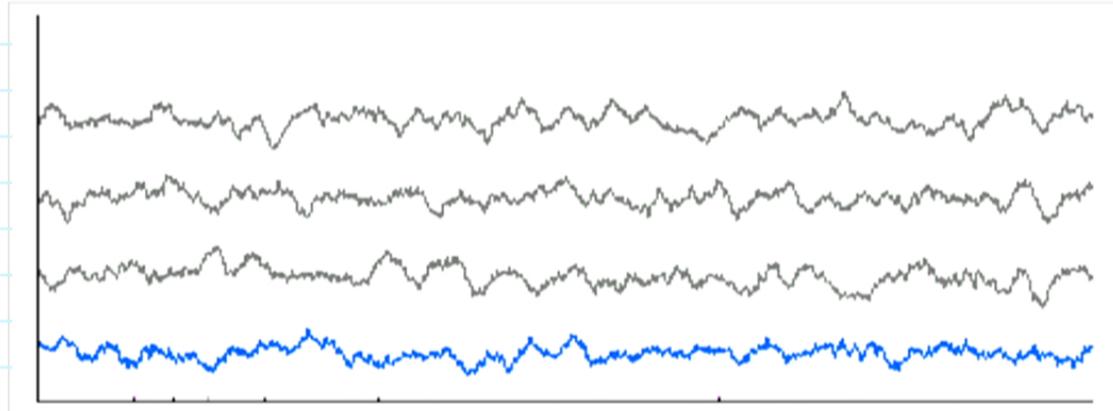
## Intuition:

\* Consider water waves:





\* Multiple cork's oscillations are correlated

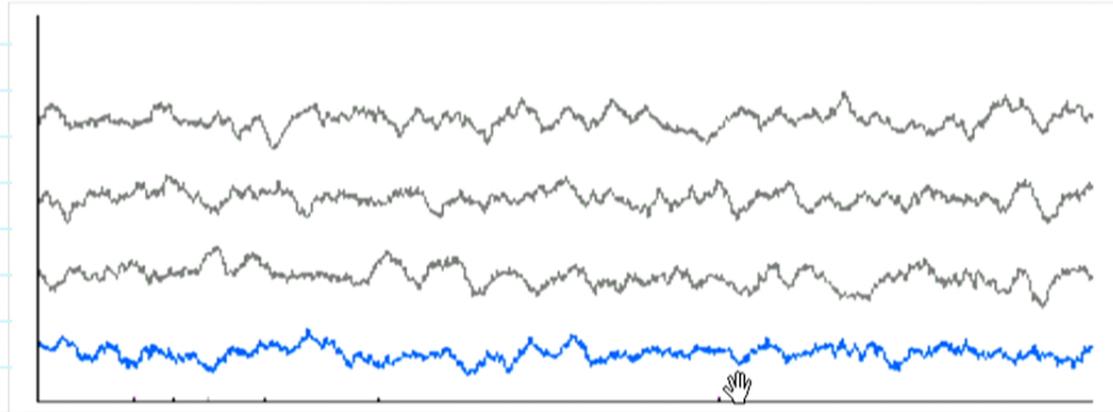


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→ System of coupled (harmonic) oscillators!

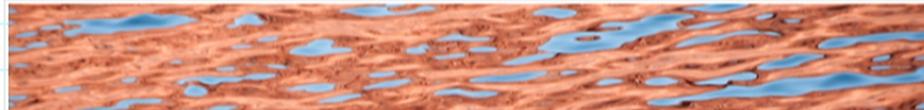


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## Plan:

1. Recall harmonic oscillators

2. ...

