Title: Topological entropies in the classical toric code model - An information theory perspective

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Abstract: <p>Concepts of information theory are increasingly used to characterize collective phenomena in condensed matter systems, such as the use of entanglement entropies to identify emergent topological order in interacting quantum many-body systems. Here we employ classical variants of these concepts, in particular Renyi entropies and their associated mutual information, to identify topological order in classical systems.</p>

<p>Like for their quantum counterparts, the presence of topological order can be identified in such classical systems via a universal, subleading contribution to the prevalent volume and boundary laws of the classical Renyi entropies. We demonstrate that an additional subleading O(1) contribution generically arises for all Renyi entropies S(n) with n â‰¥ 2 when driving the system towards a phase transition, e.g. into a conventionally ordered phase. This additional subleading term, which we dub connectivity contribution, tracks back to partial subsystem ordering and is proportional to the number of connected parts in a given bipartition. Notably, the Levin-Wen summation scheme â€“ typically used to extract the topological contribution to the Renyi entropies â€“ does not fully eliminate this additional connectivity contribution in this classical context. This indicates that the distillation of topological order from Renyi entropies requires an additional level of scrutiny to distinguish topological from non-topological O(1) contributions. This is also the case for quantum systems, for which we discuss which entropies are sensitive to these connectivity contributions. We showcase these findings by extensive numerical simulations of a classical variant of the toric code model, for which we study the stability of topological order in the presence of a magnetic field and at finite temperatures from a Renyi entropy perspective.</p>

Topological entropies in the classical toric code model

An information theory perspective

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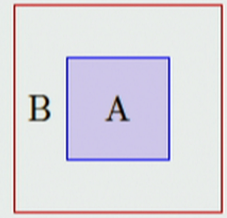
Condensed matter seminar – Perimeter Institute

1. Classical bipartite entropies
2. Classical toric code
3. Numerical approach
4. Results
   - Phase transition
   - Finite temperature crossover
5. Outlook on 3D problem

Entanglement entropy

Entanglement of **many-body** system in ground state

- bipartition the system into parts A and B
- quantify entropy of subsystem A



Entanglement entropies

$$S(A) = -\mathrm{Tr}\left(\rho_A \ln \rho_A\right)$$

generalized (Renyi) entropies

$$S_n(A) = \frac{1}{1-n} \ln \mathrm{Tr} \rho_A^n$$

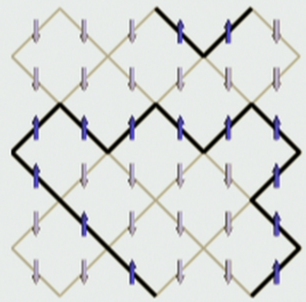
accessible by Monte Carlo methods for integer n

Scaling with **boundary** ℓ of A

$$S_n(A) = a \cdot \ell + \ldots$$

- $b \cdot \ln \ell$
  - corner terms
  - Goldstone modes in gapless phases
- $-\gamma$
  - **topological order**

Classical Toric Code



Quantum

$$H = -J_p \sum_{p \in P} \prod_{i=1}^{4} \sigma^x_{p_i} - J_v \sum_{v \in V} \prod_{i=1}^{4} \sigma^z_{v_i}$$

Groundstate wavefunction

$$|\psi_{\text{G.S.}}\rangle = \frac{1}{\sqrt{N_{\mathcal{L}}}} \sum_{\mathcal{L}} |\mathcal{L}\rangle$$

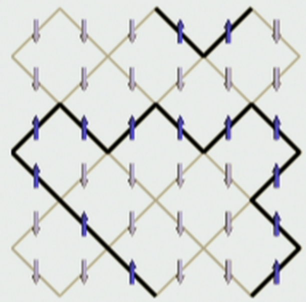
loop gas

↓

perturbation by **loop tension** (magnetic field)

A. Y. Kitaev, Annals of Physics 303 (2003)

Classical Toric Code



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Groundstate wavefunction

$$|\psi_{\text{G.S.}}\rangle = \frac{1}{\sqrt{N_{\mathcal{L}}}} \sum_{\mathcal{L}} |\mathcal{L}\rangle$$

