Title: Multitude of percolative orders: Infinite cascades of geometric phase transitions

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Abstract: The evolution of many kinetic processes in 1+1 dimensions results in 2D directed percolative landscapes. The active phases of those models possess numerous hidden geometric orders characterized by distinct percolative patterns. From Monte-Carlo simulations of the directed percolation (DP) and the contact process (CP) we demonstrate the emergence of those patterns at specific critical points as a result of continuous phase transitions.These geometric transitions belong to the DP universality class and their nonlocal order parameters are the capacities of corresponding backbones. The multitude of conceivable percolative ordering patterns implies the existence of infinite cascades of such transitions in the models considered. We conjecture that such cascades of transitions is a generic feature of percolation as well as many other transitions with non-local order parameters.

Multitude of Percolative Orders: Infinite Cascades of Geometric Transitions

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Southern Federal University, Russia

References:
1. J. Phys. A: Math. Theor. ***48****, 135003 (2015)*
2. arXiv:1507.08026

Supported by:

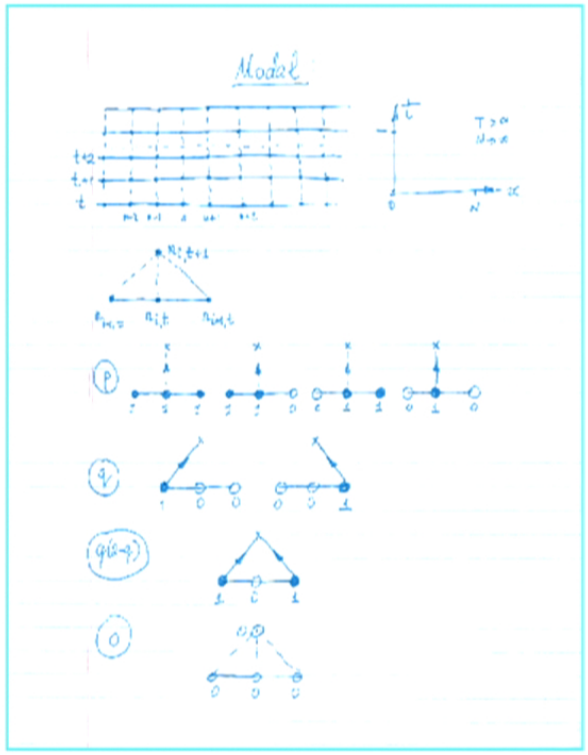
Outline:

- *Replication Model of Probabilistic Cellular Automation (PCA). Some definitions*
- *Percolation. Basic notions*
- *Hidden percolation in replication model of PCA*
- *Percolation, percolation, percolation…*
  Infinite cascades of geometric transitions:
  Directed percolation, contact process, replication model, ….
- *Conclusions*
- *Future works*

Model of 1D Kinetic Contact Process with Parallel Update = Probabilistic Cellular Automation (PCA)

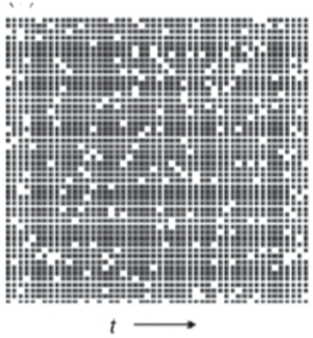
Table 1.

n_{i-1}, n_i, n_{i+1}	111, 110, 011, 010	100, 001	101	000
Probability $(n_i = 1)$	p	q	$q(2-q)$	0



Possible Phases:

1. Active (Alive) Phase
   Black: active (n=1) sites
   White: dead (n=0) sites



2. Absorbing (Dead, Empty) Phase
   -- Inverse of the above

Active-Absorbing phases: *Non-equilibrium Phase Transition*

Applications:

Statistical physics

Critical phenomena

Quantitative finance

Ecology

Biology

Disease spreading

Population replication

Networks

Percolation:

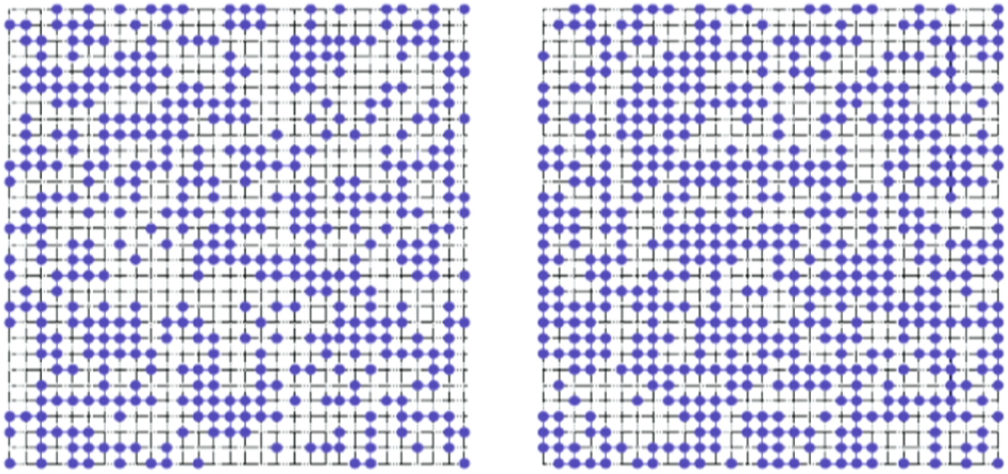


Fig. 1. $2D$ site percolation. Shown are lattice realizations below (*left*) and above (*right*) the classical percolation threshold $p_c = 0.592746$

Schubert, G., Fehske, H.: *Quantum Percolation in Disordered Structures*. Lect. Notes Phys. **762**, 135–162 (2009)
DOI 10.1007/978-3-540-85428-9_5

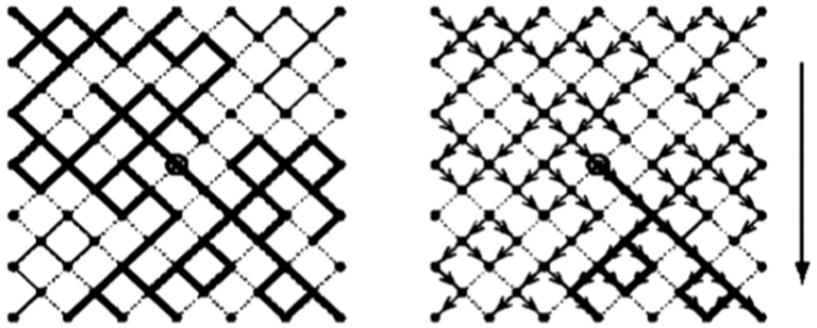


Figure 10. Isotropic and directed bond percolation on a diagonal square lattice with free boundary conditions. Open (closed) bonds are represented by solid (dotted) lines. In both cases a cluster, indicated by bold lines, is generated from the lattice site in the centre. In the directed case the spreading agent is restricted to follow the sense of the arrows, leading to a directed cluster of connected sites.

