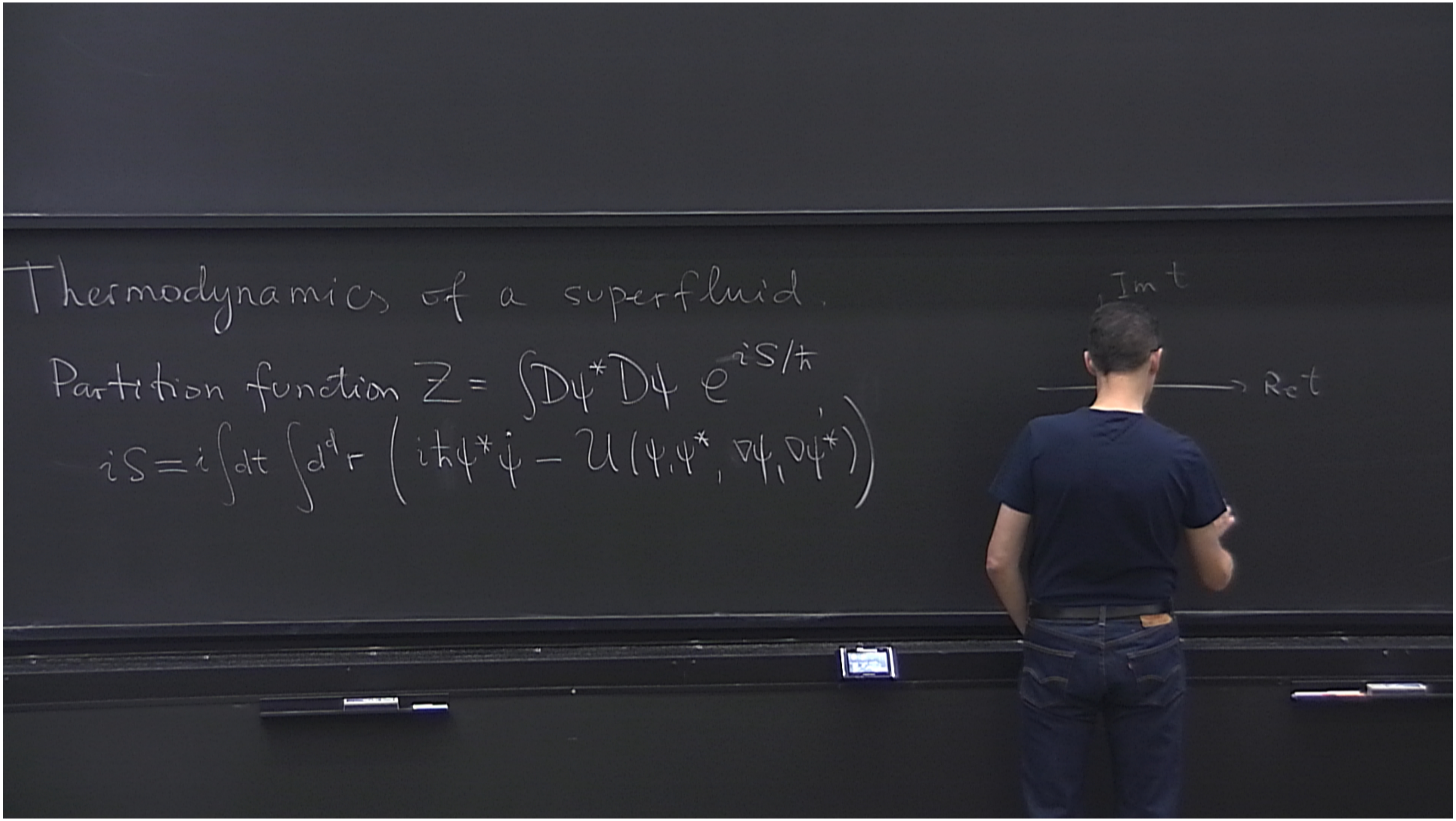
Title: PSI 2015/2016 Condensed Matter - Oleg Tchernyshyov - Lecture 12

Date: Nov 24, 2015 10:45 AM

URL: http://pirsa.org/15110053

Abstract:


Thermodynamics of a superfluid.
Partition function $Z = \int D\psi^* D\psi \, e^{iS/\hbar}$
$iS = i\int dt \int d^d r \left(i\hbar \psi^* \dot{\psi} - U(\psi, \psi^*, \nabla\psi, \nabla\psi^*) \right)$
Im t
Re t

Thermodynamics of a superfluid.

