Title: Entanglement Entropy Scaling Laws and Eigenstate Thermalization in Many-Particle Systems

Date: Nov 10, 2015 03:30 PM

URL: http://pirsa.org/15110085

Abstract: <p>While entanglement entropy of ground states usually follows the area law, violations do exist, and it is important to understand their origin. In 1D they are found to be associated with quantum criticality. Until recently the only established examples of such violation in higher dimensions are free fermion ground states with Fermi surfaces, where it is found that the area law is enhanced by a logarithmic factor. In Ref. [1], we use multi-dimensional bosonization to provide a simple derivation of this result, and show that the logarithimic factor has a 1D origin. More importantly the bosonization technique allows us to take into account the Fermi liquid interactions, and obtain the leading scaling behavior of the entanglement entropy of Fermi liquids. The central result of our work is that Fermi liquid interactions do not alter the leading scaling behavior of the entanglement entropy, and the logarithmic enhancement of area law is a robust property of the Fermi liquid phase. In sharp contrast to the fermioic systems with Fermi surfaces, quantum critical (or gapless) bosonic systems do not violate the area law above 1D (except for the case discussed below). The fundamental difference lies in the fact that gapless excitations live near a single point (usually origin of momentum space) in such bosonic systems, while they live around an (extended) Fermi surface in Fermi liquids. In Ref. [2], we studied entanglement properties of some specific examples of the so called Bose metal states, in which bosons neither condense (and become a superfluid) nor localize (and</p>

<p>insulate) at T=0. The system supports gapless excitations around ``Bose surfaces", instead of isolated points in momentum space. We showed that similar to free Fermi gas and Fermi liquids, these states violate the entanglement area law in a logarithmic fashion. Compared to ground states, much less is known concretely about entanglement in</p>

<p>(highly) excited states. Going back to free fermion systems, in [3] we show that there exists a duality relation between ground and excited states, and the area law obeyed by ground state turns into a volume law for excited states, something that is widely expected but very hard to prove. Most importantly, we find in appropriate limits the reduced density matrix of a subsystem takes the form of thermal density matrix, providing an explicit example of the eigenstate thermalization hypothesis. Our work [3] explicitly demonstrates how statistical physics emerges from entanglement in a single eigenstate.</p>

<p> </p>

<p>[1] Entanglement Entropy of Fermi Liquids via Multi-dimensional Bosonization, Wenxin Ding, Alexander Seidel, Kun Yang, Phys. Rev. X 2,</p>

<p>011012 (2012).</p>

<p> </p>

<p>[2] Violation of Entanglement-Area Law in Bosonic Systems with Bose</p>

<p>Surfaces: Possible Application to Bose Metals, Hsin-Hua Lai, Kun Yang, N.</p>

<p>E. Bonesteel, Phys. Rev. Lett. 111, 210402 (2013).</p>

<p> </p>

<p>[3] Entanglement entropy scaling laws and eigenstate thermalization in free fermion systems, Hsin-Hua Lai, Kun Yang, Phys. Rev. B 91,081110 (2015)</p>

Summary of Results

- Ground state EE mostly obeys area law in high D, even in critical systems; free Fermi gas is the only known example that violates area law logarithmically, until recently.
- We use high-D bosonization to show that interacting fermions in the Fermi liquid phase, EE violates area law in a manner identical to free Fermi gas; super-criticality (W. Ding, A. Seidel and KY, PRX 12).
- We find a similar area-law violation in Bose liquids with Bose surfaces (H.-H. Lai, KY and N. Bonesteel, PRL 13).
- Shape of these critical surfaces may be determined by inspecting the coefficients of the logarithm (H.-H. Lai and KY, arXiv 15).
- EE of highly excited states (with finite excitation energy density) are expected to follow volume law, but very few explicit results available. We find it is **dual** to ground state area law, and of the **same** origin in free fermion systems. We further show that **thermalization** emerges under appropriate conditions (H.-H. Lai and KY, PRB 15).

Perhaps statistical mechanics and thermal entropy emerge from entanglement in a single eigenstate!

Free fermion system an example of this Eigenstate Thermalization Hypothesis.

(H.-H. Lai and KY, PRB 15)

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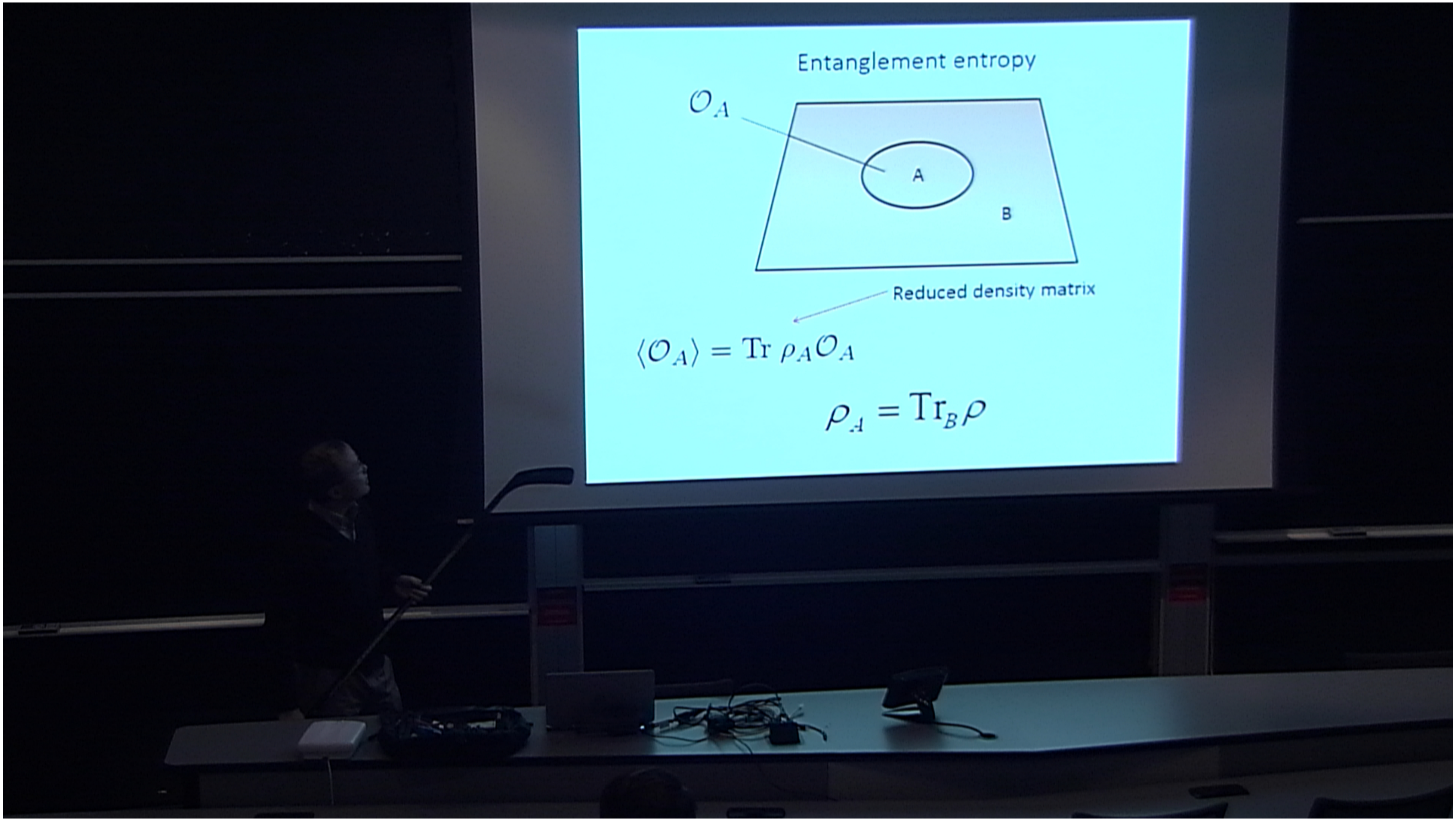
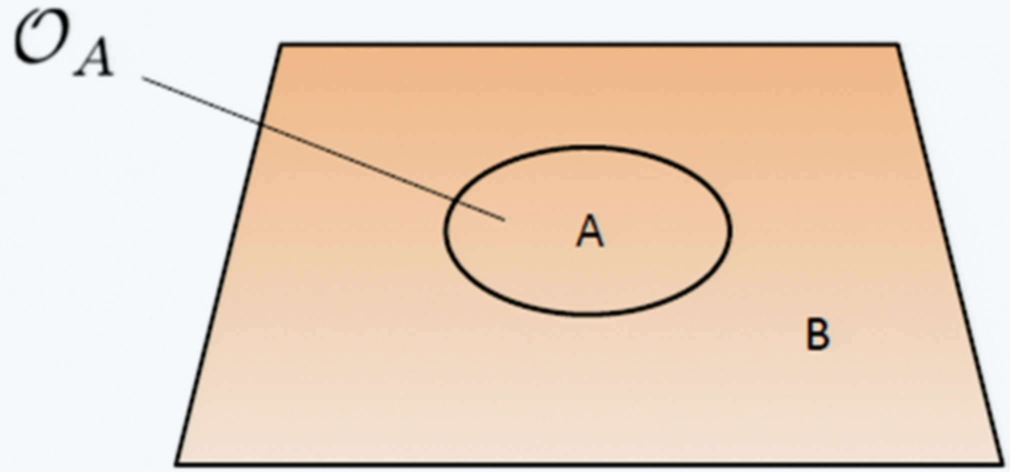

Entanglement entropy
$\mathcal{O}_A$
A
B
Reduced density matrix
$\langle \mathcal{O}_A \rangle = \mathrm{Tr}\ \rho_A \mathcal{O}_A$
$\rho_A = \mathrm{Tr}_B \rho$