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2. Relativistic fields
3. 2nd quantization
4. The harmonic oscillators of fields & their vacuum fluctuations

## 1. Harmonic oscillators

Classical:

▢ Hamiltonian:  $H = \frac{p^2}{2} + \frac{\omega^2}{2} q^2$

▢ Eqs of motion:  $\dot{p} = -\omega^2 q, \quad \dot{q} = p$

# 1. Harmonic oscillators

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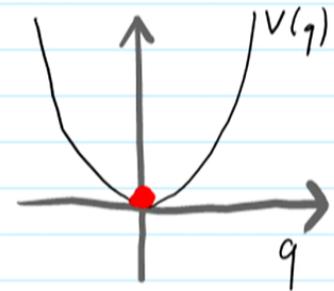
□ Hamiltonian:  $H = \frac{p^2}{2} + \frac{\omega^2}{2} q^2$

□ Eqs of motion:  $\dot{p} = -\omega^2 q$ ,  $\dot{q} = p$

□ Lowest energy solution: (later relevant for "vacuum")

$$q(t) = 0, p(t) = 0$$

i.e.,  $H(t) = 0$  for all  $t$ :



- $H$  and Eqs of motion unchanged.
- But, the canonically conjugate pairs of variables (here,  $q$  and  $p$ ) no longer commute:

□ Hamiltonian:  $\hat{H} = \frac{\hat{p}^2}{2} + \frac{\omega^2}{2} \hat{q}^2$

□ Eqs of motion:  $\dot{\hat{p}} = -\omega^2 \hat{q}, \quad \dot{\hat{q}} = \hat{p}$

□ And now:

$$[\hat{q}(t), \hat{p}(t)] = i\hbar 1$$

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□ And now:

$$[\hat{q}(t), \hat{p}(t)] = i\hbar 1$$

□  $\Rightarrow \hat{q}(t), \hat{p}(t), \hat{H}$  etc are operator-valued.

□ Lowest energy solution now?

The lowest energy state,  $|\psi_0\rangle$ ,  
obeys:

$$\hat{H}|\psi_0\rangle = E_0|\psi_0\rangle$$

□ Lowest energy solution now?

The lowest energy state,  $|\psi_0\rangle$ , obeys:

$$\hat{H}|\psi_0\rangle = E_0|\psi_0\rangle$$

$$\text{with } E_0 = \frac{1}{2}\hbar\omega$$

□ We notice:

Lowest energy is elevated! Why?

(Later for quantum fields  $\Rightarrow$  nonzero vacuum energy)

□ Lowest energy state  $|\psi_0\rangle$ ?

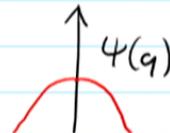
Consider eigenbasis  $|q\rangle$  of  $\hat{q}$ :

$$\hat{q}|q\rangle = q|q\rangle \quad \text{for } q \in \mathbb{R}$$

$$\langle q|q'\rangle = \delta(q-q')$$

Then, recall:

$$\psi_0(q) = \langle q|\psi_0\rangle = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{\omega}{2\hbar}q^2}$$



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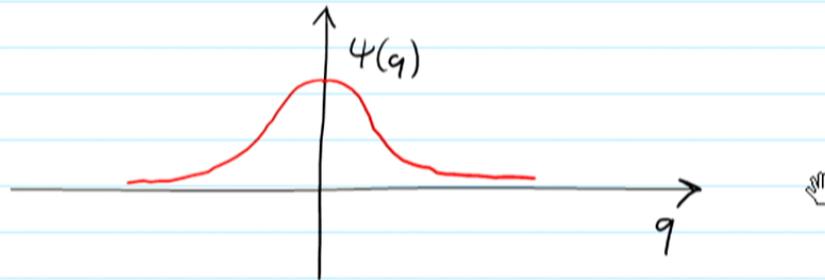
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□ Is oscillator at resting position  $q=0$ ?

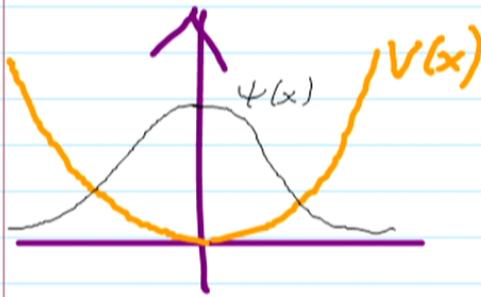
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$$\bar{q} = \langle \psi_0 | \hat{q} | \psi_0 \rangle = \int_{-\infty}^{+\infty} \psi_0^*(q) q \psi_0(q) dq = 0$$

i.e. the position expectation vanishes, as in classical mechanics.

