Title: Spectral networks with spin and wild BPS spectra

Date: Dec 15, 2015 02:00 PM

URL: http://pirsa.org/15120041

Abstract: <p>The BPS spectrum of d=4 N=2 field theories in general contains not only hyper and vector-multipelts but also short multiplets of particles with arbitrarily high spin. These BPS states of higher spin reveal quite a peculiar behavior, so sometimes they are called "wild"</p>

<p>states. In this talk we would try to discuss a small refinement of the asymptotic study (spectral network technique) of tt* equations arising in an effective theory on 2d defects in N=2 4d SYM theory capturing spin information and apply it to study some properties of wild BPS spectra.</p>

This talk is based on:

1. D. Galakhov, P. Longhi, G. W. Moore, "Spectral Networks with Spin", **Commun.Math.Phys. 340 (2015) 1, 171-232**, arXiv:1408.0207
2. D. Galakhov, P. Longhi, T. Mainiero, G. W. Moore, A. Neitzke, "Wild Wall Crossing and BPS Giants", **JHEP 1311 (2013) 046**, arXiv:1305.5454

Outline

1. $\mathcal{N} = 2$ SUSY algebra and BPS spectrum
2. tt^*-equations and parallel transport
3. Stokes' lines and spectral networks
4. Path algebra
5. Wild states pheno
6. Example: Kronecker quiver

$\mathcal{N} = 2$ SUSY algebra

The centrally extended $\mathcal{N} = 2$ SUSY algebra is given by supercharges

$$\begin{aligned} \left\{ Q^A_\alpha, \bar{Q}_{\dot\beta B} \right\} &= 2\sigma^\mu_{\alpha\dot\beta} P_\mu \delta^A_B \\ \left\{ Q^A_\alpha, Q^B_\beta \right\} &= 2\epsilon_{\alpha\beta}\epsilon^{AB}\bar{Z} \\ \left\{ \bar{Q}_{\dot\alpha A}, \bar{Q}_{\dot\beta B} \right\} &= -2\epsilon_{\alpha\beta}\epsilon_{AB} Z \end{aligned} \tag{1}$$

We could form the following generators:

$$R^A_\alpha = \zeta^{-\frac{1}{2}} Q^A_\alpha + \zeta^{\frac{1}{2}} \sigma^0_{\alpha\dot\beta} \bar{Q}^{\dot\beta A}, \quad \{R^A_\alpha, R^B_\beta\} = 4(E - \mathrm{Re}(Z/\zeta))\epsilon_{\alpha\beta}\epsilon^{AB} \tag{2}$$

The central charge is non-zero in the presence of a non-trivial field configuration:

$$Z = -\frac{1}{g^2} \int_{S^2_{R\to\infty}} \langle \mathrm{Tr}\left[(-iF + *F)\phi\right] \rangle = a_i n_i + a_i^{(D)} m_i \tag{3}$$

Charge sector $\gamma = (n_i, m_i)$, where $i = 1, \ldots, \mathrm{rank}\ G$.
BPS states – short reps: $\zeta = \mathrm{Arg}\ Z$, $M = |Z|$

Protected spin character

To define spin content we use a protected spin character or refined BPS index. As representations of algebra $so(3) \oplus su(2)_R$ states are of two kinds

Long reps: $\rho \otimes \rho \otimes \mathfrak{h}$ Short reps: $\rho \otimes \mathfrak{h}$

Where $\rho = (\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$

We define

$$\Omega(\gamma; u|y) := \frac{1}{y - y^{-1}} \mathrm{Tr}_{H_\gamma}(2J_3) y^{2J_3}(-y)^{2I_3} = \mathrm{Tr}_{\mathfrak{h}} y^{2J_3}(-y)^{2I_3} \tag{4}$$

