Title: The AGT conjecture for gauge theories on A-type ALE spaces, K-theoretic/Cohomological Hall algebras and sheaves on stacks

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Abstract: K-theoretical/Cohomological Hall algebras, associated with the stack of zero-dimensional sheaves on $\mathbb{C}^2$, play a prominent role in the proof, given by Schiffmann and Vasserot, of the AGT conjecture for

(pure) gauge theories on $\mathbb{C}^2$. In the present talk I will describe K-theoretic/Cohomological Hall algebras associated with stacks of torsion sheaves on A-type ALE spaces and their representations; the study of these algebras will provide eventually an approach to prove the AGT conjecture for (pure) gauge theories on A-type ALE spaces. (Work in progress with Olivier Schiffmann.)

AGT conj. for gauge th. on A-type ALE spaces, K-th/G-HAs and sheaves on stacks

1. Motivation: AGT conj. for $\mathbb{C}^2$ + SV proof
2. Crash course in Hall algebras
3. K-HA$_{\mathbb{C}^2}$, K-HA$_{[\mathbb{C}^2/\mathbb{Z}_k]}$, K-HA$_{\mathbb{P}^1_{a,b}}$
4. AGT conj. for A-type ALE spaces

End sheaves on stacks

1. Motivation

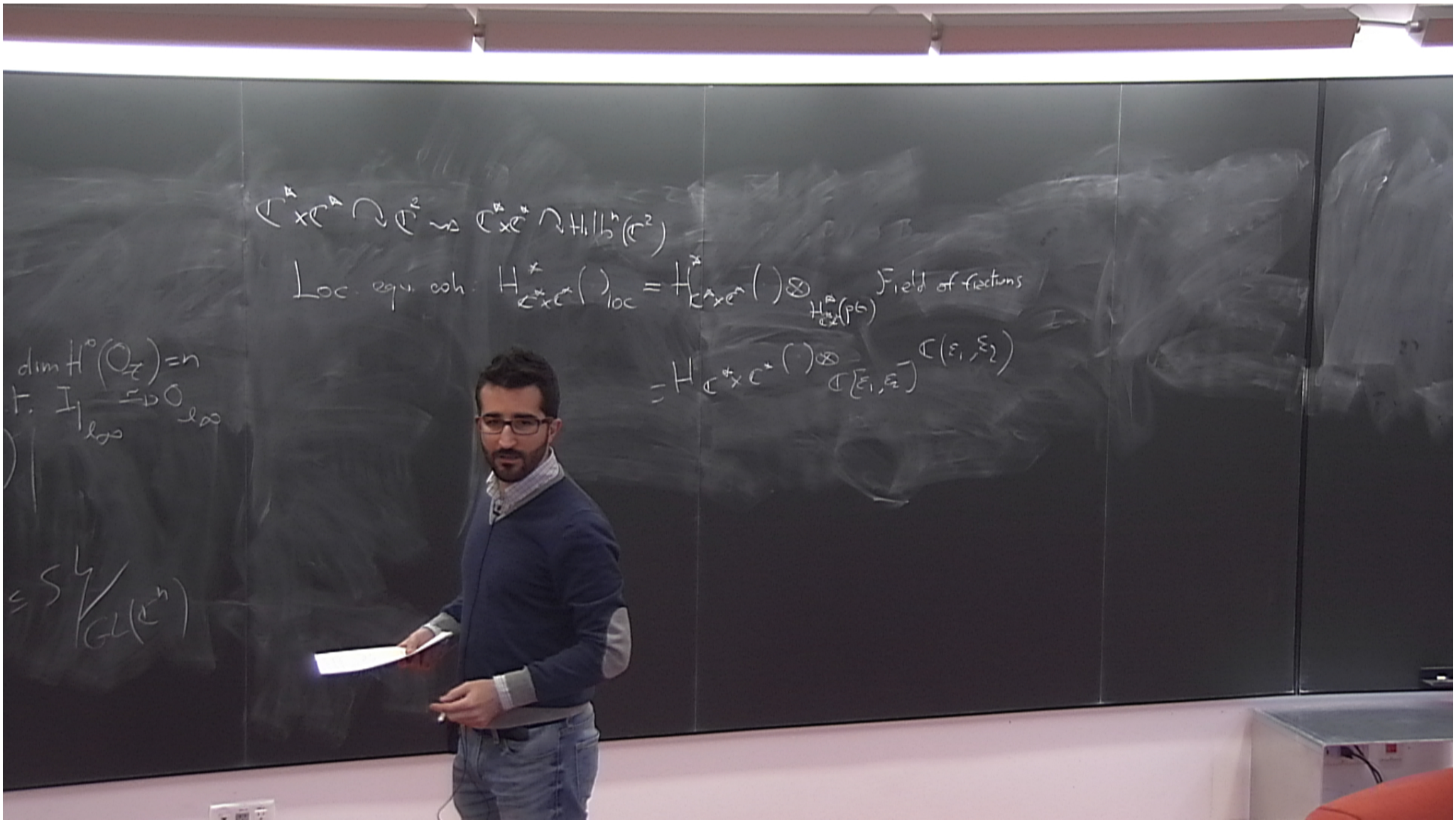
$\mathbb{C}^n \times \mathbb{C}^n \circlearrowleft$

$\mathrm{Hilb}^n(\mathbb{C}^2)$ = Hilbert scheme of n-points on $\mathbb{C}^2$

= moduli space par. 0-dim $Z \subset \mathbb{C}^2$ with $\dim H^0(\mathcal{O}_Z) = n$

= " " " ideal sheaves $I \subset \mathcal{O}_{\mathbb{P}^2}$ s.t. $I|_{\ell_\infty} \xrightarrow{\sim} \mathcal{O}_{\ell_\infty}$

$$= \left\{ (A, B, i) \in \mathrm{End}(\mathbb{C}^n)^{\oplus 2} \oplus \mathrm{Hom}(\mathbb{C}, \mathbb{C}^n) \;\middle|\; \begin{array}{l} \cdot\ [A, B] = 0 \\ \cdot\ \nexists\, 0 \neq S \subsetneq \mathbb{C}^n,\ \begin{array}{l} A(S) = S \\ B(S) = S \end{array},\ \mathrm{Im}\, i \subseteq S \end{array} \right\} \Big/ GL(\mathbb{C}^n)$$


$\mathbb{C}^*\times\mathbb{C}^* \curvearrowright \mathbb{C}^2 \rightsquigarrow \mathbb{C}^*\times\mathbb{C}^* \curvearrowright \mathrm{Hilb}^n(\mathbb{C}^2)$
Loc. equ. coh. $H^*_{\mathbb{C}^*\times\mathbb{C}^*}(\ )_{loc} = H^*_{\mathbb{C}^*\times\mathbb{C}^*}(\ )\otimes_{H^*_{\mathbb{C}^*\times\mathbb{C}^*}(pt)}$ Field of fractions
$= H_{\mathbb{C}^*\times\mathbb{C}^*}(\ )\otimes_{\mathbb{C}[\xi_1,\xi_2]} \mathbb{C}(\xi_1,\xi_2)$
$\dim H^0(\mathcal{O}_Z)=n$
$\mathrm{Tr}\ I|_{\ell_\infty} \to \mathcal{O}_{\ell_\infty}$
$= S^n/GL(\mathbb{C}^n)$

$\mathbb{C}^*\times\mathbb{C}^* \curvearrowright \mathbb{C}^2 \rightsquigarrow \mathbb{C}^*\times\mathbb{C}^* \curvearrowright \mathrm{Hilb}^n(\mathbb{C}^2)$

Loc. equ. coh.: $H^*_{\mathbb{C}^*\times\mathbb{C}^*}(\ )_{loc} = H^*_{\mathbb{C}^*\times\mathbb{C}^*}(\ )\otimes_{H^*_{\mathbb{C}^*\times\mathbb{C}^*}(pt)}$ Field of fractions $\mathbb{C}(\varepsilon_1,\varepsilon_2)$

$= H_{\mathbb{C}^*\times\mathbb{C}^*}(\ )\otimes_{\mathbb{C}[\varepsilon_1,\varepsilon_2]}$

Heisenberg algebra $h_{\mathbb{C}^2}$: Lie alg. generated by $p_n(\alpha)$, $n\in\mathbb{Z}\setminus\{0\}$, $\alpha\in H^*_{\mathbb{C}^*\times\mathbb{C}^*}(\mathbb{C}^2)_{loc}$, c

satisfying

$[p_n(\alpha), c]=0$, $[p_n(\alpha), p_m(\beta)] = n\delta_{n+m,0}\langle\alpha,\beta\rangle c$