From H. Hinrichsen, Adv. Phys. (2000)

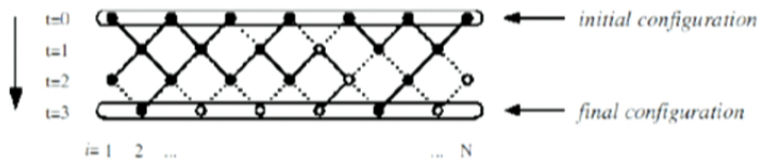


Figure 11. Directed bond percolation in $1+1$ dimensions interpreted as a time-dependent stochastic process. Open (closed) bonds are indicated by solid (dashed) lines. Filled (hollow) circles denote active (inactive) sites. The configuration of the horizontal row at $t = 0$ is the initial state. Starting from a fully occupied initial state the model 'evolves' through intermediate configurations according to the dynamic rules of equation (80) and reaches a final state at $t = 3$.

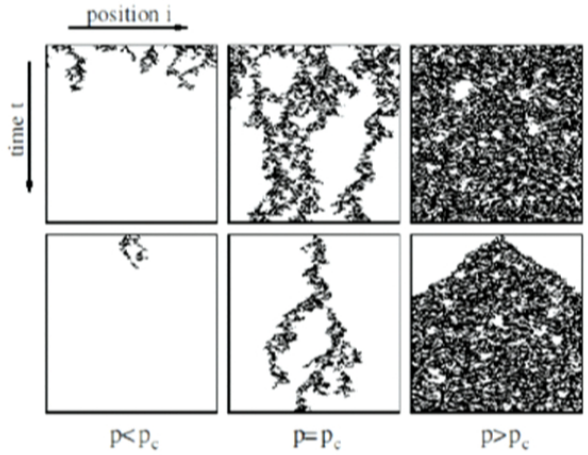


Figure 12. Directed bond percolation in $1+1$ dimensions starting from random initial conditions (top) and from a single active seed (bottom). Each horizontal row of pixels represents four updates. As can be seen, critical DP is a reaction-limited process.

From H. Hinrichsen, Adv. Phys. (2000)

From H. Hinrichsen, Adv. Phys. (2000)

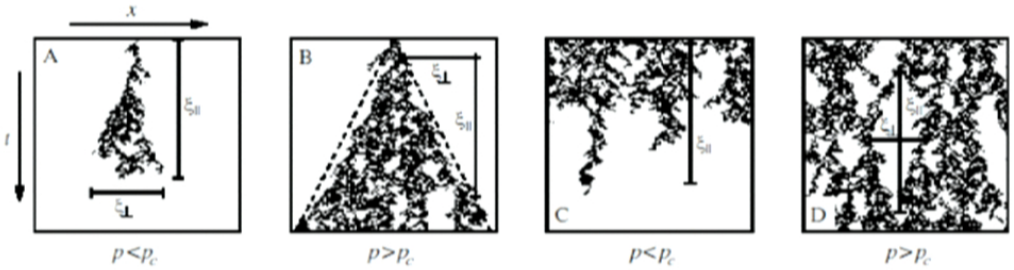


Figure 19. Interpretation of the correlation lengths $\xi_\perp$ and $\xi_\parallel$ in an almost critical $(1+1)$-dimensional DP process below (left) and above criticality (right). In panels (a) and (b) a cluster is grown from a single active seed while in panel (c) a fully occupied lattice is used as the initial state. Panel (d) shows a stationary DP process in the active phase. The indicated length scales $\xi_\perp$ and $\xi_\parallel$ must be interpreted as averages over many independent realizations.

$$\xi_\perp \sim |p - p_c|^{-\nu_\perp}, \quad \xi_\parallel \sim |p - p_c|^{-\nu_\parallel}$$

The order parameter of a spreading process is the density of active sites

$$\rho(t) = \left\langle \frac{1}{N} \sum_i s_i(t) \right\rangle,$$

$$\rho^{\text{stat}} \sim (p - p_c)^\beta,$$

$$\rho(t) \sim t^{-\beta/\nu_\parallel} f(\Delta t^{1/\nu_\parallel}, t^{d/z}/N),$$

Janssen-Grassberger Conjecture:

A Model should belong to the DP Universality Class If:

(1) The model displays a continuous phase transition from a *fluctuating* active phase into a *unique* absorbing state.
(2) The transition is characterized by a *positive one-component* order parameter.
(3) The dynamic rules involve only *short-range* processes.
(4) The system has no special attributes such as additional symmetries or quenched randomness.

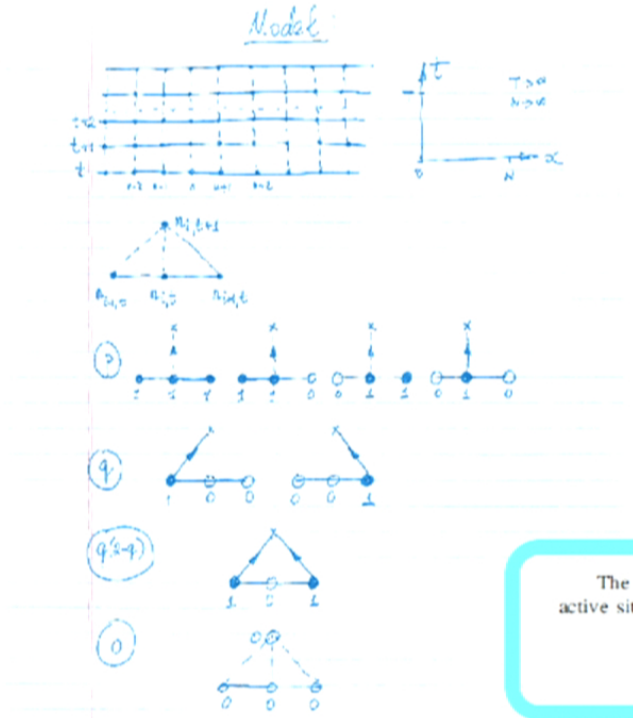
H.K. Janssen, Z. Phys. B (1981)

