Title: PSI 2015/2016 QED & Renormalization Extra Course - Lecture 3

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Abstract:

$\mathcal{O}(\hbar)$ QED II

1. $g-2$
2. IR Divergences

$$H = \frac{\vec{p}^{\,2}}{2m^2} + V(r) + \frac{e}{2m}\vec{B}\left(\vec{L} + g\vec{S}\right)$$

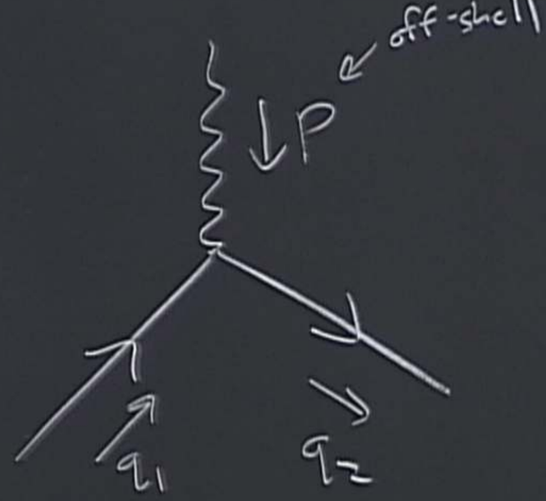
can derive from non-relativistic limit of $(i\not{D} - m)\psi$

$$(\not{D}^2 + m^2)\psi = 0$$

$$\Big(D_\mu^2 + m^2 + \underbrace{\tfrac{e}{2} F_{\mu\nu}\sigma^{\mu\nu}}_{= -e\begin{pmatrix}(\vec{B}+i\vec{E})\vec{\sigma} & \\ & (\vec{B}-i\vec{E})\vec{\sigma}\end{pmatrix}}\Big)\psi = 0 \qquad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

in Weyl rep

$$\vec{S} = \frac{\vec{\sigma}}{2}$$



$$= -ie\,\bar{u}(q_2)\gamma^\mu u(q_1)\,\epsilon_\mu$$

Gordon identity: $\bar{u}(q)\,\underline{i\sigma^{\mu\nu}}$

$$(q_\nu)\,\epsilon_\mu(p) = i\,\epsilon_\mu \mathcal{M}_0^\mu$$

$$\frac{)\,i\sigma^{\mu\nu}(q_\nu - p_\nu)u(p)}{2m} = \bar{u}(q)\frac{i^2}{4m}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right)(q_\nu - p_\nu)u(p)$$

$$= \frac{1}{4m}\bar{u}(q)\left(-\gamma^\mu(\not{q} - \not{p}) + (\not{q} - \not{p})\gamma^\mu\right)u(p)$$

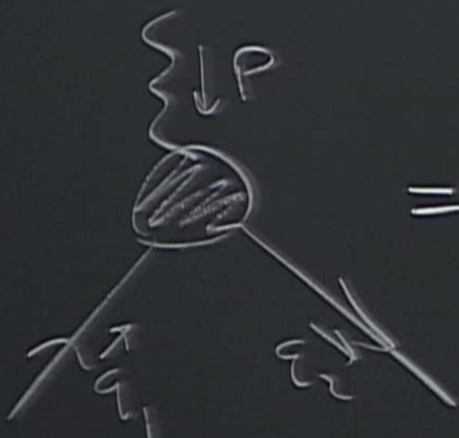
$$(\not{p} - m)u(p) = 0$$

$$= \frac{1}{4m}\bar{u}(q)\left(-\gamma^\mu \not{q} + 2m\gamma^\mu - \not{p}\gamma^\mu\right)u(p)$$

$$= \frac{1}{4m}\bar{u}(q)\left(4m\gamma^\mu - 2q^\mu - 2p^\mu\right)u(p)$$

$$= -\frac{e}{2m}\left(q_1^\mu + q_2^\mu\right)\bar{u}(q_2)\,\bar{u}(q_1) - \frac{e}{2m}\, i\,\bar{u}(q_2)\, p_\nu \sigma^{\mu\nu} u(q_1)$$

coefficient of this term determines g

$=$ 

by mom. cons.

