Title: BMS Symmetry in Three Dimensions

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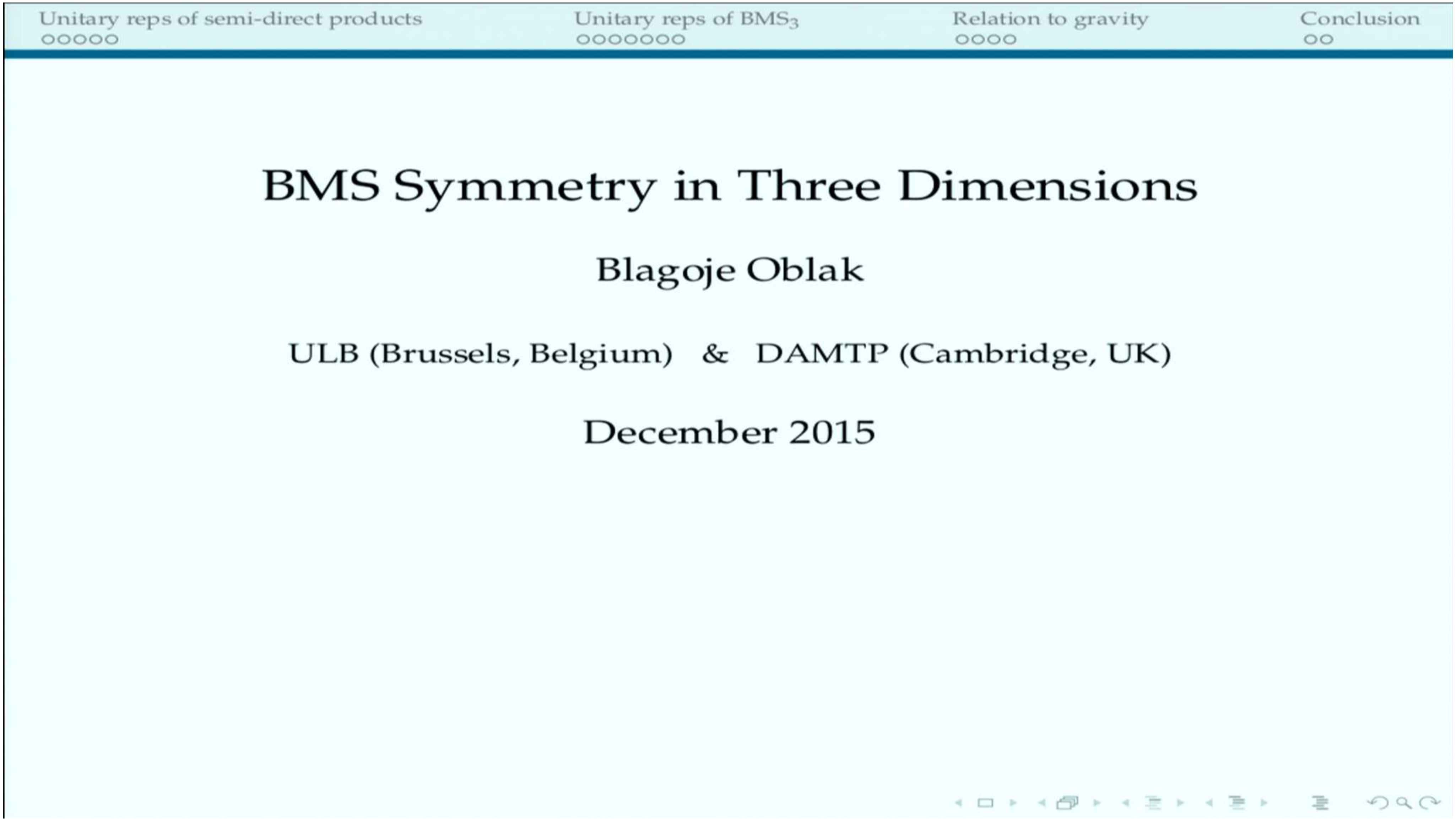
Abstract: <p>In this talk I will sketch the relation between unitary representations of the BMS3 group and three-dimensional, asymptotically flat gravity. More precisely, after giving an exact definition of the BMS group in three dimensions, I will argue that its unitary representations are classified by orbits of CFT stress tensors under conformal transformations. These stress tensors, in turn, can be interpreted as Bondi mass aspects for asymptotically flat metrics. I will also show how one can compute characters of the BMS3 group, which coincide with suitable gravitational one-loop partition functions</p>



BMS Symmetry in Three Dimensions

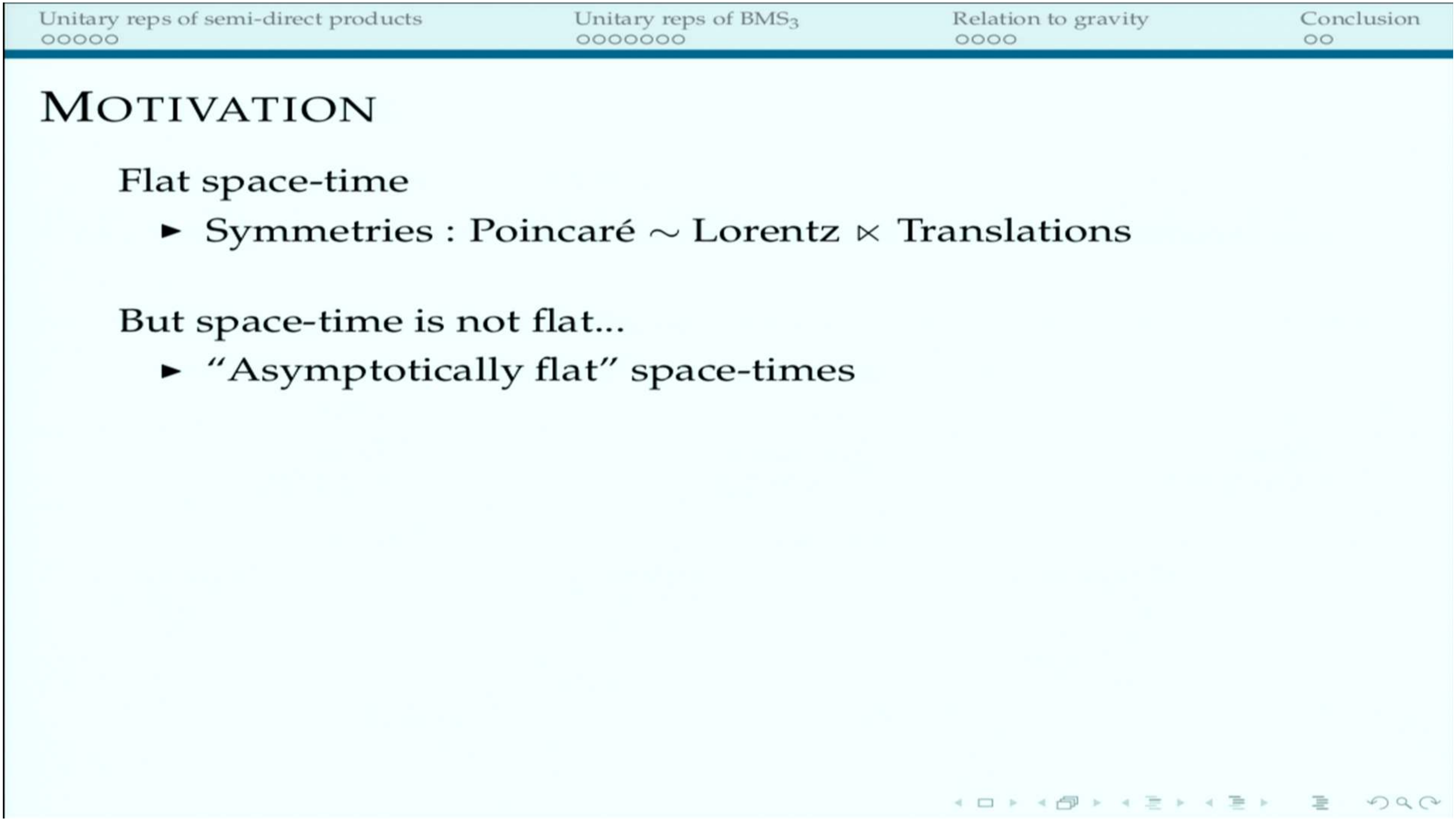
Blagoje Oblak

ULB (Brussels, Belgium) & DAMTP (Cambridge, UK)

December 2015

MOTIVATION

Flat space-time



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- Symmetries : Poincaré $\sim$ Lorentz $\ltimes$ Translations

But space-time is not flat...