Entanglement entropy



Reduced density matrix

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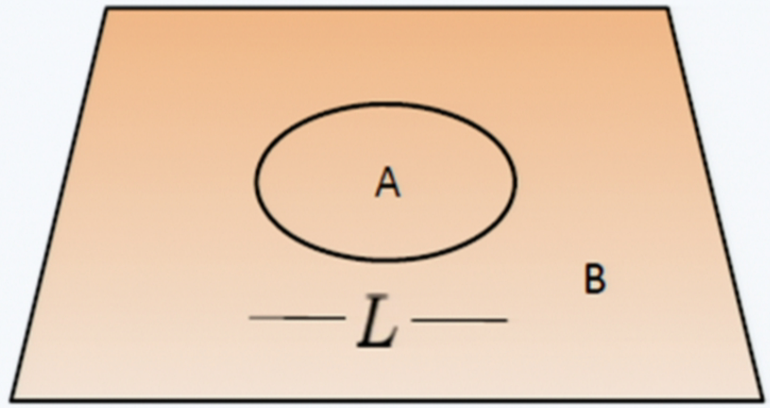
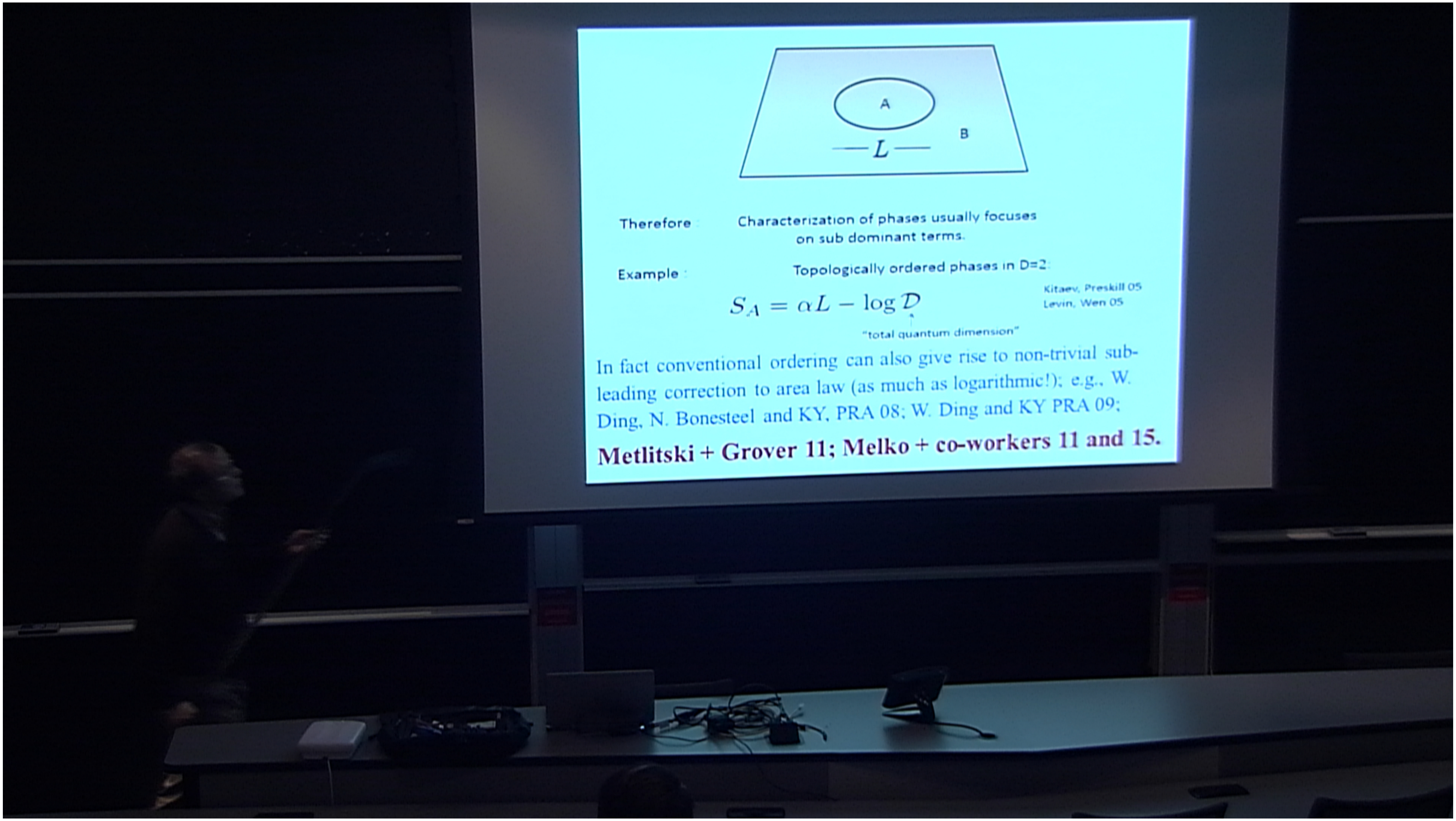
Entanglement entropy as characteristic of phases of matter

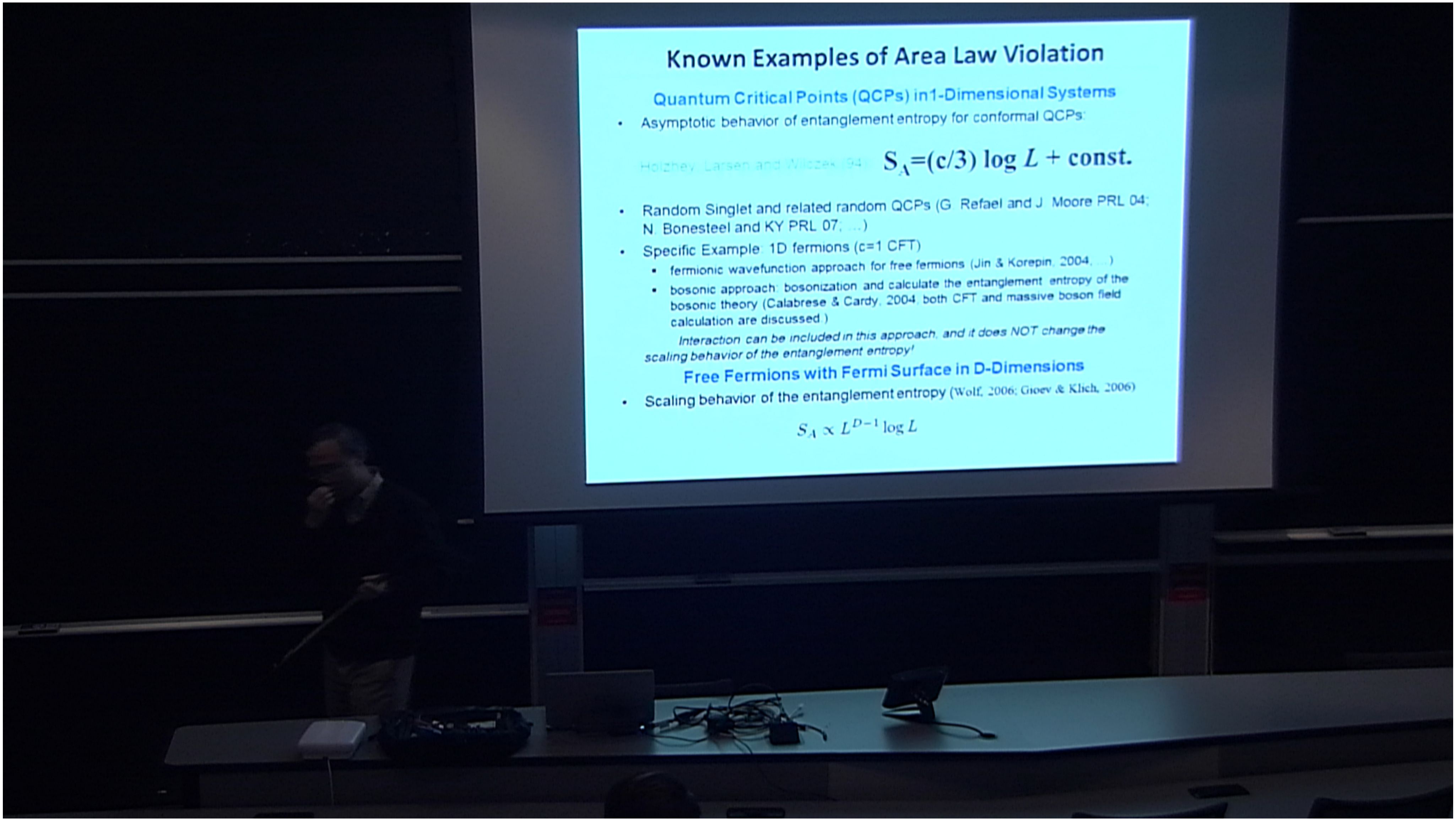


Cf. black hole entropy (Bombelli el at. 86', Holzhey el at.94')

Typically : $S_A \propto L^{D-1} \propto \text{surface area}$

Examples : Ground states of gapped local Hamiltonians
Most gapless Hamiltonians


A
B
L
Therefore : Characterization of phases usually focuses on sub dominant terms.
Example : Topologically ordered phases in D=2:
$S_A = \alpha L - \log \mathcal{D}$
Kitaev, Preskill 05
Levin, Wen 05
"total quantum dimension"
In fact conventional ordering can also give rise to non-trivial sub-leading correction to area law (as much as logarithmic!); e.g., W. Ding, N. Bonesteel and KY, PRA 08; W. Ding and KY PRA 09;
Metlitski + Grover 11; Melko + co-workers 11 and 15.


Known Examples of Area Law Violation
Quantum Critical Points (QCPs) in1-Dimensional Systems
Asymptotic behavior of entanglement entropy for conformal QCPs:
Holzhey, Larsen and Wilczek (94)
S_A=(c/3) log L + const.
Random Singlet and related random QCPs (G. Refael and J. Moore PRL 04; N. Bonesteel and KY PRL 07, ...)
Specific Example: 1D fermions (c=1 CFT)
fermionic wavefunction approach for free fermions (Jin & Korepin, 2004, ...)
bosonic approach: bosonization and calculate the entanglement entropy of the bosonic theory (Calabrese & Cardy, 2004, both CFT and massive boson field calculation are discussed.)
Interaction can be included in this approach, and it does NOT change the scaling behavior of the entanglement entropy!
Free Fermions with Fermi Surface in D-Dimensions
Scaling behavior of the entanglement entropy (Wolf, 2006; Gioev & Klich, 2006)
$S_A \propto L^{D-1} \log L$

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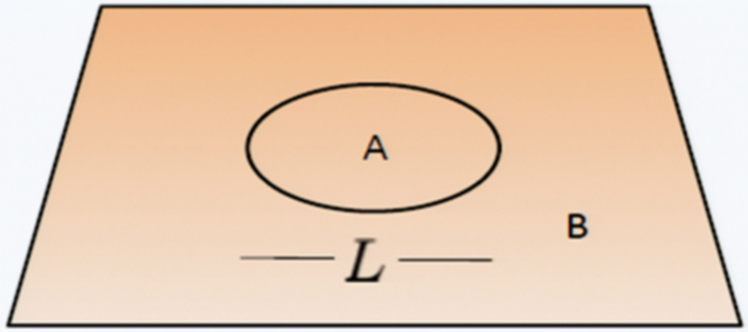

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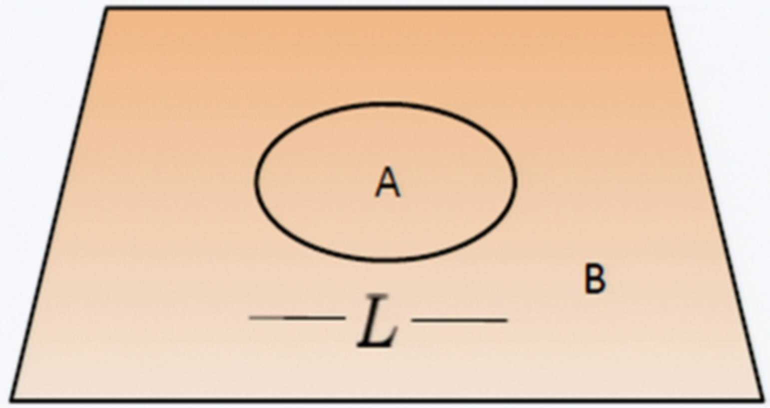