P. Grassberger, Z. Phys. B (1982)

Phase Diagram of Proposed Replication Model – Results:

Table 1.

n_{i-1}, n_i, n_{i+1}	111, 110, 011, 010	100, 001	101	000
Probability ($n_i = 1$)	p	q	$q(2-q)$	0



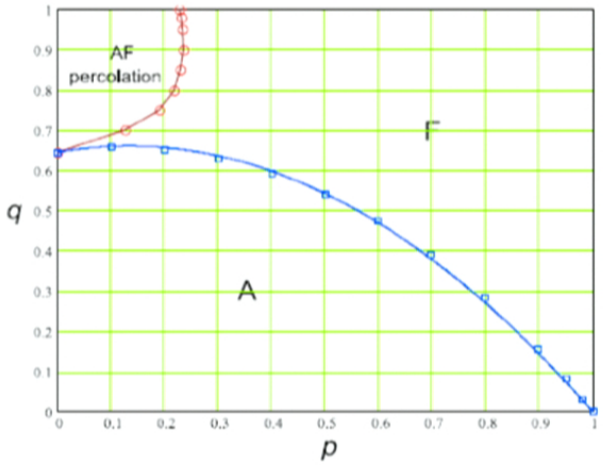


Figure 4. Phase diagram from the Monte Carlo simulations. Circles and squares are the MC data, the lower blue line is the parabolic approximation equation (20), the upper red line represent approximate boundary between the AF percolated and the ferromagnetic phases. The both lines correspond to the continuous phase transitions.

$$q = (1-p)\left(q_{\mathrm{BDP}} + ap\right), \quad q_{\mathrm{BDP}} \approx 0.645, \quad a \approx 0.88.$$

Two Active Phases!!!

The order parameter for A–F transition is the infinite-time limit of the concentration of active sites

$$\rho(t) = \sum_{i=1}^{N} n_{t,i}/N.$$

A-F Transition: Critical Scaling, DP Universality

samples with $N=10^3$ sites and monitor $\rho(t)$ near-critical relaxation during 2000 time steps averaging these data over 1000 trials. Critical relaxation we studied on samples with $N=100$, 200, 400 sites for $T=5\times10^5$, 1.5×10^5, 5×10^6 time steps averaging data over 2×10^4, 10^4 and 5×10^3 trials correspondingly. The example of such simulations establishing the validity of critical scaling is shown in figure 5. We get the index α describing critical relaxation of the order parameter, correlation length index $\nu_\parallel$ for time-direction and dynamical index $z=\nu_\parallel/\nu_\perp$.

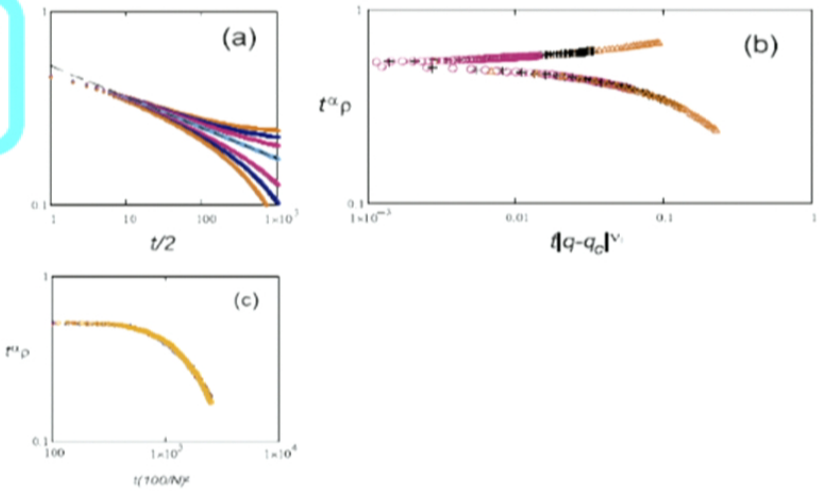
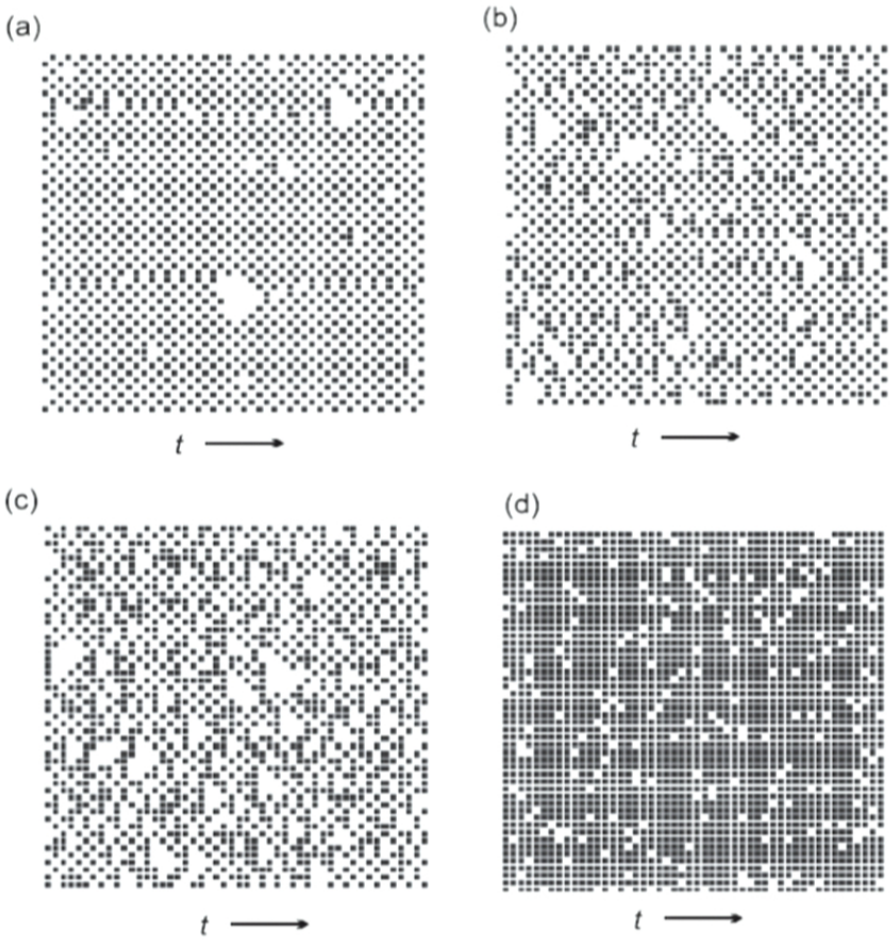