- "Asymptotically flat" space-times

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But space-time is not flat...

- "Asymptotically flat" space-times
- Symmetries : BMS $\sim$ Superrotations $\ltimes$ Supertranslations

[Bondi *et al.* 1962, Barnich *et al.* 2009]

PLAN OF THE TALK

1. **Unitary reps of semi-direct products**
2. Representations of BMS_3
3. Relation to gravity

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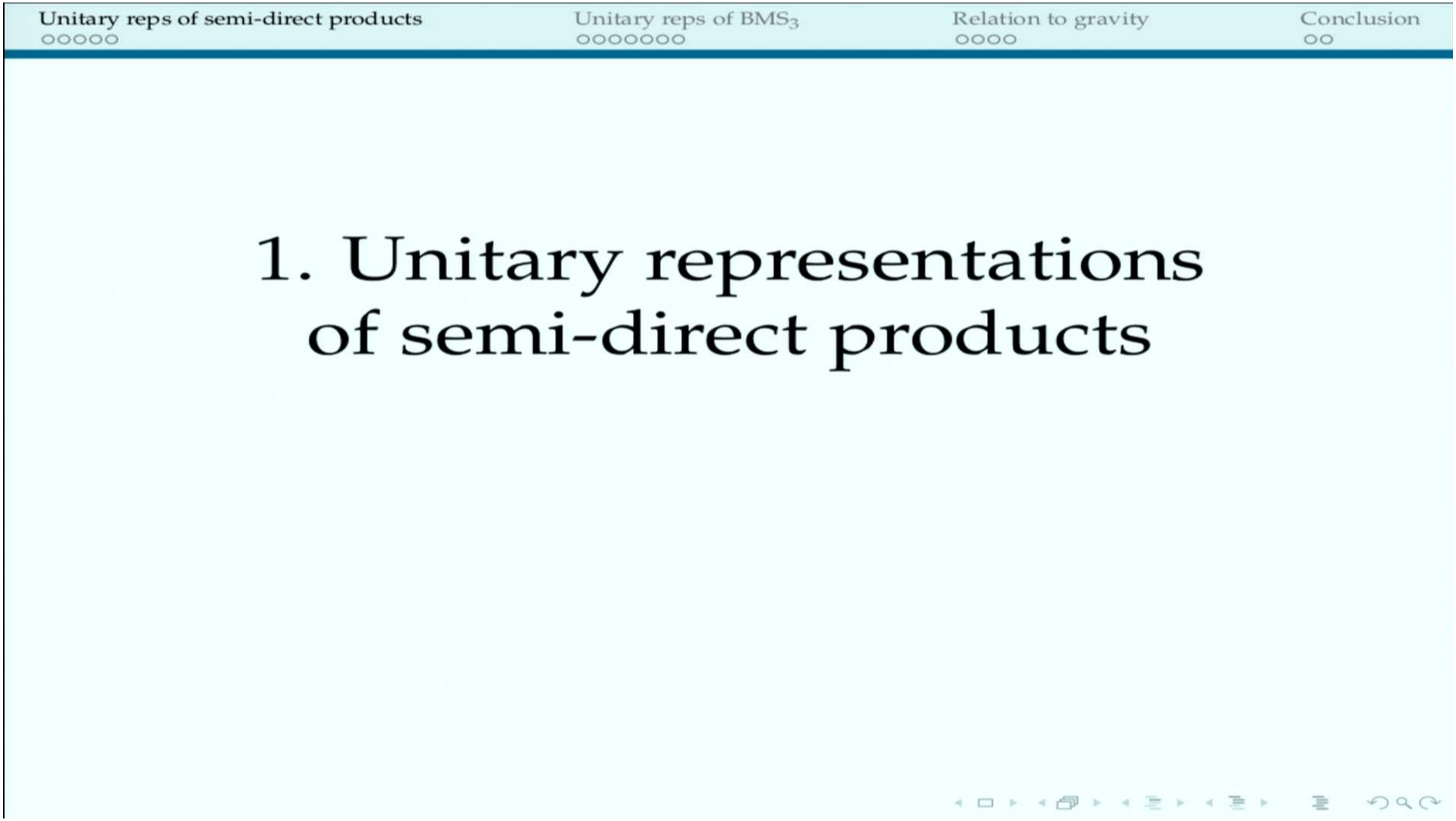