Classical

$$H = -J_v \sum_{v \in V} \prod_{i=1}^{4} \sigma_{v_i}$$

Partition function

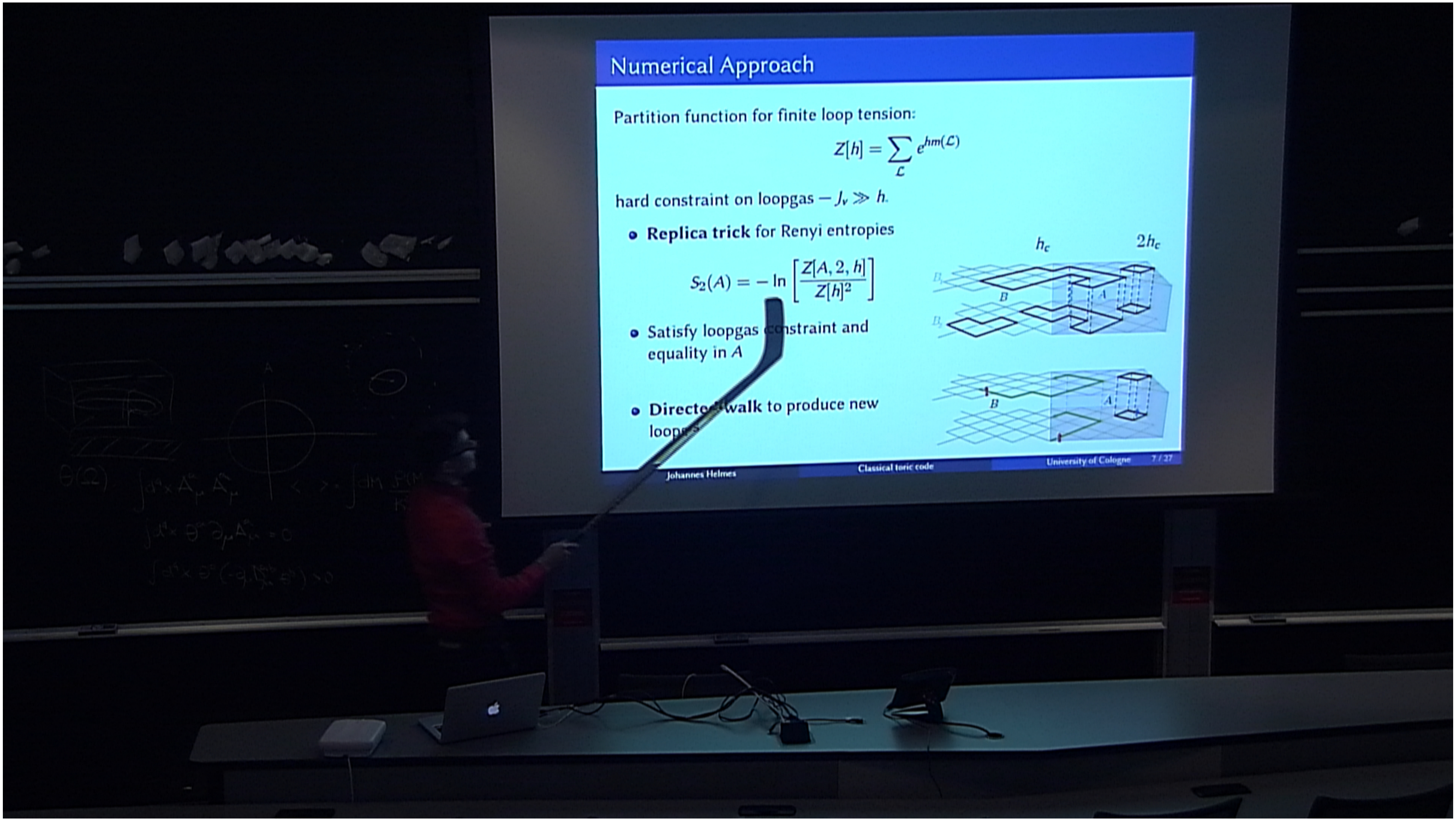
$$Z = \sum_{\mathcal{L}} 1$$

loop gas

↓

perturbation by **loop tension** (magnetic field)

A. Y. Kitaev, Annals of Physics 303 (2003)


Numerical Approach
Partition function for finite loop tension:
$$Z[h] = \sum_{\mathcal{L}} e^{hm(\mathcal{L})}$$
hard constraint on loopgas $- J_v \gg h$.
Replica trick for Renyi entropies
$$S_2(A) = -\ln\left[\frac{Z[A,2,h]}{Z[h]^2}\right]$$
Satisfy loopgas constraint and equality in A
Directed walk to produce new loops
h_c
$2h_c$
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Numerical Approach

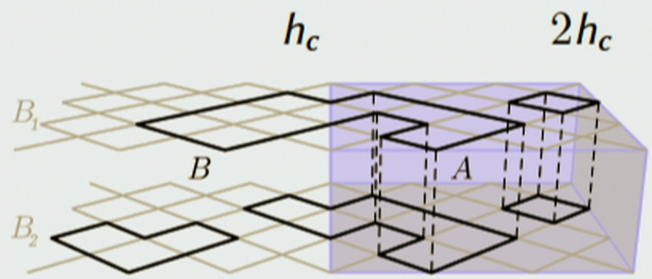
Partition function for finite loop tension:

$$Z[h] = \sum_{\mathcal{L}} e^{hm(\mathcal{L})}$$

hard constraint on loopgas $- J_v \gg h$.