$$= \bar{u}(q_2)\left[f_1 \gamma^\mu + \cancel{f_2 p^\mu} + f_3 q_1^\mu + f_4 q_2^\mu\right] \cdot u(q_1)$$

f_i scalars

$$0 = p_\mu M^\mu$$

$$= f_1 \, \underbrace{\cancel{\bar{u} \not{p} u}}_{}{}^{\,0} + p \cdot q_1 f_3 \bar{u} u + p \cdot q_2 f_4 \bar{u} u$$

$$\bar{u} \not{p} u = \bar{u} (\not{q}_2 - \not{q}_1) u = \bar{u} (m - m) u = 0$$

$$p \cdot q_1 = q_2 \cdot q_1 - m^2 = - p \cdot q_2$$

$$f_3 = f_4$$

$$iM^\mu = (-ie)\,\bar{u}(q_2)\left[F_1\left(\frac{p^2}{m^2}\right)\gamma^\mu + \frac{i\sigma^{\mu\nu}}{2m}\,p_\nu F_2\left(\frac{p^2}{m^2}\right)\right]u(q_1)$$

tells us about g

p^2, m^2, $\underbrace{q_1 \cdot q_2}_{p^2 = 2m^2 - 2q_1\cdot q_2}$, $\not{q}_1$, $\not{q}_2 \to m$

tree level $F_1 = 1$

$F_2 = 0$

$$\left(\frac{p^2}{m^2}\right)\gamma^\mu + \frac{i\sigma^{\mu\nu}}{2m} p_\nu F_2\left(\frac{p^2}{m^2}\right)\Big] u(q_1)$$

tells us about $g = 2 + 2F_2(0)$

$$^2,\ m^2,\ q_1 \cdot q_2,\ \not{q}_1,\ \not{q}_2 \to m$$

$$p^2 = 2m^2 - 2q_1 \cdot q_2$$

$$= (-ie)^3 \int \frac{d^4k}{(2\pi)^4}\, \bar{u}(q_2)\, \gamma^\nu \frac{i(\not p + \not k + m)}{\underbrace{(p+k)^2 - m^2 + i\epsilon}_{A}}\, \gamma^\mu\, \frac{i(\not k \ldots}{k^2 - \ldots}$$

$$= -e^3\, \bar{u}(q_2) \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\nu(\not p + \not k + m)\gamma^\mu(\not k + m)\gamma_\nu}{ABC}$$

$$) \gamma^\nu \frac{i(\not p + \not k + m)}{\underbrace{(p+k)^2 - m^2 + i\epsilon}_{A}} \gamma^\mu \frac{i(\not k + m)}{\underbrace{k^2 - m^2 + i\epsilon}_{B}} \gamma^\alpha u(q_1) \frac{-i g_{\nu\alpha}}{\underbrace{(k-q_1)^2 + i\epsilon}_{C}}$$

$$\frac{\gamma^\nu(\not p + \not k + m)\gamma^\mu(\not k + m)\gamma_\nu}{ABC} u(q_1)$$

$$= (-ie)^3 \int \frac{d^4k}{(2\pi)^4} \bar{u}(q_2) \gamma^\nu \frac{i(\not{p}+\not{k}+m)}{\underbrace{(p+k)^2-m^2+i\epsilon}_{A}} \gamma^\mu \frac{i(\not{k}+m)}{\underbrace{k^2-m^2+i\epsilon}_{B}}$$

$$= -e^3 \bar{u}(q_2) \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\nu(\not{p}+\not{k}+m)\gamma^\mu(\not{k}+m)\gamma_\nu}{ABC} u(q_1)$$

$$\frac{1}{ABC} = 2\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1) \frac{1}{[xA+yB+zC]^3}$$

$$\frac{-igva}{\underbrace{(k-q_1)^2+i\varepsilon}_{C}}$$

$$xA+yB+zC=(x+y+z)k^2-(x+y)m^2+2k\cdot(yp-zq_1)$$
$$+yp^2+zq_1^2+i\varepsilon$$
$$=(k^\mu+yp^\mu-zq_1^\mu)^2-\Delta+i\varepsilon$$
$$\Delta=-xyp^2+(1-z)^2m^2$$