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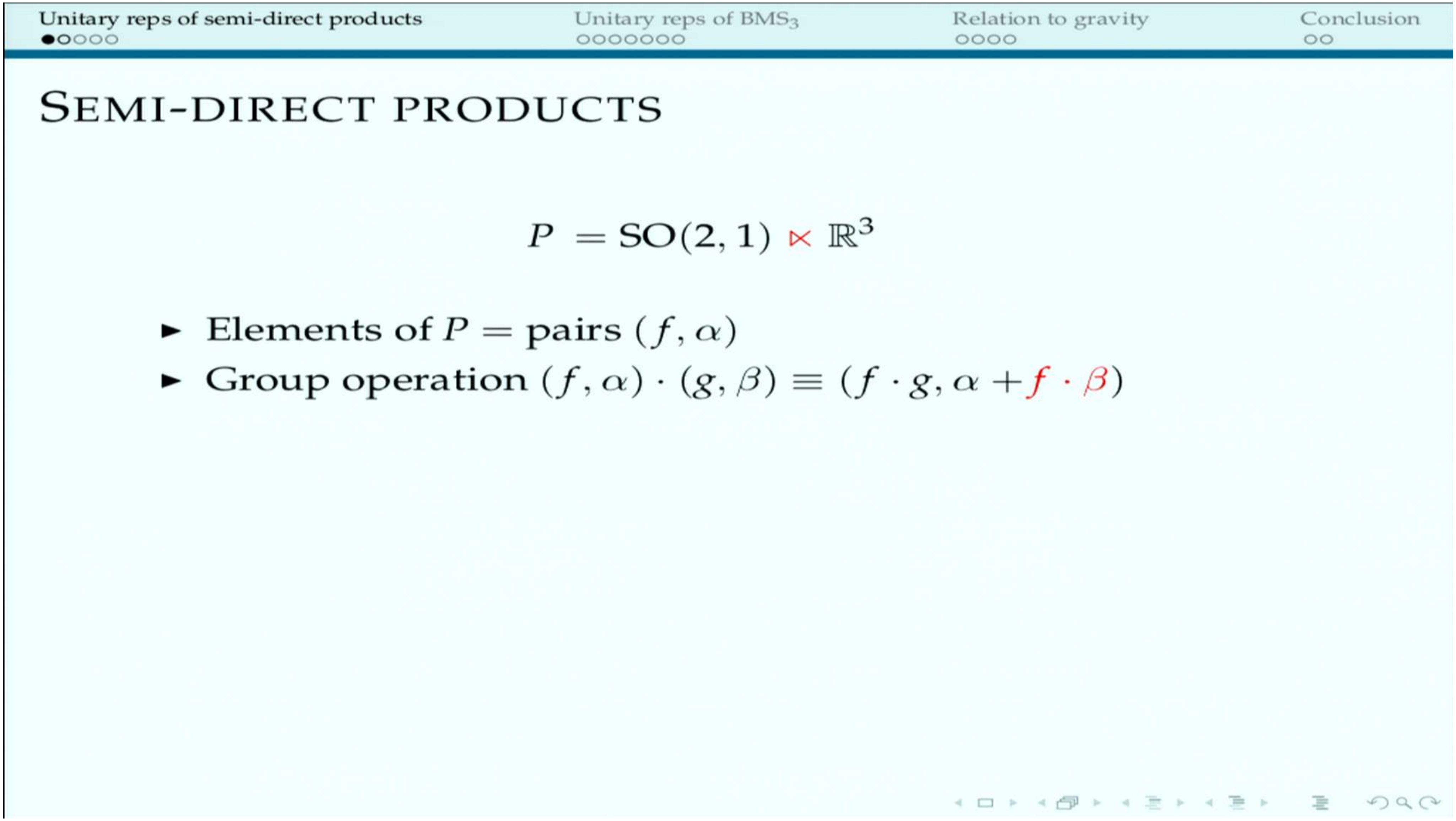
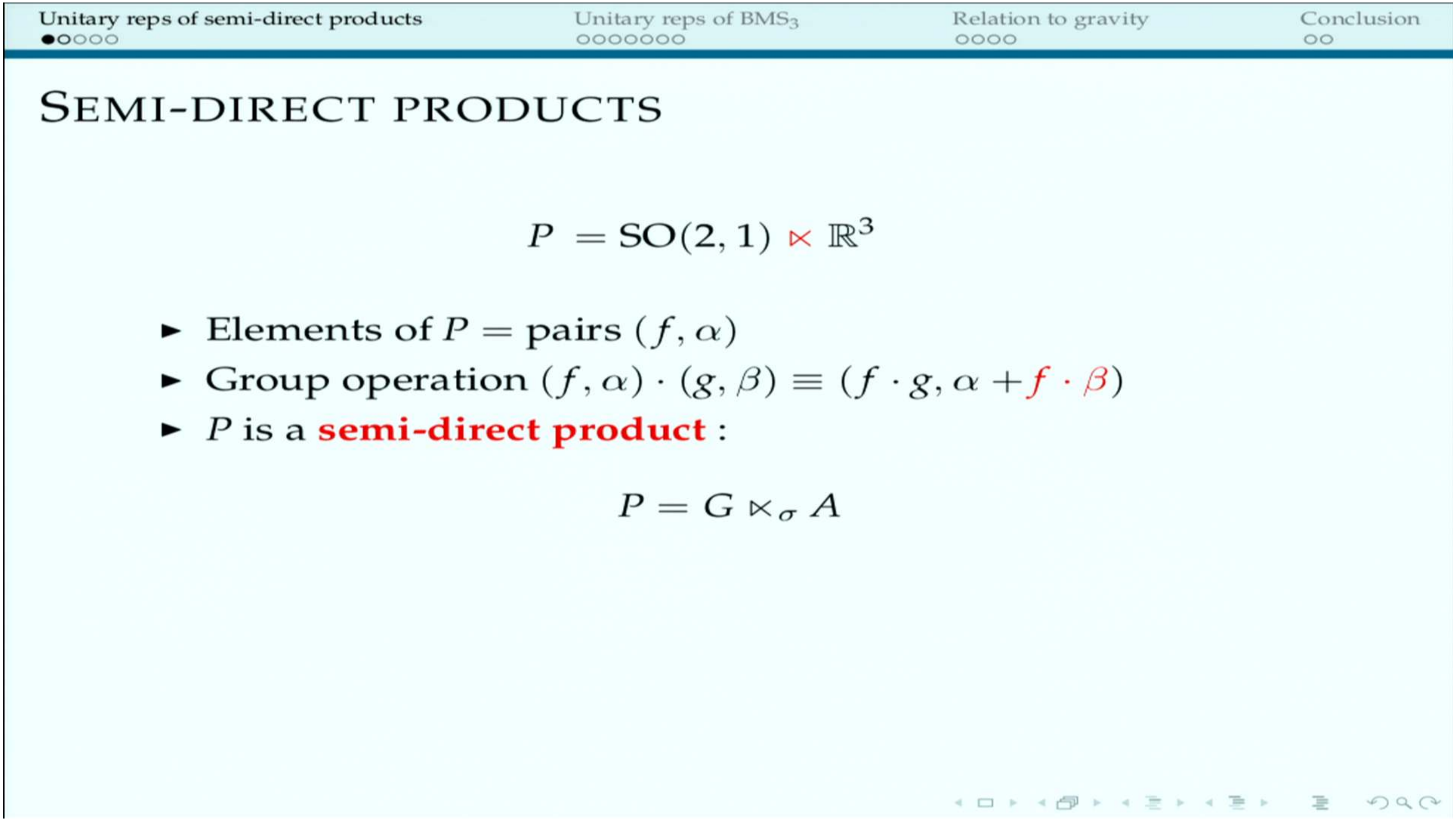


SEMI-DIRECT PRODUCTS

$$P = SO(2,1) \ltimes \mathbb{R}^3$$

- Elements of $P =$ pairs (f, α)
- Group operation $(f, \alpha) \cdot (g, \beta) \equiv (f \cdot g, \alpha + f \cdot \beta)$



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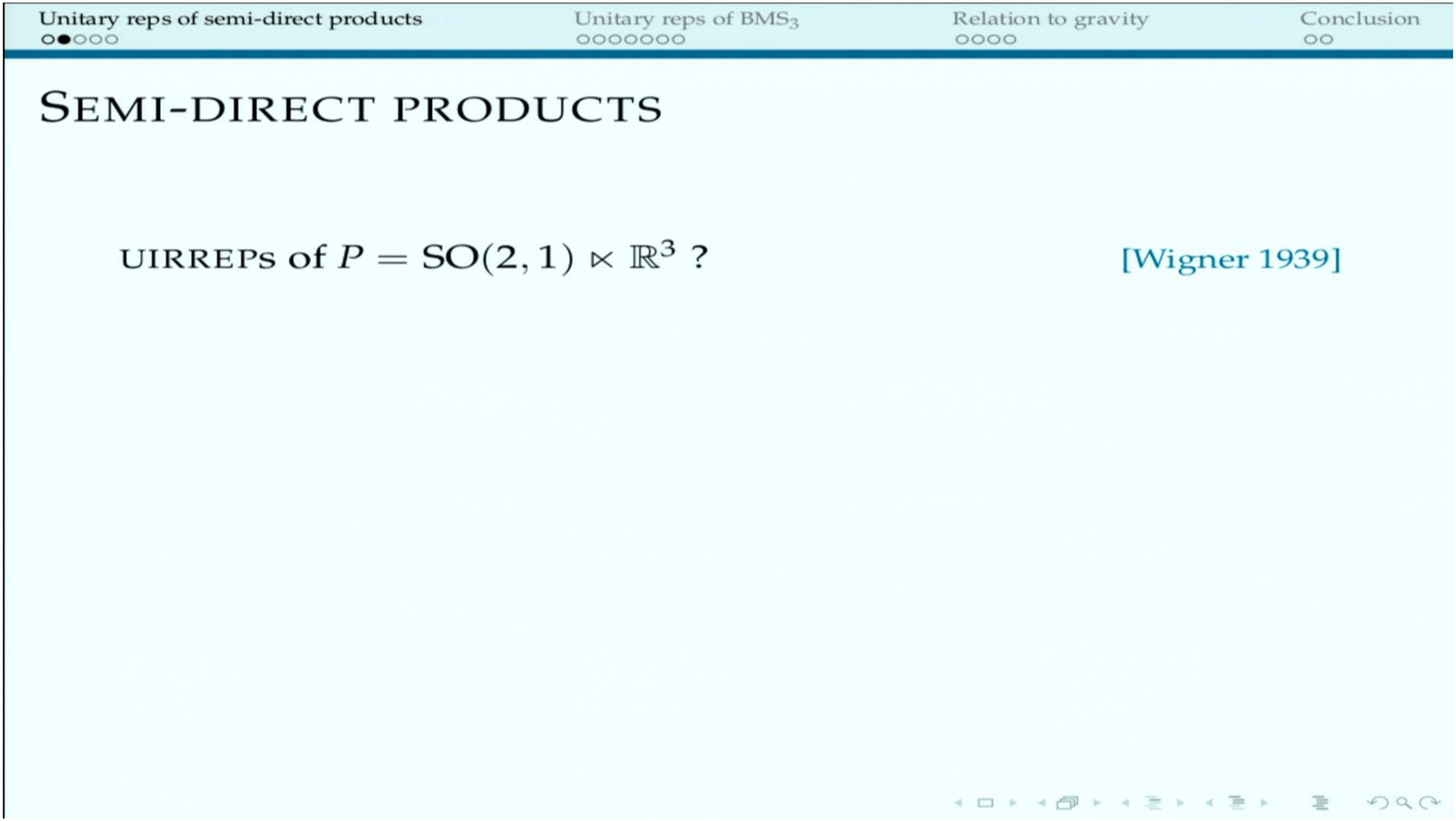