□ But, there are quantum fluctuations!



$$\Delta q = \langle \psi_0 | (\hat{q} - \bar{q})^2 | \psi_0 \rangle^{1/2} = \sqrt{\frac{\hbar}{2m}}$$

i.e., actual measurements yield values spread around  $q=0$ .  
(Note:  $\Rightarrow$  plausible why energy is elevated)

## Plan:

1. Recall harmonic oscillators ✓
2. Relativistic fields
3. 2nd quantization
4. Harmonic oscillators in fields  $\Rightarrow$  vacuum fluctuations

## 2. Relativistic fields

□ How to make the Schrödinger equation, say

choose simple case  
without a potential



$$i\hbar \frac{\partial}{\partial t} \psi(x,t) = -\frac{\hbar^2}{2m} \Delta \psi(x,t) \quad (S)$$

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$$i\hbar \frac{\partial}{\partial t} \psi(x,t) = -\frac{\hbar^2}{2m} \Delta \psi(x,t) \quad (S)$$

relativistically covariant?

Laplacian:  $\Delta = \sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2}$

□ Klein & Gordon:

Recall:  $p_i = -i\hbar \frac{\partial}{\partial x_i}$  and  $E = i\hbar \frac{\partial}{\partial t}$ , i.e., the

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Schrödinger equation can be written in this form:

$$E\psi = \frac{\vec{p}^2}{2m} \psi, \text{ i.e.:}$$

$$\vec{p}^2 = \sum_{i=1}^3 p_i^2$$

$$E = \frac{\vec{p}^2}{2m}$$

$$\text{i.e. } E = \frac{1}{2} m \dot{x}^2$$

But special relativity demands:

$$\frac{E^2}{c^2} - \vec{p}^2 = m^2 c^2 \quad (\text{Namely: } p_\mu p^\mu = m^2 c^4)$$

$$\text{i.e.: } (-\hbar^2 \partial^2 + \hbar^2 \Delta) \psi = m^2 c^2 \psi$$

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□ This "Klein Gordon equation" is usually written as:

$$\left( \frac{\partial^2}{\partial t^2} - \Delta + m^2 \right) \Psi = 0$$

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Or, also  $(\square + m^2) \Psi = 0$  with d'Alembertian  $\square = \partial_\mu^2 - \Delta$

□ Nonrelativistic limit ok?

Must show that KG eqn reduces

to Schrödinger eqn for small momenta:

Choose positive energy solution:

$$E = \sqrt{m^2 c^4 + \vec{p}^2 c^2}$$

Taylor expansion for small  $\vec{p}^2$ : (or large  $c$ )

$$E = m c^2 + \frac{1}{2} \frac{c^2}{\sqrt{\vec{p}^2 c^2 + m^2 c^4}} \Big|_{\vec{p}^2=0} \vec{p}^2 + \mathcal{O}((\vec{p}^2)^2)$$

$$\Rightarrow E = m c^2 + \frac{\vec{p}^2}{2m} + \mathcal{O}((\vec{p}^2)^2)$$

→ For small momenta the KE is

⇒ For small momenta the K.G. eqn becomes the Schrödinger eqn:

$$E\psi = \left( \frac{\vec{p}^2}{2m} + mc^2 \right) \psi$$

i.e.:  $i\hbar \frac{\partial}{\partial t} \psi = \left( -\frac{\hbar^2}{2m} \Delta + mc^2 \right) \psi$

Note: We obtain an extra term:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \underbrace{mc^2}$$

In QM irrelevant: (use Heisenberg picture)

$$\partial_t \left( \frac{1}{2m} \dots \right)$$

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In QM irrelevant: (use Heisenberg picture)

$$i\hbar \frac{d}{dt} \hat{f} = [\hat{f}, \hat{H} + \text{const } 1] = [\hat{f}, \hat{H}]$$

### Remarks:

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Namely:

Require the negative energy solutions to propagate backwards in time: anti-particles!  
They look like travelling forward in time with opposite properties.

