Title: Entanglement entropy from thermodynamic entropy in one higher dimension

Date: Dec 08, 2015 11:00 AM

URL: http://pirsa.org/15120033

Abstract: <p>In recent years, entanglement has become a new frontier with applications across several fields in physics. Nevertheless, simple conceptual pictures and practical ways to quantify entanglement in many-body systems have remained elusive even for the simplest models. In this talk, I will consider entanglement and Renyi entropies as well as quantum (mutual, tripartite, etc.) information in a quantum field theory. For free field theories, I will show that quantum entropies and information can be computed and understood by analogy with the thermal Casimir effect in one higher dimension. Furthermore, I will introduce a geometrical picture for the quantum (mutual, tripartite) information as a sum over polymers establishing a connection to purely entropic effects that prove useful in deriving information inequalities. Finally, I will show that similar ideas may be extended beyond free field theories.</p>

Entanglement entropy from thermodynamic entropy in one higher dimension

Mohammad Maghrebi
Joint Quantum Institute
Joint Center for Quantum Information and Computer Science
University of Maryland

Perimeter Institute
December 2015

M.F. Maghrebi, H. Reid, Phys. Rev. Lett. **114**, 151602 (2015)
M.F. Maghrebi, arXiv:1510.00018

Entanglement

What is entanglement?

$$|\Psi\rangle \neq |\psi_A\rangle \otimes |\psi_{\tilde{A}}\rangle$$

$$|\Psi\rangle = |\psi_A\rangle \otimes |\psi_{\tilde{A}}\rangle + |\chi_A\rangle \otimes |\chi_{\tilde{A}}\rangle \longrightarrow$$ A and Ã entangled

Entanglement

What is entanglement?

$$|\Psi\rangle \neq |\psi_A\rangle \otimes |\psi_{\tilde{A}}\rangle$$

$$|\Psi\rangle = |\psi_A\rangle \otimes |\psi_{\tilde{A}}\rangle + |\chi_A\rangle \otimes |\chi_{\tilde{A}}\rangle \longrightarrow$$ A and Ã entangled

Why do we care about entanglement?

A central subject in

- Quantum Information
- High Energy Physics and Black Holes
- Condensed Matter Physics
- AMO Physics: Entangled atoms, photons, quantum dots, …

Entanglement

What is entanglement?

$$|\Psi\rangle \neq |\psi_A\rangle \otimes |\psi_{\tilde{A}}\rangle$$

$$|\Psi\rangle = |\psi_A\rangle \otimes |\psi_{\tilde{A}}\rangle + |\chi_A\rangle \otimes |\chi_{\tilde{A}}\rangle \longrightarrow$$ A and Ã entangled

Why do we care about entanglement?

A central subject in

- Quantum Infor
- High Energy Ph
- Condensed Ma
- AMO Physics: E tum dots, ...

Entanglement Frontier

Entanglement entropy

State of a subsystem

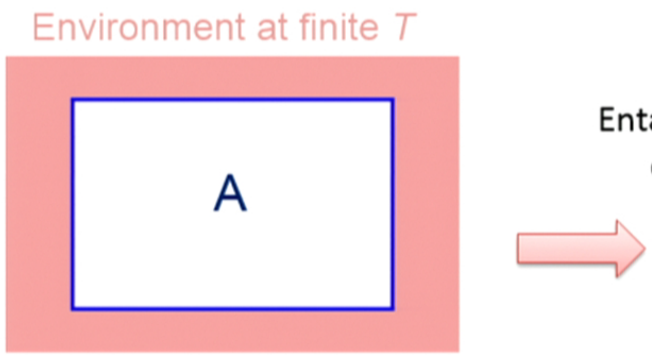
$$\rho_A = \mathrm{Tr}_{\tilde{A}}(|\Psi\rangle\langle\Psi|)$$

- A measure of entanglement:

Von Neumann (entanglement) entropy

$$S_A = -\mathrm{Tr}(\rho_A \log \rho_A)$$

Entanglement vs thermodynamic entropy



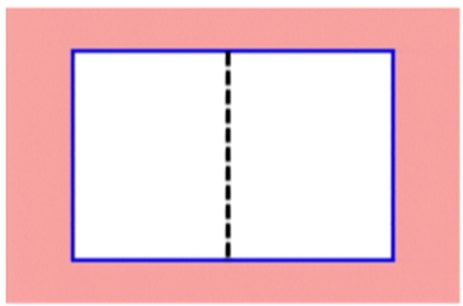
Entanglement entropy — Thermodynamic entropy