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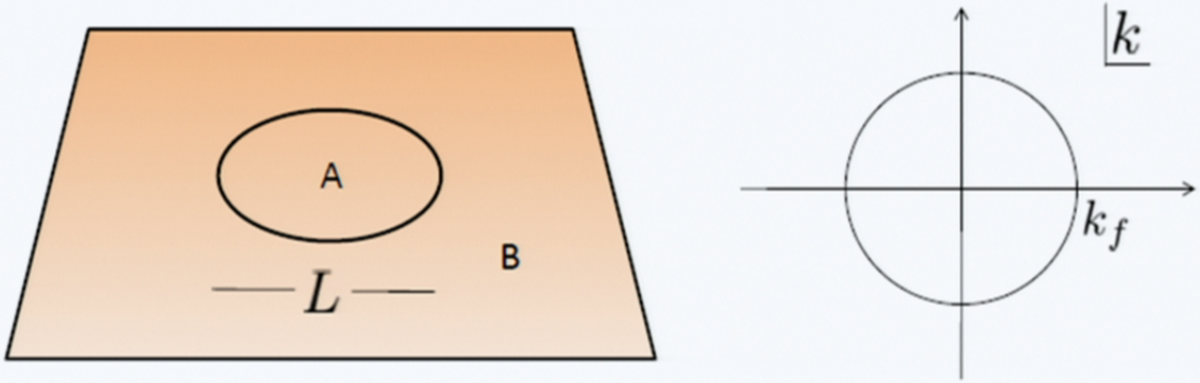
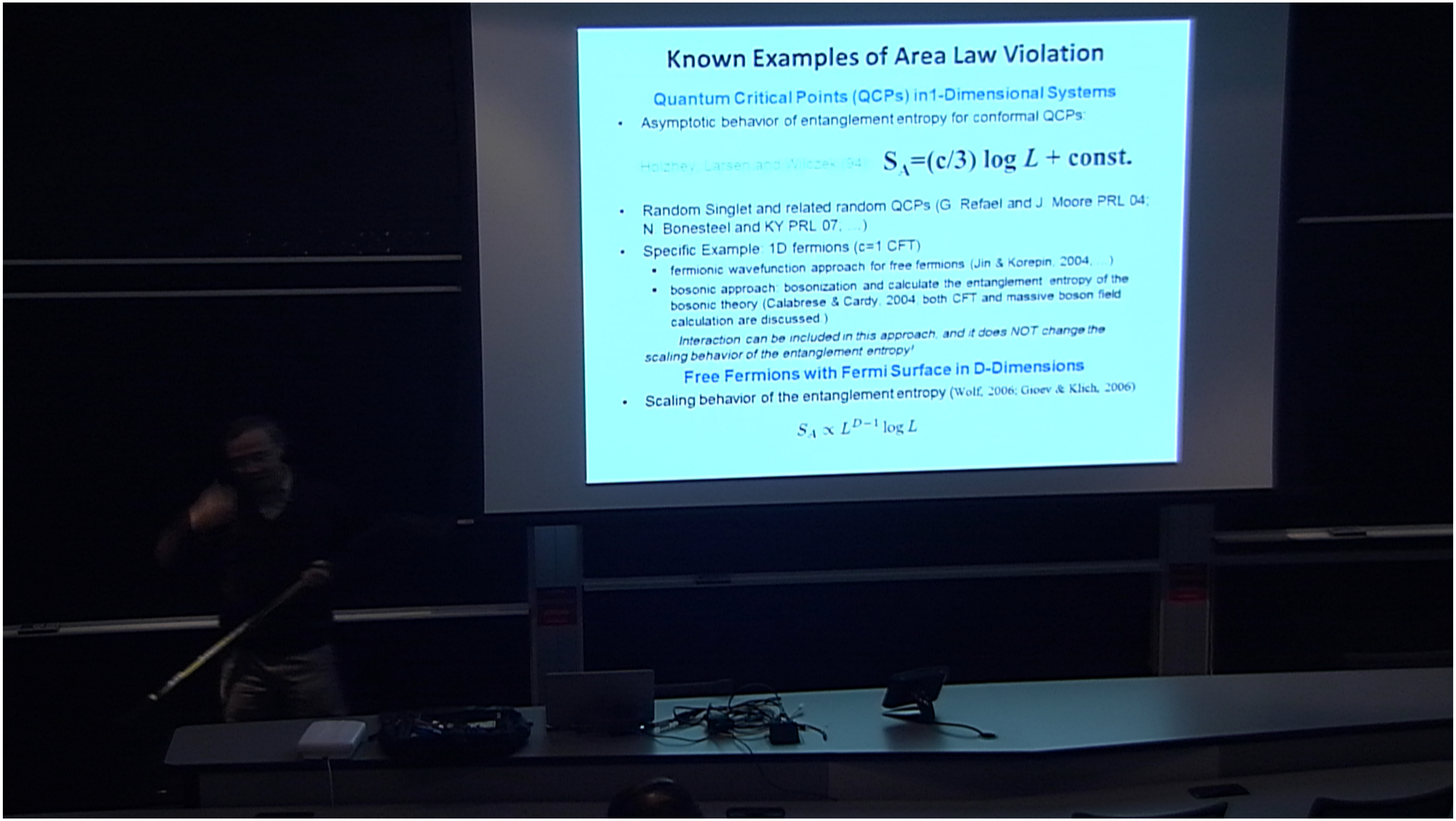
A More Accurate Formula



$$S_A = \frac{1}{12}\frac{\log L}{(2\pi)^{D-1}}\int_{\partial A}\int_{\partial\Gamma}|d\Sigma_{\boldsymbol{x}}\cdot d\Sigma_{\boldsymbol{k}}|$$
$$\propto L^{D-1}\log L$$

Gioev, Klich 06, using **Widom's conjecture**



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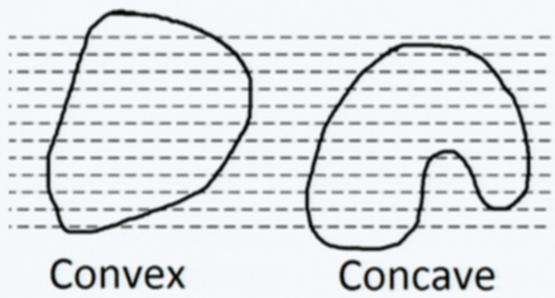
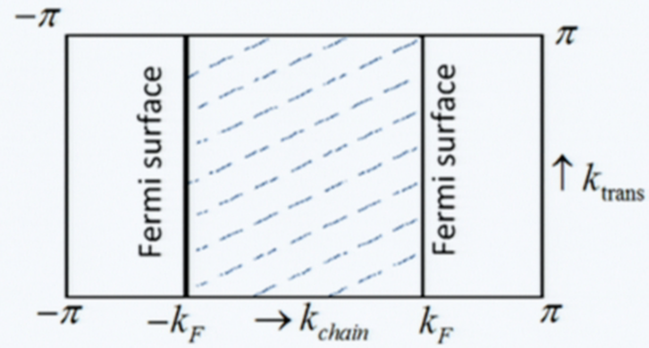
Intuitive Understanding of G&K's Result via a Toy Model

Decoupled parallel 1D chains





- Each intersection contributes an entanglement entropy of order $(1/3)\log L$
  L is the characteristic length scale of the subsystem. The contribution due to the shape adds up to of order L^{D-1} which is sub-leading.
- Sum over all the chains:

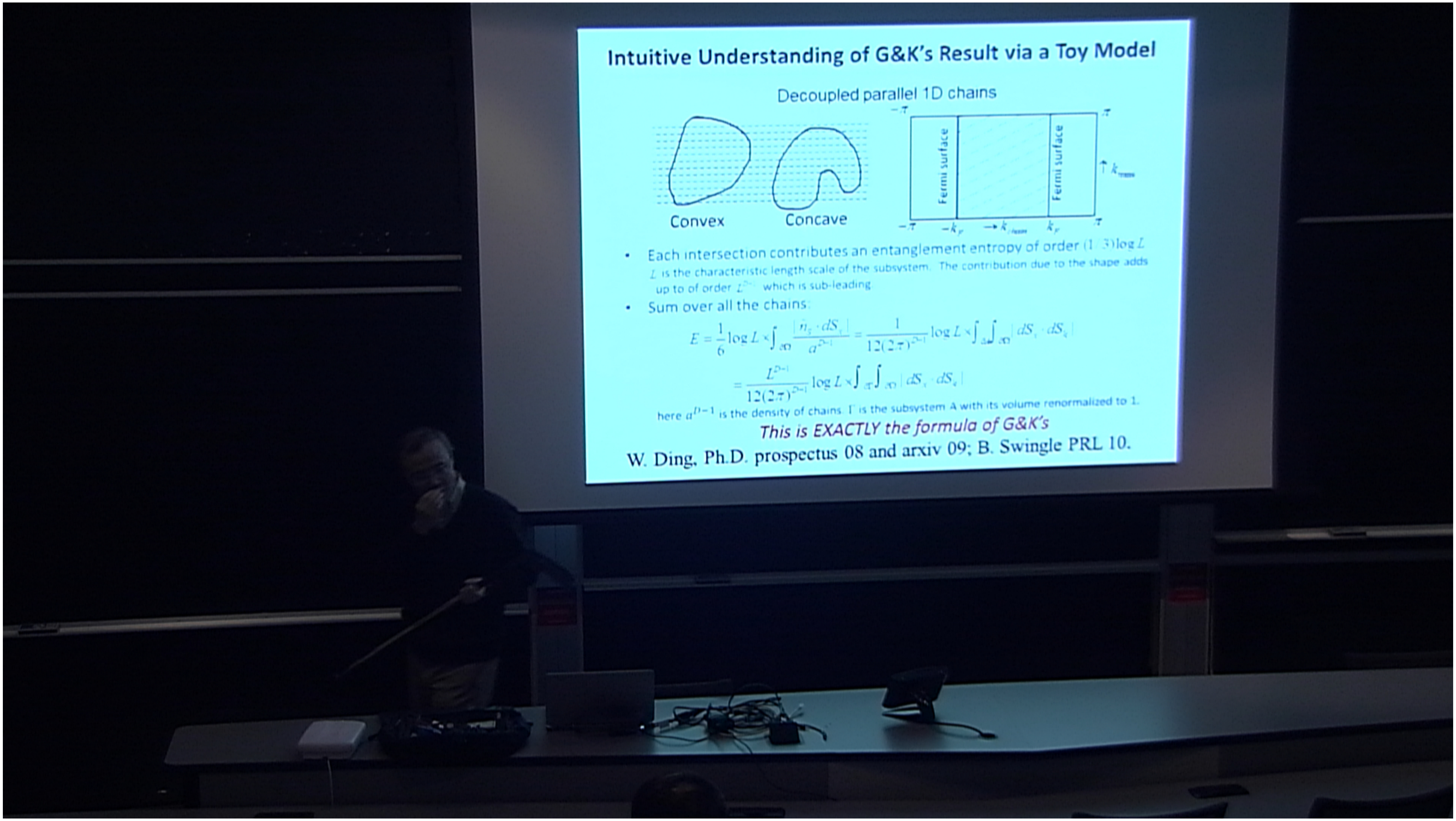
$$E = \frac{1}{6}\log L \times \int_{\partial\Omega} \frac{|\hat{n}_S \cdot dS_x|}{a^{D-1}} = \frac{1}{12(2\pi)^{D-1}} \log L \times \int_{\partial A}\int_{\partial\Omega} |dS_x \cdot dS_k|$$

$$= \frac{L^{D-1}}{12(2\pi)^{D-1}} \log L \times \int_{\partial\Gamma}\int_{\partial\Omega} |dS_x \cdot dS_k|$$

here a^{D-1} is the density of chains. Γ is the subsystem A with its volume renormalized to 1.