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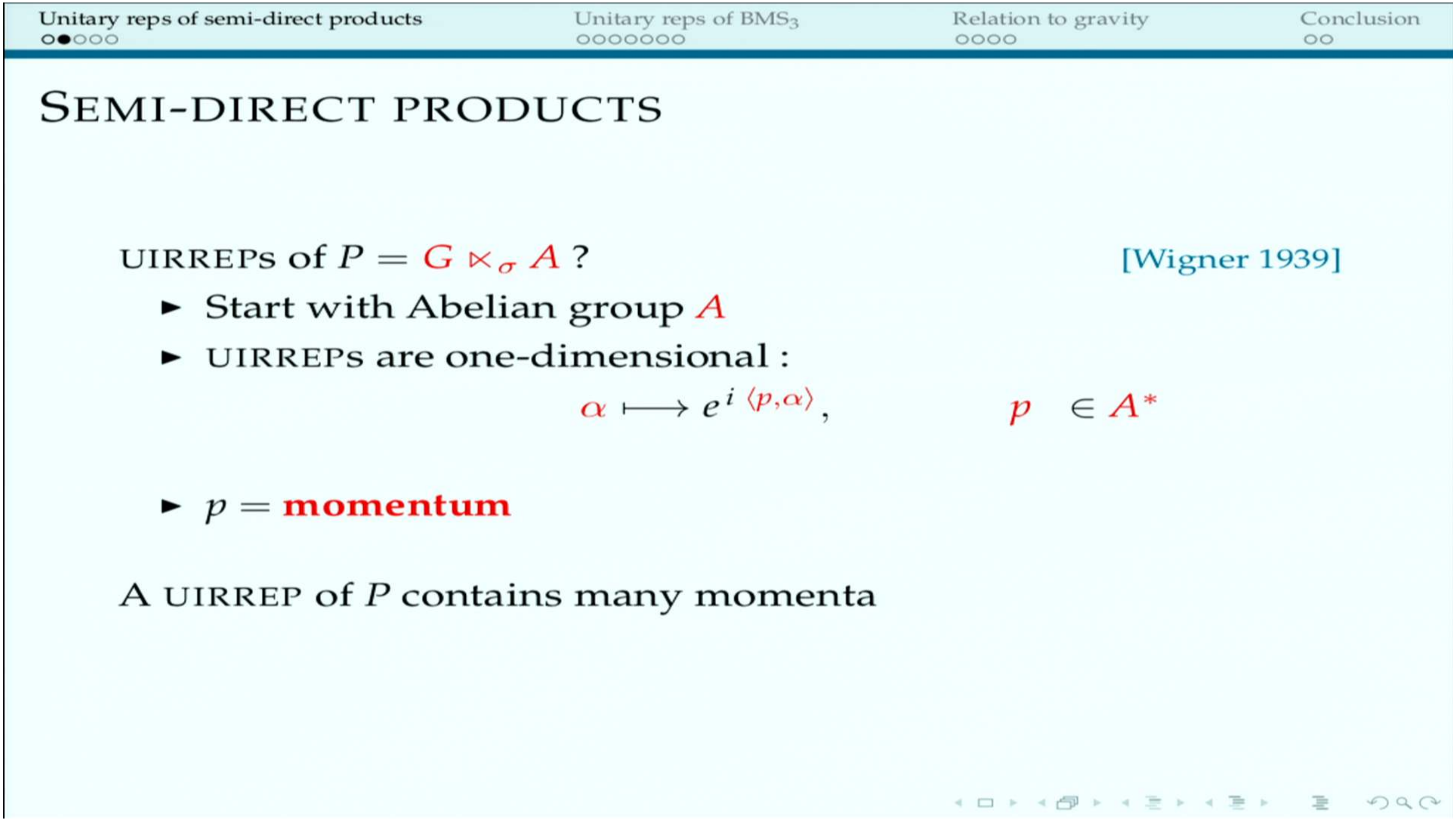

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A UIRREP of P contains many momenta

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Define $\langle f \cdot p, \alpha\rangle \equiv \langle p,\, \sigma_{f^{-1}}\alpha\rangle$

- **Orbit** of p : $\mathcal{O}_p \equiv \{f \cdot p \,|\, f \in G\}$

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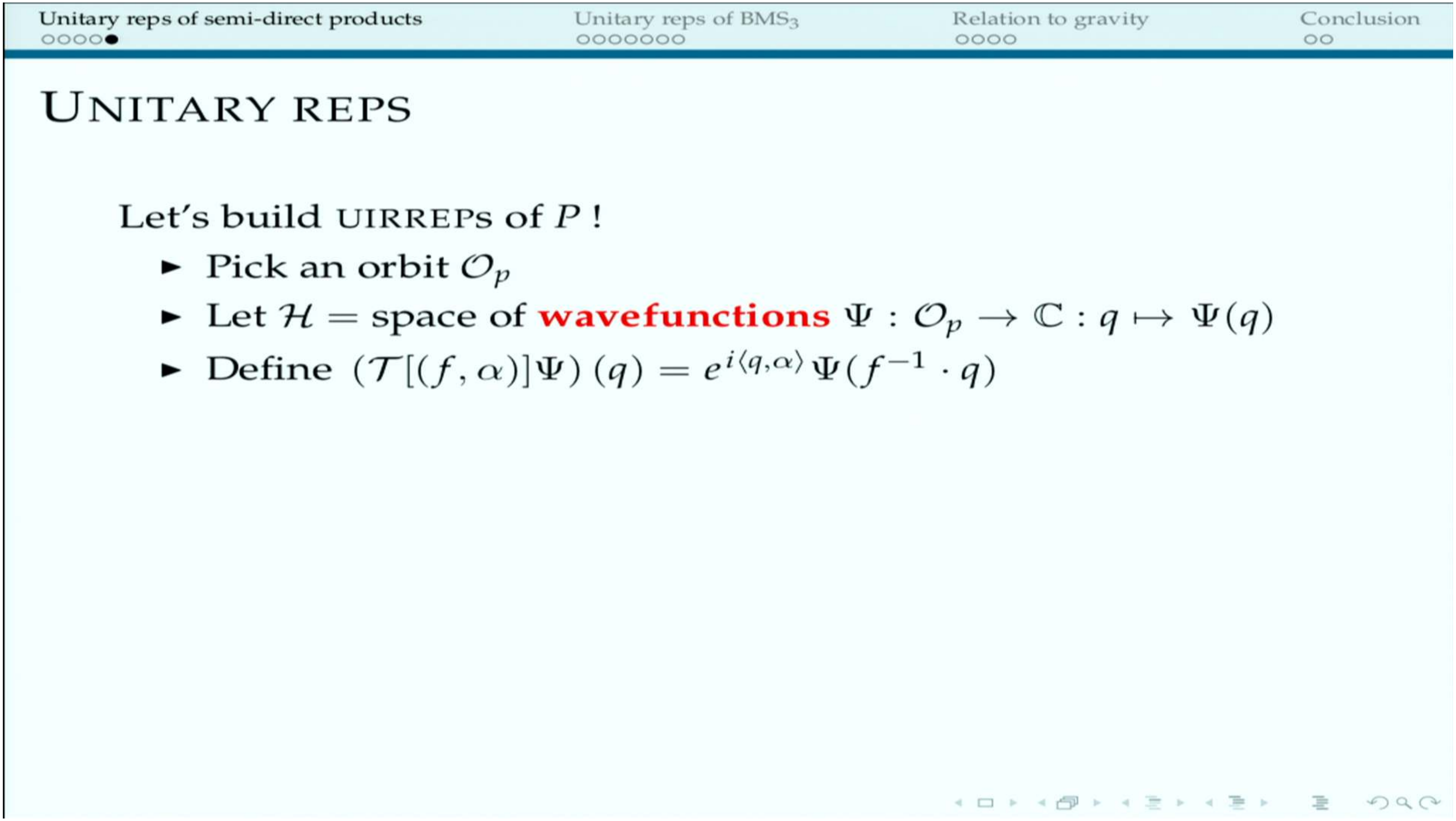