Figure 5. (a) MC simulations of $\rho(t)$ relaxation for $p=0.5$ and series of q near $q_c\approx0.542$, from top to bottom: $q=0.546$, 0.544, 0.543, 0.542, 0.538, 0.536, 0.534. Dashed line corresponds to power law $0.52/t^{\alpha}$,with $\alpha=0.16$. (b) Collapse of the curves from (a) onto a single scaling function. Fitting gives the values of $\nu_\parallel$ and q_c shown in table 2. (c) Relaxation of the order parameter for chains with numbers of sites $N=100$ (x), $N=200$ (+), $N=400$ (o) at $q=q_c$ collapsed onto a single scaling function yields the index z.

NB: A-F Transition ⇔ Percolation Transition

Table 2. Critical points q_c and critical indices for the transition between the absorbing (A) and ferromagnetic (F) phases for several values of p. The last row presents the known DP indices [1] for comparison.

p	q_c	α	$\nu_\parallel$	z	$\beta=\alpha\nu_\parallel$	$\nu_\perp=\nu_\parallel/z$
0.8	0.2821(66)	0.16(01)	1.71(01)	1.54(7)	0.27(4)	1.10(5)
0.5	0.54151(05)	0.16(01)	1.71(4)	1.57(6)	0.27(4)	1.08(8)
0.2	0.6510(58)	0.16(01)	1.72(1)	1.56(6)	0.27(5)	1.09(9)
0	0.6446(6)	0.16(01)	1.72(5)	1.56(4)	0.27(6)	1.10(3)
DP	0.6447(7)	0.159(5)	1.73(4)	1.58(1)	0.27(6)	1.09(1)



AF Percolative Phase

F (active) Phase

Figure 7. Regions of the active states stable configurations in two different active phases ($q = 0.9$). The AF percolated phase: (a) $p = 0.01$, (b) $p = 0.1$; The ferromagnetic phase: (c) $p = 0.3$, (d) $p = 0.9$. Black sites are filled.

AF (Hidden) Percolation: Duality Transformation

1. No naïve mean-field AF order parameter !!
2. Local AF parameters defined on the pairs of neighboring spatial sites:

$$\lambda_{2j,2\tau} = n_{2j+1,2\tau} - n_{2j,2\tau},$$
$$\lambda_{2j+1,2\tau+1} = n_{2j+2,2\tau+1} - n_{2j+1,2\tau+1},$$
$$j = 0,1\ldots, N/2 - 1,\ \tau = 0,1\ldots, T/2 - 1.$$

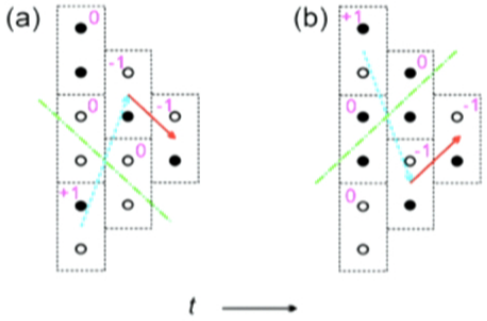


Figure 9. Percolation between different contacting AF clusters. Black circles are filled sites. Dash-dotted lines separate clusters; directed solid arrows indicate the nearest hoppings (bonds of the dual lattice in figure 8(a)) which provide percolation within AF clusters; directed dotted arrows indicate the next-nearest hoppings (additional bonds of the decorated dual lattice in figure 8(b)) which provide percolation between AF clusters. The numbers shown inside the cells are the values of λ.

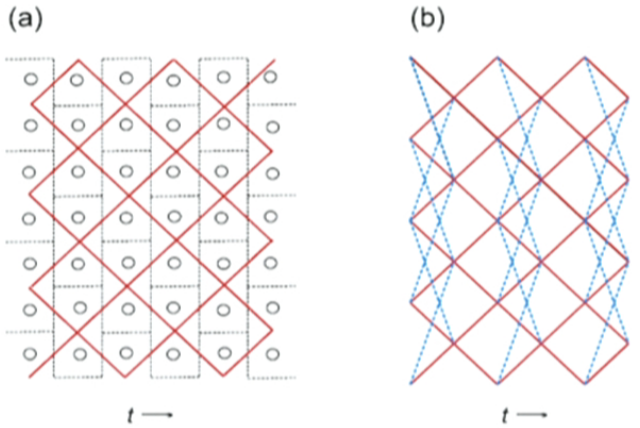


Figure 8. (a) Separation of the original lattice onto the two-site cells (dotted lines) and the dual square lattice where the local AF variables λ reside (solid lines); (b) decorated dual lattice with additional bonds (dashed lines) providing the percolation between different AF clusters.