$$k^\mu\to k^\mu-yp^\mu+zq_1^\mu$$

$$N^\mu = -2\bar{u}(q_2)\left[\not{k}\gamma^\mu\not{p} + \not{k}\gamma^\mu\not{k} + m^2\gamma^\mu - 2m(2k^\mu + p^\mu)\right]u(q_1)$$

numerator

using e.g. $\gamma^\nu\gamma^\mu\gamma_\nu = -2\gamma^\mu$

numerator

$$\int \frac{d^4k}{(2\pi)^4} \frac{k^\mu}{(k^2-\Delta)^3} = 0 \quad \text{by symmetry}$$

$$\int \frac{d^4k}{(2\pi)^4} \frac{k^\mu k^\nu}{(k^2-\Delta)^3} = c \int \frac{d^4k}{(2\pi)^4} \frac{g^{\mu\nu} k^2}{(k^2-\Delta)^3}$$

$$g_{\mu\nu} k^\mu k^\nu = k^2 = c\, g_{\mu\nu} g^{\mu\nu} \Rightarrow c = \frac{1}{4}$$

$$\text{in } N^\mu \quad k^\mu k^\nu \to \frac{1}{4} g^{\mu\nu} k^2$$

$$N^\mu = -2\bar{u}(q_2)\left[\cancel{\not k \gamma^\mu \not p}^{\,\to 0} + \not k \gamma^\mu \not k + m^2\gamma^\mu - 2m(\cancel{2k^\mu}^{\,\to 0} + p^\mu)\right]u(q_1)$$

$\uparrow$ numerator

using e.g. $\gamma^\nu \gamma^\mu \gamma_\nu = -2\gamma^\mu$

$$\int \frac{d^4k}{(2\pi)^4} \frac{k^\mu}{(k^2-\Delta)^3} = 0 \quad \text{by symmetry}$$

$$\int \frac{d^4k}{(2\pi)^4} \frac{k^\mu k^\nu}{(k^2-\Delta)^3} = c \int \frac{d^4k}{(2\pi)^4} \frac{g^{\mu\nu}k^2}{(k^2-\Delta)^3}$$

$$g_{\mu\nu}k^\mu k^\nu = k^2 = c\, g_{\mu\nu}g^{\mu\nu} \Rightarrow c = \frac{1}{4}$$

in N^μ: $\quad k^\mu k^\nu \to \frac{1}{4} g^{\mu\nu} k^2$

$$= \left[-\frac{1}{2}k^2 + (1-x)(1-y)p^2 + (1-4z+z^2)m^2\right] \bar{u}(q_2)\gamma^\mu u(q_1)$$

contributes to F_1

$$+ im\, z(1-z)\, p_\nu \bar{u}(q_2)\sigma^{\mu\nu}u(q_1) \rightarrow \text{Contribute to } F_2$$

$$+ m(z-2)(x-y)\, p^\mu \bar{u}(q_2)u(q_1) \rightarrow \text{should be zero by Ward Identity}$$

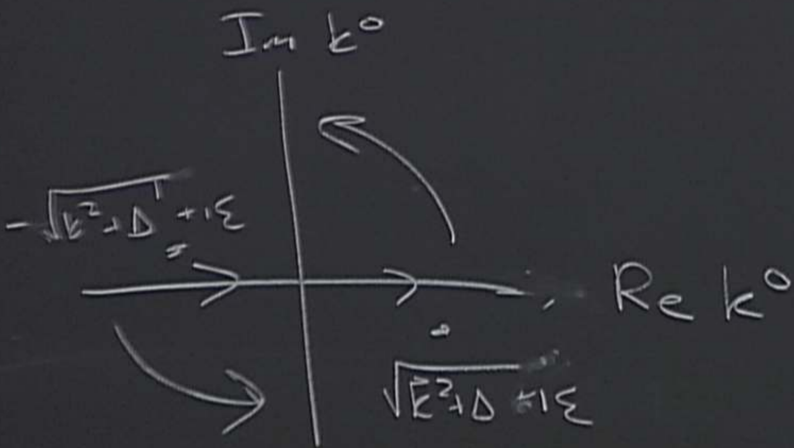
problem term is odd under $x \leftrightarrow y$