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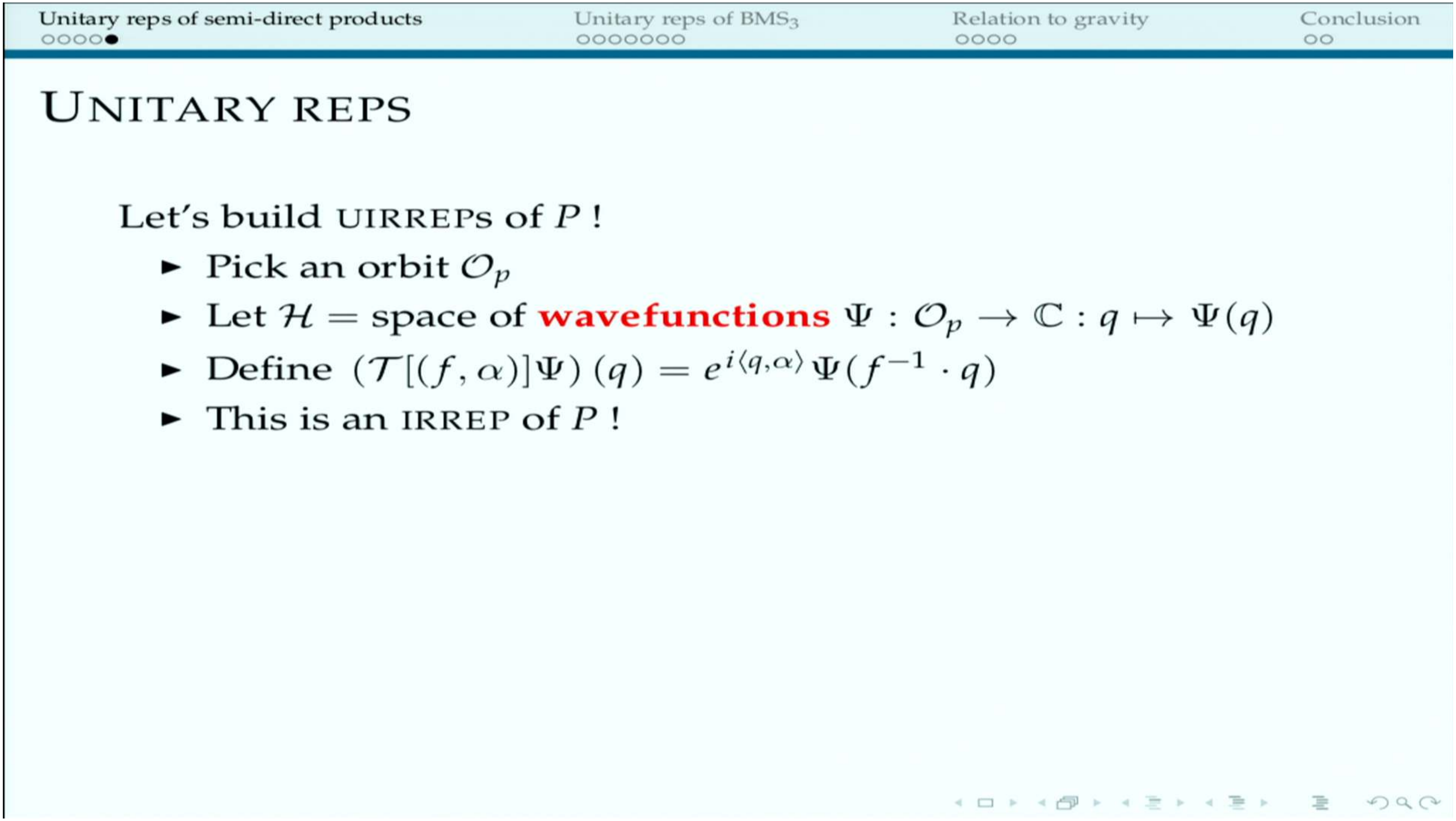

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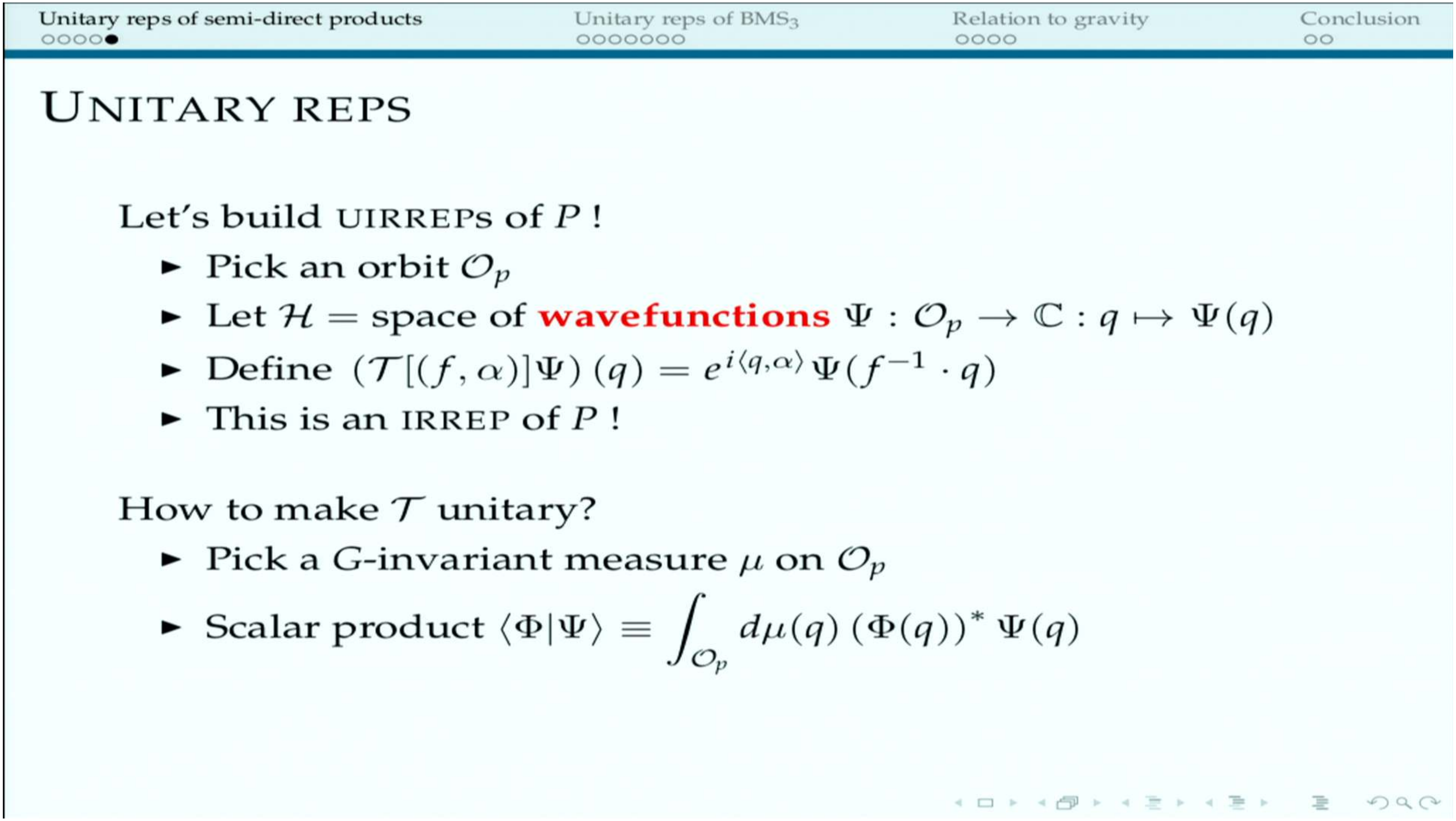