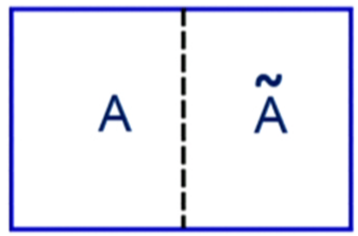
$$S_A = S_{\mathrm{Th}}$$

Entanglement vs thermodynamic entropy

Entanglement entropy | Thermodynamic entropy

$$S_A = S_{\mathrm{Th}}$$

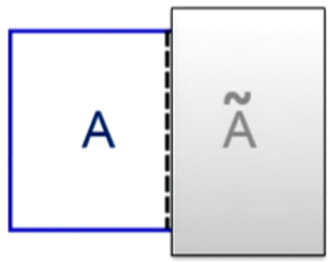




Thermodynamic entropy is extensive

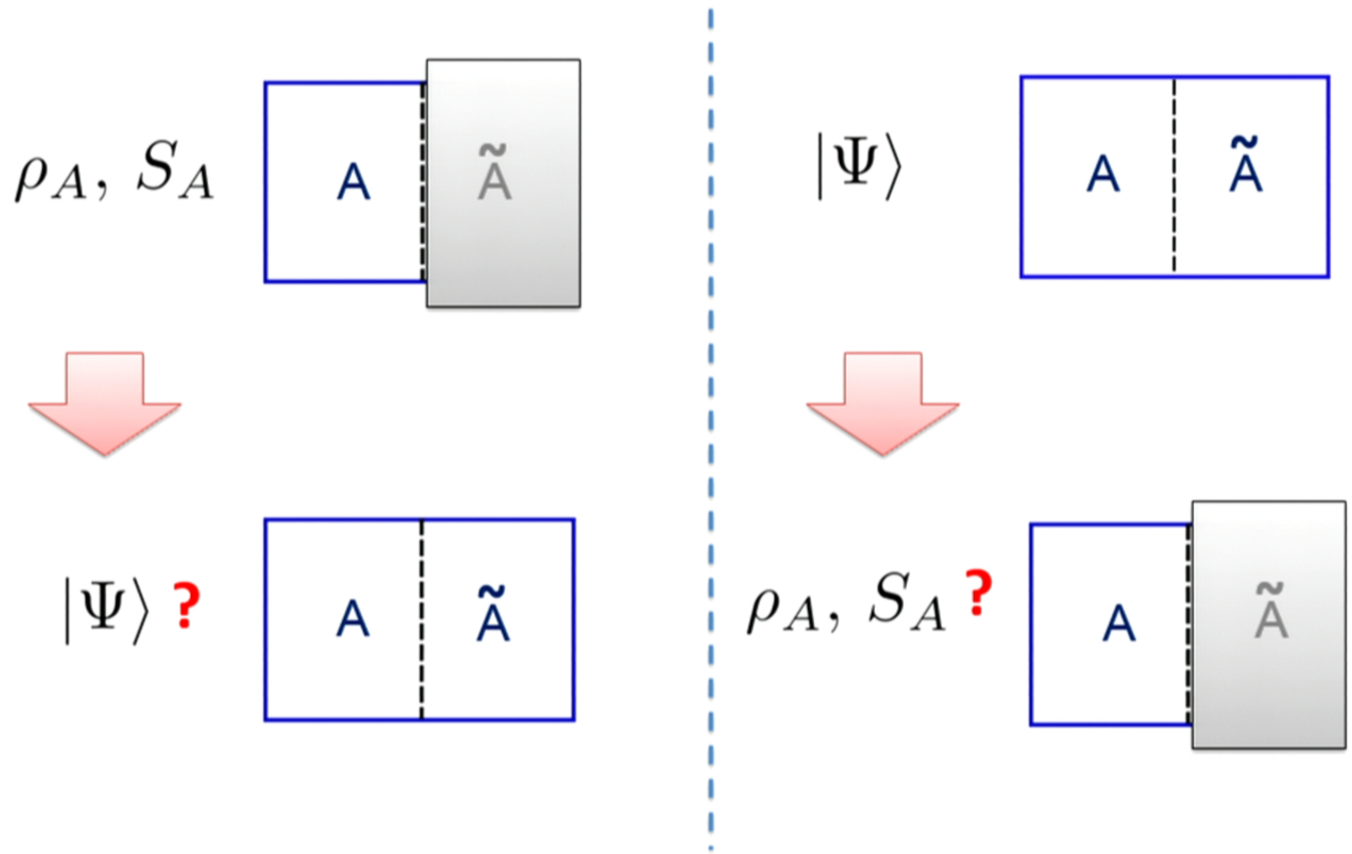
Entanglement entropy

$\rho_A,\, S_A$



$|\Psi\rangle$ **?**

Entanglement entropy



Entanglement entropy

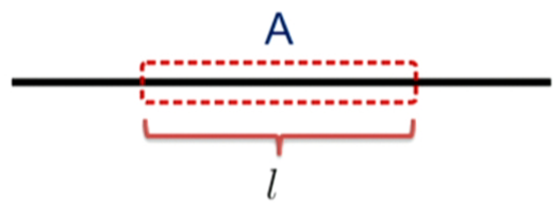
Information about many-body or field theory

- 1+1 conformal field theory

$$S_A \sim (c/3)\log l$$

↓

Central charge



Holzhey, Larsen, Wilczek '94

$S_{A\cup B}$ ⟶ Information content of CFT

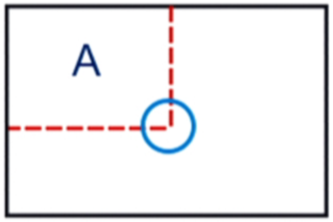
Calabrese, Cardy, Tonni '11
Calabrese, Cardy, Tonni '13

Entanglement entropy

Information about many-body or field theory

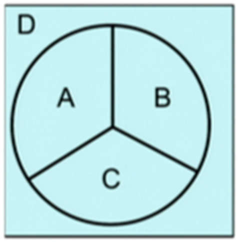
- d-dim conformal field theory

$$S_A\Big|_{\text{corner}} \longrightarrow \text{d-dim CFT}$$



Casini, Huerta '07
. . .
Melko's group
Myers' group

- Topological quantum field theory



Kitaev, Preskill '06
Levin, Wen '06

- And more . . .

Entanglement entropy

Free field theories?

All information content → 2-point functions

$$\langle \phi(x)\phi(y)\rangle \equiv G(x-y)$$

Entanglement entropy is hard to compute!

Hertzberg, Wilczek 2011
Klebanov, Pufu, Sachdev, Safdi 2012
Cardy 2013
. . . . . .

Entanglement entropy

Free field theories?

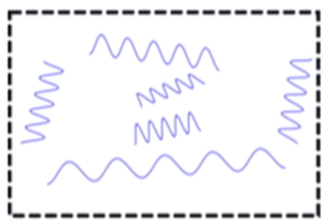
All information content → 2-point functions

$$\langle \phi(x)\phi(y)\rangle \equiv G(x-y)$$

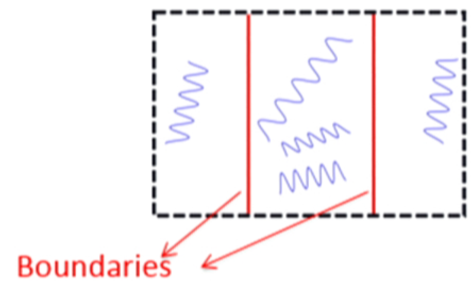
Entanglement entropy is hard to compute!

Hertzberg, Wilczek 2011
Klebanov, Pufu, Sachdev, Safdi 2012
Cardy 2013
. . . . . .

Vacuum of a free QFT is simple

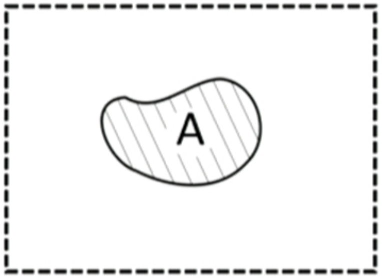


Casimir energy, not so simple!



Entanglement entropy

- Entanglement entropy depends on geometrical, and topological, properties of A



- ❑ **Computational scheme?**
- ❑ **Conceptual and pictorial scheme?**

Outline

- Free field theory (with or without coupling to matter)
- Connections with ***thermal Casimir effect***
- Mutual information of disjoint regions
  from ***thermodynamic entropy*** in one higher dimension
- A ***polymer*** interpretation of the mutual, tripartite, … information
- Beyond free field theories?