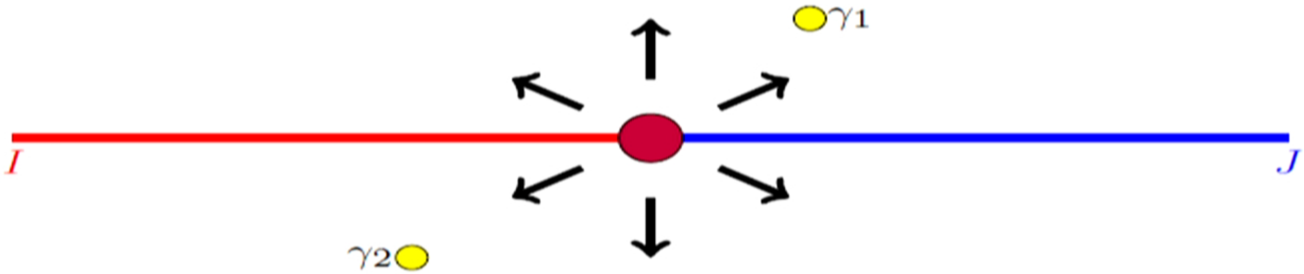
For example:

- Hypermultiplet: $\Omega(y) = 1$
- Vectormultiplet: $\Omega(y) = y + y^{-1}$

2d degefects, tt^*-equations and parallel transport

We follow [Gaiotto-Moore-Neitzke '11 & '12]
2d defects preserve a half of $\mathcal{N} = 2$ SUSY in 4d, so they are parameterized by some phase ζ. There are N vacua.
The preserved ζ-SUSY becomes $(2, 2)$ on the defect worldsheet.
η – electric parameters, 2d θ-angles
α – magnetic parameters, $A \sim \alpha d\varphi$

Interfaces:



Central charge of this configuration:

$$Z_{IJ} = \mathcal{W}^I - \mathcal{W}^J + \sum_a \gamma_a \cdot \begin{pmatrix} \vec{a} \\ \vec{a}^{(D)} \end{pmatrix} \tag{5}$$

Partition function:

$$\Psi_{IJ} = \langle I | J \rangle \tag{6}$$

tt^*-equations – flat connections:

$$\nabla_a = \partial_{t_a} + A_a + \beta\zeta C_a, \quad \bar{\nabla}_a = \partial_{\bar{t}_a} + \bar{A}_a + \beta\zeta^{-1}\bar{C}_a \tag{7}$$

$$\nabla_a \Psi = 0 \tag{8}$$

Just a taste:

- Liouville CFT:

$$\left(b^{-2}\partial_z^2 - \sum_i \left(\frac{\Delta_i}{(z-q_i)^2} + \frac{1}{z-q_i}\partial_{q_i}\right)\right)\Psi = 0 \tag{9}$$

- WZW model:

$$\left(\partial_z - \frac{1}{\kappa+N}\sum_i \frac{\rho_z(T^\alpha)\otimes\rho_{q_i}(T^\alpha)}{z-q_i}\right)\Psi = 0 \tag{10}$$

Major problem: Abelianization map

$$(\text{“}\hbar\text{”}\partial_z - \mathcal{A}(z,\mathrm{prm}))\Psi = 0 \Rightarrow \Psi = \left(\mathcal{P}\exp\ \text{“}\hbar^{-1}\text{”}\int \mathcal{A}\right)\Psi_0 \tag{11}$$

Asymptotic behavior:

$$\Psi \sim e^{\text{“}\hbar^{-1}\text{”} \int^{z} dW} \tag{12}$$

This reduces to an algebraic equation, spectral cover – Seiberg-Witten curve (Σ):

$$\text{Det}(dW - \mathcal{A}dz) = 0 \tag{13}$$

For example (pure $SU(3)$) $(\pi : \Sigma \to \mathcal{C})$:

$$x^3 - \frac{u_2}{z}x + \left(\frac{1}{z^2} + \frac{u_3}{z^3} + \frac{1}{z^4}\right) = 0, \quad dW = x\,dz \tag{14}$$