This is EXACTLY the formula of G&K's

W. Ding, Ph.D. prospectus 08 and arxiv 09; B. Swingle PRL 10.



Intuitive Understanding of G&K's Result via a Toy Model
Decoupled parallel 1D chains
Convex
Concave
Fermi surface
Fermi surface
$-\pi$
$-k_F$
k_F
π
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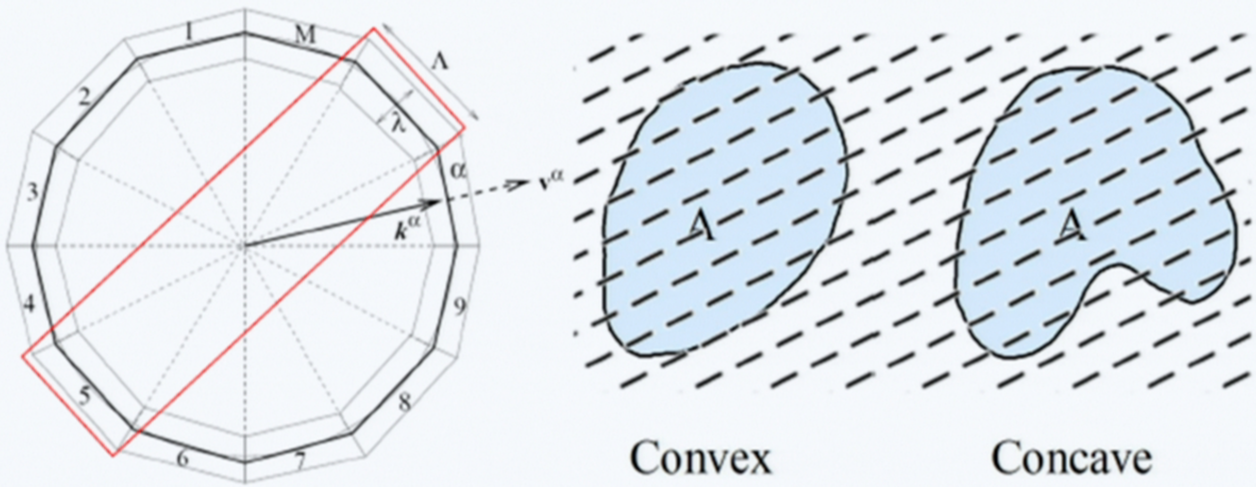
Entanglement entropy & bosonization of the Fermi surface



Chain density:

$$S_A = \sum_{\text{patch pairs}} \frac{1}{6}\frac{\log L}{(2\pi)^{d-1}} \int_{\partial A} \left| dS_{\mathbf{x}} \cdot dS_{\mathbf{k}} \right|$$

$$= \frac{1}{12}\frac{\log L}{(2\pi)^{d-1}} \int_{\partial A} \int_{\text{Fermi surface}} \left| dS_{\mathbf{x}} \cdot dS_{\mathbf{k}} \right|$$

$$\frac{1}{a^{d-1}} = \left(\frac{\Lambda}{2\pi}\right)^{d-1} = \left| \hat{n}_{\Sigma_{\mathbf{x}}} \cdot dS_{\mathbf{k}} \right|$$

GK formula!

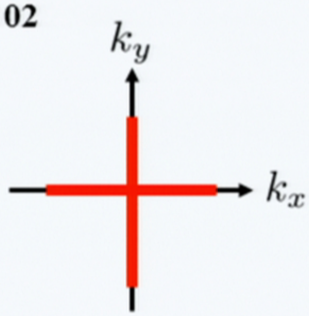
Obviously existence of **extended** Fermi surface(s) crucial for area law violation. What about bosonic systems?

Bosons are different; in general they either localize or Bose condense, resulting in Godstone modes that are gapless **isolated** points. No area law violation in these cases. Emergent Fermi surface possible in spin liquids, leading to area law violation (Zhang, Grover and Vishwanath 11).

Simplest Example of Bose Metal: Exciton Bose Liquid (EBL)

A. Paramekanti , L. Balents, and M. Fisher, PRB 02

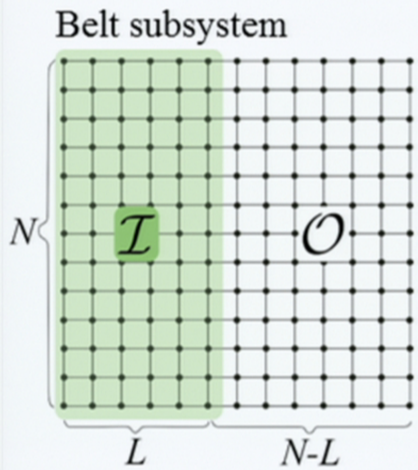
- **Low energy theory: Free bosons with an energy dispersion which vanishes linearly on a locus of points in *k*-space.**



$\omega^2 \sim |k_x k_y|^2$

Bose surfaces (where gapless bosonic excitations live) along $k_x=0$, $k_y=0$

Entanglement Entropy of a Belt Subsystem



Anti-periodic boundary condition

$$V_{\vec{r}=(x,y)} = -V_{(x+N,y)} = -V_{(x,y+N)} = V_{(x+N,y+N)}$$

Partial Fourier transform along y-axis

$$\tilde{q}_{x,k_y} = \frac{1}{\sqrt{N}} \sum_y e^{ik_y y} q_{x,y}, \qquad \tilde{p}_{x,k_y} = \frac{1}{\sqrt{N}} \sum_y e^{ik_y y} p_{x,y}$$

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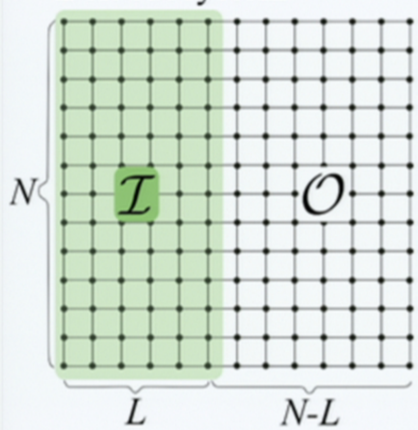

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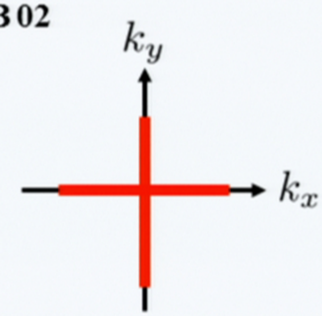


$$\omega^2 \sim |k_x k_y|^2$$

Bose surfaces (where gapless bosonic excitations live) along $k_x=0$, $k_y=0$

- **Bosonic harmonic oscillator with factorized dispersion**

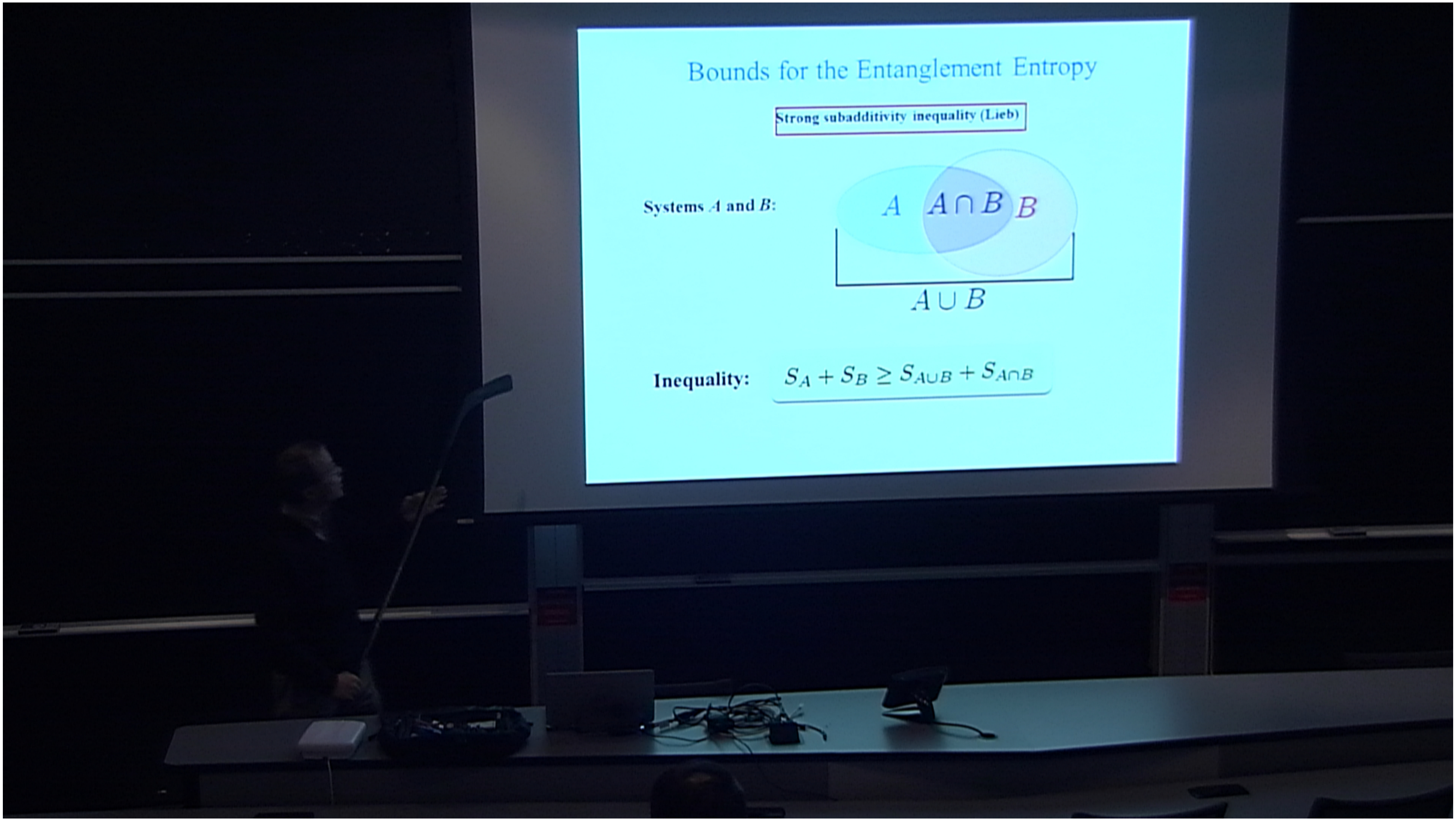
$$\omega^2 = |f_x(k_x) f_y(k_y)|^2$$

$$H = \frac{1}{2}\sum_j p_j^2 + \frac{1}{2}\sum_{j,k} q_j V_{jk} q_k$$

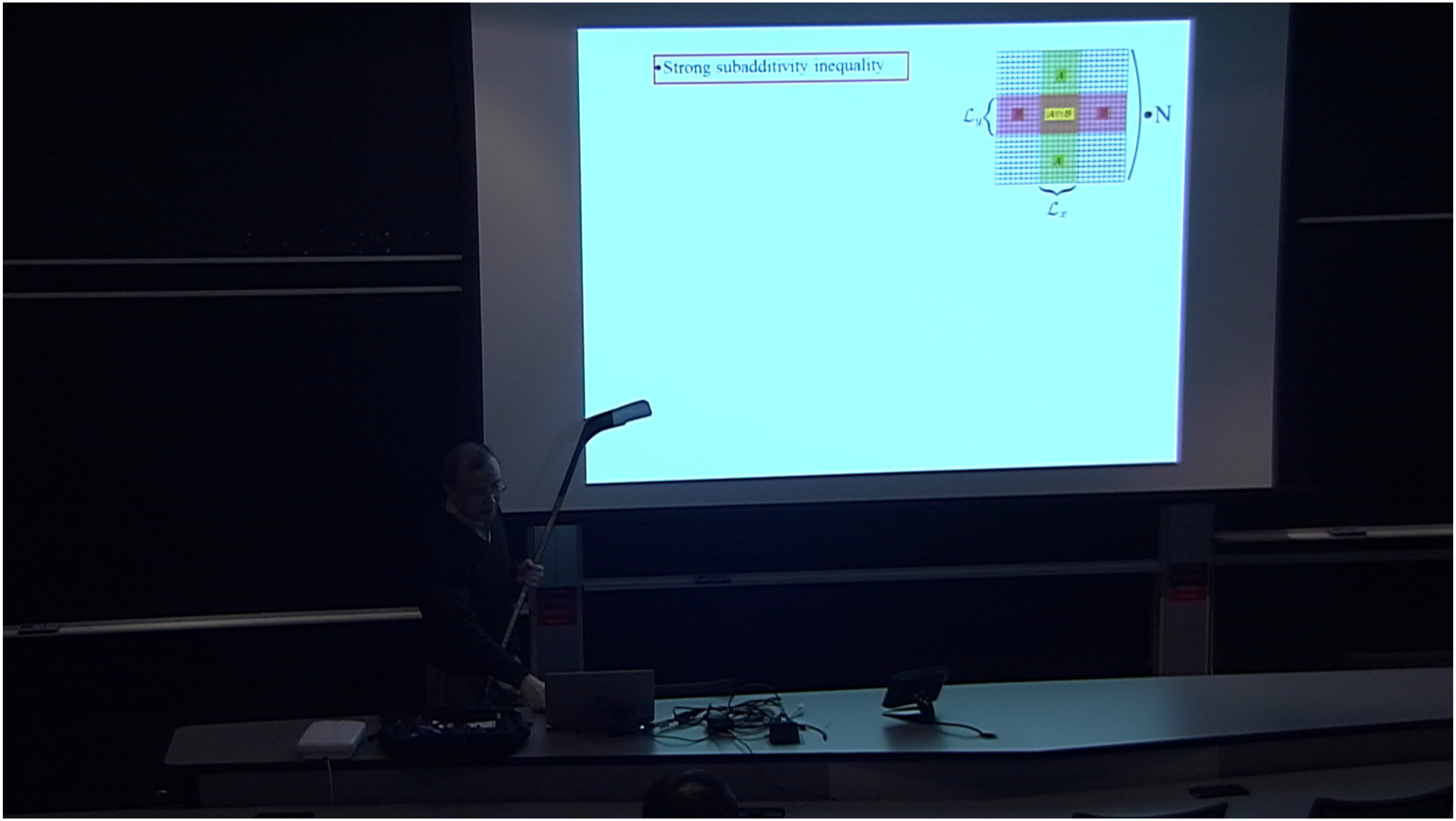
***$f_x(k_x)$*, *$f_y(k_y)$*: Periodic functions which vanish linearly as k_x, $k_y \to 0$, such as *$sin(k_x)$*, *$sin(k_y)$*.**