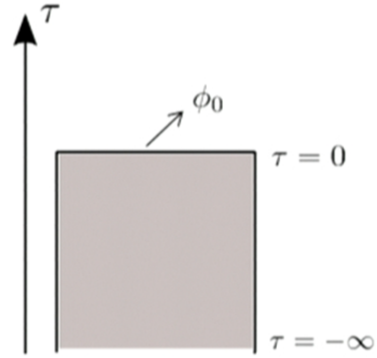
Density matrix from path integral

Ground state wavefunction:

$$|\mathrm{G.S.}\rangle \sim e^{-H\,T}|0\rangle \qquad \text{for} \quad T \to \infty$$

Path integral

$$\Psi\left[\phi_0\right] \propto \int D\phi \; e^{-\int_{\tau=-\infty}^{\tau=0} d\tau \, L[\phi]}$$



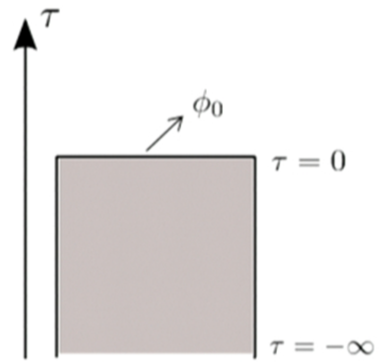
Density matrix from path integral

Ground state wavefunction:

$$|\text{G.S.}\rangle \sim e^{-H\,T}|0\rangle \qquad \text{for} \quad T \to \infty$$

Path integral ↓

$$\Psi\left[\phi_0\right] \propto \int D\phi \; e^{-\int_{\tau=-\infty}^{\tau=0} d\tau \, L[\phi]}$$



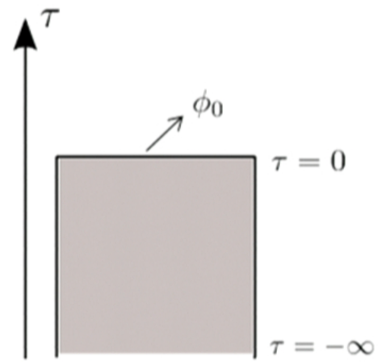
Density matrix from path integral

Ground state wavefunction:

$$|\mathrm{G.S.}\rangle \sim e^{-H\,T}|0\rangle \qquad \text{for} \quad T \to \infty$$

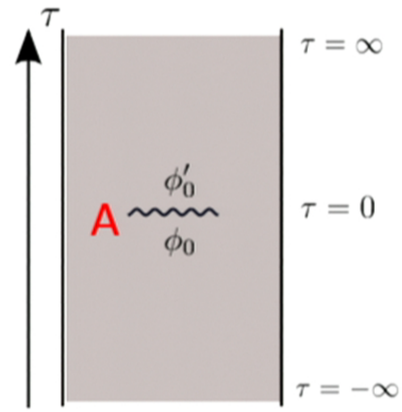
Path integral

$$\Psi\left[\phi_0\right] \propto \int D\phi \; e^{-\int_{\tau=-\infty}^{\tau=0} d\tau L[\phi]}$$



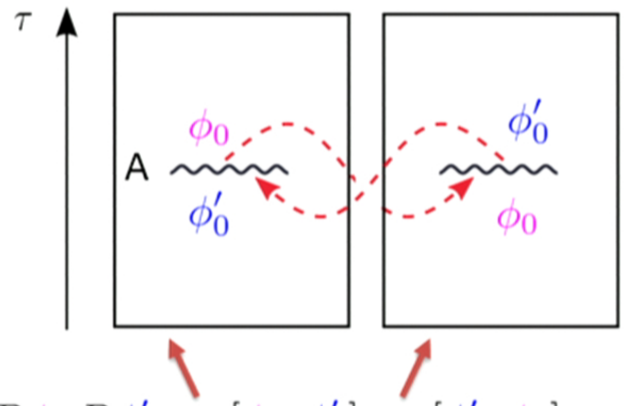
- Reduced density matrix of A

$$\rho_A(\phi_0, \phi_0') \propto \int D\phi \; e^{-S[\phi]}$$



Renyi entropy

$$\mathrm{Tr}\left(\rho_A^2\right) =$$



$$\int D\phi_0 \, D\phi_0' \;\; \rho_A[\phi_0, \phi_0'] \;\; \rho_A[\phi_0', \phi_0]$$

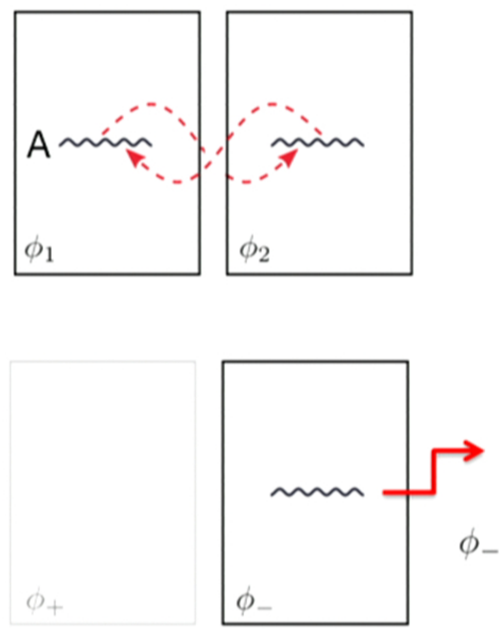
Free Scalar Field

Model**: Free scalar field theory**

Action $= \int dt\ d^d x\ (\partial_t \phi)^2 - (\nabla\phi)^2 - M^2\phi^2 - V(x)\phi^2$