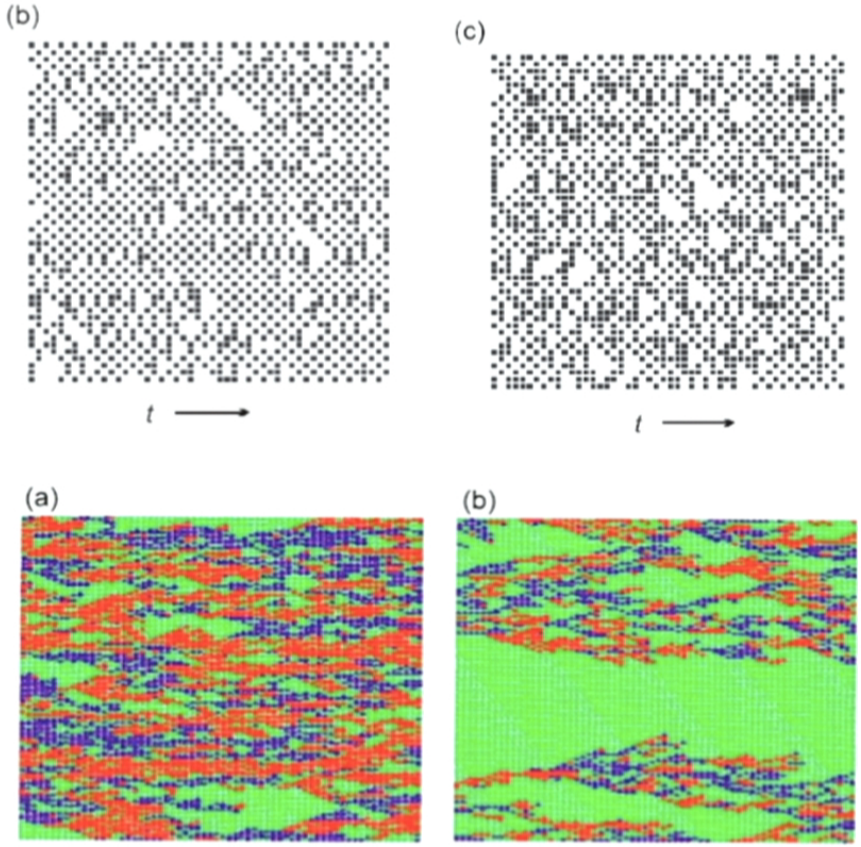
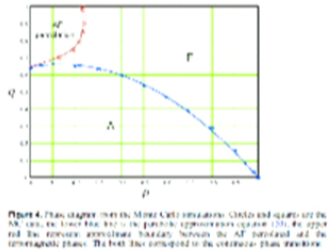


Figure 10. Percolation patterns for $q = 0.9$, $p = 0.15$ (a) and $q = 0.9$, $p = 0.22$ (b). Red and blue squares are percolating AF cells with $\lambda = 1$ and $\lambda = -1$, respectively. The green background stands either the for the non-AF cells with $\lambda = 0$, or for the disconnected AF cells.



AF (Hidden) Percolation: Critical Scaling, DP Universality

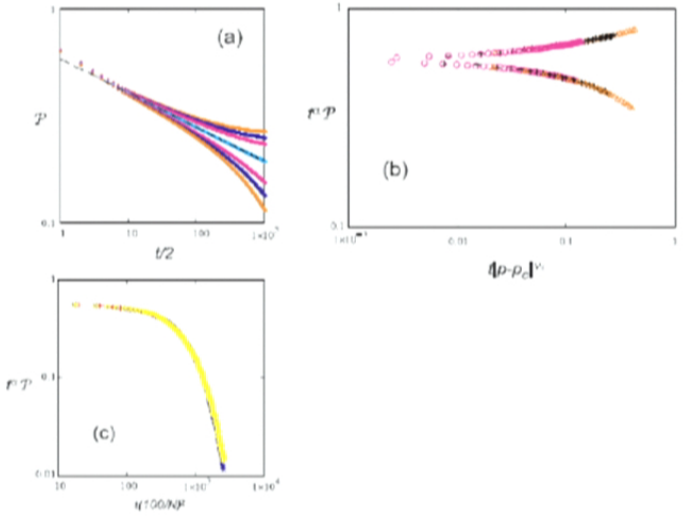
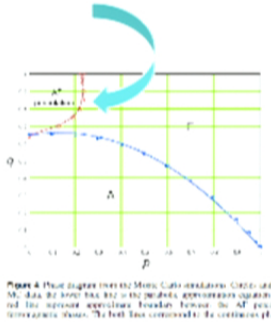


Figure 11. (a) MC simulations of $P(t)$ relaxation for $q = 0.8$ and series of p near $p_c \approx 0.218$, from top to bottom: $p = 0.21$, 0.212, 0.214, 0.218, 0.222, 0.224, 0.226. Dashed line corresponds to power law $0.58/t^{\delta}$ with $\delta = 0.16$. (b) Collapse of the curves from (a) onto a single scaling function. Fitting gives the values of $\nu_{\parallel}$ and p_c shown in table 3. (c) Relaxation of the AF order parameter $P(t)$ for chains with numbers of sites $N = 100$ (x), $N = 200$ (+), $N = 400$ (o) at p_c collapsed onto a single scaling function yields the index z.

Table 3. Critical points p_c and critical indices for several values of q for the AF percolation transition.

q	p_c	α	$\nu_{\parallel}$	z	$\beta = \alpha\nu_{\parallel}$	$\nu_{\perp} = \nu_{\parallel}/z$
0.7	0.1258(2)	0.16(01)	1.75(7)	1.56(5)	0.28(1)	1.12(3)
0.8	0.2181(2)	0.16(01)	1.74(7)	1.55(8)	0.28(01)	1.12(1)
0.9	0.2364(65)	0.16(01)	1.74(9)	1.58(2)	0.28(01)	1.10(6)
1	0.2274(5)	0.16(01)	1.74(2)	1.60(5)	0.27(9)	1.08(5

NB: Crossover of average length of AF domains!



More (hidden) Percolation Patterns

Directed Percolation

Model's transfer probabilities $P(n_{i,t+1}\,|n_{i-1,t}, n_{i+1,t})$. $n_{i,t} = 0, 1$:

$$P(1\,|0,1) = P(1\,|1,0) = p, \quad P(1\,|1,1) = p(2-p),$$
$$P(1\,|0,0) = 0, \quad P(0\,|a,b) = 1 - P(1\,|a,b).$$