$\dim H^0(\mathcal{O}_Z)=n$

$\mathrm{tr}\ I_{\ell_\infty} \to 0_{\ell_\infty}$

$= S^n/GL(\mathbb{C}^n)$

$H^{*}_{\mathbb{C}^{*}\times\mathbb{C}^{*}}(\mathbb{C}^2)_{loc}, c$

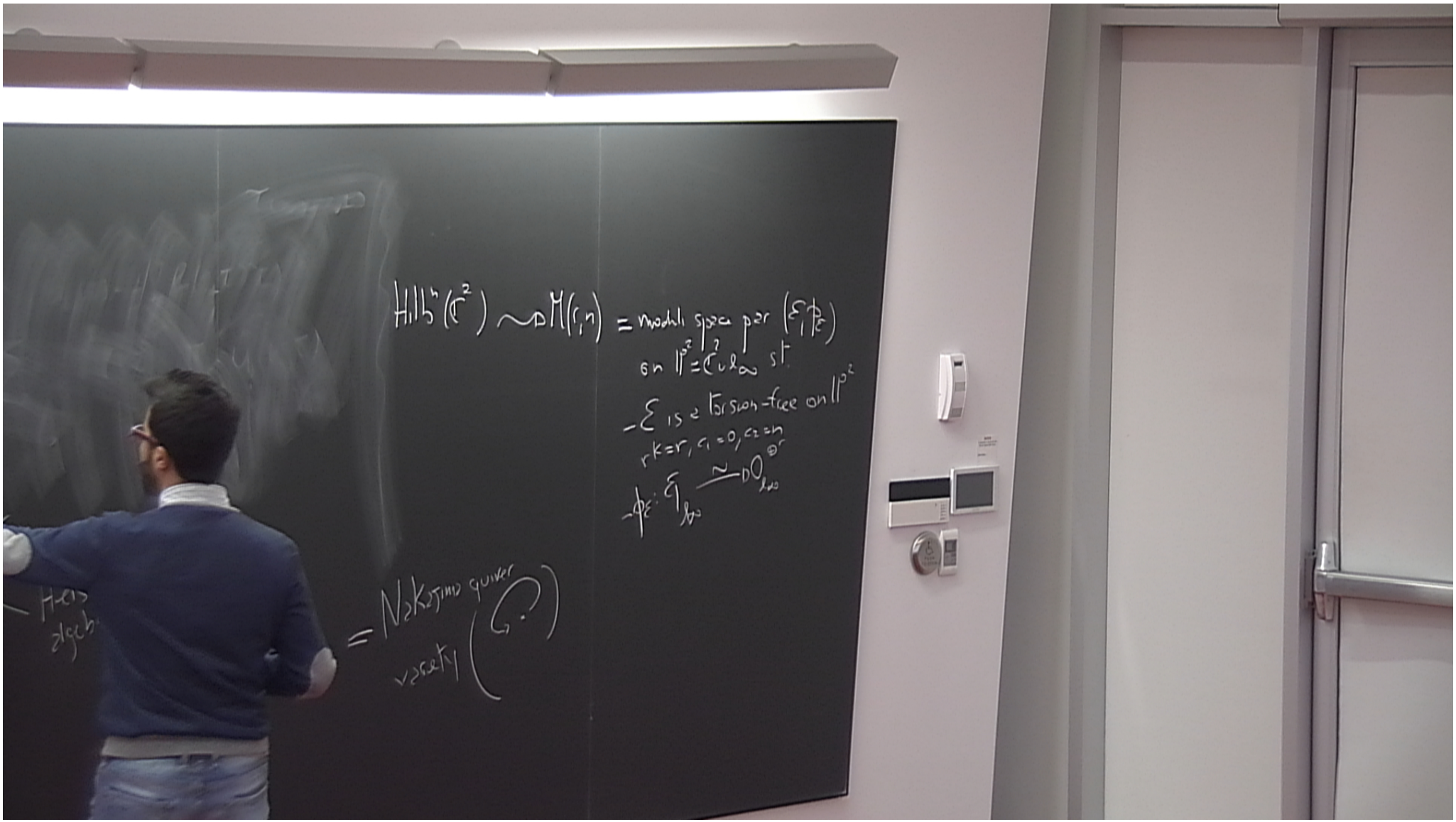
$\alpha, \beta > c$

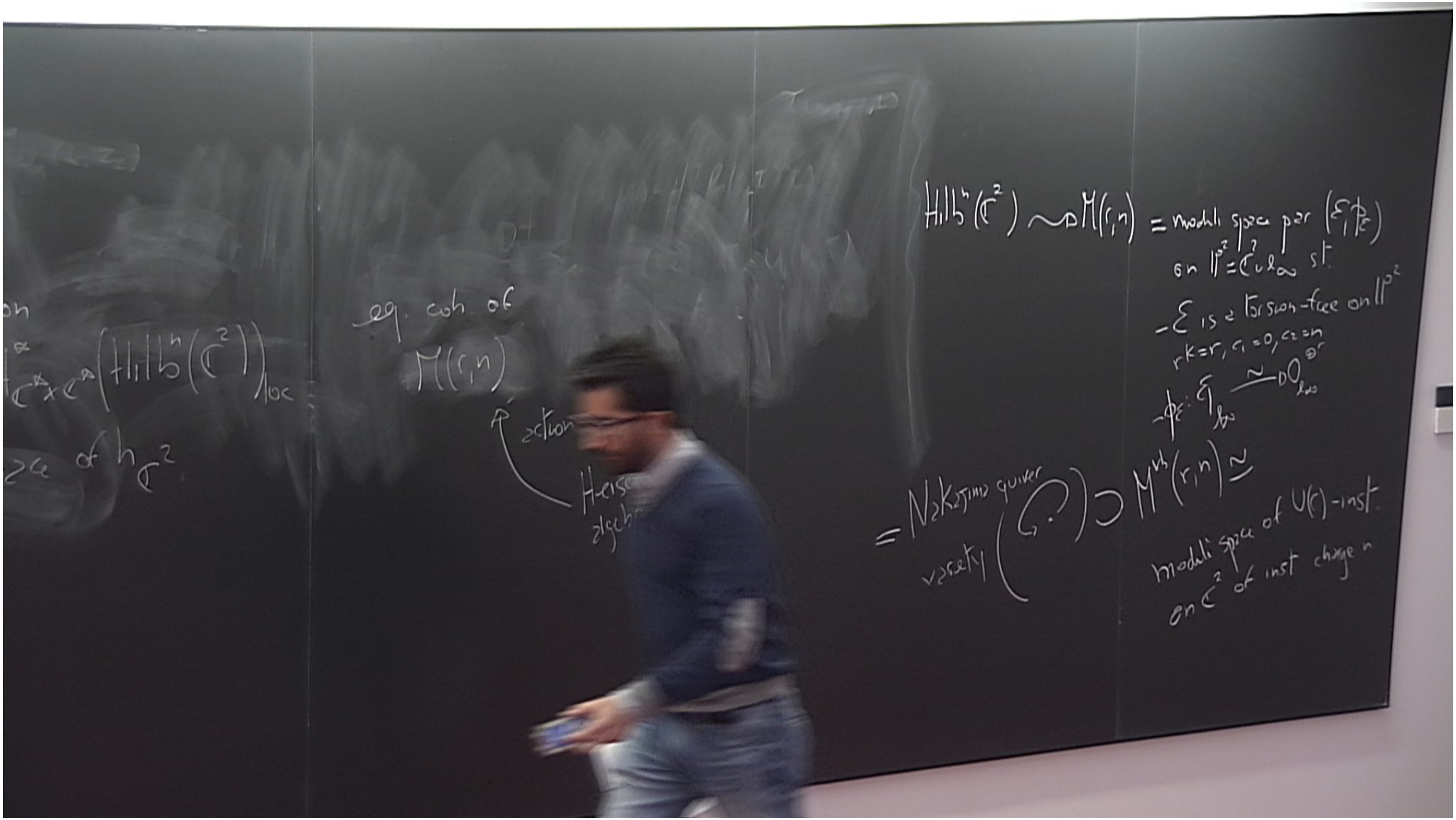
Theorem (Nakajima, G.)