➢ **Second Renyi entropy**

$$e^{-R_2(A)} =$$



New basis

$\phi_\pm = \phi_1 \pm \phi_2$

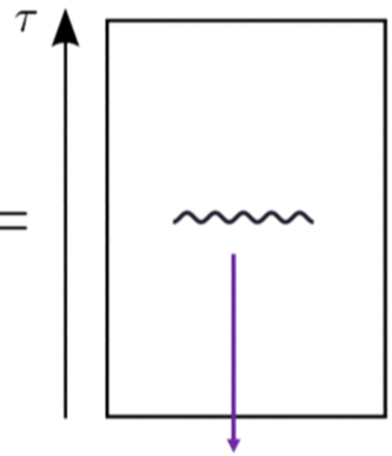
$$=$$

Boundary conditions

$\phi_-(0^+) = -\phi_-(0^-)$

Renyi entropy from partition function

$$e^{-R_2(A)} = \quad = e^{-\Delta F(A)}$$



d dim

Quantum

$k_B T = 0$

Boundary conditions

$\phi_-(0^+) = -\phi_-(0^-)$

d+1 dim

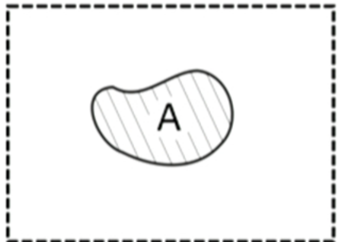
Classical

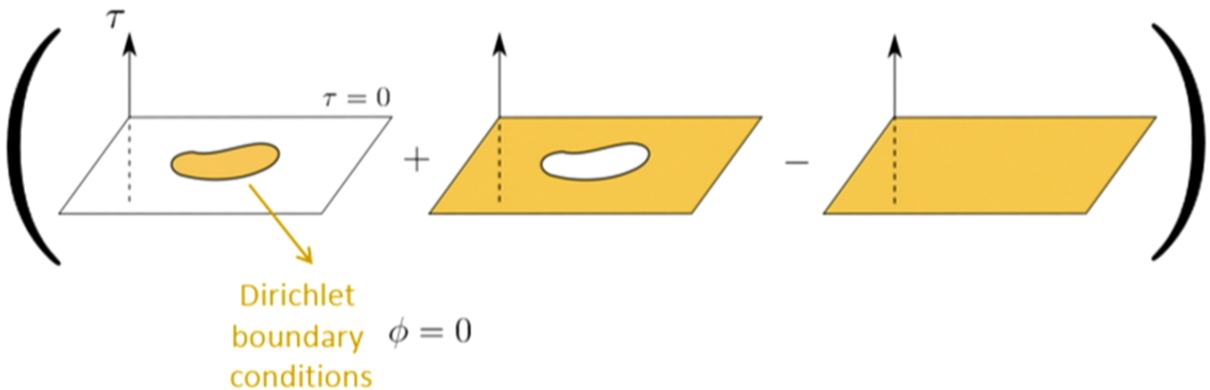
$k_B T = 1$

$$\mathcal{H} = \int d^{d+1}x \; (\nabla_E \, \phi)^2$$

Renyi entropy from free energy

MM '15

R_2 

$= \Delta F$ 

Free Scalar Field

Model**: Free scalar field theory**

Action $= \int dt\, d^dx\, (\partial_t \phi)^2 - (\nabla\phi)^2 - M^2\phi^2 - V(x)\phi^2$

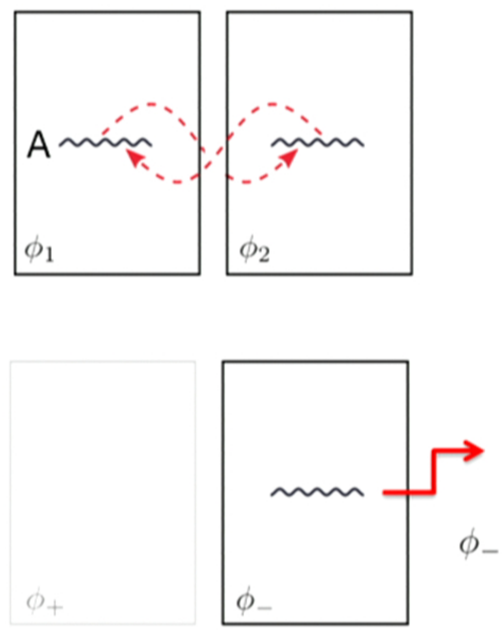
- **Second Renyi entropy**

$e^{-R_2(A)} =$

New basis

$\phi_\pm = \phi_1 \pm \phi_2$

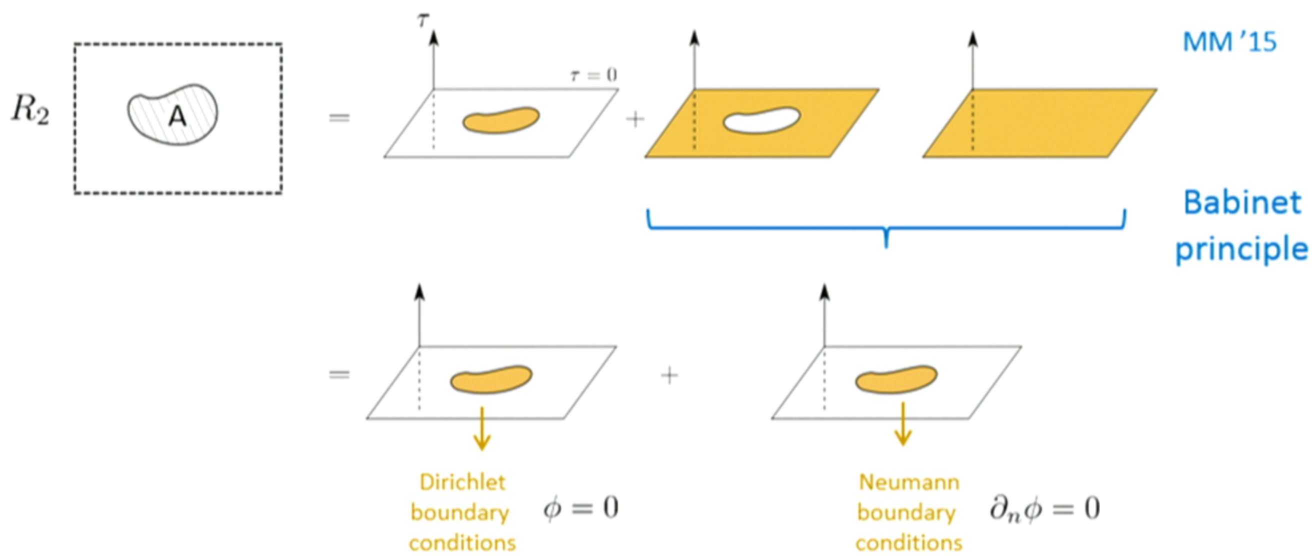
$=$



Boundary conditions

$\phi_-(0^+) = -\phi_-(0^-)$

Renyi entropy from free energy

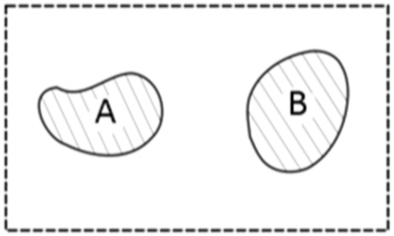


Mutual information

MM '15

Quantum correlations between domains

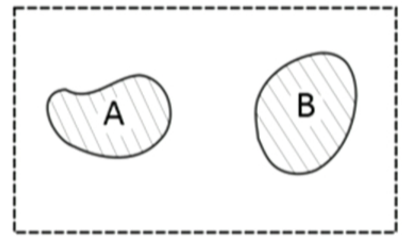
$$I_n(A,B) = R_n(A) + R_n(B) - R_n(A \cup B)$$



Mutual information

MM '15

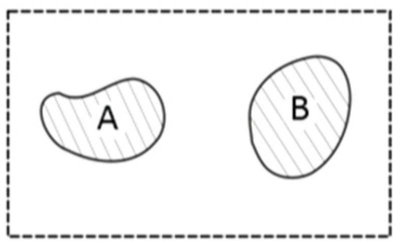
Quantum correlations between domains



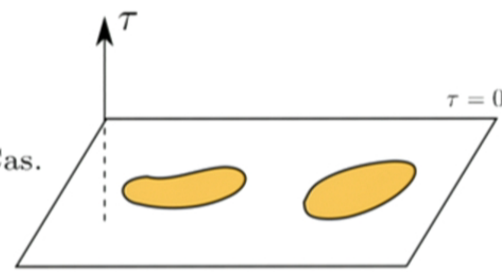
$$I_n(A,B) = R_n(A) + R_n(B) - R_n(A \cup B)$$