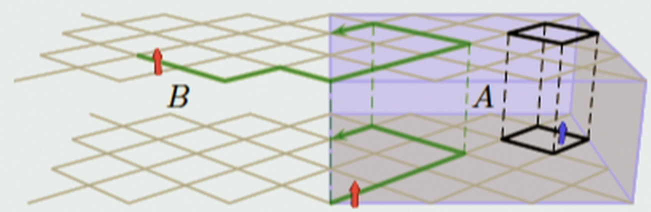
- **Replica trick** for Renyi entropies

$$S_2(A) = -\ln\left[\frac{Z[A,2,h]}{Z[h]^2}\right]$$



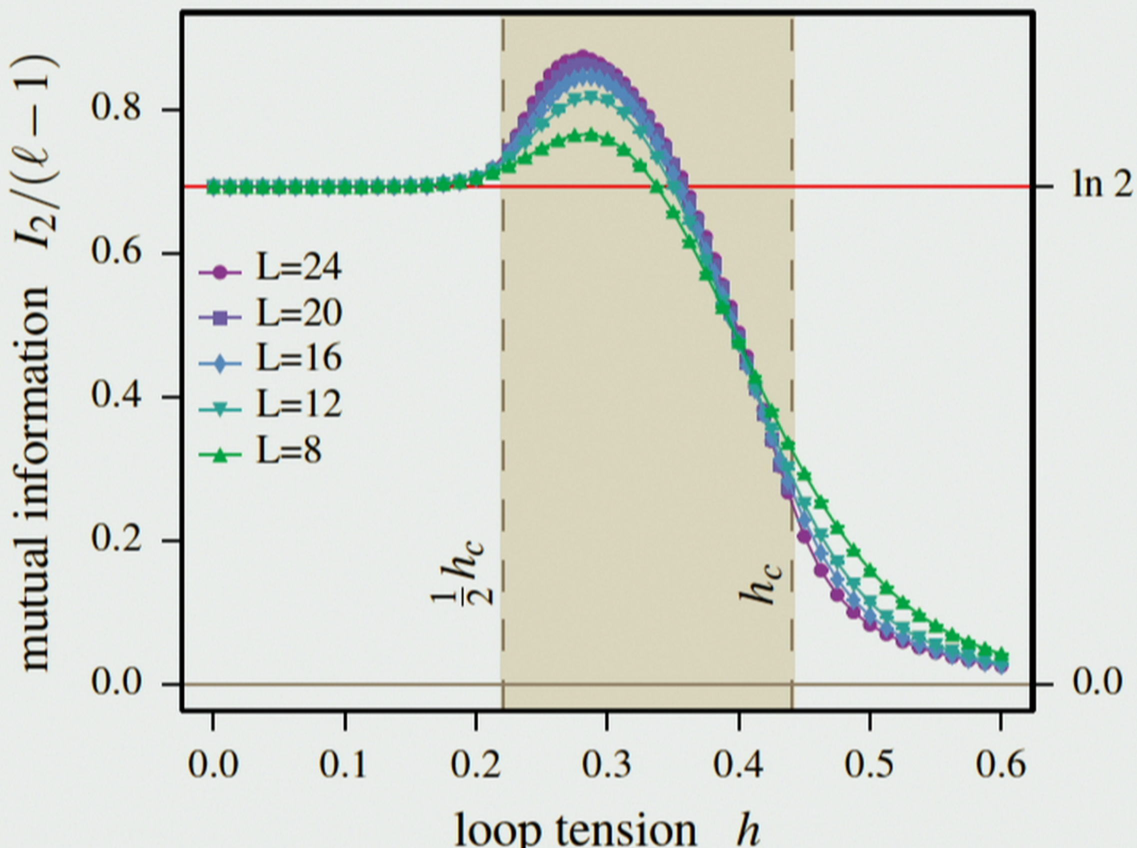
- Satisfy loopgas constraint and equality in A