We expect the following solution ($x^{(i)}$ – roots):

$$\Psi \sim \begin{pmatrix} e^{\text{“}\hbar^{-1}\text{”} \int x^{(1)} dz} & 0 & 0 \\ 0 & e^{\text{“}\hbar^{-1}\text{”} \int x^{(2)} dz} & 0 \\ 0 & 0 & e^{\text{“}\hbar^{-1}\text{”} \int x^{(3)} dz} \end{pmatrix} \tag{15}$$

NO! Stokes phenomenon gives off-diagonal contributions:

$$\Psi \sim \begin{pmatrix} e^{\text{“}\hbar^{-1}\text{”} \int x^{(1)} dz} & \star & \star \\ \star & e^{\text{“}\hbar^{-1}\text{”} \int x^{(2)} dz} & \star \\ \star & \star & e^{\text{“}\hbar^{-1}\text{”} \int x^{(3)} dz} \end{pmatrix} \tag{16}$$

We can form a formal generating function for the line defect (monodromies):

$$F(u, \zeta, \gamma_c; y) = \sum_{\gamma_h \in \Gamma} \overline{\underline{\Omega}}(u, \gamma_c + \gamma_h; y) Y_{\gamma_c + \gamma_h} \tag{17}$$

Due to spin fugacity y, Y_γ form a non-commutative algebra:

$$Y_\gamma Y_{\gamma'} = y^{\langle \gamma, \gamma' \rangle} Y_{\gamma + \gamma'} \tag{18}$$

Across stability wall at the phase ζ we loose (gain) a gas of BPS states of phase ζ:

$$Y_{\gamma_c} \mapsto Y_{\gamma_c} \prod_{\gamma_h} \prod_{m \in \mathbb{Z}} \prod_{k=-(|\langle \gamma_c, \gamma_h \rangle| - 1)}^{|\langle \gamma_c, \gamma_h \rangle| - 1} (1 + (-y)^m y^k Y_{\gamma_h})^{a_m(\gamma_h)} \tag{19}$$

Where powers $a_m(\gamma_h)$ are defined as:

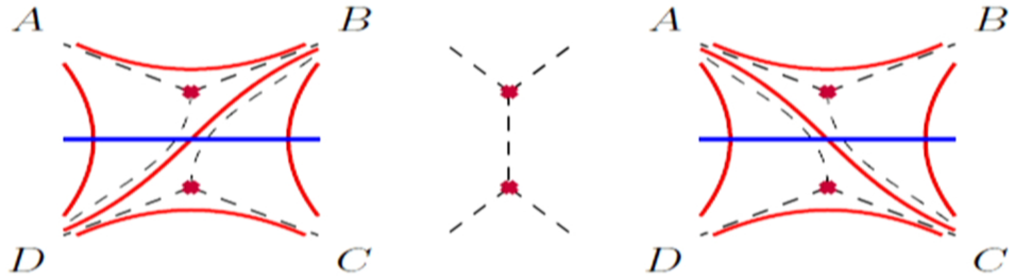
$$\Omega(\gamma_h; y) =: \sum_{m \in \mathbb{Z}} a_m(\gamma_h)(-y)^m \tag{20}$$

Stokes' lines and spectral networks

(ij)-(Anti-)Stokes lines $\mathfrak{a}$: $\forall z \in \mathfrak{a}$

$$\mathrm{Im}(\zeta \, dW_{ij}(T_z \mathfrak{a})) = 0 \tag{21}$$

As an example consider two ramification points:



In a case of **multiple** covers: **Networks**:

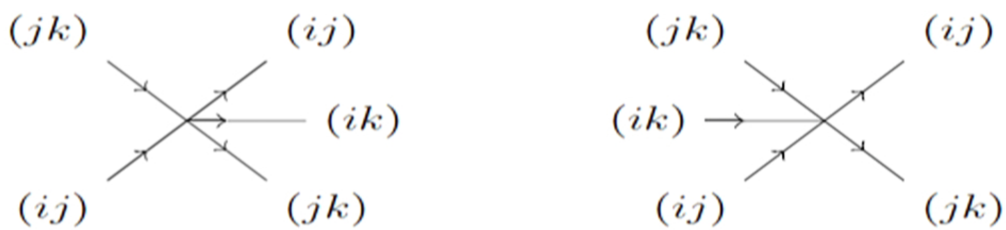
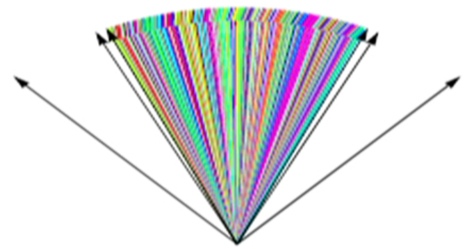


Figure: Secondary Stokes lines

Wild BPS states



The index behaves as $\Omega(\gamma,1) \sim e^{\alpha(\gamma/|\gamma|)|\gamma|} \sim e^{aE}$.
In principle it should be bounded by the entropy of a free gas in UV

$$\Omega(\gamma,1) \lesssim e^{S(E)} \sim e^{\kappa V^{\frac{1}{d}} E^{\frac{d-1}{d}}}$$

γ	$\delta(J_{\max}),\ \delta(J_{\max}-1),\ \dots$
$4\gamma_1 + 3\gamma_2$	$1, 0, 2, 2, 3, 2, 2, 0, \dots$
$7\gamma_1 + 6\gamma_2$	$1, 0, 2, 2, 5, 6, 13, 14, \dots$
$8\gamma_1 + 6\gamma_2$	$1, 0, 2, 2, 5, 6, 13, 16, \dots$
$8\gamma_1 + 7\gamma_2$	$1, 0, 2, 2, 5, 6, 13, 16, \dots$

(22)

$$g(\xi) = \prod_{m=2}^{\infty} (1-\xi^m)^{-2} = 1 + 0\xi + 2\xi^2 + 2\xi^3 + 5\xi^4 + 6\xi^5 + 13\xi^6 + \dots$$

Path algebra

So we have:

$$\pi : \begin{array}{c} \Sigma \\ \text{SW curve} \\ e^{\int x^{(i)} dz} \end{array} \longrightarrow \begin{array}{c} \mathcal{C} \\ \text{UVcurve} \\ \mathcal{P}\exp \int \mathcal{A} \end{array} \tag{23}$$

Let us think abstractly:

$$\mathscr{Y} \sim e^{\int x^{(i)} dz}, \quad \mathfrak{U} \sim \mathcal{P}\exp \int \mathcal{A} \tag{24}$$

For paths on Σ:

1. If two paths a and b are regular-homotopic then

$$\mathscr{Y}_a = \mathscr{Y}_b \tag{25}$$

2. Concatenation rule

$$\mathscr{Y}_a \mathscr{Y}_b = \begin{cases} \mathscr{Y}_{a \circ b}, & \text{if a concatenation } a \circ b \text{ exists,} \\ 0, & \text{otherwise} \end{cases} \tag{26}$$

There are **no more** natural **a priori** rules for this algebra.

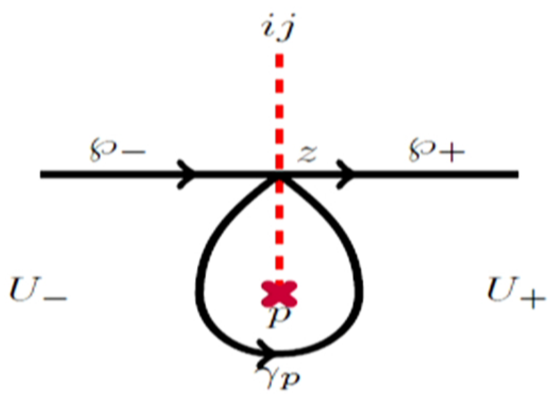
Detour rules

The WKB lines cut $\mathcal{C}$ into charts $\mathcal{C} \setminus (\mathrm{WKB-lines}) = \coprod_i U_i$