Common wisdom:
Two-state model
$p_c = 0.6447$

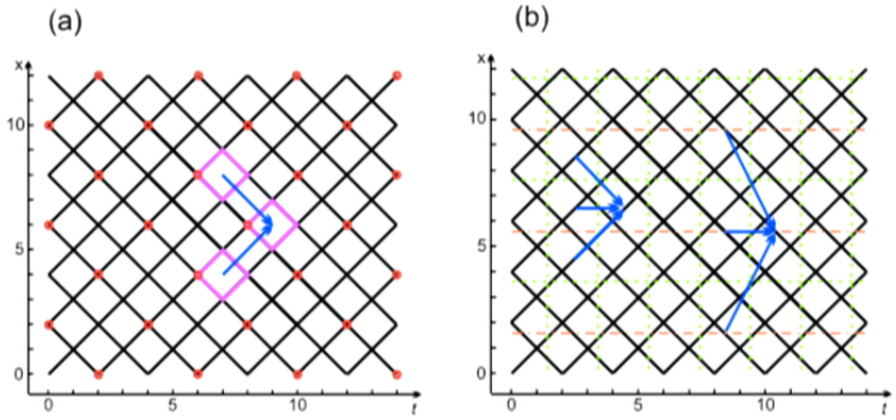


FIG. 1: *Examples of coarse-grained lattices (backbones) for the BDP process.* The original tilted square lattice for the BDP process with spacing $a = \sqrt{2}$ is shown by solid lines on both panels. (a) The tilted square sublattice with the spacing $2a$ and the sites denoted by red circles. Those circles also indicate the leftmost nodes of plaquettes (magenta squares) discussed in the text. (b) Division of the original lattice into 4-site cells (dashed lines) and into 2-site cells (dashed and dot-dashed lines). Colored arrows show the bonds for percolation on resulting coarse-grained lattices.

Table 1. Critical points of geometric phase transitions where different backbones appear. The parameter h designate the presence (h = 1) or absence (h = 0) of horizontal bonds in percolative backbone.

Sublattices		2-site hubs			Plaquettes		4-site hubs		
Spacing	p_c	f	h	p_c	f		f	h	p_c
2a	0.663(5)	1	1	p_{BDP}=0.6447	1	p_{BDP}=0.6447	2	1	p_{BDP}=0.6447
4a	0.677(6)	1	0	0.646(7)	2	0.646(5)	2	0	0.646(4)
		2	1	0.656(2)	3	0.671(2)	3	1	0.657(4)
		2	0	0.696(3)	4	0.742(5)	3	0	0.682(7)
							4	1	0.710(2)
							4	0	0.759(5)

Percolative Order Parameters:

Order Parameter: Capacity of the backbone P(t)
~ number of Connected Backbone Sites (CBS)
~ string operator (in terms of occupation numbers)

$$2N_{\mathrm{CBS}}(t)/N = \langle \mathcal{P}_{i,t} \rangle = P(t)$$

$$\mathcal{P}_{i,t} = \sum_{\boldsymbol{\sigma}} \mathcal{O}_{i,t}(\boldsymbol{\sigma})$$

$$\mathcal{O}_{i,t}(\boldsymbol{\sigma}) \equiv n_{2i,2t} \prod_{\tau=1}^{t-1} n_{2\left(i+\sum_{k=\tau}^{t-1}\sigma_k\right),2\tau}$$

For the backbone obtained by coarse-graining "renormalized" occupation number is defined:

$$\nu_{j,\tau} \equiv \Theta\Big(\sum_{i,t\in \mathrm{cell}} n_{i,t} - f\Big)$$

Table 1. Critical points of geometric phase transitions where different backbones appear. The parameter h designate the presence (h = 1) or absence (h = 0) of horizontal bonds in percolative backbone.

Sublattices		2-site hubs			Plaquettes		4-site hubs		
Spacing	p_c	f	h	p_c	f		f	h	p_c
2a	0.663(5)	1	1	p_{BDP}=0.6447	1	p_{BDP}=0.6447	2	1	p_{BDP}=0.6447
4a	0.677(6)	1	0	0.646(7)	2	0.646(5)	2	0	0.646(4)
		2	1	0.656(2)	3	0.671(2)	3	1	0.657(4)
		2	0	0.696(3)	4	0.742(5)	3	0	0.682(7)
							4	1	0.710(2)
							4	0	0.759(5)

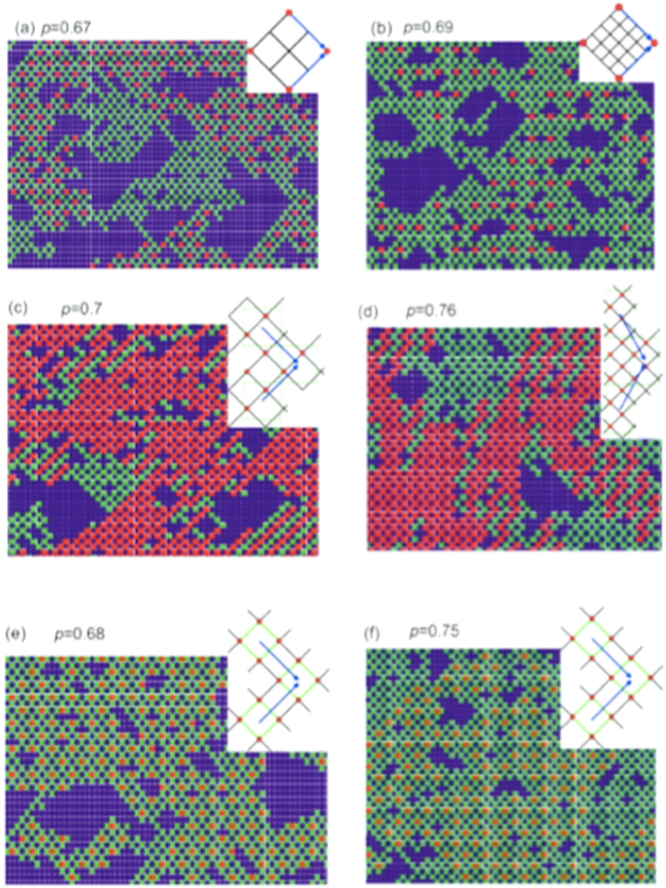
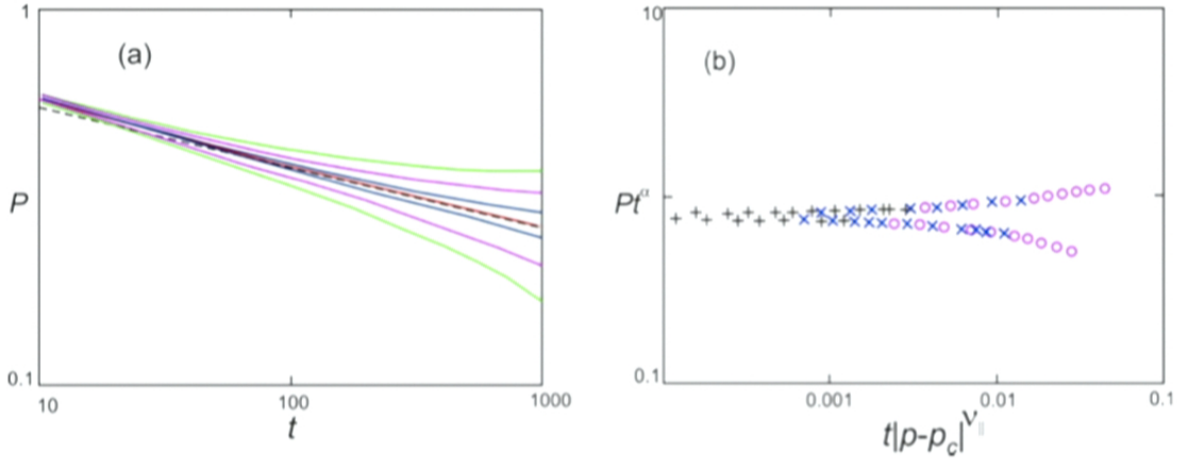