There exists an action of $h_{\mathbb{C}^2}$ on

$$H_{\mathbb{C}^2}(1) = \bigoplus_n H^{*}_{\mathbb{C}^{*}\times\mathbb{C}^{*}}(\mathrm{Hilb}^n(\mathbb{C}^2))_{loc}$$

s.t. $H_{\mathbb{C}^2}(1) \simeq$ Fock space of $h_{\mathbb{C}^2}$


Hilb^n(C^2) ~ M(r,n) = moduli space par (E, φ_E)
on P^2 = C ∪ l_∞ s.t.
- E is a torsion-free on P^2
rk = r, c_1 = 0, c_2 = n
- φ_E : E|_{l_∞} ~ O_{l_∞}^{⊕r}
= Nakajima quiver variety


Hilb^n(C^2) ⇝ M(r,n) = moduli space par (E, φ_E)
on P^2 = C ∪ ℓ_∞ s.t.
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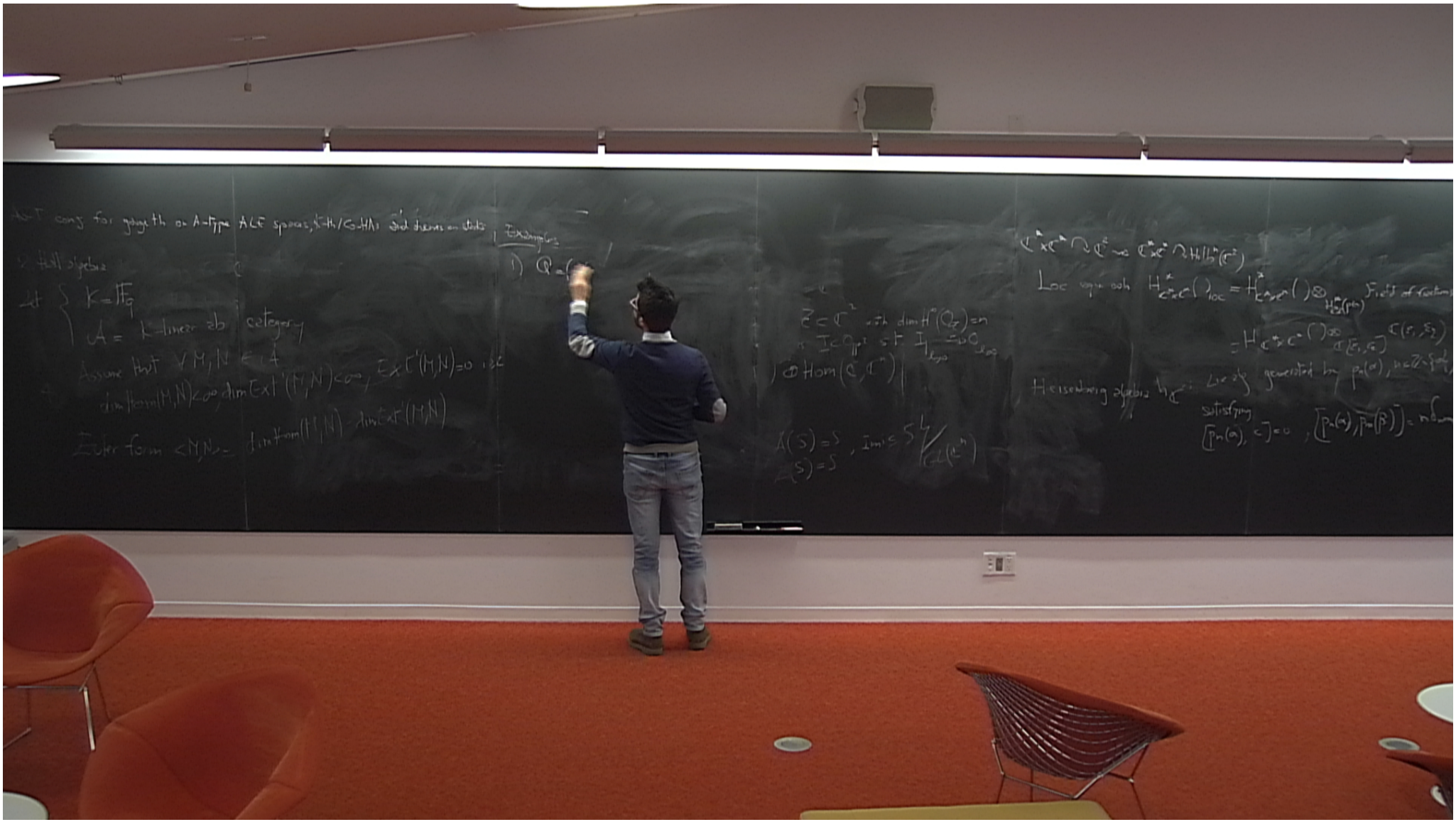
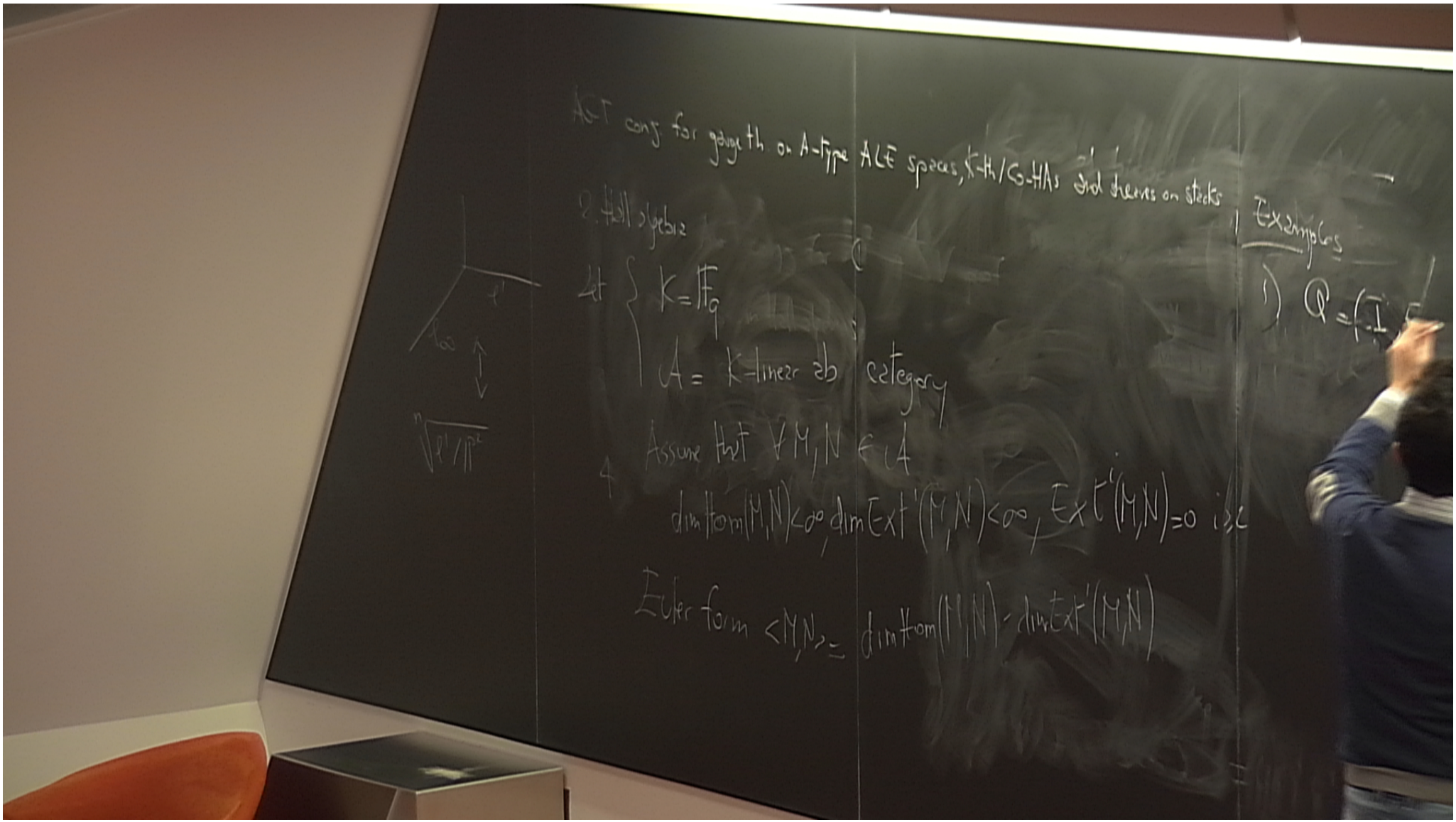
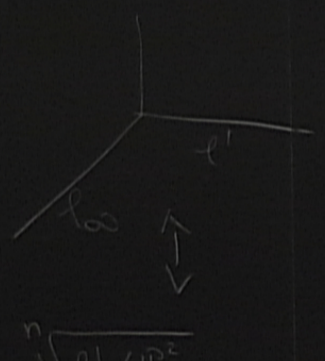
rk = r, c_1 = 0, c_2 = n
- φ_E : E|_{ℓ_∞} ≃ O_{ℓ_∞}^{⊕r}
= Nakajima quiver variety ( ) ⊃ M^{NS}(r,n) ≃
moduli space of U(r)-inst. on C^2 of inst charge n
eq. coh. of
M(r,n)
action
Heis
algebra