1. If $\wp \subset U_i$ we define the parallel transport

$$\mathfrak{U}_{\wp} := \sum_{a \in \pi^{-1}(\wp)} \mathscr{Y}_a \tag{27}$$

2. For intersection we define the re-gluing function as:



$$\begin{aligned} &\wp \cap U_{\pm} = \wp_{\pm} \\ &\partial \gamma_p = \pi_j^{-1}(z) - \pi_i^{-1}(z) \\ &\boxed{\mathfrak{U}_{\wp} = \mathfrak{U}_{\wp_+}(1 + \mathscr{Y}_{\gamma_p})\mathfrak{U}_{\wp_-}} \end{aligned} \tag{28}$$

The flatness condition for the formal parallel transport requires the equivalence of parallel transports along paths D and $A_1 B_1 C_1$ on $\mathcal{C}$ depicted on this figure.

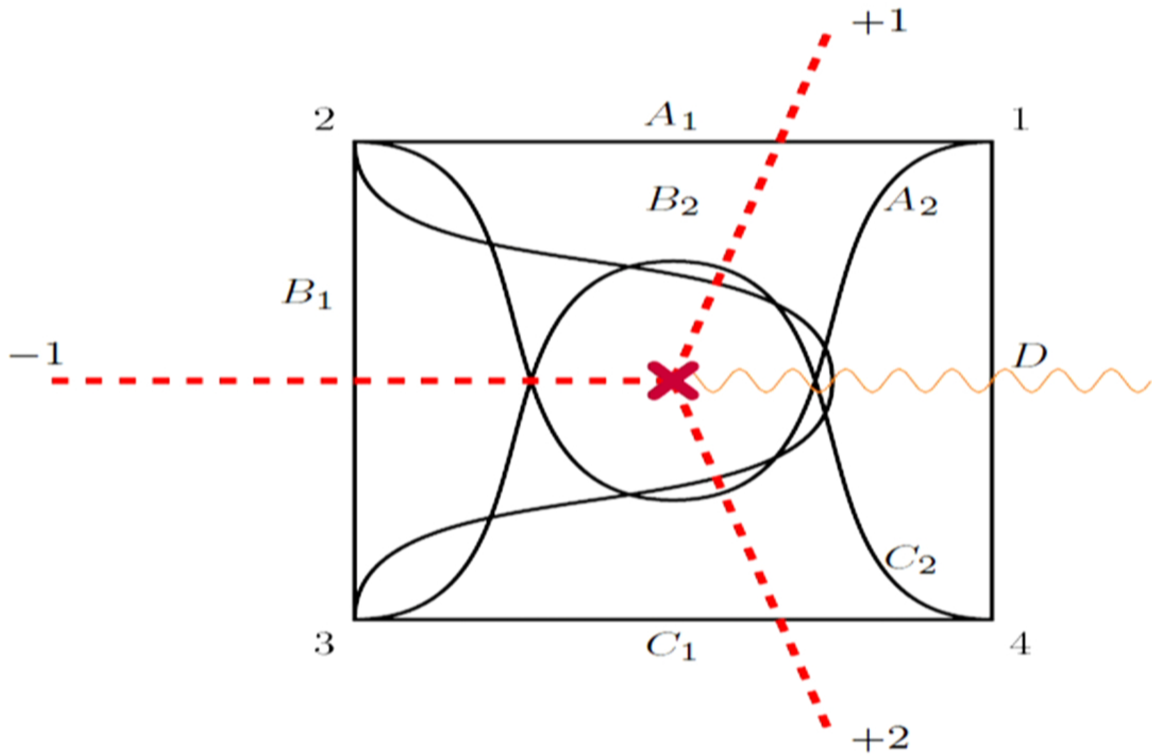
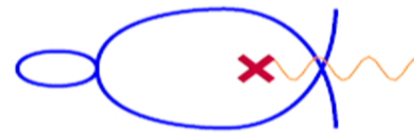