FIG. 3: *BDP patterns.* The fragments of steady patterns of the BDP process (green squares) with backbones (red squares). (a) $p = 0.67$, $2a$ sublattice backbone, (b) $p = 0.68$, $4a$ sublattice backbone (c) $p = 0.68$, 2-site cell backbone with next-nearest bonds, (d) $p = 0.76$, 4-site cell backbone with next-nearest bonds, (e) $p = 0.68$, plaquette backbone with $f = 3$, red squares are centers of percolative plaquettes, (f) $p = 0.75$, plaquette backbone with $f = 4$.

Critical Scaling: DP universality class



BDP on the 2*a sublattice*. (a) MC simulations of relaxation of the BDP sublattice percolation capacity *P(t)* for series of *p* near $p_c \approx 0.6635$, from top to bottom: *p=0.6665, 0.665, 0.664, 0.6635, 0.663, 0.662, 0.661*. The dashed line corresponds to the power law $0.8/t^{\alpha}$, with $\alpha = 0.16$. *(b)* Collapse of the curves from (a) onto a single scaling function. Fitting gives the values of $\nu_{||} \approx 1.731$ and $p_c = 0.6635$

Contact Process:

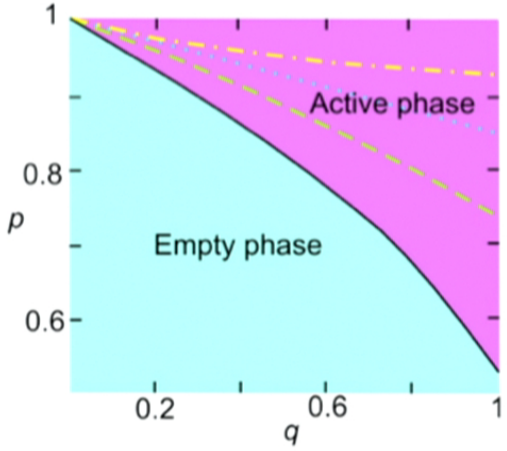
Model's transfer probabilities $P(n_{i,t+1}\,|n_{i-1,t}, n_{i+1,t})$.

$P(1\,|0,0,1) = P(1\,|1,0,0) = q/2, \quad P(1\,|1,0,1) = q, \quad P(1\,|*,1,*) = p,$
$P(1\,|0,0,0) = 0, \quad P(0\,|a,b,c) = 1 - P(1\,|a,b,c)$

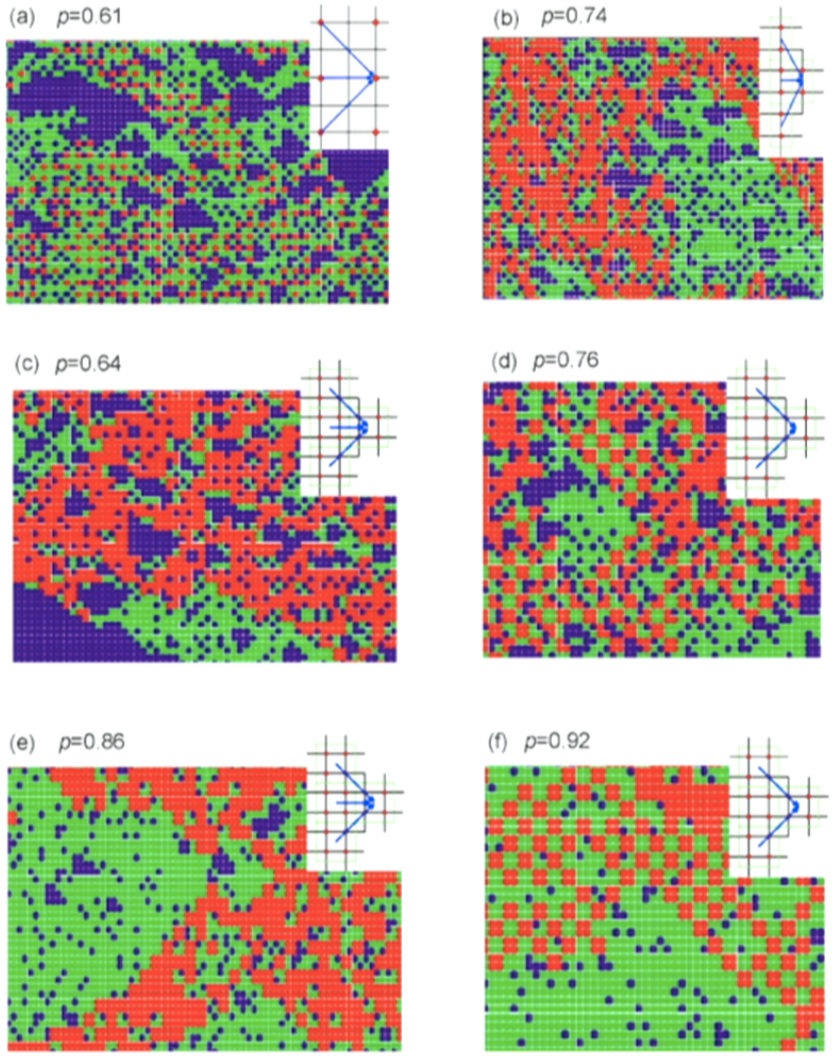
The model undergoes a continuous absorbing-active phase transition of the DP class at

$$p_{cp}(q) \approx 1 - aq - bq^3, \quad a = 0.3, \quad b = 0.17, \qquad (1)$$

with the OP $\rho(\infty) = 0$ at $p < p_{cp}(q)$ and $\rho(\infty) \neq 0$ at $p > p_{cp}(q)$. Here $\rho(t) = \langle \sum_i n_{i,t} \rangle / N$.