➢ Independent of UV cutoff

$$I_2(A,B) \longrightarrow - \sum_{\text{Dir, Neu}} \underbrace{\Delta F(A \cup B) - \Delta F(A) - \Delta F(B)}_{\text{Thermal Casimir energy!}}$$

I_2 

$$= - \sum_{\text{Dir, Neu}} E_{\text{Th. Cas.}}$$

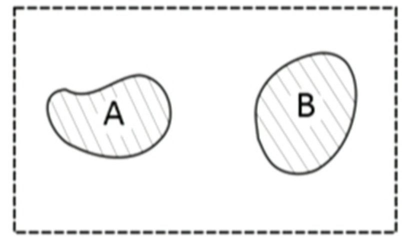


Mutual information

MM '15

Quantum correlations between domains

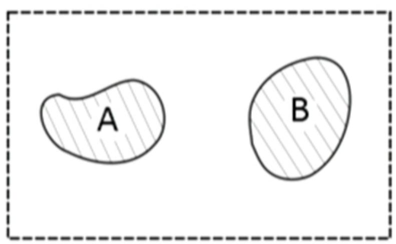
$$I_n(A,B) = R_n(A) + R_n(B) - R_n(A \cup B)$$



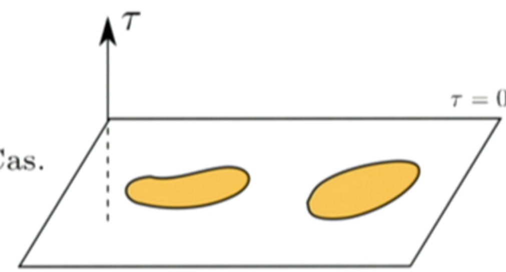
➢ Independent of UV cutoff

$$I_2(A,B) \longrightarrow -\sum_{\text{Dir, Neu}} \underbrace{\Delta F(A\cup B) - \Delta F(A) - \Delta F(B)}_{\text{Thermal Casimir energy!}}$$

I_2

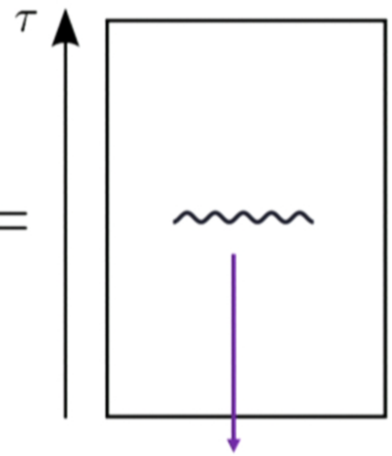


$$= -\sum_{\text{Dir, Neu}} E_{\text{Th. Cas.}}$$



Renyi entropy from partition function

$$e^{-R_2(A)} =$$



$$= e^{-\Delta F(A)}$$

d dim

Quantum

$k_B T = 0$

Boundary conditions

$$\phi_-(0^+) = -\phi_-(0^-)$$

d+1 dim

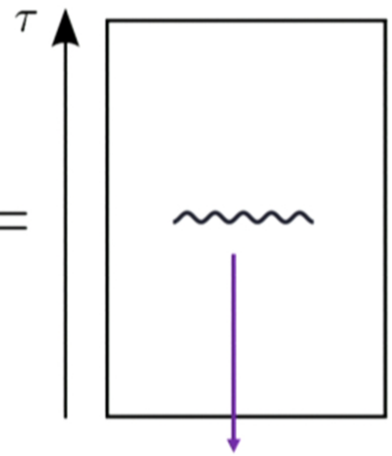
Classical

$k_B T = 1$

$$\mathcal{H} = \int d^{d+1}x \, (\nabla_E \phi)^2$$

Renyi entropy from partition function

$$e^{-R_2(A)} =$$



$$= e^{-\Delta F(A)}$$

d dim

Quantum

$k_B T = 0$

Boundary conditions

$$\phi_-(0^+) = -\phi_-(0^-)$$

d+1 dim

Classical

$k_B T = 1$

$$\mathcal{H} = \int d^{d+1}x \; (\nabla_E \, \phi)^2$$

Mutual information from Casimir energy

What can we learn from it?

❑ **Computational** ❑ Conceptual ❑ Pictorial

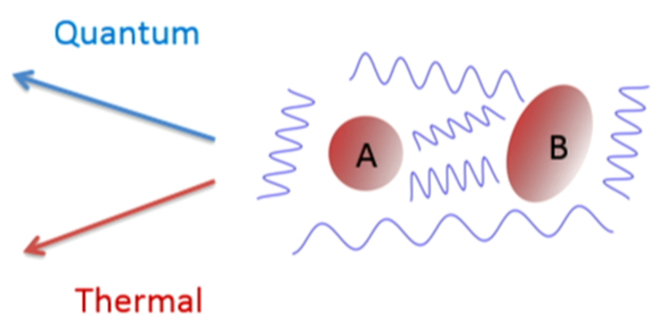
Casimir energy from electro-dynamics(statics):

Casimir energy

from scattering theory

Thermal Casimir energy

from capacitance matrices



$$E_{\mathrm{Th.\,Cas.}} \sim k_B T \;\mathrm{Tr}\log[1 - \mathbf{C}_A \mathbf{G}\, \mathbf{C}_B \mathbf{G}]$$

Mutual information from Casimir energy

What can we learn from it?

❑ **Computational** ❑ Conceptual ❑ Pictorial

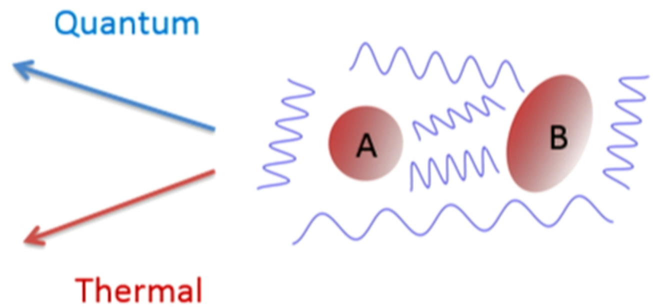
Casimir energy from electro-dynamics(statics):

Casimir energy

from scattering theory

Thermal Casimir energy

from capacitance matrices



$$E_{\mathrm{Th.\,Cas.}} \sim k_B T \;\mathrm{Tr}\log[1 - \mathbf{C}_A \mathbf{G}\, \mathbf{C}_B \mathbf{G}]$$

Mutual information from Casimir energy

What can we learn from it?

❑ **Computational** ❑ Conceptual ❑ Pictorial

MM '15

Mutual information from **capacitances** in one higher dimension!