$$V_{jk} = V^x_{x_j - x_k} V^y_{y_j - y_k}$$

***short-ranged* oscillator coupling!**



Bounds for the Entanglement Entropy
Strong subadditivity inequality (Lieb)
Systems A and B:
A
$A \cap B$
B
$A \cup B$
Inequality:
$S_A + S_B \geq S_{A \cup B} + S_{A \cap B}$


Strong subadditivity inequality
N

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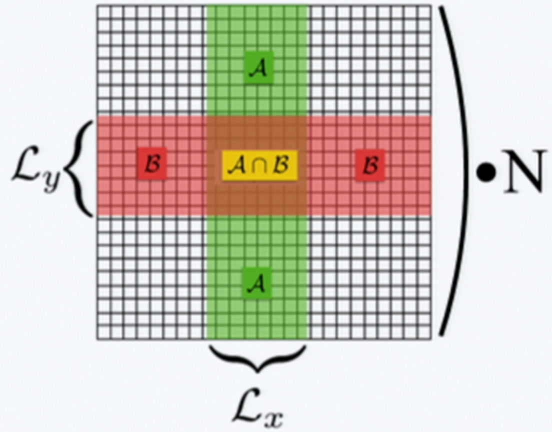
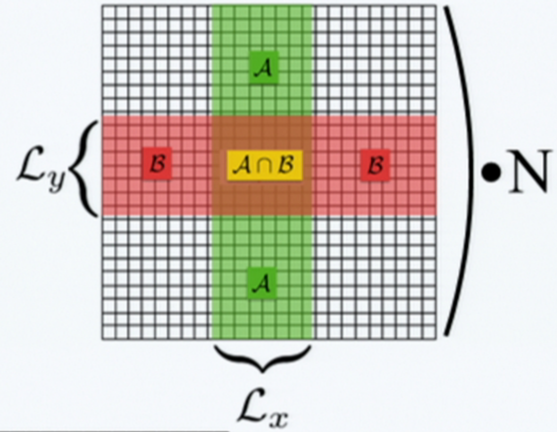


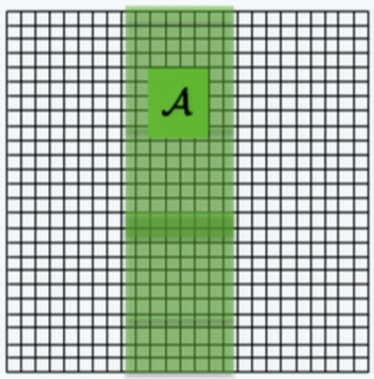
- Strong subadditivity inequality

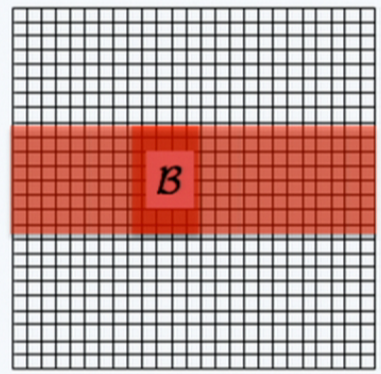


- Upper bound:

$$S_A + S_B \geq S_{A\cup B} + S_{A\cap B} \geq S_{A\cap B}$$
$$\Rightarrow \frac{N}{3}\ln(\mathcal{L}_x\mathcal{L}_y) \geq S_{A\cap B} \equiv S_{\Box}$$

- Lower bound:



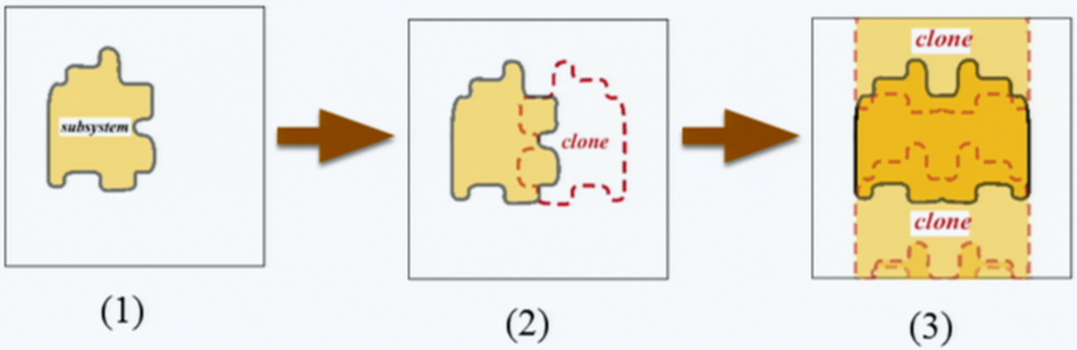


$$S_1 + S_2 + \ldots + S_{\lceil n_y \rceil} \geq S_{1\cup 2\cup \ldots \lceil n_y \rceil} = S_{\mathcal{A}}$$
$$\Rightarrow S_{\Box} \geq \frac{\mathcal{L}_y}{3}\ln \mathcal{L}_x$$

Argument for arbitrary shape

For a non-rectangular system with at least one boundary parallel to a Cartesian axis

Take EBL for example: **Lattice symmetries (translation and mirror)** + Strong subadditivity inequality



(H.-H. Lai, KY and N. Bonesteel, PRL 13)

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(1) (2) (3)

For subsystems with general smooth boundary: No rigorous argument

Arbitrarily sliced

- **$S_{L\cup R}$ is likely to show logarithmic enhancement.**
- **Results generalisable to similar models with arbitrary Bose surfaces.**

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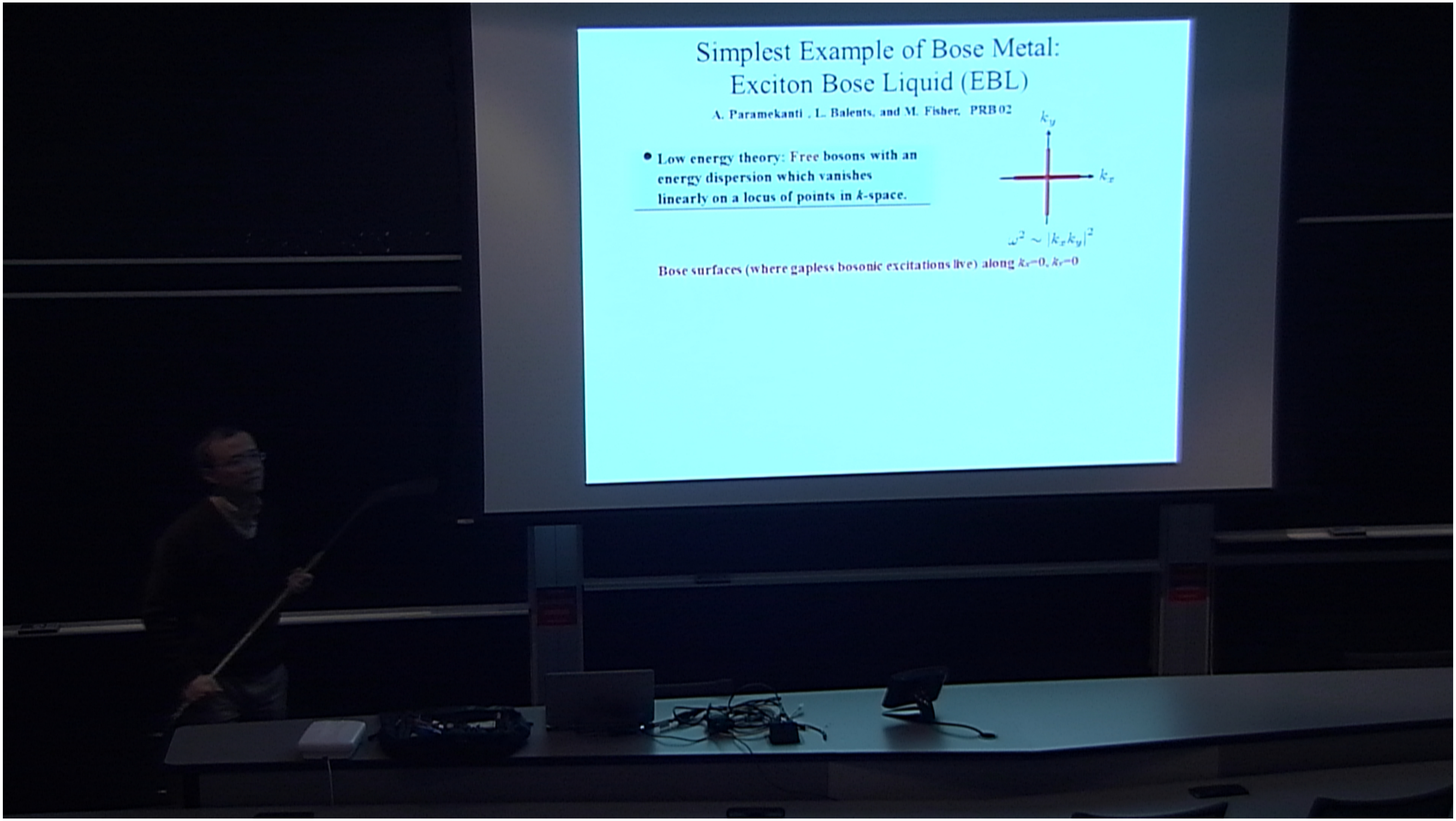
Summary Thus Far
• Logarithmic violations of area law established for free boson/harmonic oscillator models with Bose surfaces.
• Can be viewed as bosonic version of Wolf/Gieov-Klich results on free fermions; domain of area law violation in high-D expanded.
• Challenge: Generalize to interacting systems.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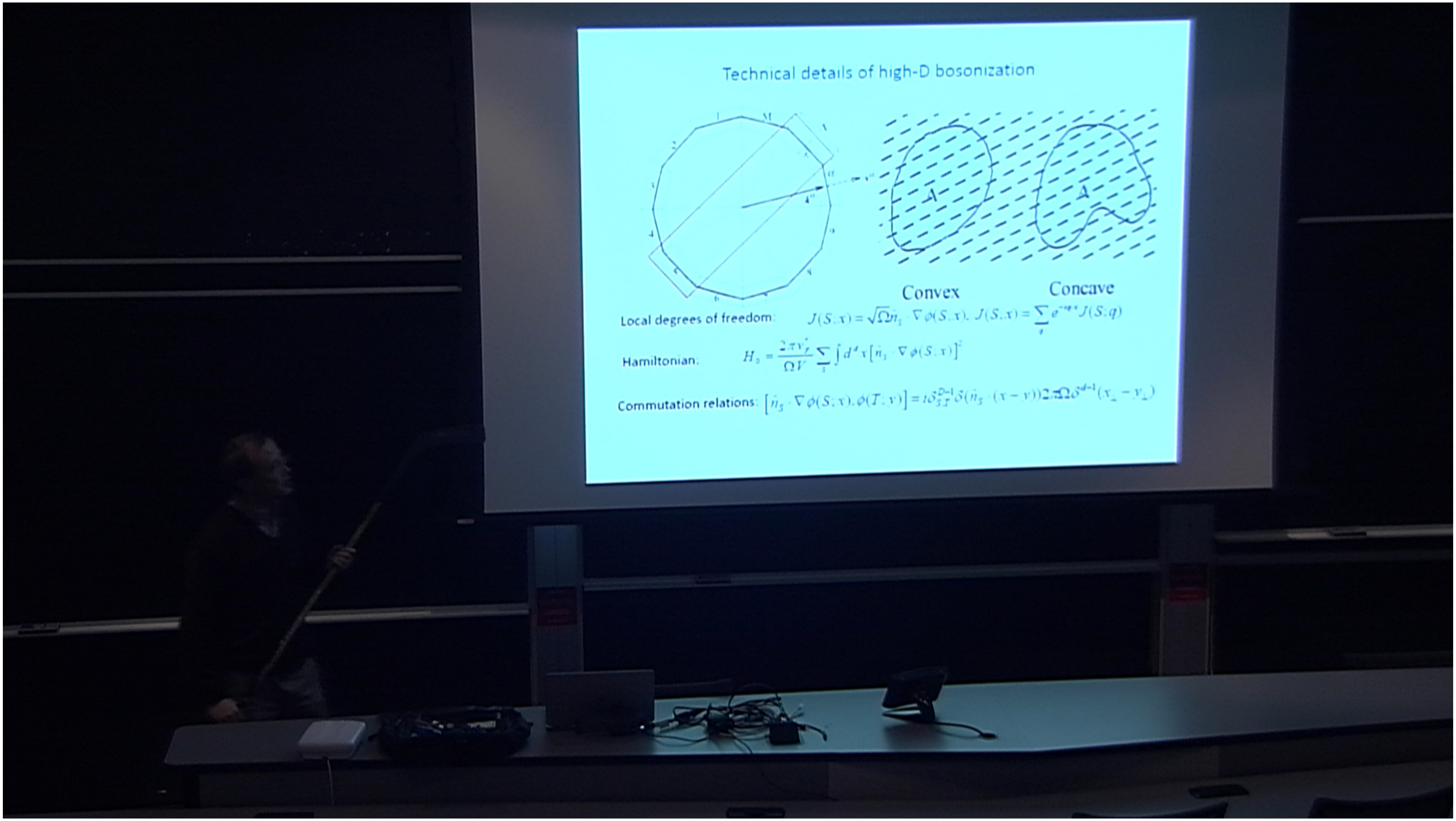
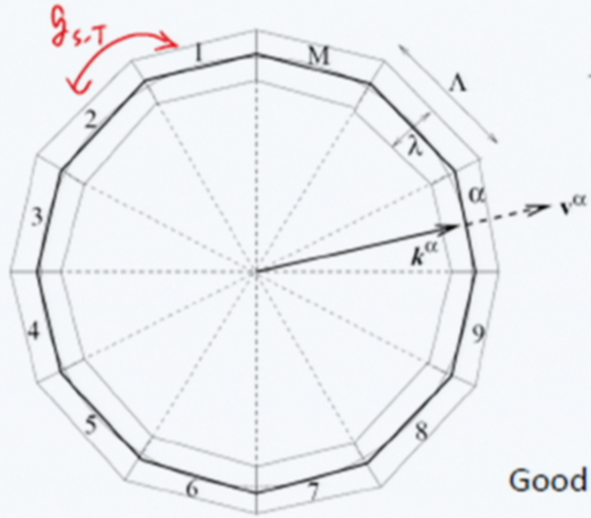