Examples
Hall algebra
Euler form


AGT conj for gauge th on A-type ALE spaces, K-th/G-HAs and theories on stacks
2. Hall algebra
Let K = $\mathbb{F}_q$
A = K-linear ab category
Assume that ∀ M, N ∈ A
dim Hom(M,N) < ∞, dim Ext¹(M,N) < ∞, Ext^i(M,N) = 0 i ≥ 2
Euler form ⟨M,N⟩ = dim Hom(M,N) − dim Ext¹(M,N)
Examples
1) Q =

AGT conj. for gauge th. on A-type ALE spaces, K-th / G-HAs and sheaves on stacks



2. Hall algebra

Let $\begin{cases} K = \mathbb{F}_q \\ \mathcal{A} = K\text{-linear ab. category} \end{cases}$

Assume that $\forall M, N \in \mathcal{A}$

$\dim \mathrm{Hom}(M,N) < \infty$, $\dim \mathrm{Ext}^1(M,N) < \infty$, $\mathrm{Ext}^i(M,N) = 0$ $i \geq 2$

Euler form $\langle M, N \rangle = \dim \mathrm{Hom}(M,N) - \dim \mathrm{Ext}^1(M,N)$

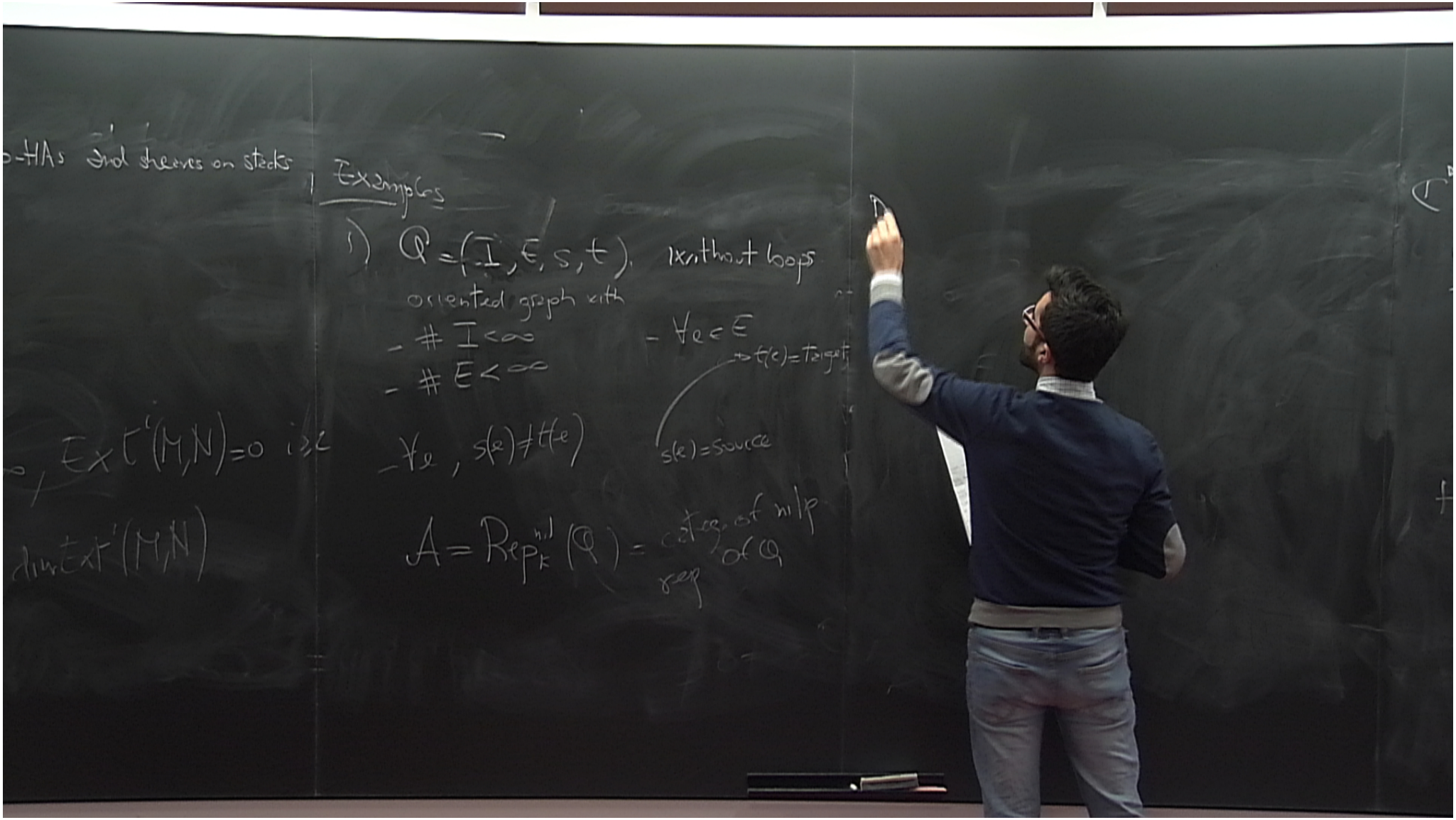
Examples

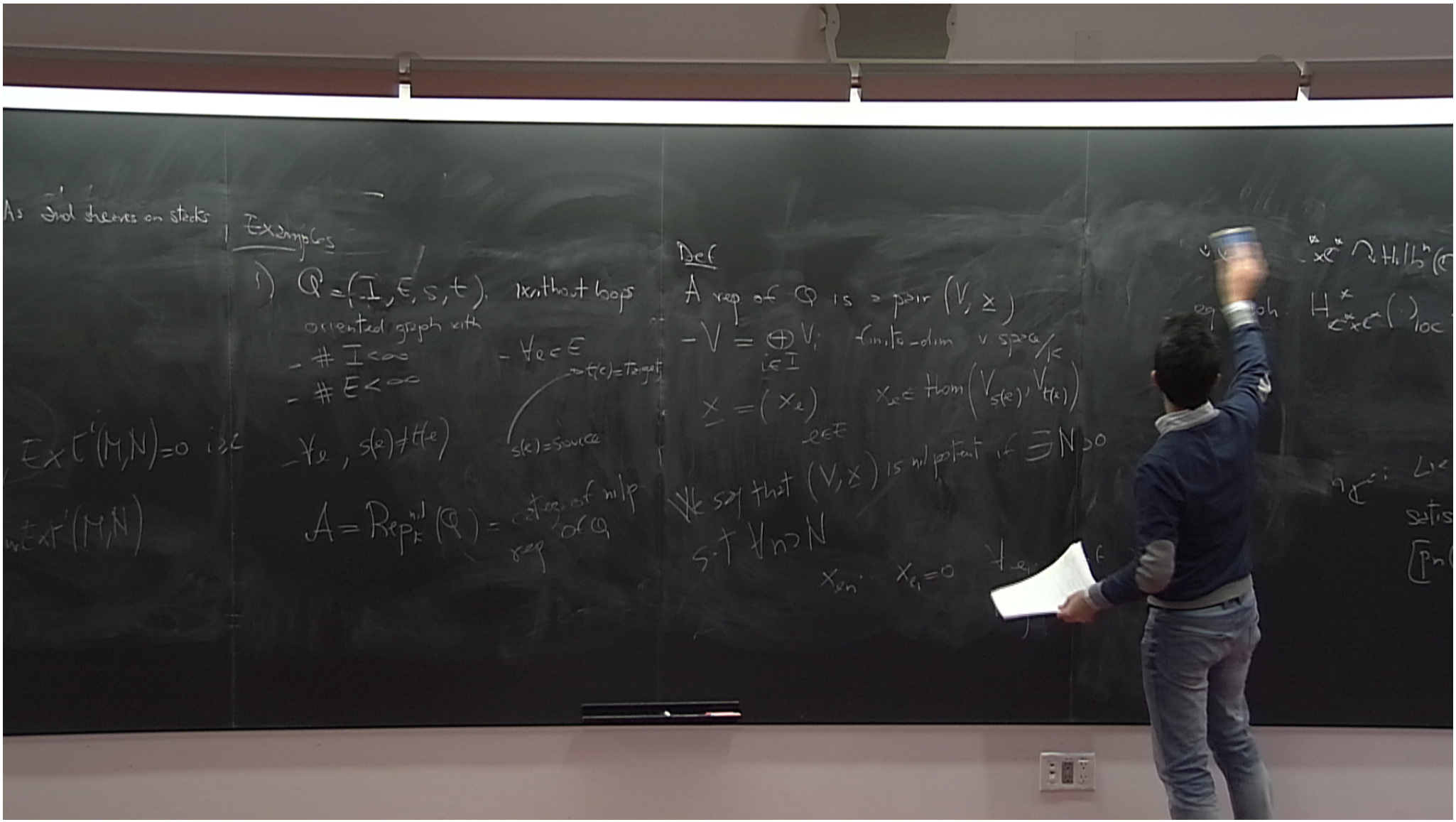
1) $Q = ($

orient

- # I

- # E


-HAs and sheaves on stacks
Examples
1) Q = (I, E, s, t), I without loops
oriented graph with
- # I < ∞
- # E < ∞
- ∀e ∈ E
t(e) = Target
s(e) = source
- ∀e, s(e) ≠ t(e)
A = Rep_k^nil (Q) = cat. of nilp rep of Q
Ext^i(M,N) = 0 i ≥ 2
dim Ext^1(M,N)