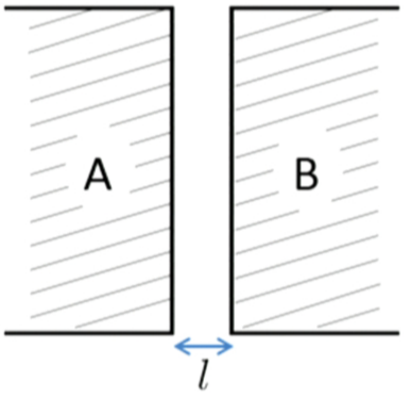
Mutual information from Casimir energy

What can we learn from it?

❑ Computational ❑ **Conceptual** ❑ Pictorial

Thermal Casimir energy → **Force !**

$$f = \frac{d}{dl} I_2(A, B)$$



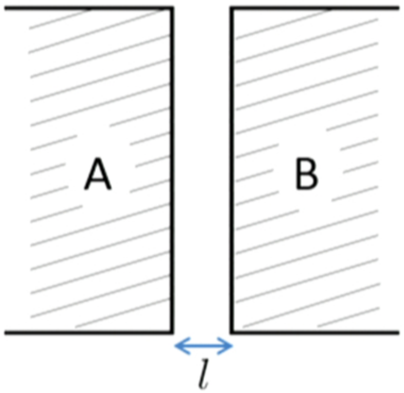
Mutual information from Casimir energy

What can we learn from it?

❑ Computational　　❑ **Conceptual**　　❑ Pictorial

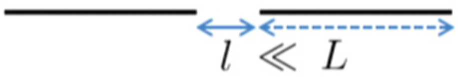
Thermal Casimir energy → **Force !**

$$f = \frac{d}{dl} I_2(A, B)$$



➢ On dimensional grounds $f \sim k_B T \frac{\text{area}}{l^d}$

$$I_2 \sim \begin{cases} \text{area}/l^{d-1}, & d > 1 \\ \log(L/l), & d = 1 \end{cases}$$



Mutual information from Casimir energy

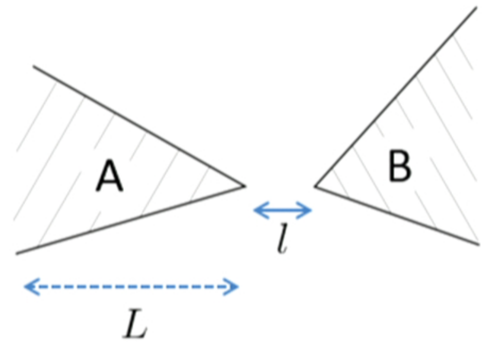
What can we learn from it?

MM '15

❑ Computational **❑ Conceptual** ❑ Pictorial

➢ On dimensional grounds $f \sim k_B T \frac{1}{l}$

$$I_2 \sim \log(L/l)$$



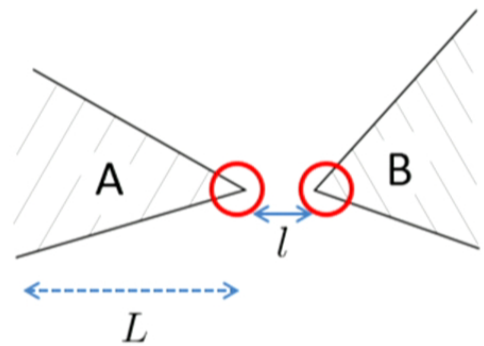
Mutual information from Casimir energy

What can we learn from it?

MM '15

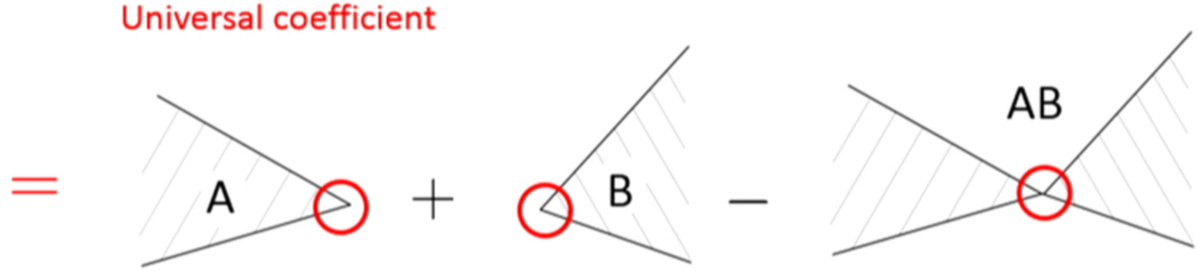
❑ Computational ❑ **Conceptual** ❑ Pictorial

➢ On dimensional grounds $f \sim k_B T \frac{1}{l}$



$I_2 \sim \log(L/l)$

Universal coefficient



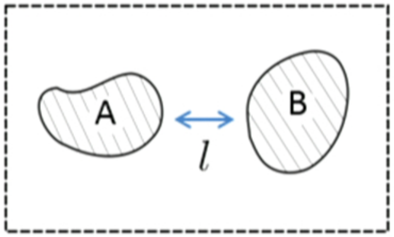
Mutual information from Casimir energy

What can we learn from it?

MM '15

❑ Computational   ❑ **Conceptual**   ❑ Pictorial

Monotonicity of information?



Mutual information from Th. entropy

Every physicist who is any good knows six or seven different representations for the same physics.

Richard P. Feynman

What can we learn from it?

- ❑ Computational
- ❑ Conceptual
- ❑ **Pictorial**

Mutual information → Thermodynamic free energy

But why thermodynamic *entropy*?!

Mutual information from Th. entropy

Every physicist who is any good knows six or seven different representations for the same physics.

Richard P. Feynman

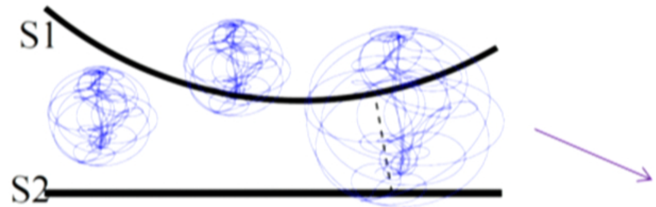
What can we learn from it?

- ❑ Computational
- ❑ Conceptual
- ❑ **Pictorial**

Mutual information → Thermodynamic free energy

But why thermodynamic *entropy*?!

Vacuum energy: A sum over closed loops or polymers!



Contributes to Casimir energy!

Mutual information from Th. entropy

What can we learn from it?