Technical details of high-D bosonization
Convex
Concave
Local degrees of freedom:
Hamiltonian:
Commutation relations:

Add Fermi liquid interactions



$$H[\phi(\boldsymbol{S};\boldsymbol{x})] = \frac{2\pi v_F^*}{\Omega V}\int d^2x \Big[\sum_{\boldsymbol{S}}(\partial_{\boldsymbol{S}}\phi(\boldsymbol{S};\boldsymbol{x}))^2 + \sum_{\boldsymbol{S},\boldsymbol{T}} g_{\boldsymbol{S},\boldsymbol{T}}\partial_{\boldsymbol{S}}\phi(\boldsymbol{S};\boldsymbol{x})\partial_{\boldsymbol{T}}\phi(\boldsymbol{T};\boldsymbol{x})\Big]$$

Can be diagonalized in terms of new modes $\tilde{\phi}(\boldsymbol{S};\boldsymbol{x})$

Good News: Diagonalized Hamiltonian looks the same as the non-interacting Hamiltonian

Bad News: Relation between $\phi(\boldsymbol{S};\boldsymbol{x})$ and $\tilde{\phi}(\boldsymbol{S};\boldsymbol{x})$ is very non-local!

$$\phi(\boldsymbol{S};\boldsymbol{x}) = \tilde{\phi}(\boldsymbol{S};\boldsymbol{x}) + \int d\boldsymbol{y}\sum_{l} f(\boldsymbol{S},l;\boldsymbol{x}-\boldsymbol{y})\tilde{\phi}(l;\boldsymbol{y}),$$

Solution: Make use of the quadratic nature of the bosonized Hamiltonian without diagonalizing it. Relate entanglement entropy to Green's function

Entanglement entropy via replica Green's function: a roadmap

Replica trick: $S_A = -\lim_{n\to 1} \mathrm{Tr} \frac{\partial}{\partial n} \rho_A^n = -\lim_{n\to 1} \frac{\partial}{\partial n} \frac{Z_n(A)}{Z^n}$

↓

For quadratic theory: $\frac{\partial}{\partial m^2} \log Z_n = -\frac{1}{2} \int d^{d+1}x G^{(n)}(x,x)$

price: need a mass term

↓

Solve for $G^{(n)}(x,y)$ on the replica manifold

price: special geometry

↓

$$S_A = -\lim_{n\to 1} \frac{\partial}{\partial n} e^{-\frac{1}{2}\int_{1/L} dm^2 \int d^2x (G^{(n)}(x,x) - nG^{(1)}(x,x))}.$$

Entanglement entropy of Fermi liquids

$$G(S,T;x,y) = G_0(S,T;x,y)$$
$$+\int d^3z G_0(S,S;x,z) \times \Big(\sum_l (h_2(l,T)\partial_\tau^2 + f_1(l,T)\partial_l\partial_T)G(l,T;z,y)\Big)$$

Structure at any given order:

-- An intra-patch term, where all patch-indices are identical. This term is proportional to

$$\log L \times \sum_S \int dx_{S_\perp} \delta(x_{S_\perp} - y_{S_\perp})\big|_{y\to x}$$

Similar to the zeroth order, and could thus renormalize the leading log(L) term. However, its coefficient is formally identical to one that would appear in a 1D theory, where we know it must vanish. We checked this in detail at 1st and 2nd order.

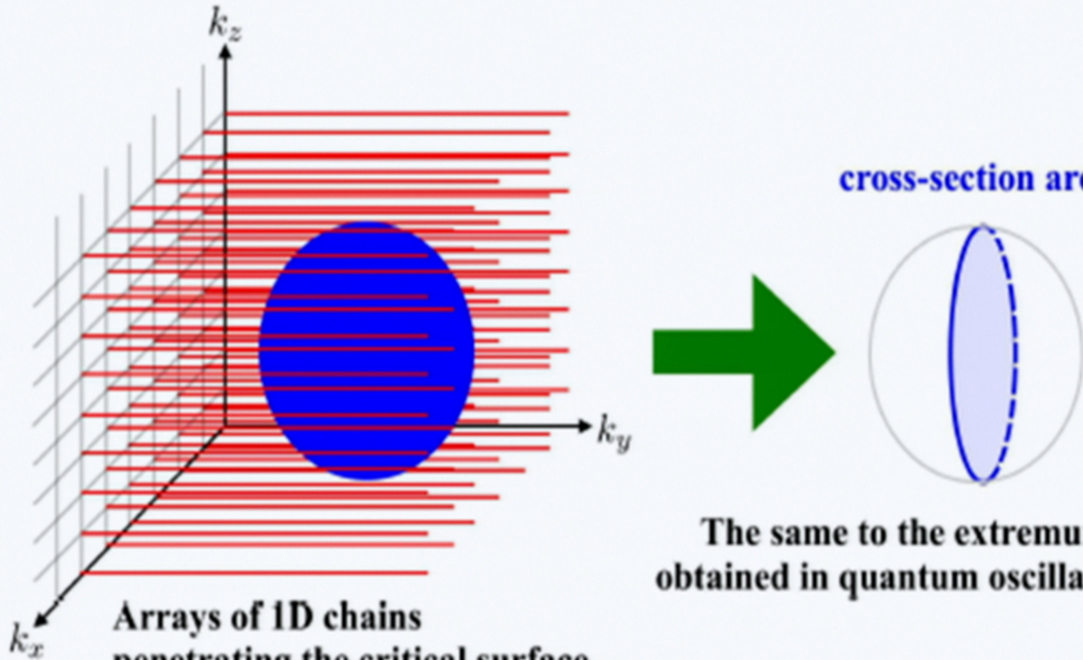
-- An inter-patch term which can be shown to be sub-leading in L using scaling arguments.

$\log L \times$ area law unrenormalized by interactions!

See W. Ding, A Seidel and KY, PRX 12 for calculational details.