$$\underbrace{\int dx\,dy\,dz\,\delta(x+y+z-1)}_{\text{even under } x\leftrightarrow y} \frac{N^\mu}{(k^2-\Delta)^3}$$

even under $x \leftrightarrow y$

vanishes by symmetry

$$F_2\left(\frac{q^2}{m^2}\right) = \frac{2m}{e}\left(4ie^3 m\right)\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\int$$



$$k^0 = ik_E^0$$

$$\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-\Delta)^3} \rightarrow -i\int$$

$$\int\frac{d^dk_E}{(2\pi)^4}\frac{1}{(k^2+\Delta)^3} = \frac{1}{(2}$$

$$1)\int \frac{d^4k}{(2\pi)^4}\,\frac{z(1-z)}{(k^2-\Delta+i\varepsilon)^3}$$

$$\int \frac{d^4k_E}{(2\pi)^4}\,\frac{1}{(k_E^2+\Delta)^3}$$

$$=\frac{1}{(2\pi)^d}\,\frac{1}{2}\,\Delta^{\frac{d}{2}-3}\,\frac{\Gamma(3-\frac{d}{2})\Gamma(\frac{d}{2})}{\Gamma(3)}\,\frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}\underset{d\to 4}{=}\frac{1}{32\pi^2\Delta}$$

$$\Delta = -xyp^2 + (1-z)^2 m^2$$

$$\Delta\big|_{p=0} = (1-z)^2 m^2$$

$$F_2(0) = \frac{\alpha}{\pi} \int_0^1 dz \int_0^1 dy \int_0^1 dx\, \delta(x+y+z-1) \frac{z}{1-z} \qquad \alpha = \frac{e^2}{4\pi}$$

$$= \frac{\alpha}{\pi} \int_0^1 dz \int_0^{1-z} dy\, \frac{z}{1-z}$$

$$= \frac{\alpha}{\pi} \int_0^1 dz\, z$$

$$= \boxed{\frac{\alpha}{2\pi}}$$

$$g = 2 + 2F_2(0)$$

$$\boxed{g-2 = \frac{\alpha}{\pi} + \mathcal{O}(\alpha^2)}$$

$$\frac{g-2}{2} = 0.00\underbrace{1159652180}_{10\text{ digits!}}73(28)$$

$$F_1\left(\frac{p^2}{m^2}\right) = 1 + f\left(\frac{p^2}{m^2}\right)$$

$$f\left(\frac{p^2}{m^2}\right) = -2ie^2 \int dx\,dy\,dz\,\delta(x+y+z-1) \int \frac{d^4k}{(2\pi)^4}$$

$$\int \frac{d^4k}{(2\pi)^4} \frac{\cancel{k^2 - 2(1-x)(1-y)p^2} - 2(1-4z-z^2)m^2}{\left[k^2 - \left(m^2(1-z^2) - \cancel{xyp^2}\right)\right]^3}$$

$$\delta(x+y+z-1) \qquad \frac{1}{(1-z^2)^3} \propto \int_0^1 dz \, \frac{z-1}{(1-z^2)^3}$$

$$\frac{\ldots p^2 - 2(1-4z-z^2)m^2}{\ldots -z^2) - xy\,p^2)]^3}$$

$$\propto \int_0^1 dz\, \frac{z-1}{(1-z^2)^3} \longrightarrow \text{diverges!}$$

$$F_1\left(\frac{p^2}{m^2}\right) = 1 + f\left(\frac{p^2}{m^2}\right)$$

$$f\left(\frac{p^2}{m^2}\right) = -2ie^2 \int dx\,dy\,dz\,\delta(x+y+z-1) \int \frac{d^4k}{(2\pi)^4}$$

UV divergent! $\rightarrow$ need counterterm

IR $f(0)$ for $k^2 \approx 0$ for $z \approx 1$

$$f(0) \propto \frac{e^2}{m^4} \int dx\,dy\,dz\,\delta(x+y+z-1)$$

tomorrow: $e^+e^- \to \mu^+\mu^-$

diverges!
- new
- can't solve w/ counterterms
- rethink what our observables are