Examples
1) Q = (I, E, s, t). without loops
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A = Rep^nil_k(Q) = categ. of nilp rep of Q
Def
A rep of Q is a pair (V, x)
- V = ⊕ V_i, i ∈ I, finite-dim v. space/k
x = (x_e), e ∈ E, x_e ∈ Hom(V_s(e), V_t(e))
We say that (V, x) is nilpotent if ∃ N > 0
s.t. ∀ n > N
x_en : x_e1 = 0

of $\mathcal{A}$ is

$$\mathcal{A} = \bigoplus_{M \in \mathrm{Iso}(\mathcal{A})} \mathbb{C}[M]$$

$$M, N \mapsto \sum_{R} \frac{\#\{0 \to N \to R \to M \to 0\}}{\#\mathrm{Aut}(M)\,\#\mathrm{Aut}(N)} [R]$$

fine also Δ.

Extended Hall algebra

$$\tilde{H}_{\mathcal{A}} = H_{\mathcal{A}} \otimes_{\mathbb{C}} K_{\mathcal{A}}$$

$$K_{\mathcal{A}} = \mathbb{C}[K(\mathcal{A})]$$

Thm (Green, Ringel)

Let us introduce $S_i = (\oplus V_\ell, \underline{0})$

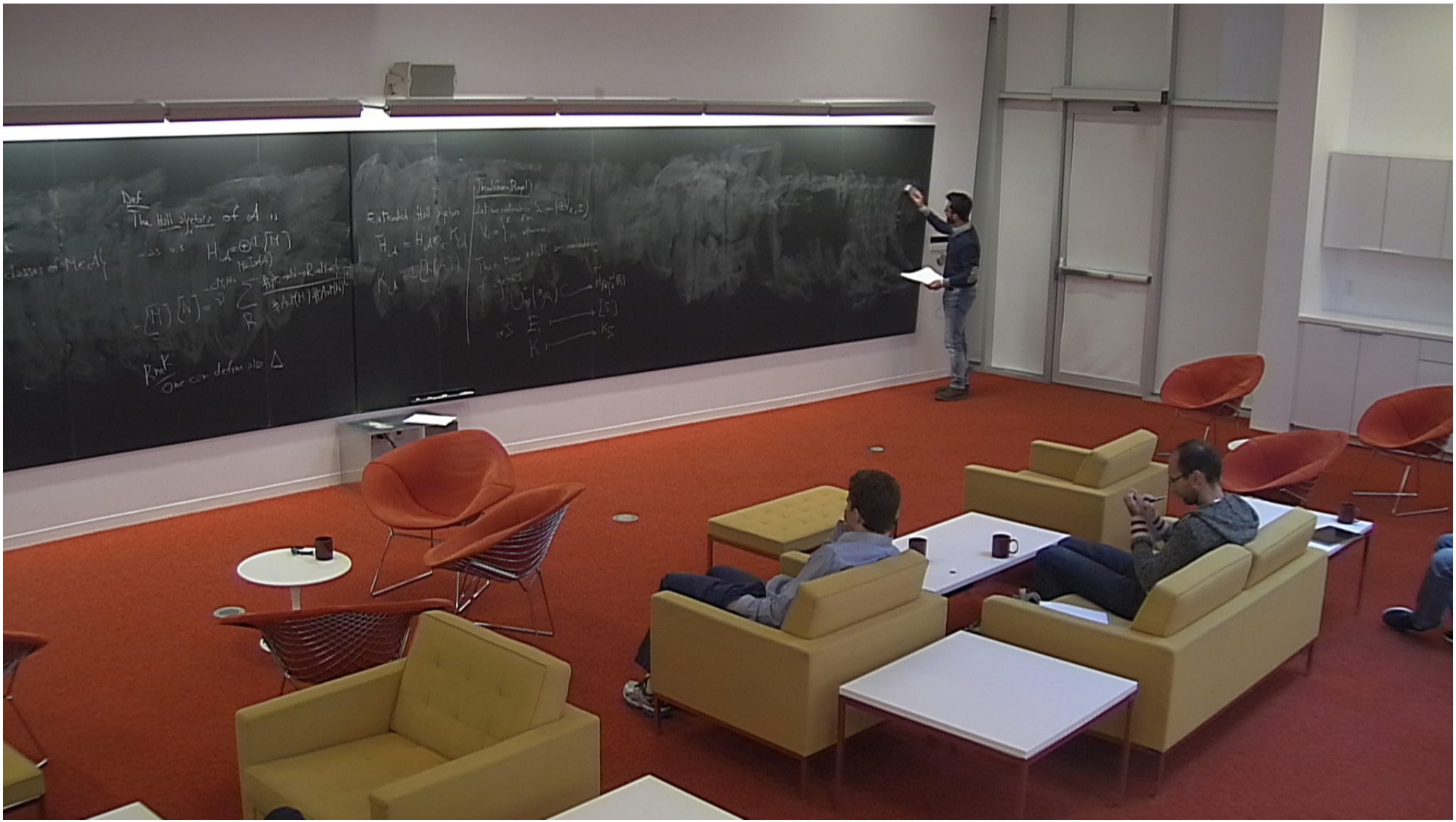
$$V_\ell = \begin{cases} k & \ell = i \\ 0 & \text{otherwise.} \end{cases}$$

Then there exists an embedding of algebras

$$U_q^+(\mathfrak{g}_Q) \hookrightarrow \tilde{H}_{\mathrm{Rep}_k^{nil}(Q)}$$

$$i \in I \quad E_i \longmapsto [S_i]$$

$$K_i \longmapsto K_{S_i}$$


Def
The Hall algebra of A is
Rmk
One can define also Δ
Extended Hall algebra
Theorem (Bapl)

Thm (Green, Riegel)

Let us introduce $S_i = (\oplus V_\ell, \underline{0})$

$V_\ell = \begin{cases} k & \ell = i \\ 0 & \text{otherwise.} \end{cases}$

Then there exists an embedding of algebras

$$U_v^+(\mathfrak{g}_Q) \hookrightarrow \tilde{H}_{\mathrm{Rep}_k^{nil}(Q)}$$

$$i \in I \quad E_i \longmapsto [S_i]$$

$$K_i \longmapsto K_{S_i}$$

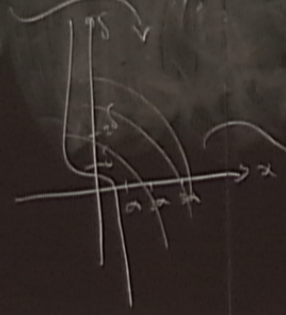
3) $X = \mathbb{P}^1$

$$K(\mathbb{P}^1) \cong \mathbb{Z}\,\bar{\mathcal{O}}_{\mathbb{P}^1} \oplus \mathbb{Z}\,\bar{\mathcal{O}}_x \xrightarrow[\psi]{\sim} \hat{Q} \quad \text{root lattice of } \hat{sl}_2$$

$$\bar{\mathcal{E}} = r\,\bar{\mathcal{O}}_{\mathbb{P}^1} + d\,\bar{\mathcal{O}}_x \qquad \bar{\mathcal{O}}_{\mathbb{P}^1} \longmapsto \alpha$$

$$\bar{\mathcal{O}}_x \longmapsto \delta$$

$$\psi\Big(\underbrace{\{r\,\bar{\mathcal{O}}_{\mathbb{P}^1} + d\,\bar{\mathcal{O}}_x \mid r>0, d\in\mathbb{Z}\}}_{\delta}, \underbrace{\{\bar{\mathcal{O}}_x \mid d>0\}}_{\hat{Q}}\Big) \longrightarrow Lb_+ \subset \hat{sl}_2$$