Determining Shape of Fermi or Bose Surfaces Using Entanglement

(H.-H. Lai and KY, arXiv 15)



Arrays of 1D chains penetrating the critical surface

The same to the extremum cross-section obtained in quantum oscillation measurement

(Another) Summary Thus Far

- Entanglement entropy area law widely obeyed by **ground states** of local Hamiltonians.
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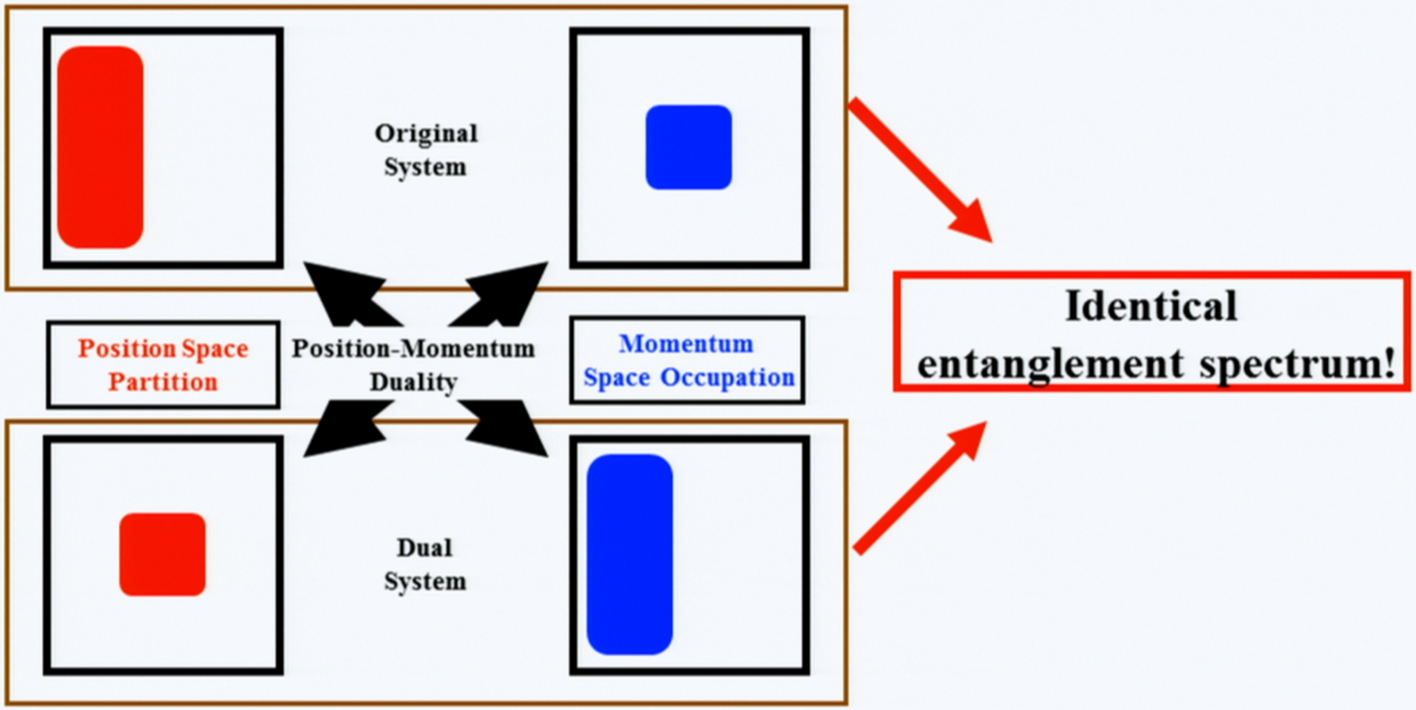
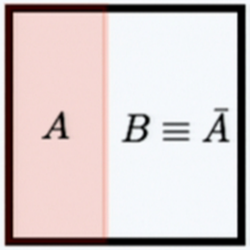
(Another) Summary Thus Far

- Entanglement entropy area law widely obeyed by **ground states** of local Hamiltonians.
- Violations of area law rare above 1D; requires Fermi or Bose surfaces. Reasonably well understood.
- **Challenge**: Highly excited states (with finite excitation energy ***density***).
- **Expectation**: entanglement entropy follow volume law, just like a random state in Hilbert space.
- **Problem**: Almost no explicit result other than limited numerics.

Position-Momentum Duality for Free Fermion States



Heuristic Derivation of Duality-I



For a Free fermion Hamiltonian: $H = \sum_{j\ell} c_j^\dagger h_{j\ell} c_\ell$

Reduced density matrix: $\rho_A = tr_B \left[|F\rangle\langle F| \right] = e^{-H_e}$

$$H_e = \sum_{j,\ell \in A} c_j^\dagger \kappa_{ej\ell} c_\ell \qquad \kappa_e = \ln \left[M^{-1} - \mathbb{1}_{V_A \times V_A} \right]$$

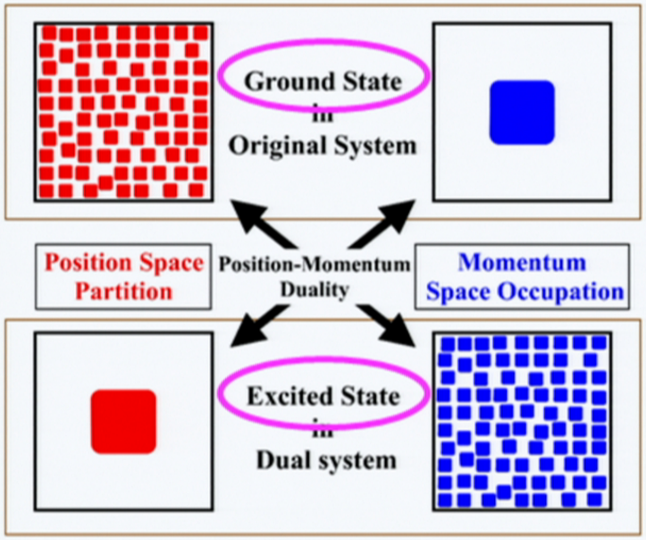
$$M_{j\ell} \equiv \langle F | c_j^\dagger c_\ell | F \rangle_A$$

M_{jl}: Two-point fermion Green's function defined in subsystem A.

In free fermion systems: A given matrix M

→ Uniquely determine the entanglement spectrum!

Duality between GS and ES



Ground State of Original system

Number of pockets: $n \simeq \frac{V}{L_\square^d}$

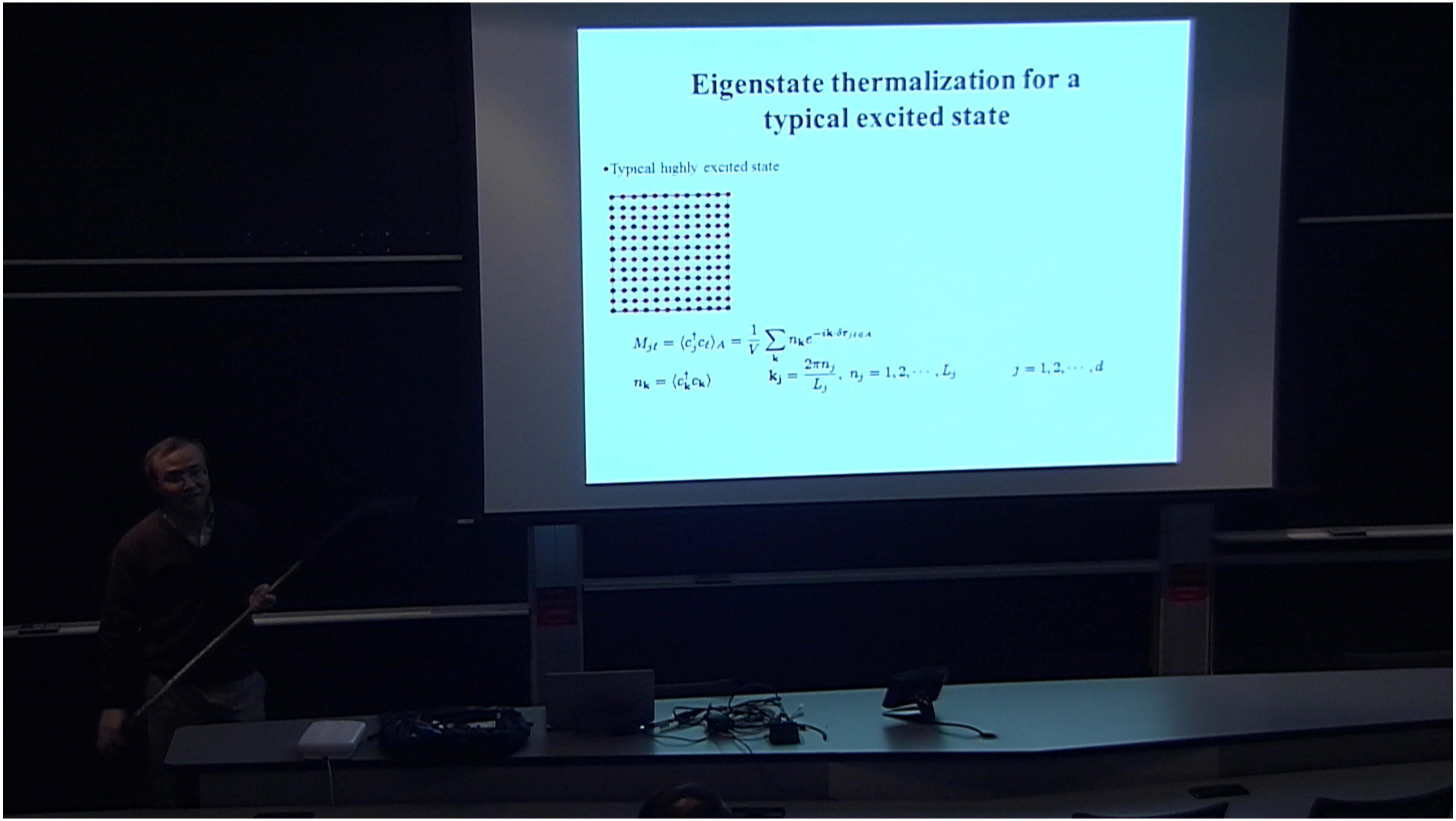
$$S_P \simeq n L_\square \ln L_\square \simeq V \frac{\ln L_\square}{L_\square^{d-1}}\Bigg|_{V \to \infty \gg L_\square^d} \sim V$$

The entanglement entropy scales with the system volume

Highly Excited State in Dual system

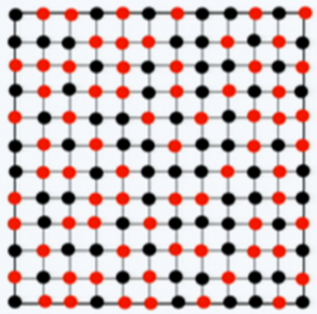
From the position-momentum duality, the entanglement entropy of the a highly excited state exhibits volume instead of area law.

The entanglement entropic area law for free fermion ground states and the volume law for highly excited states are of the same origin!


Eigenstate thermalization for a typical excited state
• Typical highly excited state
$M_{j\ell} = \langle c_j^\dagger c_\ell \rangle_A = \frac{1}{V} \sum_{\mathbf{k}} n_{\mathbf{k}} e^{-i\mathbf{k}\cdot\delta\mathbf{r}_{j\ell\in A}}$
$n_{\mathbf{k}} = \langle c_{\mathbf{k}}^\dagger c_{\mathbf{k}} \rangle$
$\mathbf{k_j} = \frac{2\pi n_j}{L_j},\ n_j = 1, 2, \cdots, L_j$
$j = 1, 2, \cdots, d$

Eigenstate thermalization for a typical excited state

- Typical highly excited state



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Coarse-Graining Process

- At thermodynamic limit, we can divide the Brillouin zone into a huge number of cells with linear size

$$\frac{1}{L} \ll \delta\mathbf{k}_{cell} \ll \frac{1}{L_A}$$

- All the momentum points within the same cell approximately share the same phase factor

$$\sum_{\mathbf{k}\in\text{same cell}} n_{\mathbf{k}} e^{-i\mathbf{k}\cdot\delta\mathbf{r}_{j\ell\in A}} \simeq N_m e^{-i\mathbf{k}_m\cdot\delta\mathbf{r}_{j\ell\in A}}$$

N_m: **Total occupation number in the cell** m

$$M_{j\ell} \simeq \frac{g}{V} \sum_m \left(\frac{N_m}{g}\right) e^{-i\mathbf{k}_m\cdot(\mathbf{r}_j-\mathbf{r}_\ell)} \equiv \frac{1}{V_{eff}} \sum_m n_m e^{-i\mathbf{k}_m\cdot(\mathbf{r}_j-\mathbf{r}_\ell)}$$

g: The total number of momentum points in a cell

Searching for the Most Probable Macrostate $\{n^*_m\}$

- Two constraints for an excited state:

$$\sum_m N_m = N, \quad \sum_m N_m \epsilon_m = E$$

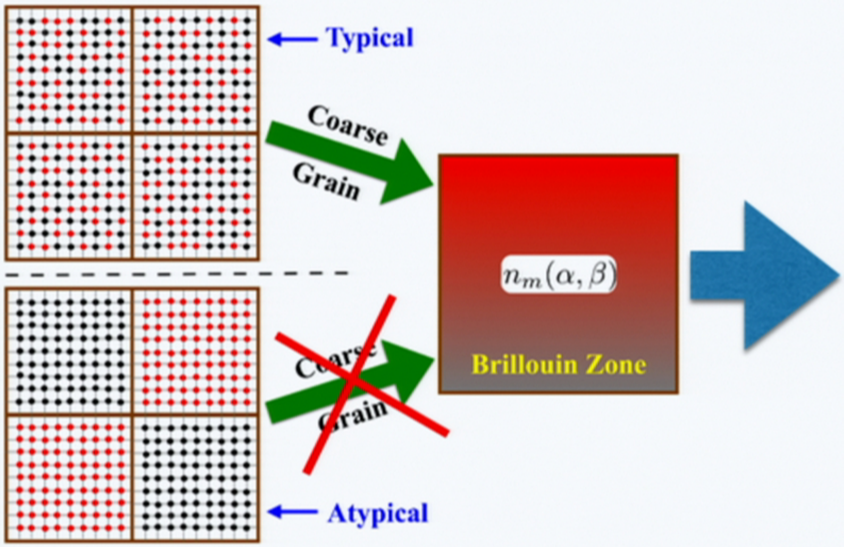
$\epsilon_m \equiv \epsilon_{\mathbf{k}_m}$: **single particle energy corresponding to a cell**

- **Introduce two Lagrange multipliers $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ and examine the fluctuation of the distribution set $\{N_m\}$**

$$n^*_m = \frac{N^*_m}{g} = \frac{1}{e^{\beta\epsilon_m + \alpha} + 1}$$

The same as the Fermi-Dirac distribution from the grand canonical thermal ensemble if we identify $\alpha = -\mu/T$ and $\beta = 1/T$!