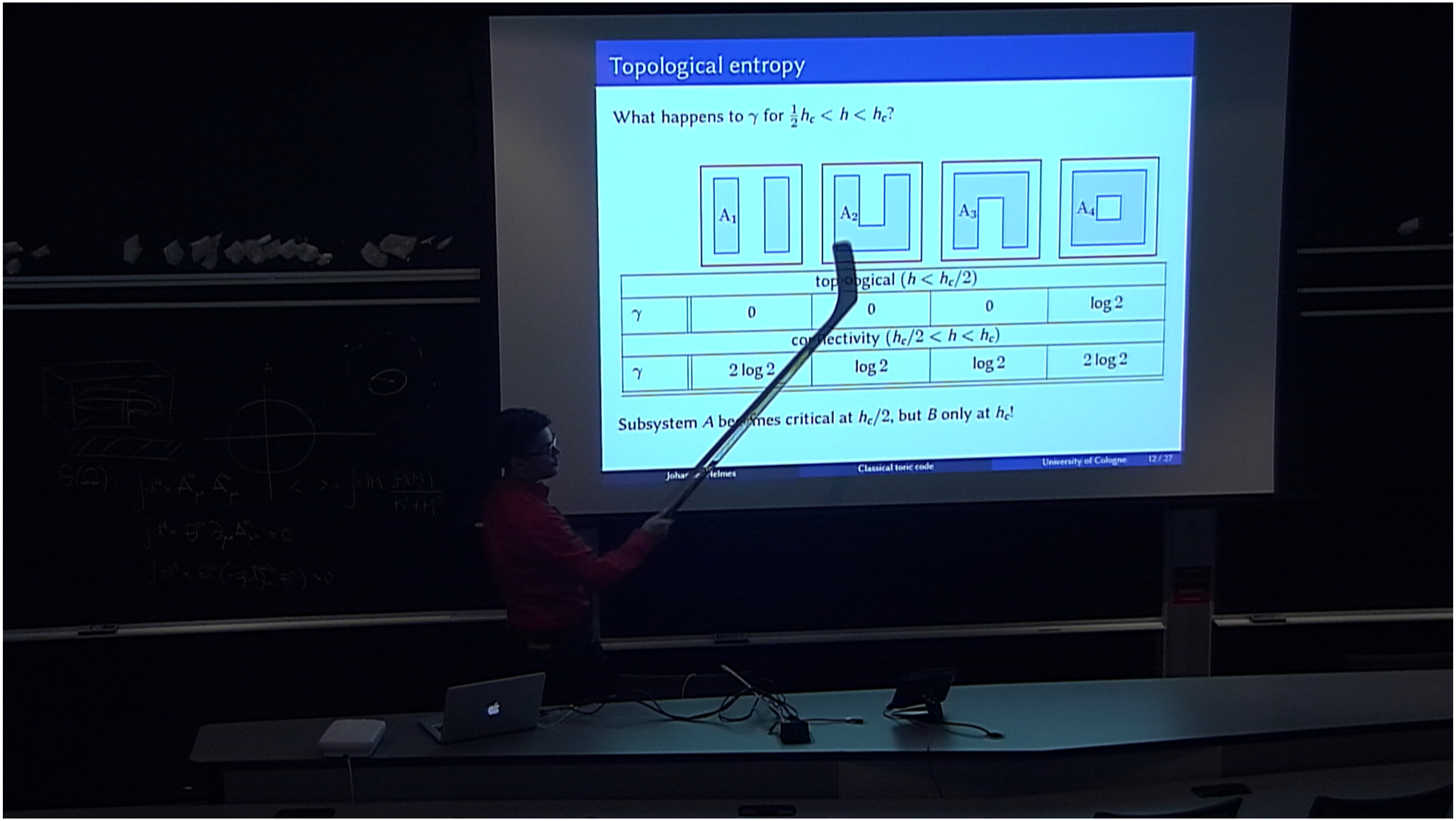
- **Directed walk** to produce new loops



Mutual information

Sanity check: **Mutual information** $I(A:B) = S_A + S_B - S_{AB}$





Topological entropy
What happens to γ for $\frac{1}{2}h_c < h < h_c$?
A1
A2
A3
A4
γ
0
0
0
log 2
γ
2 log 2
log 2
log 2
2 log 2
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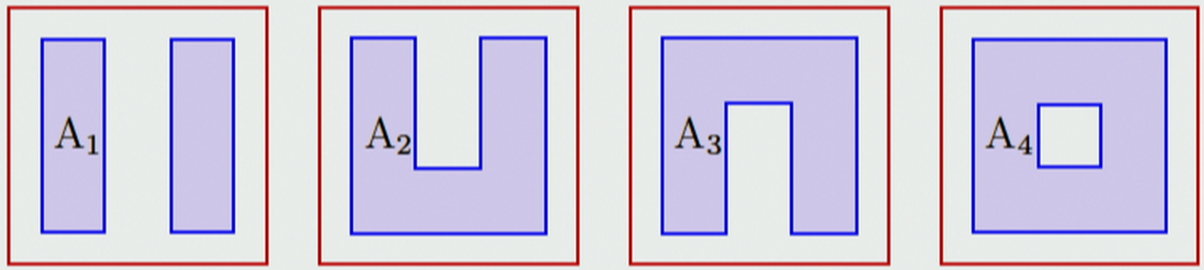
Topological entropy

What happens to γ for $\frac{1}{2}h_c < h < h_c$?



topological ($h < h_c/2$)				
γ	0	0	0	$\log 2$
connectivity ($h_c/2 < h < h_c$)				
γ	$2 \log 2$	$\log 2$	$\log 2$	$2 \log 2$

Subsystem A becomes critical at $h_c/2$, but B only at h_c!