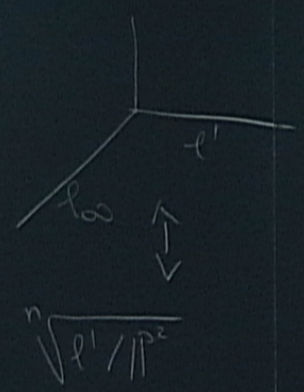
Thm

$\exists\ U_v(Lb_+) \hookrightarrow \tilde{H}_{Coh(\mathbb{P}^1)}$

$i \in \mathbb{Z} \quad E_i \longmapsto [\mathcal{O}_{\mathbb{P}^1}(i)]$

$n \in \mathbb{Z},\ n \geq 1 \quad H_n \longmapsto$ element that dep on Fourier sheaves of degree n

AGT conj. for gauge th on A-type ALE spaces, K-th / G-HAs and sheaves on stacks



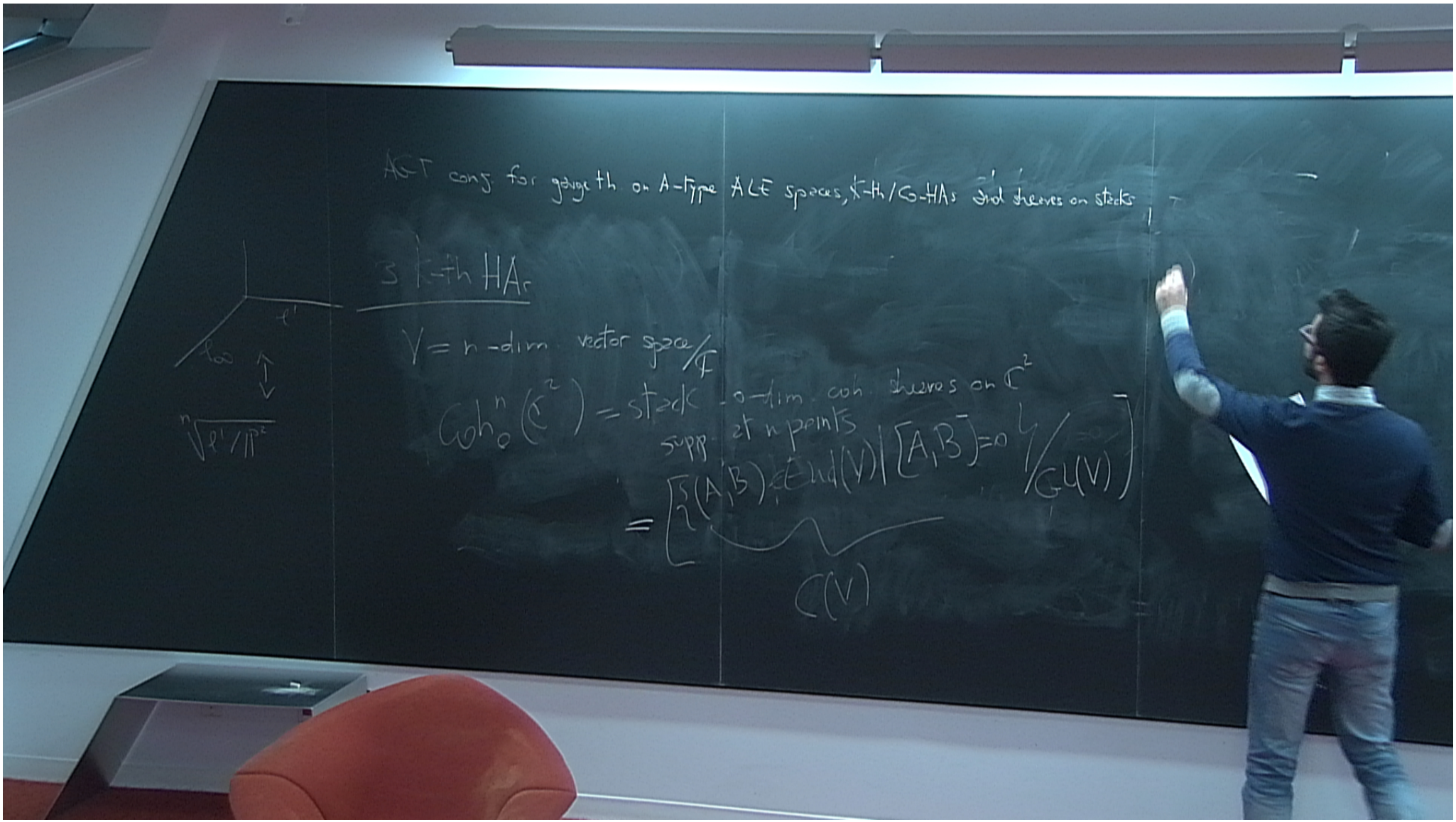
Thm (Burban-Schiffmann)

X = elliptic curve/$\mathbb{F}$

$\exists$ subalgebra $\subset \tilde{H}_{Coh(X)}$

s.t. positive part of

quantum Tor algebra of type $gl(1)$.


AGT conj for gauge th on A-type ALE spaces, K-th/G-HAs and sheaves on stacks
s K-th HAs
V = n-dim vector space/F
Coh_0^n(C^2) = stack 0-dim coh sheaves on C^2
supp at n points
= [{(A,B) ∈ End(V) | [A,B]=0} / GL(V)]
C(V)

and sheaves on stacks

s on $\mathbb{C}^2$

$B]=0$ $\quad V/GL(V)$

Def (SV)

The K-th Hall algebra of $\mathbb{C}^2$ is

$$KHA_{\mathbb{C}^2} = \varinjlim_{V} K_0^{\mathbb{C}^*\times\mathbb{C}^*}\left(\left[\mathbb{C}^{(V)}/GL(V)\right]\right)$$