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DEFINITION OF BMS_3

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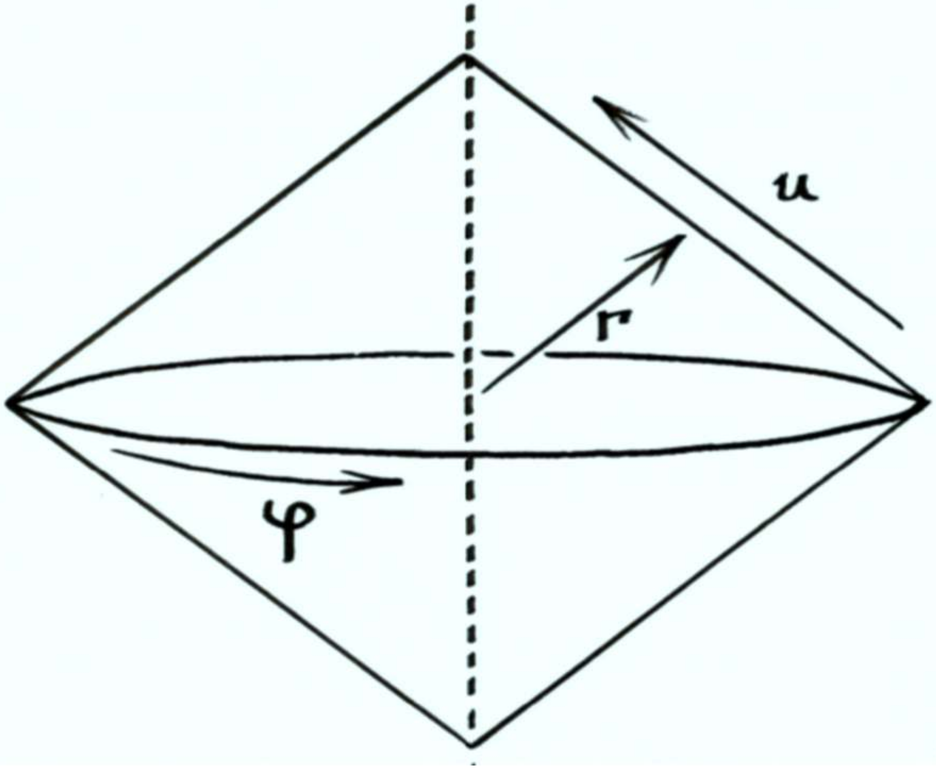

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- Aspt. flat metrics in 3D :

$$ds^2 \overset{r\to+\infty}{\sim} -du^2 - 2dudr + r^2 d\varphi^2 \; + \; \text{subleading terms}$$

- Aspt. symmetry transformations :

$$\varphi \mapsto f(\varphi), \quad u \mapsto [u+\alpha(\varphi)]\, f'(\varphi)$$

$$\swarrow \qquad\qquad\qquad\qquad \downarrow$$

Superrotations **Supertranslations**

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- Elements of BMS$_3$ = pairs (f, α)
- Group operation :
  $$(f,\alpha)\cdot(g,\beta) = (f\circ g, \alpha + \sigma_f\beta), \quad \sigma_f\beta\big|_{f(\varphi)} = f'(\varphi)\beta(\varphi)$$
- σ = tsf. law of vector fields under diffeos !
- $\alpha = \alpha(\varphi)\frac{\partial}{\partial\varphi}$

$$\boxed{\mathrm{BMS}_3 \equiv \mathrm{Diff}(S^1) \ltimes \mathrm{Vect}(S^1)}$$

[Barnich & BO 2014]

Unitary reps of BMS$_3$...

- What should we expect ?

Poincaré : exact space-time symmetry

- UIRREP $= \mathcal{H}_{\text{Poinc}} =$ Particle

BMS$_3$: aspt. space-time symmetry

- UIRREP $= \mathcal{H}_{\text{BMS}} =$ Particle dressed w/ soft gravitons $\equiv$ **BMS$_3$ particle**

- $\mathcal{H}_{\text{BMS}} = \mathcal{H}_{\text{Poinc}} \otimes \mathcal{H}_{\text{Soft grav}}$

ORBITS AND UNITARY REPS

$$\text{Unitary reps of BMS}_3 = \underbrace{\text{Diff}(S^1) \ltimes \text{Vect}(S^1)}_{G \ltimes A}\ ?$$

- $\text{Vect}(S^1)^* =$ space of **supermomenta** $p(\varphi)$:

$$\underbrace{\langle p, \alpha \rangle}_{\in \mathbb{R}} = \frac{1}{2\pi} \int d\varphi\, p(\varphi) \alpha(\varphi)$$