Schematic Illustration of Eigenstate Thermalization



$$M_{j\ell} = \frac{1}{(2\pi)^d} \int \frac{d^d\mathbf{k}}{e^{\beta\epsilon_{\mathbf{k}}+\alpha}+1} e^{-i\mathbf{k}\cdot(\mathbf{r}_j - \mathbf{r}_\ell)}$$

For "all" elements defined in the subsystem A

(Weak) Eigenstate thermalization in free Fermion systems!

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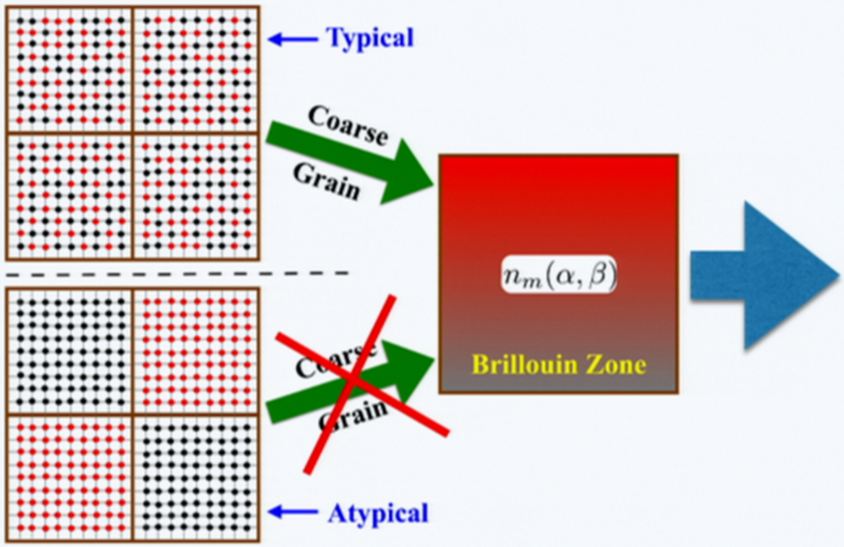

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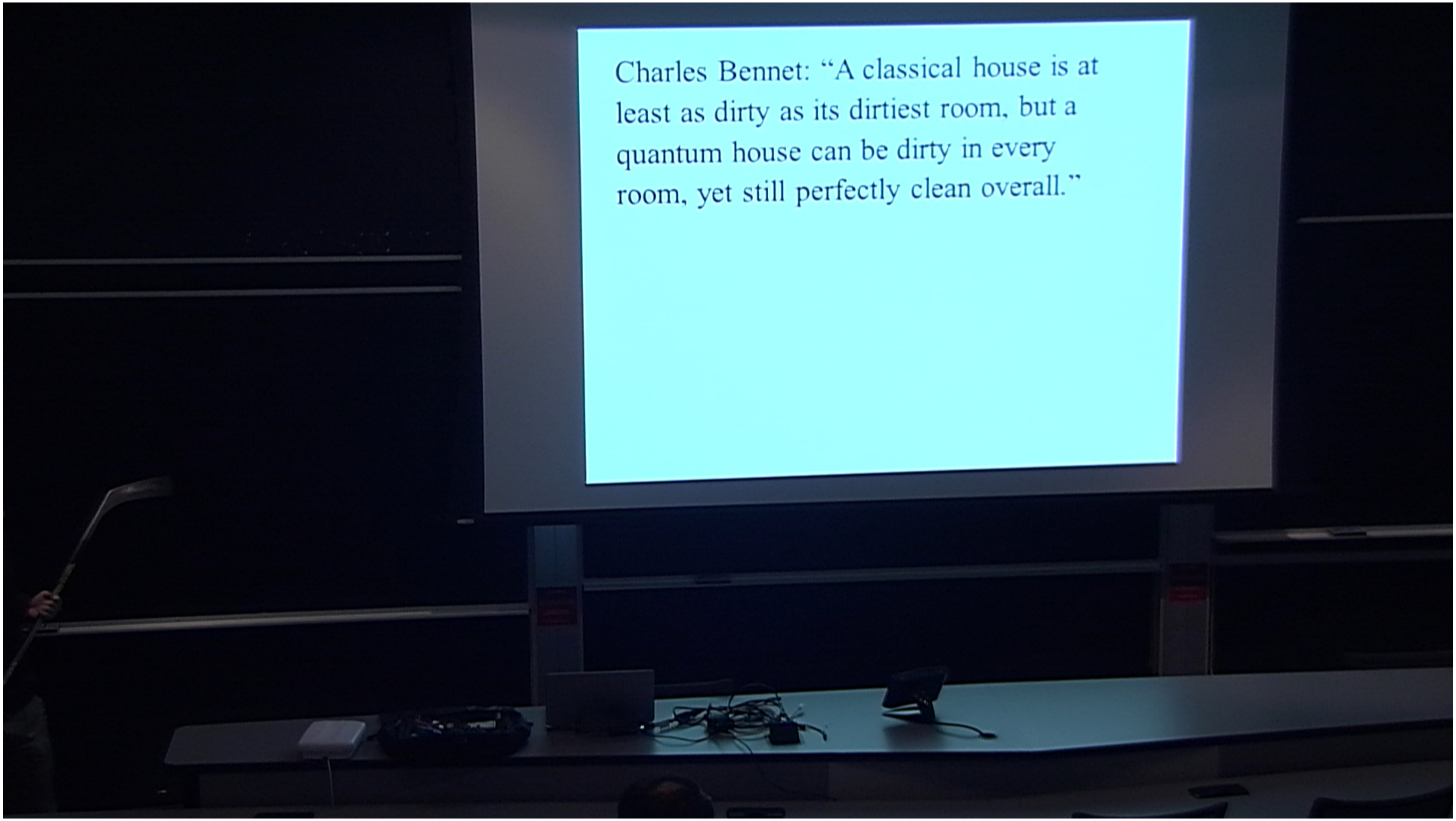
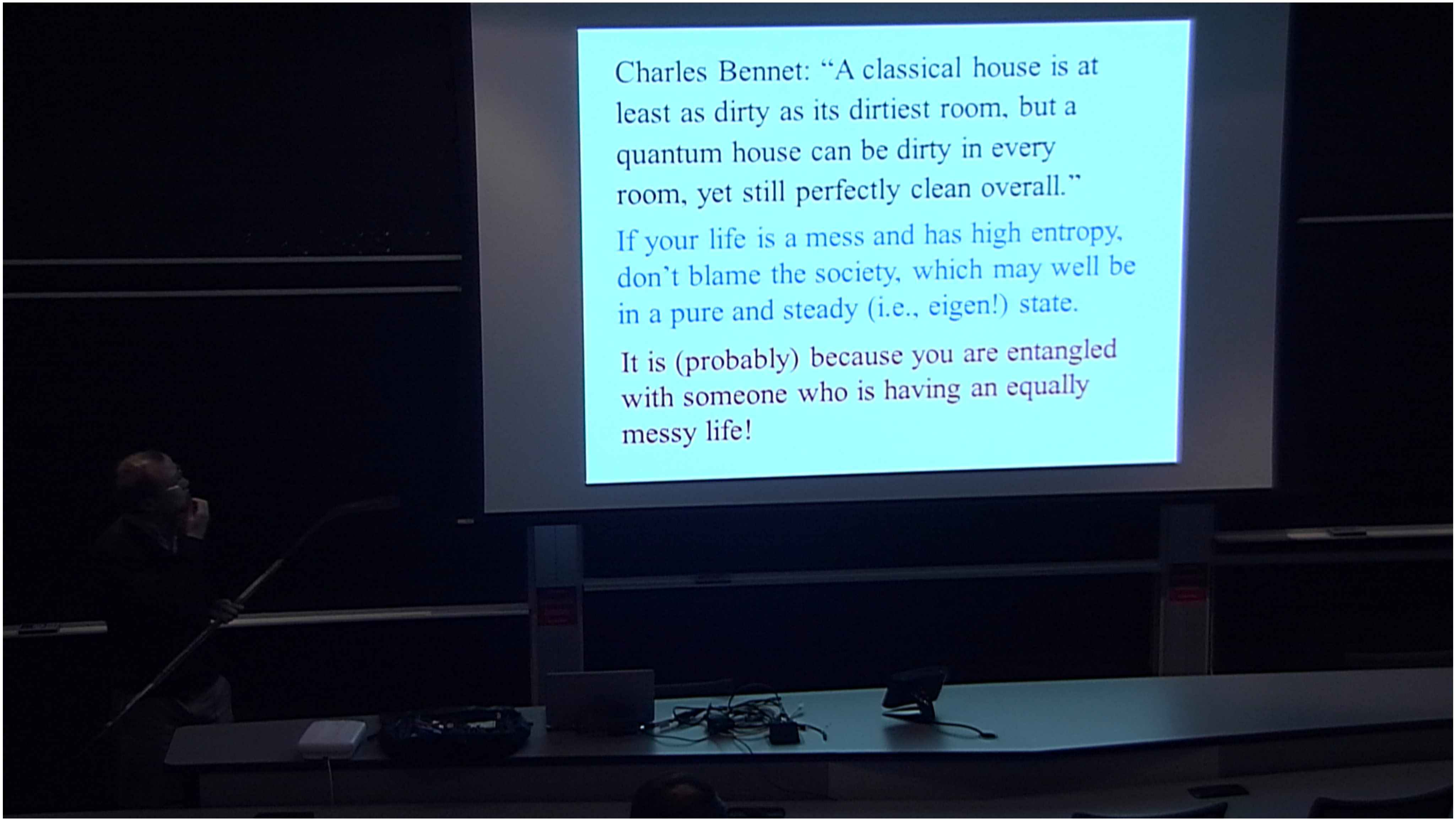

Charles Bennet: "A classical house is at least as dirty as its dirtiest room, but a quantum house can be dirty in every room, yet still perfectly clean overall."


Charles Bennet: “A classical house is at least as dirty as its dirtiest room, but a quantum house can be dirty in every room, yet still perfectly clean overall.”
If your life is a mess and has high entropy, don’t blame the society, which may well be in a pure and steady (i.e., eigen!) state.
It is (probably) because you are entangled with someone who is having an equally messy life!