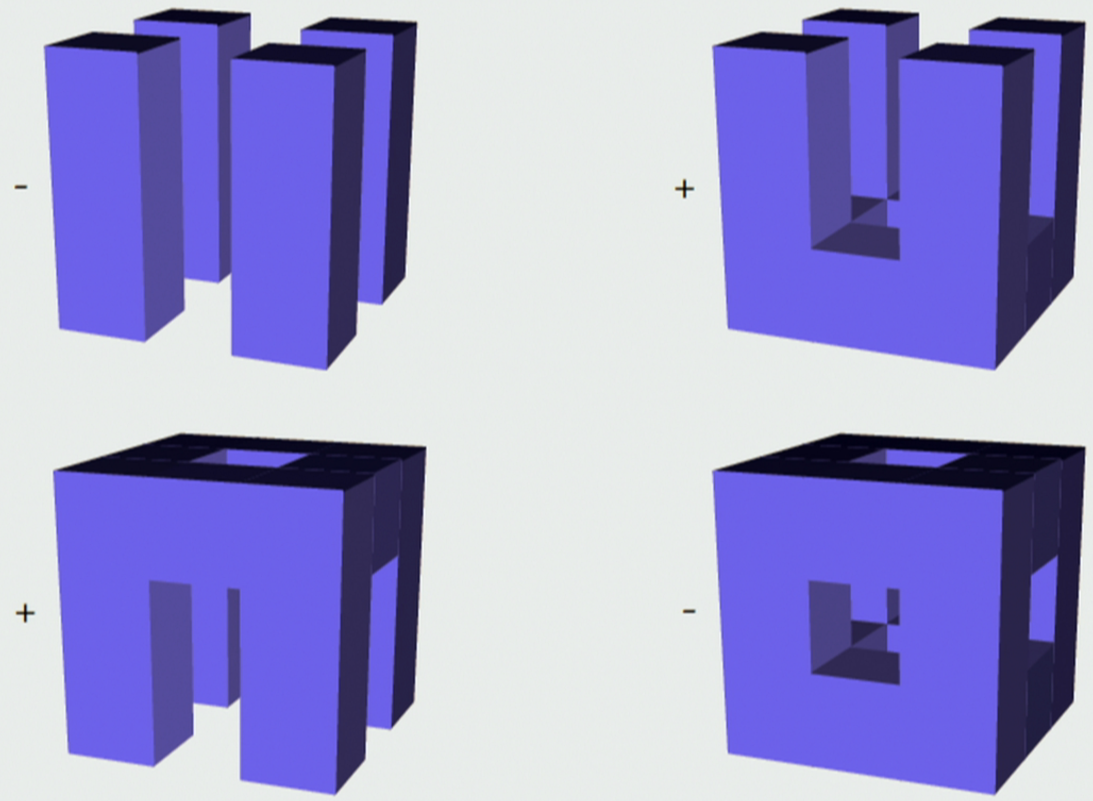
Conclusion thus far

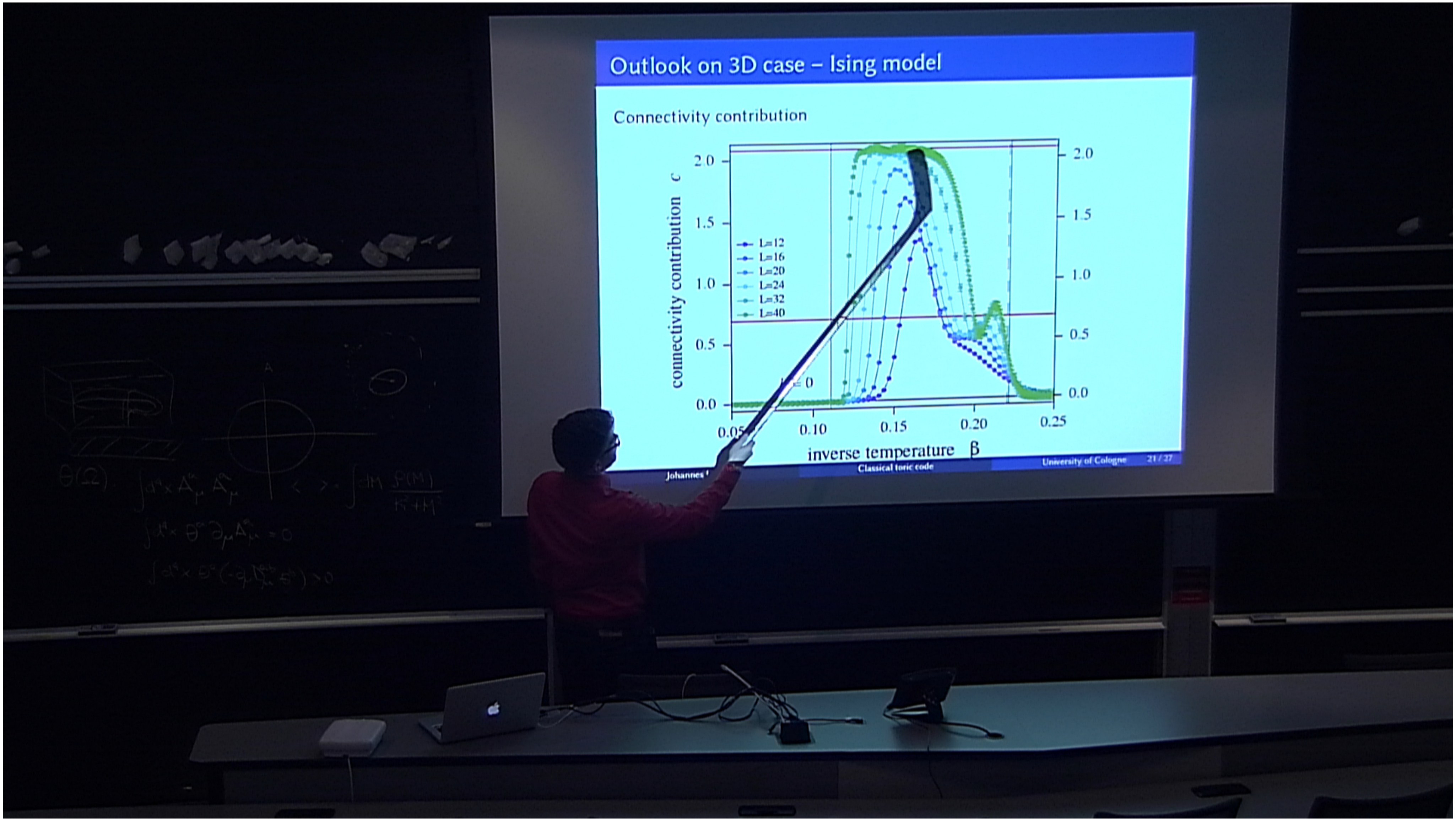
- Topological order tractable in **classical** toric code model from an information theory perspective
- Numerical accessibility of Renyi entropies in relatively **large systems**
- Detection of **O(1) connectivity contribution** in Renyi entropy
- **Levin Wen scheme** not sufficient to cancel connectivity contribution
- Relevant issue for **quantum systems** in future studies