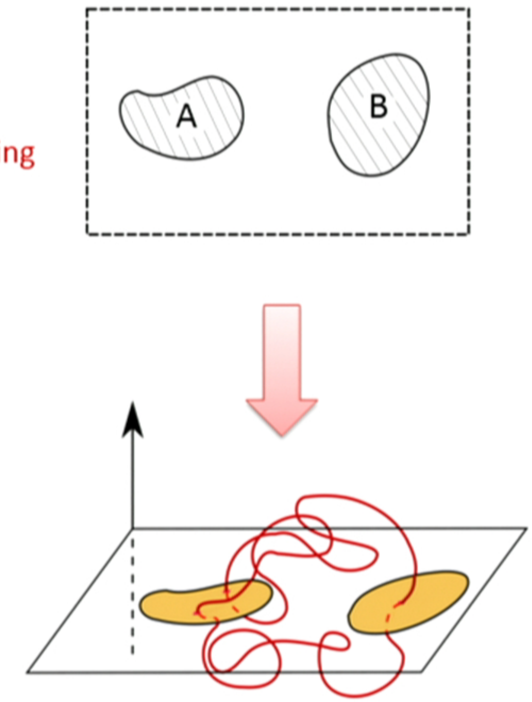
MM '15
MM, H. Reid '15

❑ Computational ❑ Conceptual ❑ **Pictorial**

➢ Polymers:

- No energetics
- Sum over configurations → Entropic

Counting



$$\Delta F = -k_B T \, \Delta S_{\rm Th}$$

Mutual information from Th. entropy

What can we learn from it?

MM '15
MM, H. Reid '15

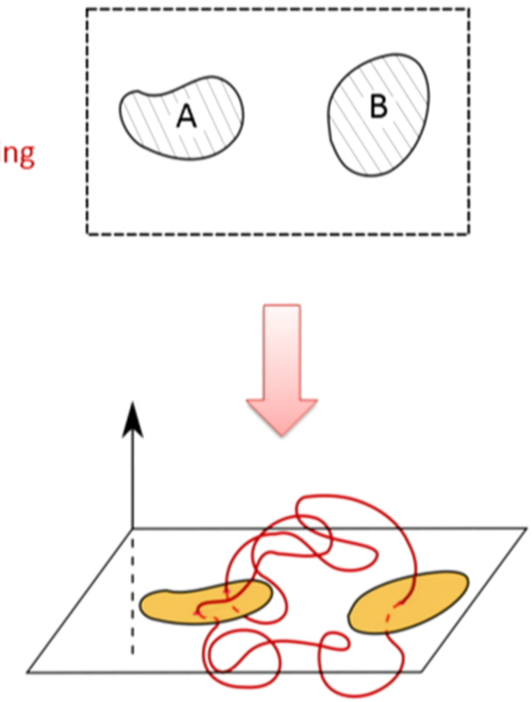
❑ Computational ❑ Conceptual ❑ **Pictorial**

➢ Polymers:

- No energetics
- Sum over configurations → Entropic

Counting

$$\Delta F = -k_B T \, \Delta S_{\rm Th}$$



Strong subadditivity property from polymers

Proof of quantum version is difficult!

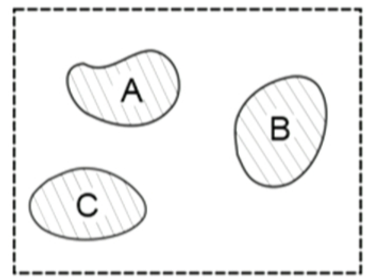
Lieb Ruskai '73

Tripartite information

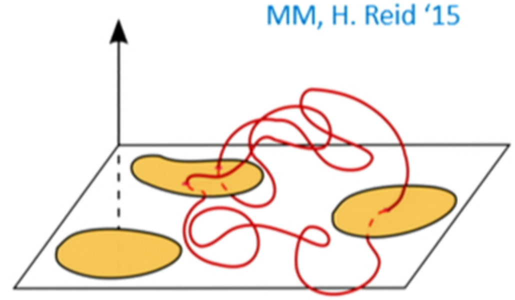
Mutual information

$$I(A,B,C) \leq I(A,B)$$

Polymers intersecting **all** A, B and C $\leq$ Polymers intersecting **both** A and B



MM '15
MM, H. Reid '15



✓ Trivial in terms of polymers !!

Strong subadditivity property from polymers

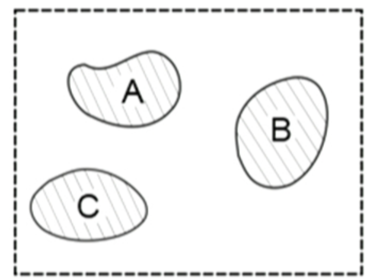
Proof of quantum version is difficult!

Lieb Ruskai '73

Tripartite information

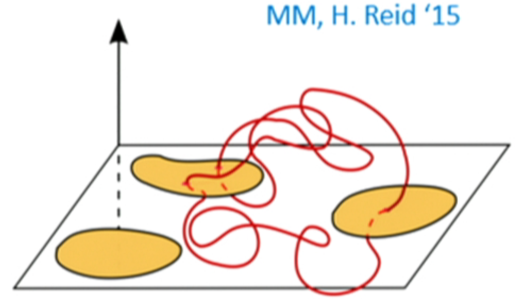
Mutual information

$$I(A,B,C) \leq I(A,B)$$



Polymers intersecting **all** A, B and C $\leq$ Polymers intersecting **both** A and B

MM '15
MM, H. Reid '15



✓ Trivial in terms of polymers !!

Mutual information from Th. entropy

Every physicist who is any good knows six or seven different representations for the same physics.

Richard P. Feynman

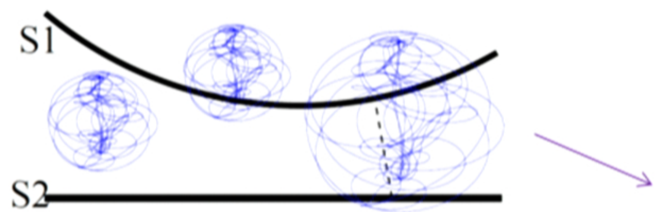
What can we learn from it?

❑ Computational ❑ Conceptual ❑ **Pictorial**

Mutual information → Thermodynamic free energy

But why thermodynamic *entropy*?!

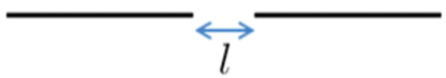
Vacuum energy: A sum over closed loops or polymers!



Contributes to Casimir energy!

Beyond Gaussian theory

Conformal field theory in 1+1D

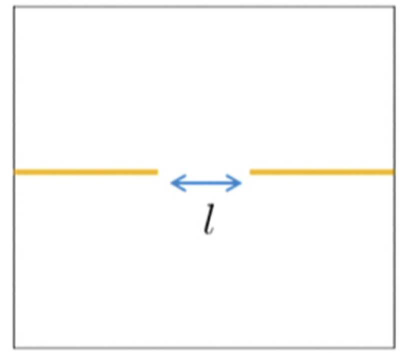


1+1 dim **Quantum**

$$k_B T = 0$$

$$\frac{d}{dl} I_2 = -\frac{c}{4}\frac{1}{l}$$

Calabrese, Cardy '04



2 dim **Classical**

$$k_B T = 1$$

For **each** conformally invariant boundary condition

$$f = -\frac{c}{8}\frac{1}{l}$$

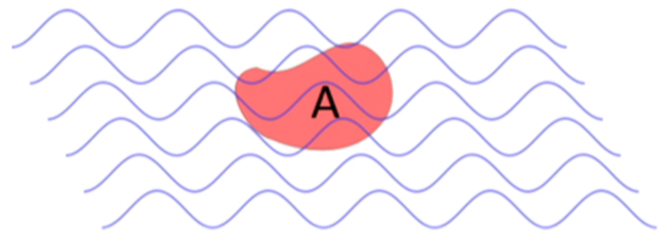
Bimonte, Emig, Kardar '15

Field coupled to matter

MM, H. Reid '15

Is the second Renyi entropy too special?