Figure: Lifts and detours

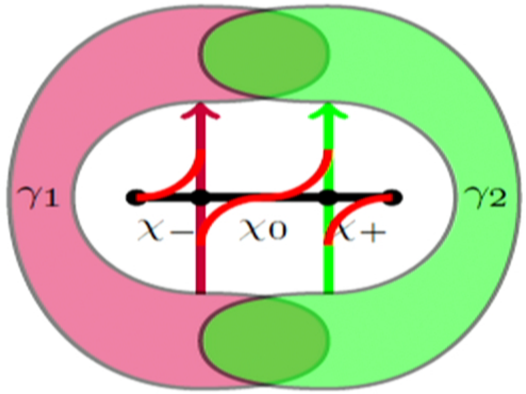
If we denote by a prime a difference between paths by a curl around the branching point, so

$a =$  , $a' =$  (29)

then the expansion of the flatness condition delivers a **new** equation for the path algebra:

$$\boxed{\mathscr{Y}_a + \mathscr{Y}_{a'} = 0} \tag{30}$$

This equation is the central source for the homological refinement.



Then we find:

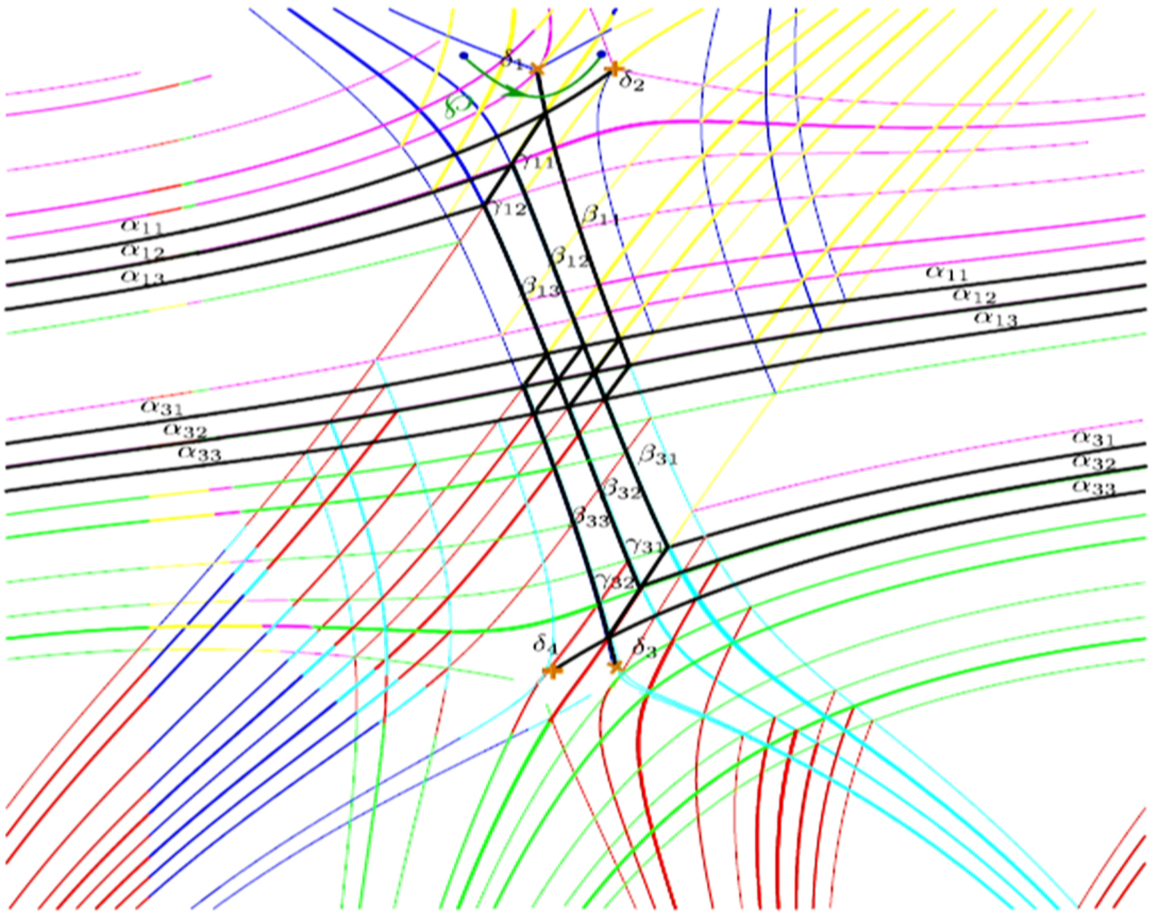
$$\mathfrak{wr}(\chi_-\gamma_1\chi_0\gamma_2\chi_+) = \mathfrak{wr}(\chi_-\gamma_1\chi_0\chi_+) + \mathfrak{wr}(\chi_-\chi_0\gamma_2\chi_+) + \langle[\gamma_1],[\gamma_2]\rangle \tag{35}$$