JH, J.-M. Stéphan, S. Trebst, PRB 92 (2015)

Outlook on 3D case – Ising model

Modified Levin-Wen scheme




Outlook on 3D case – Ising model
Connectivity contribution
connectivity contribution c
inverse temperature β
L=12
L=16
L=20
L=24
L=32
L=40
2.0
1.5
1.0
0.5
0.0
0.10
0.15
0.20
0.25
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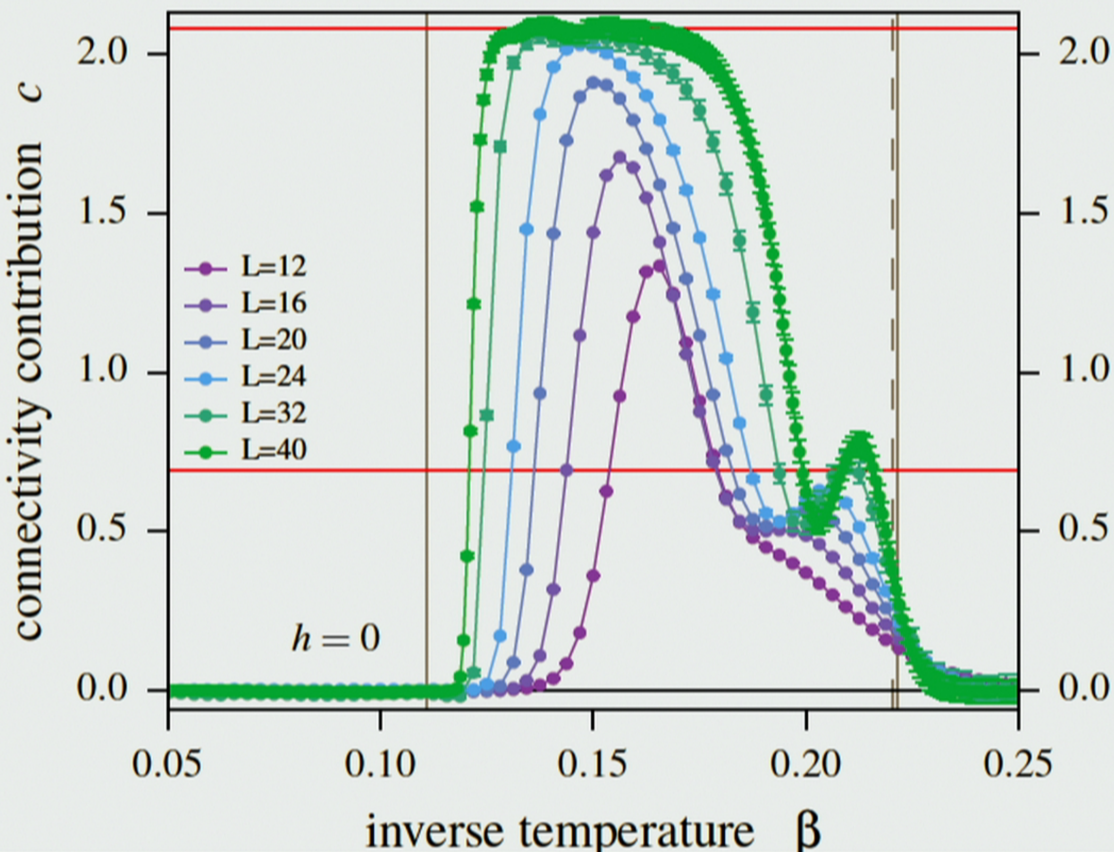
Outlook on 3D case – Ising model