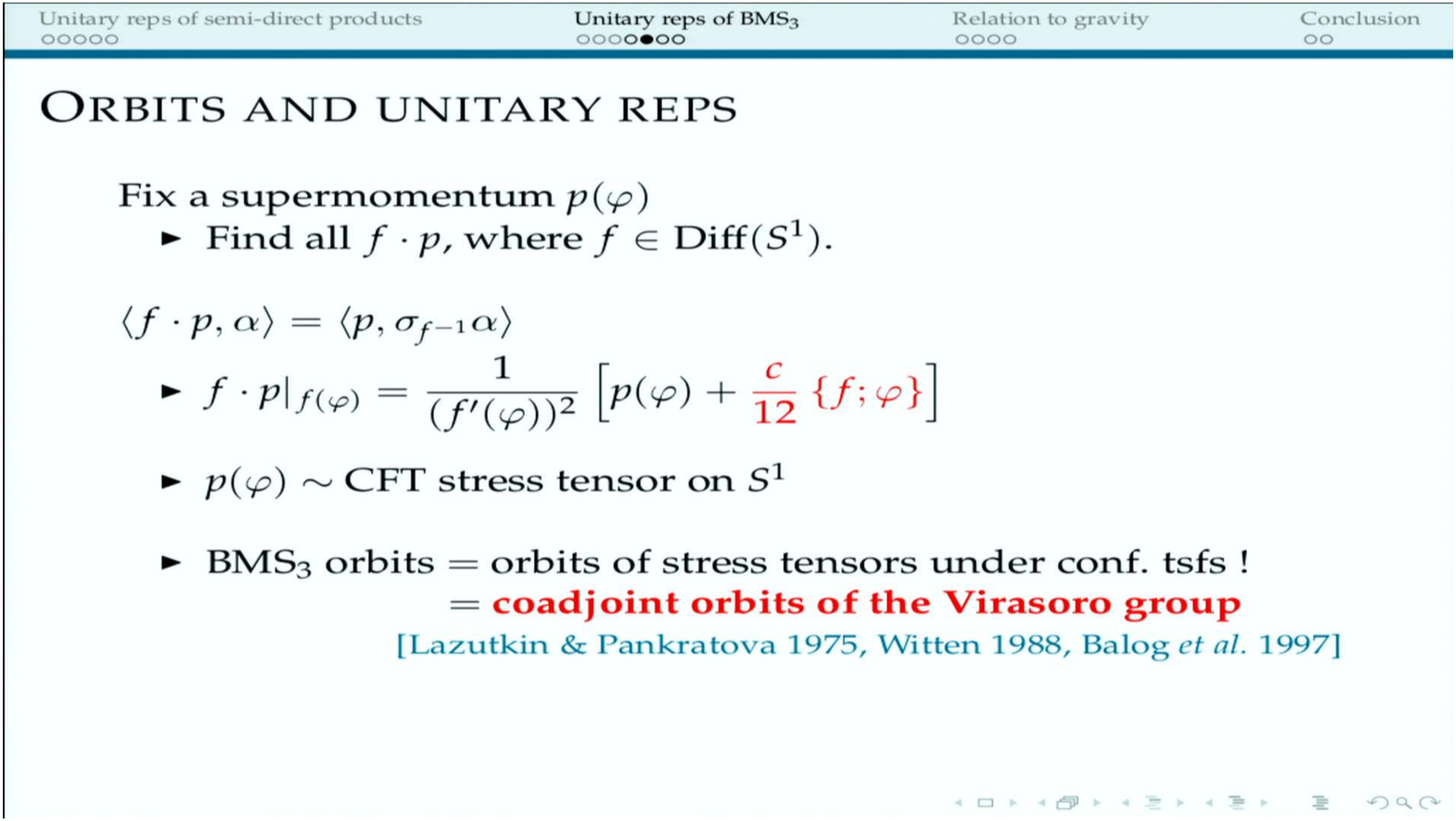
$$p(\varphi) = \sum_{n \in \mathbb{Z}} p_n\, e^{-in\varphi}$$

- $p_0 =$ **energy**

ORBITS AND UNITARY REPS

Fix a supermomentum $p(\varphi)$
- Find all $f \cdot p$, where $f \in \mathrm{Diff}(S^1)$.

$\langle f \cdot p, \alpha \rangle = \langle p, \sigma_{f^{-1}} \alpha \rangle$
- $f \cdot p|_{f(\varphi)} = \dfrac{1}{(f'(\varphi))^2} \left[p(\varphi) + \dfrac{c}{12} \{f; \varphi\} \right]$
- $p(\varphi) \sim$ CFT stress tensor on S^1



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- BMS$_3$ orbits $=$ orbits of stress tensors under conf. tsfs !
  $=$ **coadjoint orbits of the Virasoro group**
  [Lazutkin & Pankratova 1975, Witten 1988, Balog *et al.* 1997]

$$f \cdot p|_{f(\varphi)} = \frac{1}{(f'(\varphi))^2}\left[p_0 \quad + \frac{c}{12}\{f;\varphi\}\right]$$

- $\mathcal{O}_p = \{f \cdot p | f \in \mathrm{Diff}(S^1)\} \rightarrow$ complicated !
- Let's make it simple :
  1. Constant supermomentum $p(\varphi) = p_0$ ($\sim$ particle at rest)
  2. Look for stabilizer G_p

$$\frac{1}{(f'(\varphi))^2}\left[p_0 \quad + \frac{c}{12}\{f;\varphi\}\right] = p_0$$

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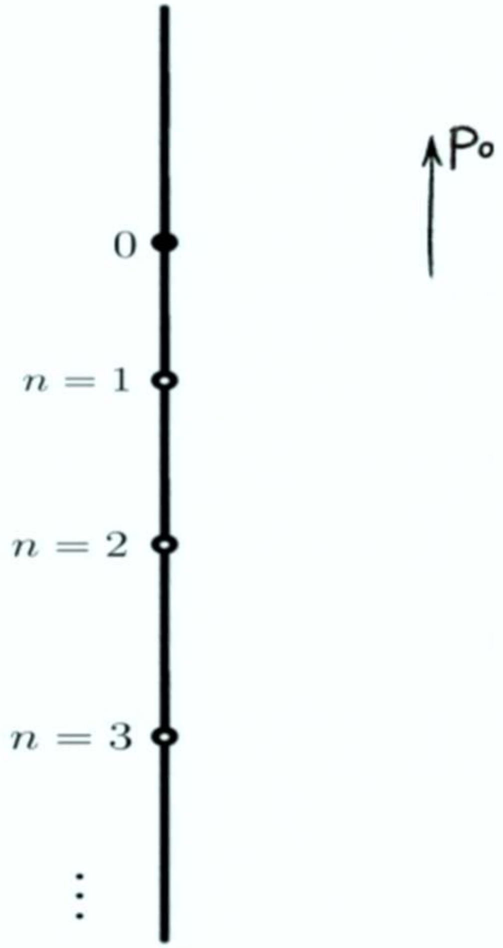
     - $\mathcal{O}_p \cong \text{Diff}(S^1)/G_p$

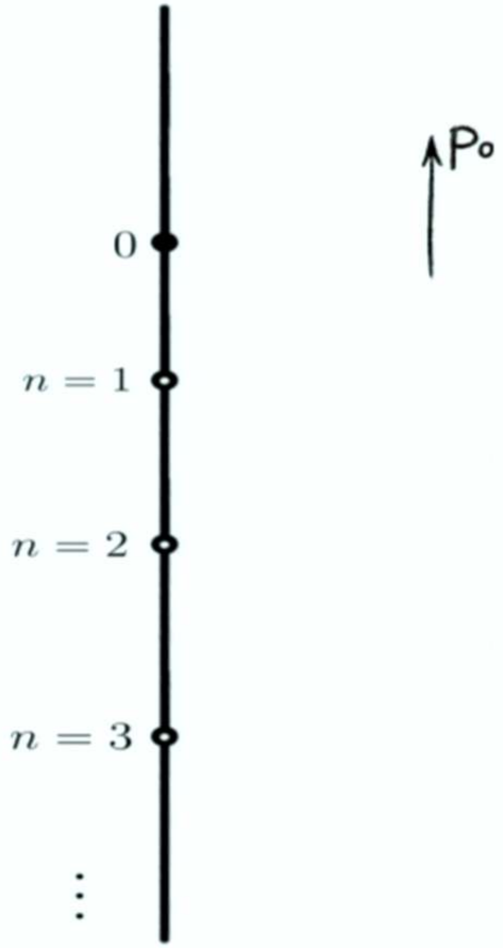
BMS_3 orbits $=$ coadjoint orbits of Virasoro group :



Generic p_0 : $\quad \mathcal{O}_p \cong \mathrm{Diff}(S^1)/S^1$

$p_0 = -n^2c/24 : \mathcal{O}_p \cong \mathrm{Diff}(S^1)/\mathrm{SO}^{(n)}(2,1)$

BMS$_3$ orbits = coadjoint orbits of Virasoro group :



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- What else ?
- Perturbations $-\dfrac{n^2c}{24} + \delta p(\varphi)$
- δp = 3-momentum under $\text{SO}^{(n)}(2,1)$
- "Poincaré orbits for each n" !