equipped with the mult given pull back - push

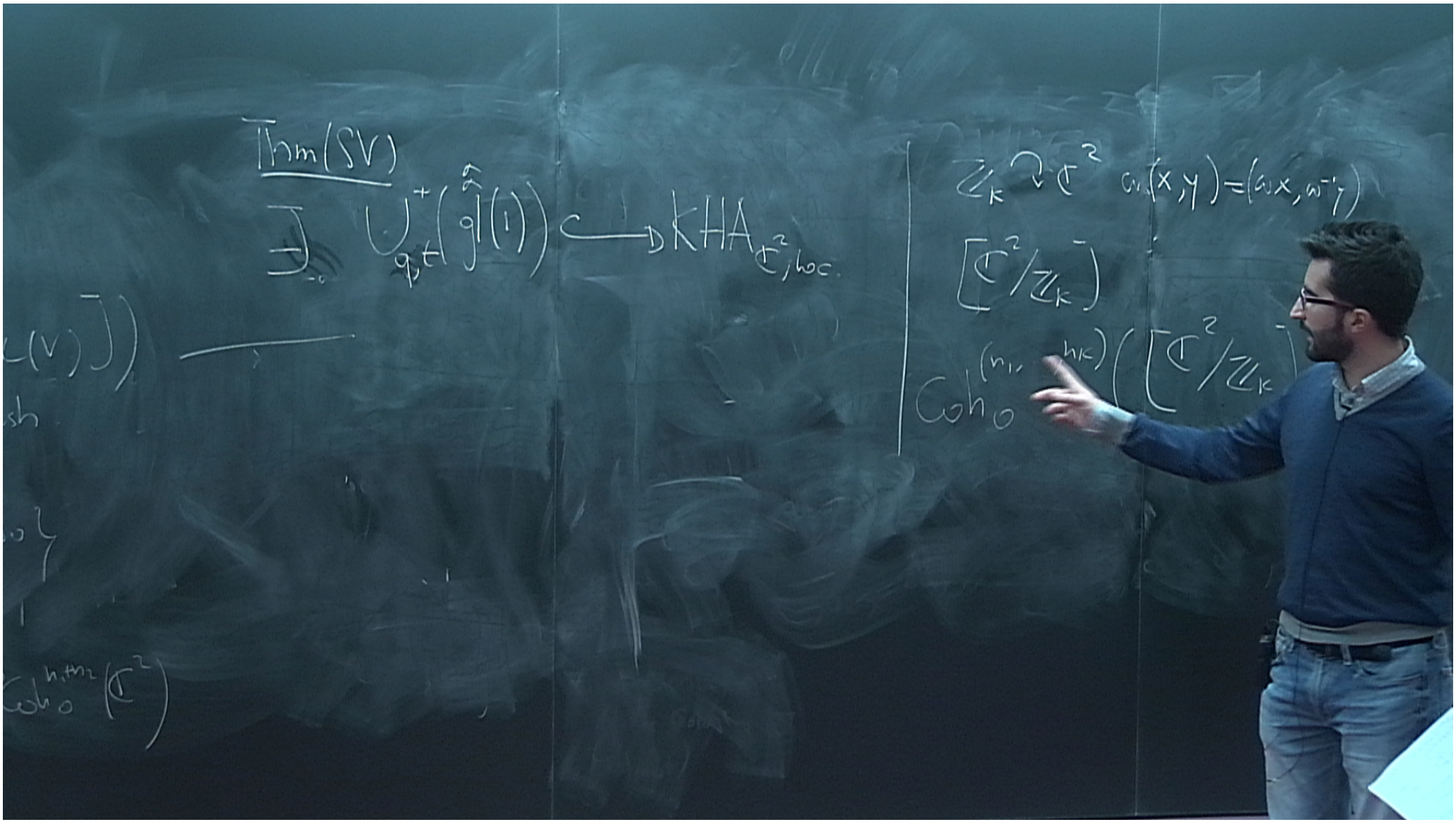
$$\{0 \to E_1 \to E \to E_2 \to 0\}$$

$$(E_1, E_2) \qquad E$$

$$Coh_0^{n_1}(\mathbb{C}^2) \times Coh_0^{n_2}(\mathbb{C}^2) \qquad Coh_0^{n_1+n_2}(\mathbb{C}^2)$$

Thm (SV)

$\exists$


Thm (SV)
$\exists$ $U_{q,t}^{+}(\hat{\mathfrak{g}}\mathfrak{l}(1)) \hookrightarrow KHA_{\mathbb{C}^2,loc}$
$\mathbb{Z}_k \curvearrowright \mathbb{C}^2$ $\omega(x,y)=(\omega x, \omega^{-1} y)$
$[\mathbb{C}^2/\mathbb{Z}_k]$
$Coh_0^{(n_1,\ldots,n_k)}([\mathbb{C}^2/\mathbb{Z}_k])$
$Coh_0^{n_1+n_2}(\mathbb{C}^2)$

<u>Thm (SV)</u>

$\exists$ $U^+_{q,t}(\hat{\mathfrak{gl}}(1)) \hookrightarrow KHA_{\mathbb{C}^2,loc}$

$\mathbb{Z}_k \curvearrowright \mathbb{C}^2$ $\omega(x,y) = (\omega x, \omega^{-1} y)$

$[\mathbb{C}^2/\mathbb{Z}_k]$

$\mathrm{Coh}_0^{(n_1,\ldots,n_k)}([\mathbb{C}^2/\mathbb{Z}_k]) = [\mathbb{C}_k(V)/\mathbb{T}$

$\mathrm{GL}(V)])$

ash

$\mathrm{Coh}_0^{n_1+n_2}(\mathbb{C}^2)$

Thm (N.)

$\exists$ $U^+_{q,t}(\hat{\mathfrak{gl}}(k)) \hookrightarrow KH_{[\mathbb{C}^2/\mathbb{Z}_k],loc}$

$\vec{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n_{\geq 0}$ $\sum k_i = k$

$U_i = [\mathbb{C}^2/\mathbb{Z}_{k_i}] \subset \mathcal{X}^\circ_{\vec{k}}$

$D_\infty = [\mathbb{P}^1/\mathbb{Z}_k] \rightsquigarrow \mathcal{X}_{\vec{k}} = \mathcal{X}^\circ_{\vec{k}} \cup D_\infty$

$\vec{v}, \vec{w} \in \mathbb{Z}^k_{\geq 0}$

$\mathcal{M}_{\vec{k}}(\vec{v}, \vec{w})$ = moduli space of framed sheaves on $\mathcal{X}_{\vec{k}}$

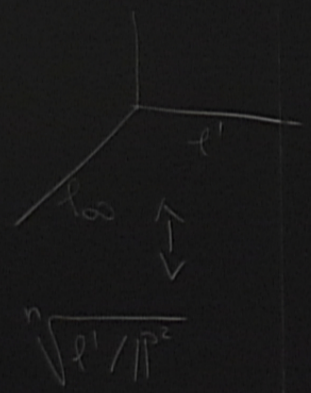
= " " pair $(\mathcal{E}, \phi_{\mathcal{E}})$ on $\mathcal{X}_{\vec{k}}$ with k-th class given by $\vec{v}$

- $\mathcal{E}$ is a torsion-free sheaf on $\mathcal{X}_{\vec{k}}$

- $\phi_{\mathcal{E}} : \mathcal{E}|_{D_\infty} \xrightarrow{\sim} \bigoplus_{i=0}^{k-1} (\mathcal{O}_{\mathbb{P}^1} \otimes R_i)^{\oplus w_i}$

i-th irr. rep of $\mathbb{Z}_k$

AGT conj. for gauge th. on A-type ALE spaces, K-th/G-HAs and sheaves on stacks



Thm. (Nekrasov $\vec{k}=(k)$, S-Schiffmann $\forall \vec{k}$) $\qquad r=\sum w_i$

Let

$$K_{\vec{k}}(\vec{w}) = \bigoplus_{\vec{v}} K_0^{T'}\left(\mathcal{M}_{\vec{k}}(\vec{v},\vec{w})\right) \qquad T=\mathbb{C}^*\times\mathbb{C}^*\times(\mathbb{C}^*)^r$$

Then there exists an action of

$$K\text{-}HA_{[\mathbb{C}^2/\mathbb{Z}_{k_1}]}\otimes\cdots\otimes K\text{-}HA_{[\mathbb{C}^2/\mathbb{Z}_{k_n}]}$$

on $K_{\vec{k}}(\vec{w})$.

Def. (S

The

K

equipped

spaces, K-th / G-HAs and sheaves on stacks

$\forall \vec{k})$ $\quad r = \sum w_i$

$\vec{w}))$ $\quad T = \mathbb{C}^* \times \mathbb{C}^* \times (\mathbb{C}^*)^r$

$[\mathbb{C}^2 / \mathbb{Z}_{r-n})$

$i = 1, \dots, n-1$

$\mathrm{Coh}^{\vec{c}_i}_{\mathbb{P}^1_{k_i, k_{i+1}}}$ = stack of coh. sheaves on $X_{\vec{k}}$ supported at $\mathbb{P}^1_{k_i, k_{i+1}}$ with the K-theory class given by $\vec{c}_i \in \mathbb{Z}^{k_i + k_{i+1}}$

$$= \bigcup_{n \geq 1} \left[S_n^{\vec{c}_i} / G_n^{\vec{c}_i} \right]$$

Thm (S - Schiffmann - $\vec{k}=(1,\dots,1)$)

The i. space

$$KH^i_{\mathbb{P}^1} = \bigoplus_{(r_i,d_i)} \left(\varinjlim_n K_0^{\mathbb{C}^\times\times\mathbb{C}^\times}\left(\left[S_n^{(r_i,d_i)} \big/ G_n^{(r_i,d_i)} \right] \right) \right)$$

has an associative algebra structure. Moreover, it acts on $H_{\vec{k}}(\vec{v},\vec{w})$