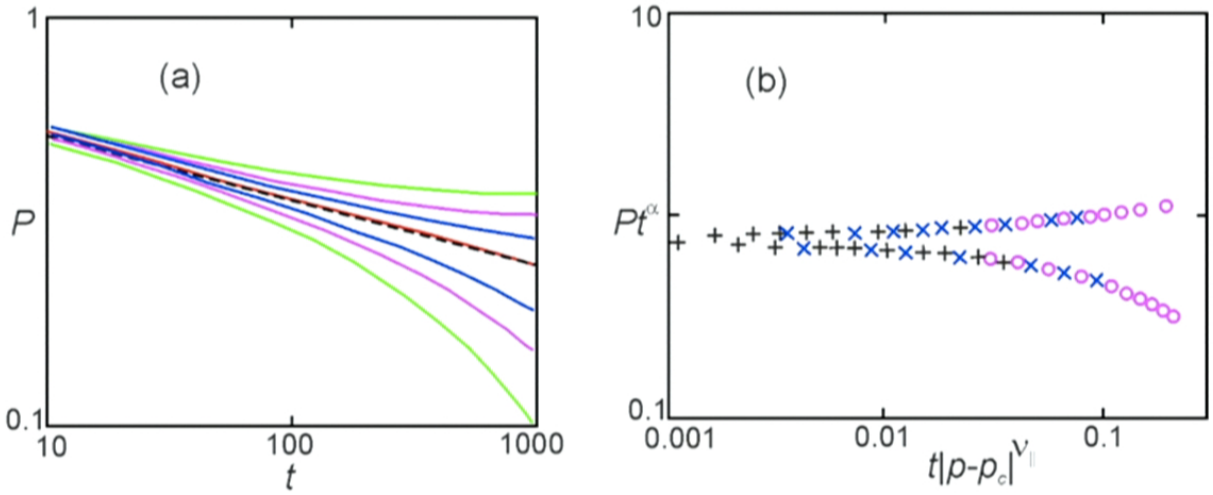
Phase diagram of contact process. The bold black curve (8) is the boundary of the absorbing-active phase transition. The boundaries for appearance of backbones with $h = 0$ are also shown. The sublattice backbone exists above dashed line, 2-site hub backbone – above dotted line and 4-site hub backbone with $f = 4$– above dashed-dotted line.



Percolative patterns of contact process.

The fragments of steady patterns of contact process at $q = 1$ (green squares) with backbones (red squares):
(a) $p = 0.61$: $2a$ sublattice backbone,
(b) $p = 0.74$: 2-site cell backbone with nearest and next-nearest bonds, $f = 2$,
(c) $p = 0.64$: 4-site cell backbone with nearest and next-nearest bonds, $f = 3$,
(d) $p = 0.76$: 4-site cell backbone with next-nearest bonds, $f = 3$,
(e) $p = 0.86$: 4-site cell backbone with nearest and next-nearest bonds, $f = 4$,
(f) $p = 0.92$: 4-site cell backbone with next-nearest bonds, $f = 4$.

Critical Scaling: DP universality class



(a) MC simulations of relaxation of the CP sublattice percolation capacity $P(t)$ with $h = 0$ for $q = 1$ and series

of p near $p_c \approx 0.73$;

From top to bottom: $p = 0.735, 0.732, 0.73, 0.7277, 0.725, 0.723, 0.72$.

Dashed line corresponds to power law law $0.75/t^{a}$, with $a = 0.16$.

(b) Collapse of the curves from (a) onto a single scaling function. Fitting gives the values

$n_{||} \approx 1.735$ and $p_c = 0.7277$-

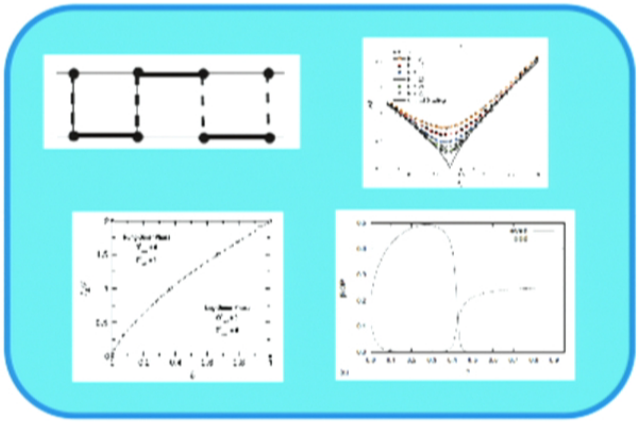
Conclusions-I:

1. 2D active (percolating) phases were studied in three models.
2. Such phases possess numerous (infinite ??) hidden geometric orders characterized by distinct percolative patterns.
3. NB: Order parameters are nonlocal (!!).
4. The cascades of continuous geometric transitions belong to the DP universality class.
5. Conjecture: cascades are a generic feature of percolation, should exist in other classical/quantum models (stat. mechanics, kinetics)

Conclusions-II: (short version)

Future work:

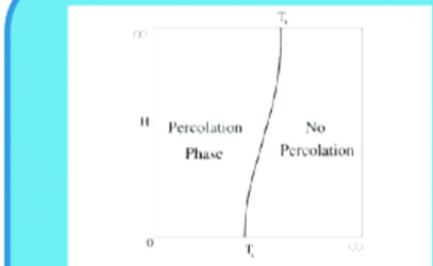
Proper Phase Transitions
Nonlocal OPs



Nonlocal OPs

Geometric Phase Transitions

Counting Phase Transitions



(Future) Unifying Theory

THE END

THANK YOU!