SUPERMOMENTUM & BONDI MASS

On-shell aspt. flat metrics

SUPERMOMENTUM & BONDI MASS

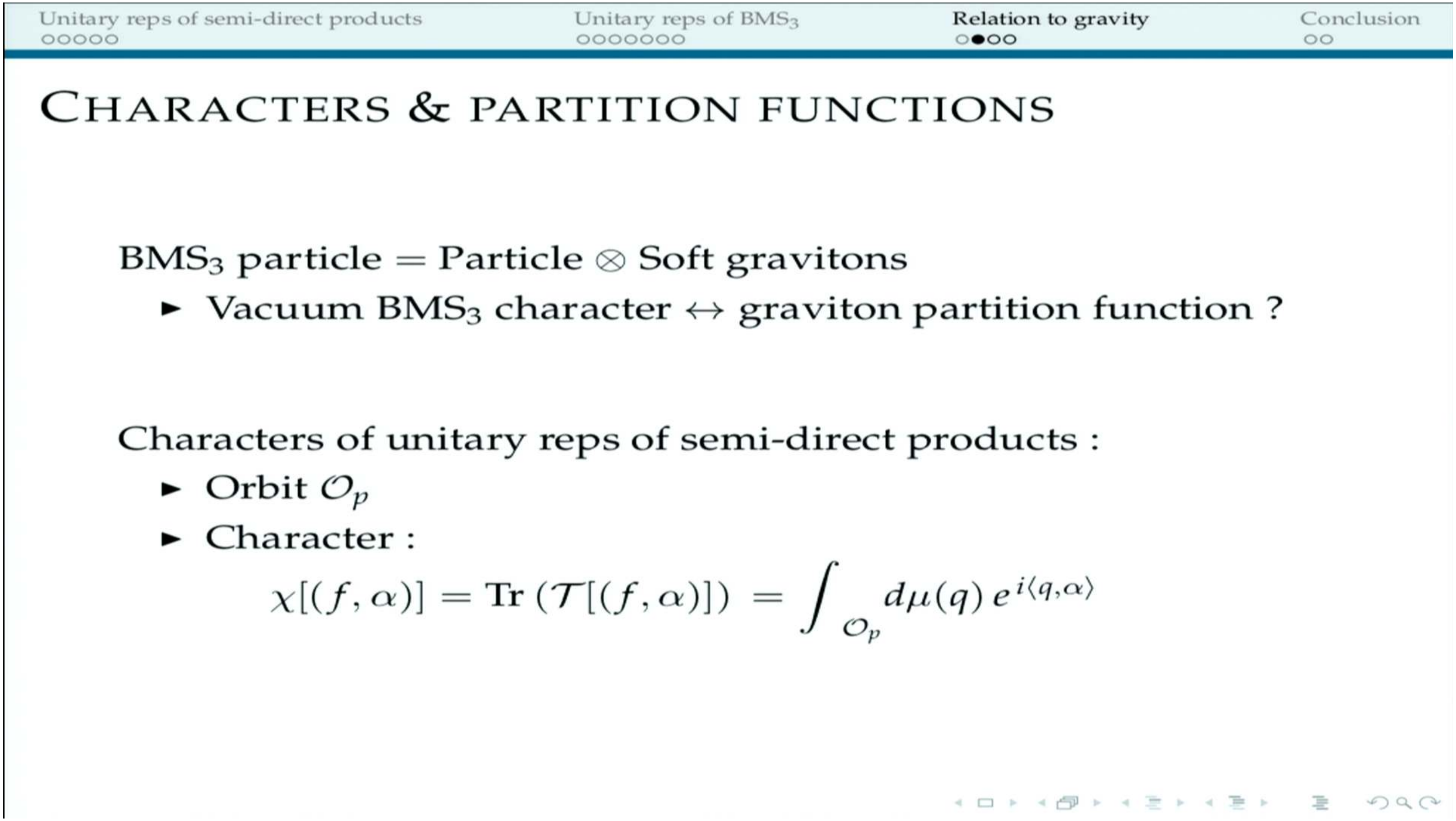
On-shell aspt. flat metrics : [Barnich & Troessaert 2010]

$$ds^2 = 8G\,p(\varphi)du^2 - 2dudr + r^2d\varphi^2 + \cdots$$

$p(\varphi)$ = Bondi mass aspect

- Action of BMS_3 on $p(\varphi)$:

$$f \cdot p|_{f(\varphi)} = \frac{1}{f'^2}\left[p + \frac{c}{12}\{f;\varphi\}\right] \quad \text{with} \quad c = 3/G$$



CHARACTERS & PARTITION FUNCTIONS

BMS$_3$ particle $=$ Particle $\otimes$ Soft gravitons

- Vacuum BMS$_3$ character $\leftrightarrow$ graviton partition function ?

Characters of unitary reps of semi-direct products :

- Orbit $\mathcal{O}_p$
- Character :

$$\chi[(f,\alpha)] = \mathbf{Tr}\left(\mathcal{T}[(f,\alpha)]\right) \;=\; \int_{\mathcal{O}_p} d\mu(q)\, e^{\,i\langle q,\alpha\rangle}$$

CHARACTERS & PARTITION FUNCTIONS

Massive BMS$_3$ particle

- $p = p_0$

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Massive BMS$_3$ particle

- $p = p_0 \to \mathcal{O}_p = \mathrm{Diff}(S^1)/S^1$

Take $f(\varphi) = \varphi + \theta$ (rotation by θ)

- Character :

$$\chi[(\mathrm{rot}_\theta, \alpha)] = \int_{\mathcal{O}_p} d\mu(q)\, \delta(q, \mathrm{rot}_\theta \cdot q)\, e^{i\langle q, \alpha\rangle}$$

$$(\mathrm{rot}_\theta \cdot q)(\varphi) = q(\varphi - \theta)$$

- The integral "localizes" to a point !

CHARACTERS & PARTITION FUNCTIONS

$$\chi_{p_0}[(\mathbf{rot}_\theta, \alpha)] \quad = e^{\,i\,p_0\alpha^0} \quad \frac{1}{\prod_{n=1}^{+\infty}|1-e^{in(\theta+i\epsilon)}|^2}$$

[BO 2015]

CHARACTERS & PARTITION FUNCTIONS

$$\chi_{\mathrm{vac}}[(\mathbf{rot}_\theta, \alpha)] = e^{\beta c/24} \frac{1}{\prod_{n=2}^{+\infty} |1 - e^{in(\theta+i\epsilon)}|^2}$$

[BO 2015]

- One-loop partition fct of gravitons on thermal flat space !

[Barnich, González, Maloney, BO 2015]

CONCLUSION

$$\text{BMS}_3 = \text{Superrotations} \ltimes \text{Supertranslations}$$

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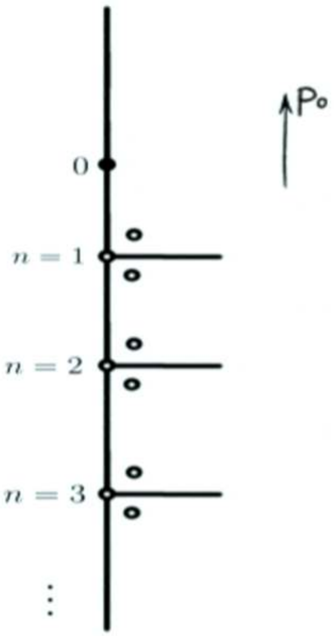
$BMS_3 = \text{Superrotations} \ltimes \text{Supertranslations}$

- "Supermomenta"
- UIRREPs classified by supermomentum orbits



Thank you !