Or **loosely** speaking,

$$\mathfrak{wr}(\gamma_1\gamma_2) = \mathfrak{wr}(\gamma_1) + \mathfrak{wr}(\gamma_2) + \langle[\gamma_1],[\gamma_2]\rangle \tag{36}$$

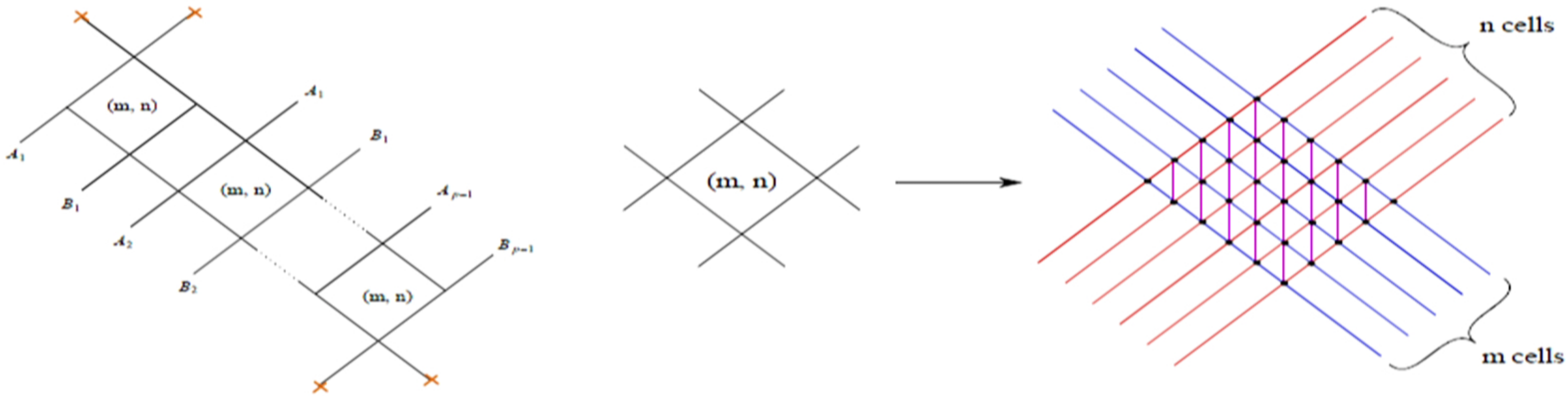
$$\begin{aligned} \mathscr{Y}_{\gamma_1}\mathscr{Y}_{\gamma_2} = \mathscr{Y}_{\gamma_1\gamma_2} \rightsquigarrow y^{\mathfrak{wr}(\gamma_1)}Y_{[\gamma_1]}y^{\mathfrak{wr}(\gamma_2)}Y_{[\gamma_2]} = y^{\mathfrak{wr}(\gamma_1\gamma_2)}Y_{[\gamma_1\gamma_2]} \\ Y_{[\gamma_1]}Y_{[\gamma_2]} = y^{\langle[\gamma_1],[\gamma_2]\rangle}Y_{[\gamma_1]+[\gamma_2]} \end{aligned} \tag{37}$$