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<u>Spin</u>	<u>Standard wave eqn</u>	<u>Examples</u>
0	Klein Gordon eqn.	Higgs, Inflaton, $\pi^0, \pi^\pm$
$1/2$	Dirac eqn.	$e^-$ , quarks, $p^+, n$

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### Higher spins?

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### Note:

- "Graviton" should be a spin 2 particle.

## Plan:

1. Recall harmonic oscillators ✓
2. Relativistic fields ✓
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4. Harmonic oscillators in fields  $\Rightarrow$  vacuum fluctuations

## 3. 2nd quantization

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□ We will 2nd quantize only the Klein Gordon equation because:

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- is only case of cosmological significance that we know of (so far).

□ Terminology: We switch from  $\psi$  to  $\phi$  and call it a "Field".

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□ Definition:

we will do the general definition later

The canonically conjugate field  $\pi(x,t)$  to  $\phi(x,t)$

is defined as:  $\pi(x,t) = \dot{\phi}(x,t)$  (analogous to  $p_i = \dot{q}_i$ )

□ Klein Gordon equation can now be written in the form:

$$\ddot{\pi}(x,t) - \Delta \phi(x,t) + m^2 \phi(x,t) = 0$$

Notice:

The K.G. equation

$$\left( \frac{\partial^2}{\partial t^2} - \Delta + m^2 \right) \phi = 0$$

( $\hbar = 1 = c$ )

does not couple  $\text{Re}(\phi)$  to  $\text{Im}(\phi)$ :  
each separately fulfills the K.G. eqn.

$\Rightarrow$  It suffices to study real-valued  $\phi$ .

Making  $\phi$  complex is then straightforward.





□ Quantization conditions:

$$[\hat{\phi}(x,t), \hat{\pi}(x',t)] = i\hbar \delta^3(x-x')$$

analogous to:

$$[\hat{q}_a(t), \hat{p}_a(t)] = i\hbar \delta_{aa'}$$

$$[\hat{\phi}(x,t), \hat{\phi}(x',t)] = 0$$

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□ We keep the equations of motion:

$$(E1) \quad \dot{\hat{\phi}}(x,t) = \hat{\pi}(x,t)$$

$$\dot{\hat{q}}_a(t) = \hat{p}_a(t)$$

$$(E2) \quad \dot{\hat{\pi}}(x,t) = -(-\Delta + m^2)\hat{\phi}(x,t)$$

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□ Note:  $\phi^*(x,t) = \phi(x,t)$  now implies hermiticity:  $\hat{\phi}^\dagger(x,t) = \hat{\phi}(x,t)$

□ Is there a Hamiltonian for 2nd quantization? **Yes!**

analogous to:

$$\hat{H} = \int_{\mathbb{R}^3} \frac{1}{2} \hat{\pi}^2(x,t) + \frac{1}{2} \hat{\phi}(x,t) (m^2 - \Delta) \hat{\phi}(x,t) d^3x$$

$$\hat{H} = \sum_a \frac{p_a^2}{2} + \frac{\omega_a^2}{2} \hat{q}_a^2$$

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yield the proper eqns of motion: E1, E2.

Indeed, e.g.:

$$i\hbar \dot{\hat{\phi}}(x,t) = [\hat{\phi}(x,t), H] = \left[ \hat{\phi}(x,t), \int_{\mathbb{R}^3} \frac{1}{2} \hat{\pi}^2(x',t) + \text{something}(\hat{\phi}) d^3x' \right]$$

$$= \frac{1}{2} \int [\hat{\phi}(x,t), \hat{\pi}(x',t)] \hat{\pi}(x',t) + \hat{\pi}(x',t) [\hat{\phi}(x,t), \hat{\pi}(x',t)] d^3x'$$

$$= \frac{i\hbar}{2} \int \delta^3(x-x') \hat{\pi}(x',t) + \hat{\pi}(x',t) \delta^3(x-x') d^3x' = \hat{\pi}(x,t) i\hbar \checkmark$$

Exercise: Prove (\*)

$$[A, B^2] = B[A, B] + [A, B]B$$