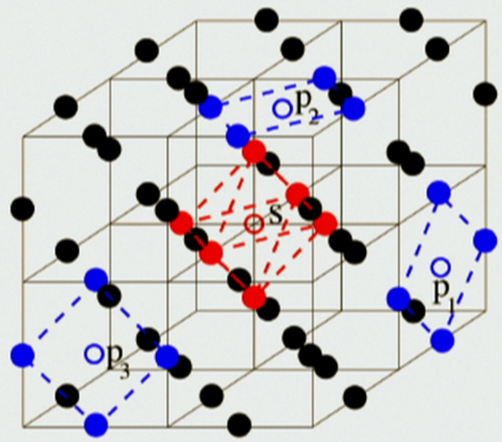
Connectivity contribution



Outlook on 3D case – toric code model

Quantum

$$H = -J_p \sum_{p \in P} \prod_{i=1}^{4} \sigma^z_{p_i} - J_v \sum_{v \in V} \prod_{i=1}^{6} \sigma^x_{v_i}$$



C. Castelnovo, C. Chamon, PRB 78 (2008)

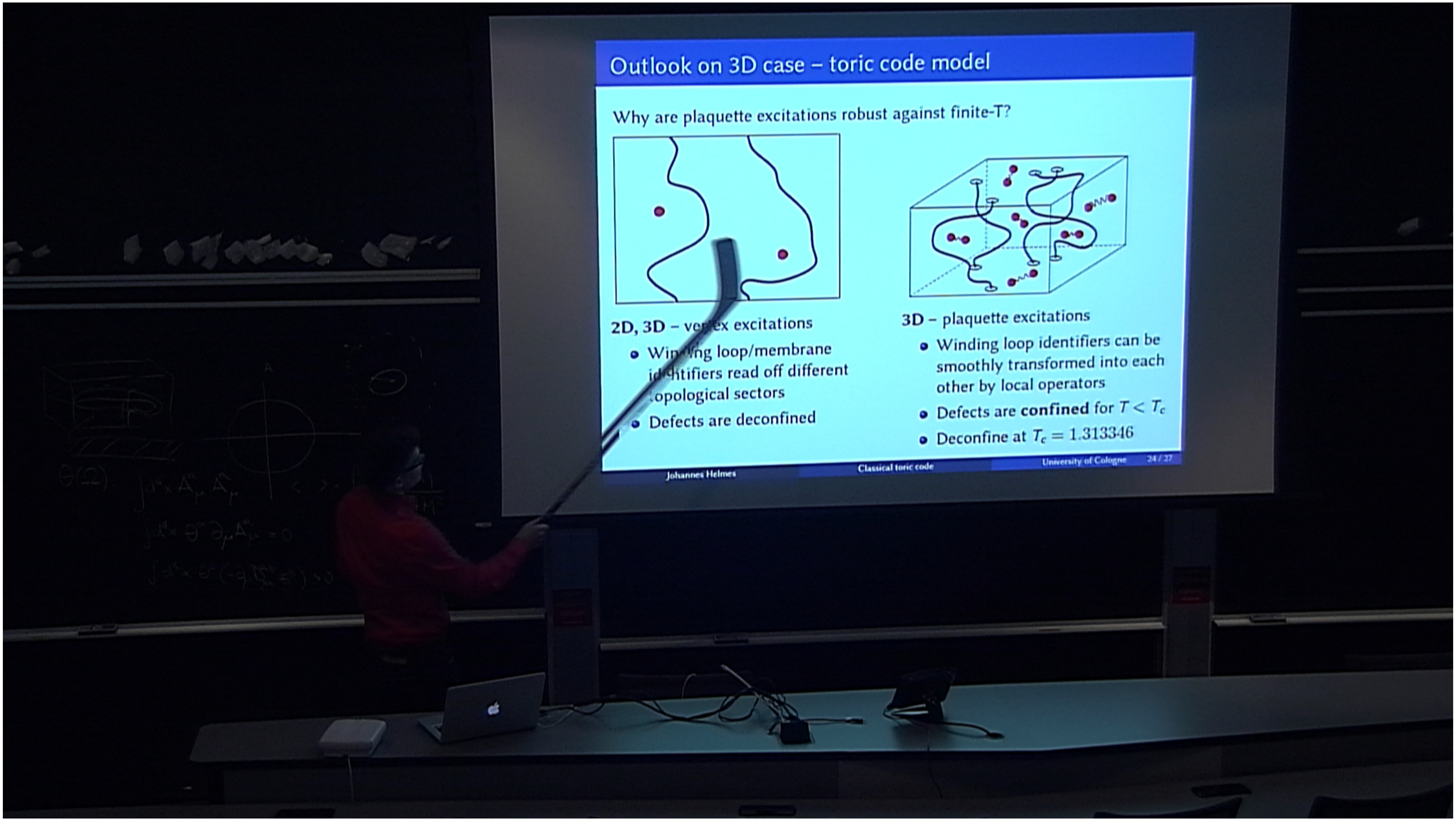
Two **different** classical derivatives:

- $H = -J_p \sum_{p \in P} \prod_{i=1}^{4} \sigma_{p_i}$ (loop structure)
- $H = -J_v \sum_{v \in V} \prod_{i=1}^{6} \sigma_{v_i}$ (membrane structure)