Action $[\phi, \psi]$



Free scalar field | Matter degrees of freedom | Interaction

$$\int (\partial\phi)^2 \quad + \quad \begin{matrix}\text{Harmonic oscillators} \\ \text{in } \psi\end{matrix} \quad + \quad \int_A \phi\psi$$

- Entanglement between field and matter?

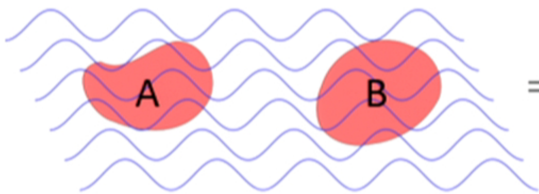
Mutual information

- Mutual information between two material bodies?

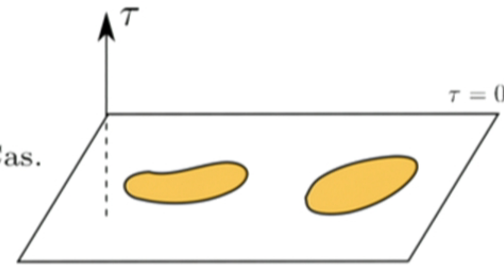
MM, H. Reid '15

From von Neumann entropy

I



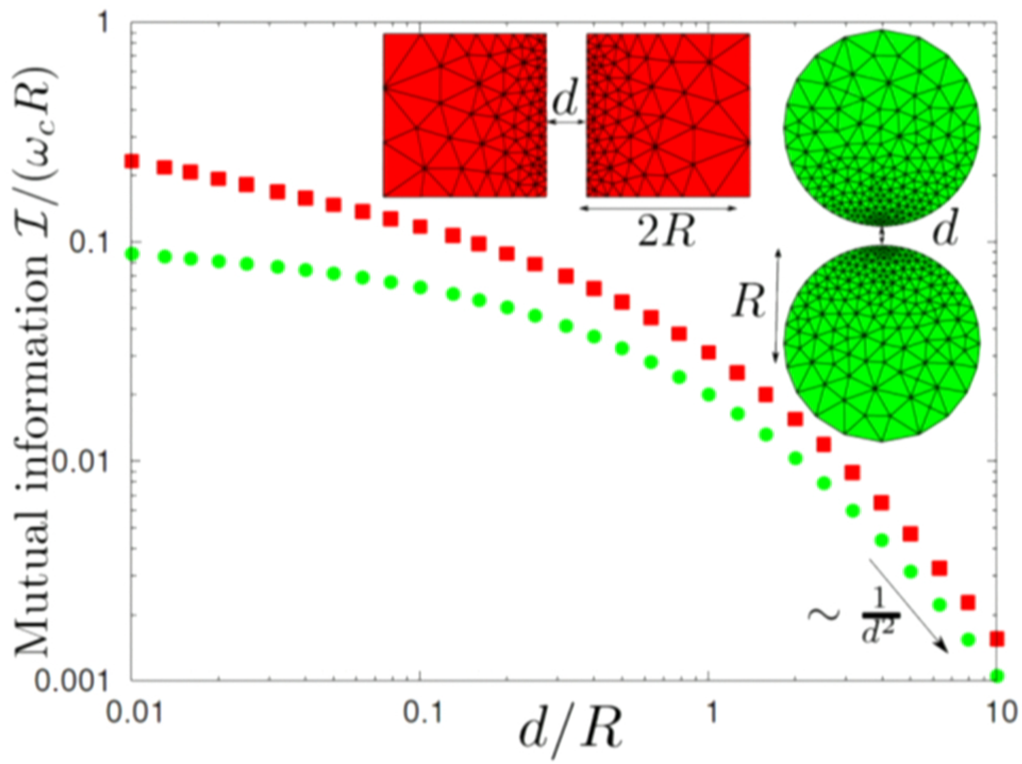
$$= -\sum_{\text{Robin}} E_{\text{Th. Cas.}}$$



A continuum of boundary conditions!

Mutual information

MM, H. Reid '15



Summary

- A quantum to classical mapping for entanglement entropy and information

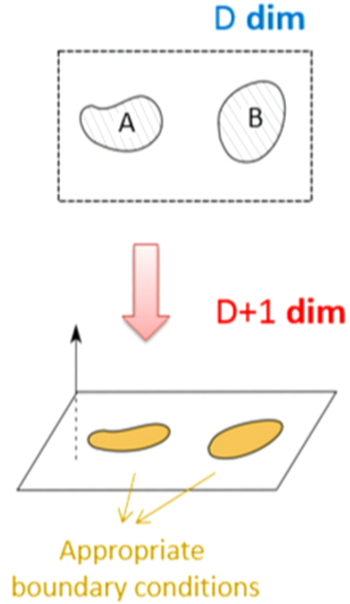
- Properties of entropy and information from classical thermodynamics

- A picture in terms of **polymers**

→ Strong subadditivity property

- Easily extended to away from criticality, external potential, finite temperature, fermions, ...

- Interacting field theories?



M.F. Maghrebi, H. Reid PRL **114**, 151602 (2015)
M.F. Maghrebi, arXiv:1510.00018

Summary

- A quantum to classical mapping for entanglement entropy and information

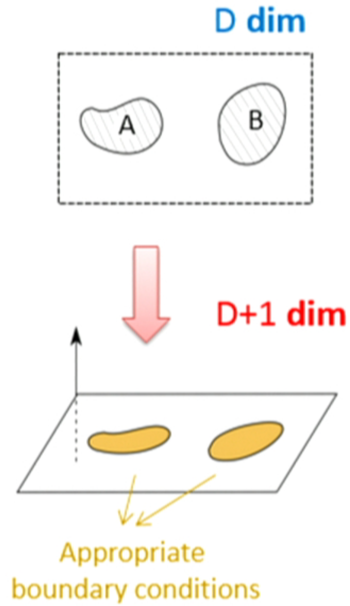
- Properties of entropy and information from classical thermodynamics

- A picture in terms of **polymers**

→ Strong subadditivity property

- Easily extended to away from criticality, external potential, finite temperature, fermions, ...

- Interacting field theories?



M.F. Maghrebi, H. Reid PRL **114**, 151602 (2015)
M.F. Maghrebi, arXiv:1510.00018