$$\mathrm{Gr}^{(r_i,d_i)}_{\mathbb{P}^1}(X_{\vec{k}})$$

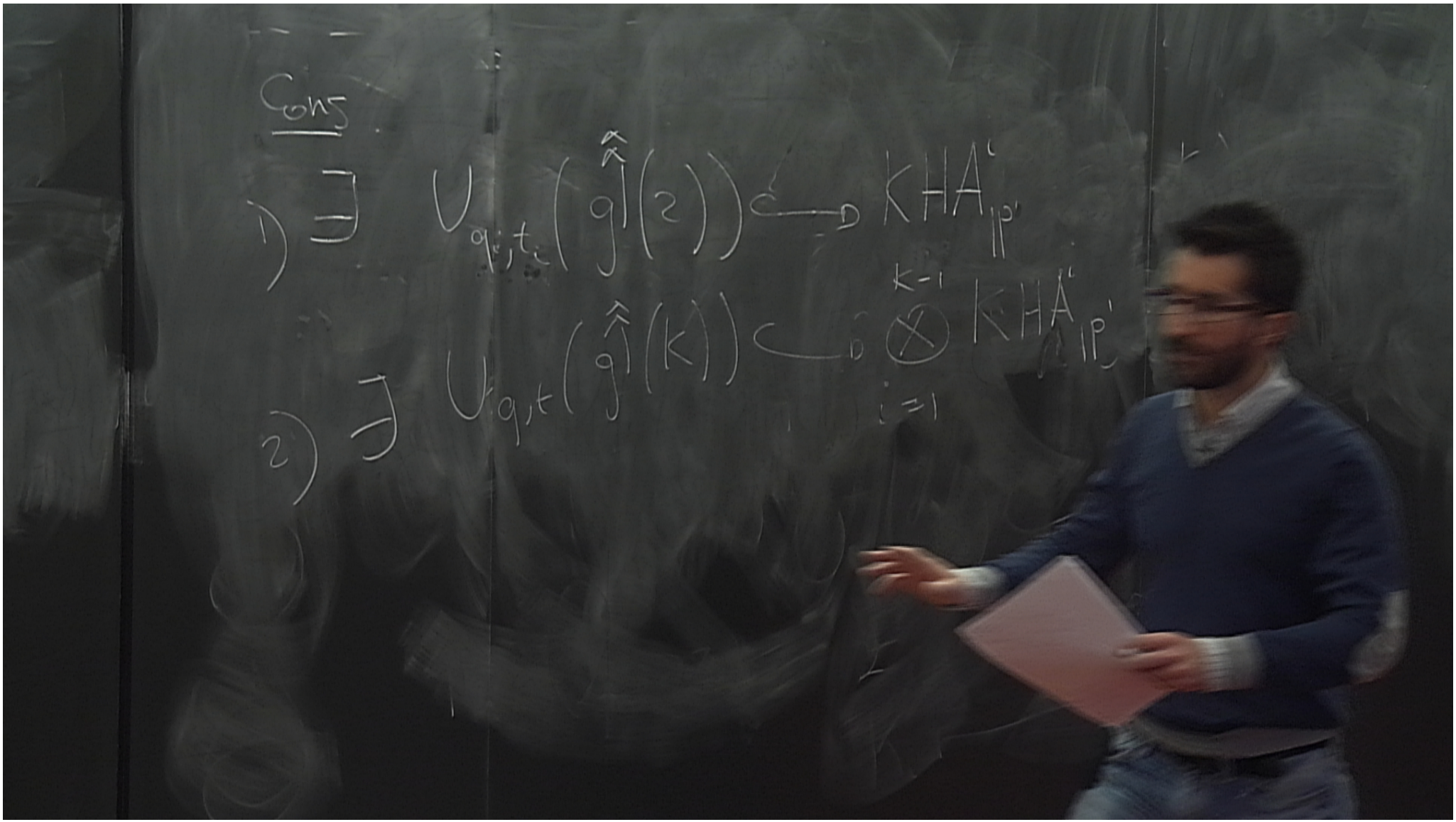
$$\uparrow$$

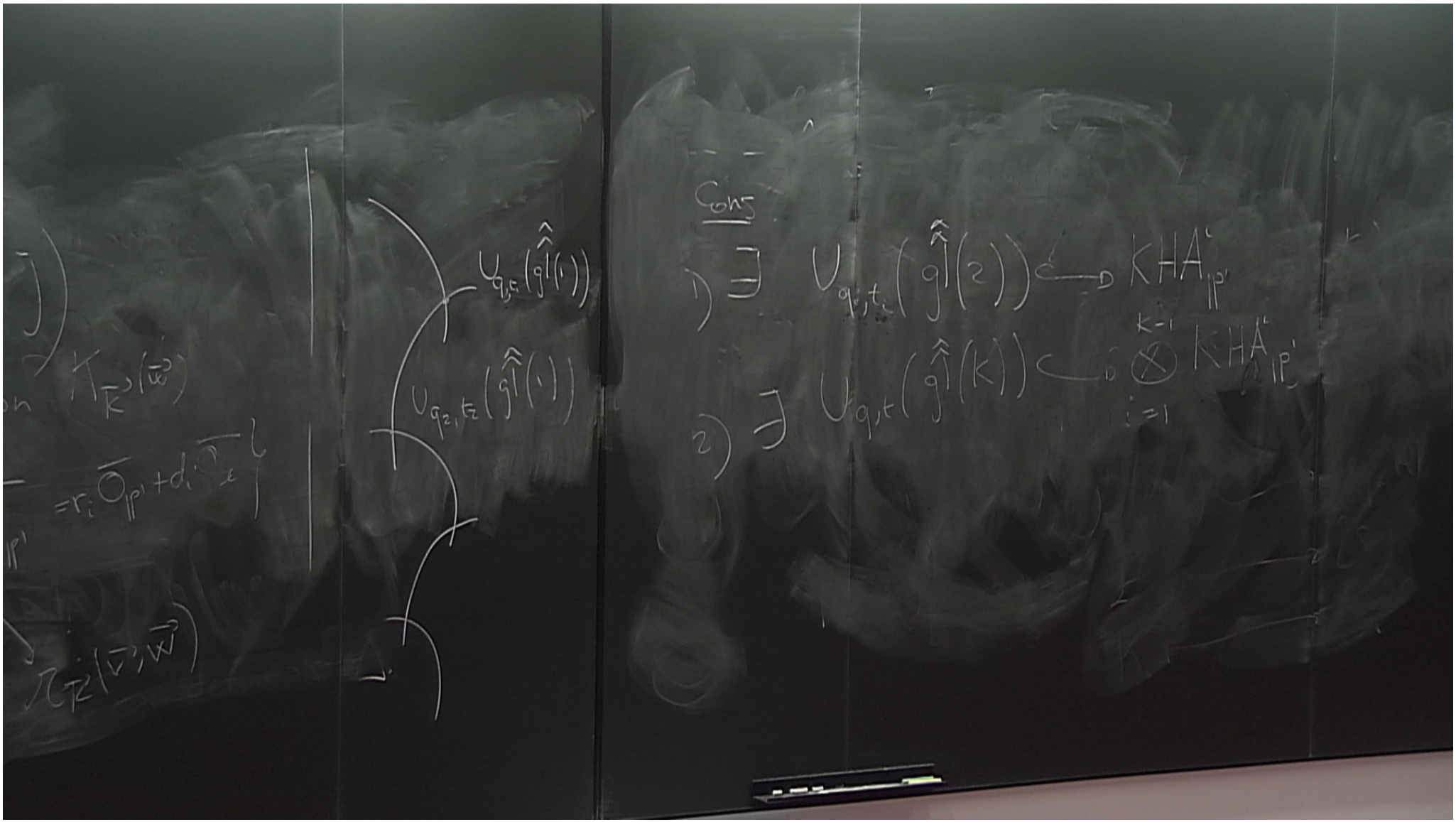
$$\left\{ \mathcal{E}_1 \subset \mathcal{E}_2 \;\middle|\; \mathrm{supp}(\mathcal{E}_2/\mathcal{E}_1) \subseteq D_1 \simeq \mathbb{P}^1,\ \overline{\mathcal{E}_2/\mathcal{E}_1}\big|_{\mathbb{P}^1} = r_i\,\mathcal{O}_{\mathbb{P}^1} + d_i\,\mathcal{O}_x \right\}$$

$$\mathcal{E}_1 \qquad\qquad \mathcal{E}_2$$

$$\Omega_{\vec{k}}(\vec{v},\vec{w})$$

$$\Omega_{\vec{k}}\left(\vec{v} + (r_i+d_i)(1,\dots,1) - r_i\vec{e}_i\right)$$


Cons
1) ∃ $U_{q_i,t_i}(\hat{\hat{g}}(2)) \hookrightarrow KHA'_{\mathbb{P}^1}$
2) ∃ $U_{q,t}(\hat{\hat{g}}(K)) \hookrightarrow \bigotimes_{i=1}^{K-1} KHA'_{\mathbb{P}^1}$


Cons
1) ∃ U_{q,t}(ĝl(2)) ↪ KHA_{P^1}
2) ∃ U_{q,t}(ĝl(k)) ↪ ⊗_{i=1}^{k-1} KHA_{P^1}
U_{q,t}(ĝl(1))
U_{q2,t2}(ĝl(1))

$$U^+_{q_1,t_1}(\hat{\mathfrak{gl}}(1)) \otimes \cdots \otimes U^+_{q_k,t_k}(\hat{\mathfrak{gl}}(1)) \xhookrightarrow{\text{Cons}} U_{q,t}(\hat{\mathfrak{gl}}(k))$$

$$\cap \qquad\qquad\qquad\qquad \cap \text{ Cons}$$

$$A_{\mathbb{C}^2} \otimes \cdots \otimes KHA_{\mathbb{C}^2,loc} \hookrightarrow \bigotimes_{i=1}^{k-1} KHA^i_{\mathbb{P}^1}$$