Example: Kronecker quiver



Generically for a quiver:

$$, \quad \alpha \in \mathbb{Z} \tag{38}$$



In the case $m = n = 1$ (slope 1) we should solve an auxiliary functional equation:

$$P(z,y) = 1 + z \prod_{s=-(p-2)}^{p-2} P(zy^{2s}, y)^{p-1-|s|} \tag{39}$$

Having its solution we can construct the generating function:

$$Q(z,y) := \prod_{s=-\frac{m-1}{2}}^{\frac{m-1}{2}} P(zy^{2s}, y) =: \prod_{n\geq 0} \prod_{m\in\mathbb{Z}} \prod_{k=1}^{n} \left(1 + (-y)^m y^{-(2k-1)} y^n z^n\right)^{a_m(n)} \tag{40}$$

And the desired protected spin character reads ($\langle\gamma_1, \gamma_2\rangle = p$):

$$\Omega(n\gamma_1 + n\gamma_2, y) = \sum_{m\in\mathbb{Z}} a_m(n)(-y)^m \tag{41}$$

Explicit examples

p=2: The equation is algebraic in this case and can be solved explicitly

$$P(z, y) = (1 - z)^{-1} \tag{42}$$

Thus

$$Q(z, y) = (1 - zy)(1 - zy^{-1}) \tag{43}$$

corresponding to the expected vectormultiplet

$$\Omega(\gamma_c, y) = y + y^{-1}, \quad \Omega(n\gamma_c, y) = 0,\ n \geq 2 \tag{44}$$

p=3: provides the first non-trivial example, since in this case equation is no longer algebraic. Nevertheless one can study its solutions perturbatively, introducing the series

$$P(z,y) = 1 + \sum_{n=1}^{\infty} \omega_n^{(p)}(y) z^n \,. \tag{45}$$

We find

$$P(z,y) = 1 + z + \left(y^{-2} + 2 + y^2\right) z^2 + \ldots \tag{46}$$

Relations allow one to extract the corresponding PSCs: denoting $\chi_s(y) = (y^{2s+1} - y^{-(2s+1)})/(y - y^{-1})$

$$\begin{aligned}
\Omega(\gamma_c, y) &= \chi_1(y) \\
\Omega(2\gamma_c, y) &= \chi_{\frac{5}{2}}(y) \\
\Omega(3\gamma_c, y) &= \chi_3(y) + \chi_5(y) \\
\Omega(4\gamma_c, y) &= \chi_{\frac{5}{2}}(y) + 2\chi_{\frac{9}{2}}(y) + \chi_{\frac{11}{2}}(y) + 2\chi_{\frac{13}{2}}(y) + \chi_{\frac{17}{2}}(y)
\end{aligned} \tag{47}$$

Thank you for your attention!

$$J^2(z) + Q\,\partial_z J(z)$$

$$\left[b^{-2}\partial_z^2 - T(z) \right] \Psi = 0$$

$$\mathrm{Pexp} \int \begin{pmatrix} 1 & T(z) \\ 0 & 1 \end{pmatrix} dz = \sum \Omega(\gamma) Y_\gamma$$

$$Y_\gamma = e^{b \oint_\gamma J(z) dz}$$

$$J^2(z) + Q\partial_z J(z) = T(z)$$

$$[J(z), J(z')] = B(z, z') + \text{higher ord. corr.}$$

$$\left[\oint_\gamma J, \oint_{\gamma'} J'\right] = \langle \gamma, \gamma' \rangle$$

$$\sum \Omega(\gamma) Y_\gamma$$

$$Y_\gamma = e^{b\oint_\gamma J(z)dz}$$

$$y = e^{\pi i b^2}$$