Breakdown of topological order at finite-T

- vertex excitations: log - scaling like in 2D
- plaquette excitations: finite-T **phase transition** at $T_c = 1.313346$

M. Caselle, M. Hasenbusch, and M. Panero, J. High Energy Phys. 03 (2003)


Outlook on 3D case – toric code model
Why are plaquette excitations robust against finite-T?
2D, 3D – vertex excitations
Winding loop/membrane identifiers read off different topological sectors
Defects are deconfined
3D – plaquette excitations
Winding loop identifiers can be smoothly transformed into each other by local operators
Defects are confined for $T < T_c$
Deconfine at $T_c = 1.313346$
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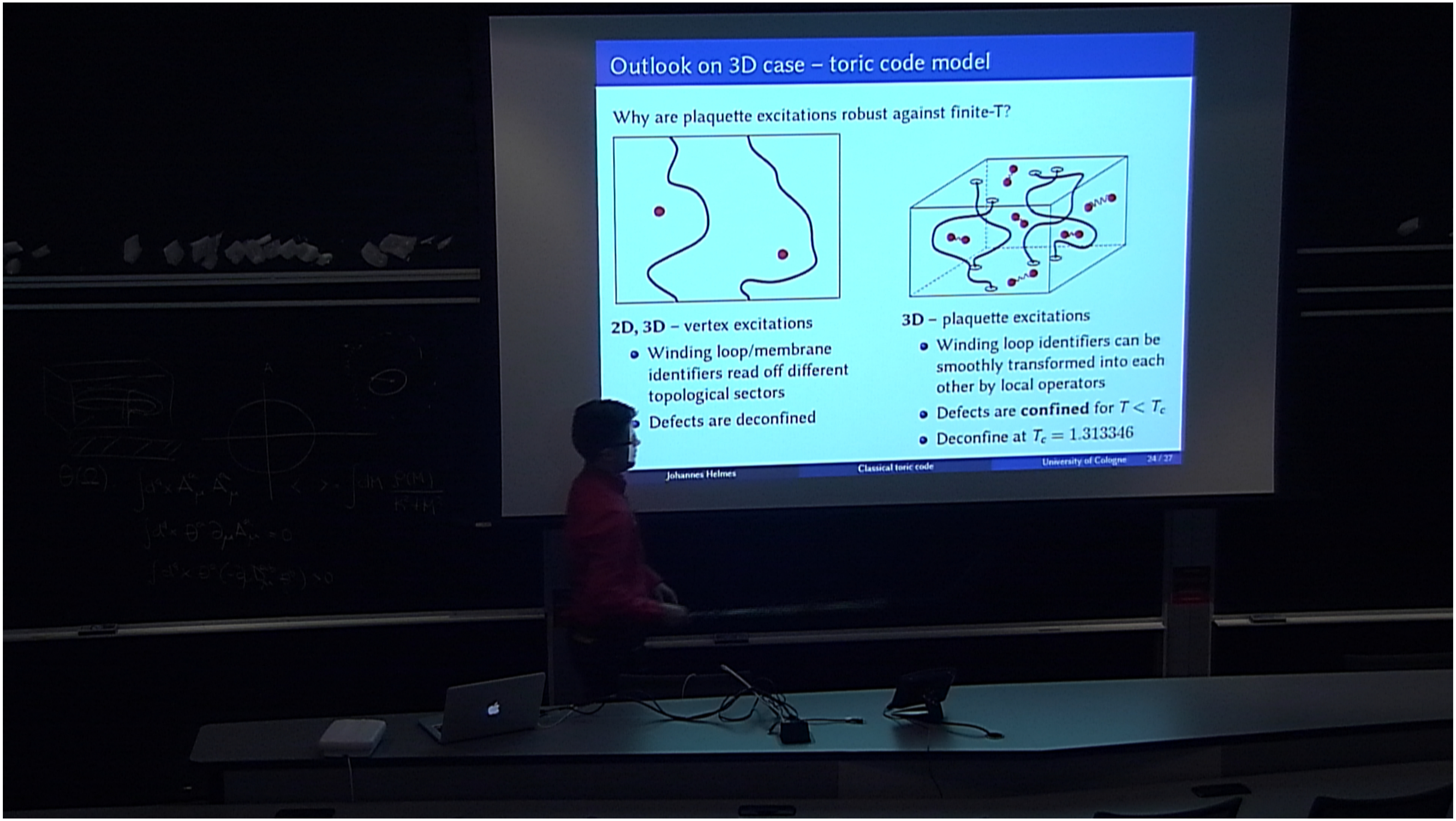
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Winding loop identifiers can be smoothly transformed into each other by local operators
Defects are confined for T < Tc
Deconfine at Tc = 1.313346
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Outlook

- ✓ Approach applicable to classical 3D systems
- ? Role of criticality of 2D boundary in 3D systems
- ? Appropriate extraction scheme for 3D toric code
- ! Need of 4D for stable finite-T error correcting code