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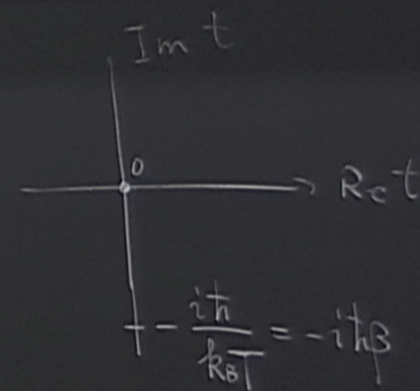
$$iS = i\int_0^{-i\hbar\beta} dt \int d^d r \left(i\hbar \psi^* \dot{\psi} - \mathcal{U}(\psi, \psi^*, \nabla\psi, \nabla\psi^*) \right)$$

$$\mathcal{U} = \frac{\hbar^2}{2m} \nabla\psi^* \cdot \nabla\psi + \frac{1}{2} g \psi^{*2} \psi^2 - \mu \psi^* \psi$$



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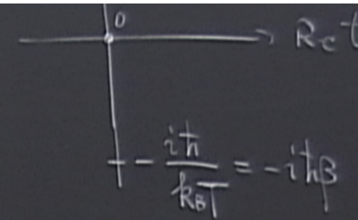
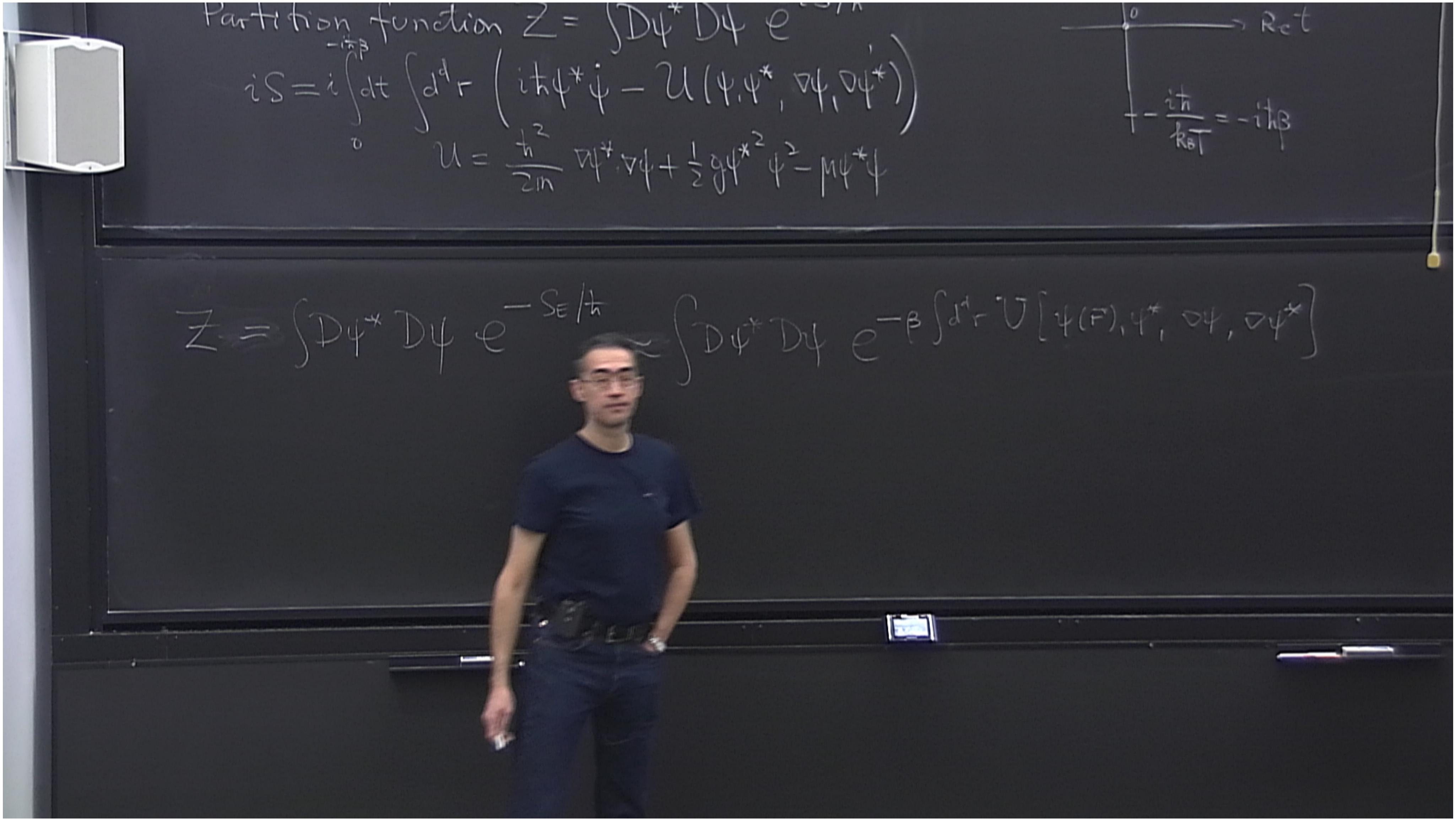
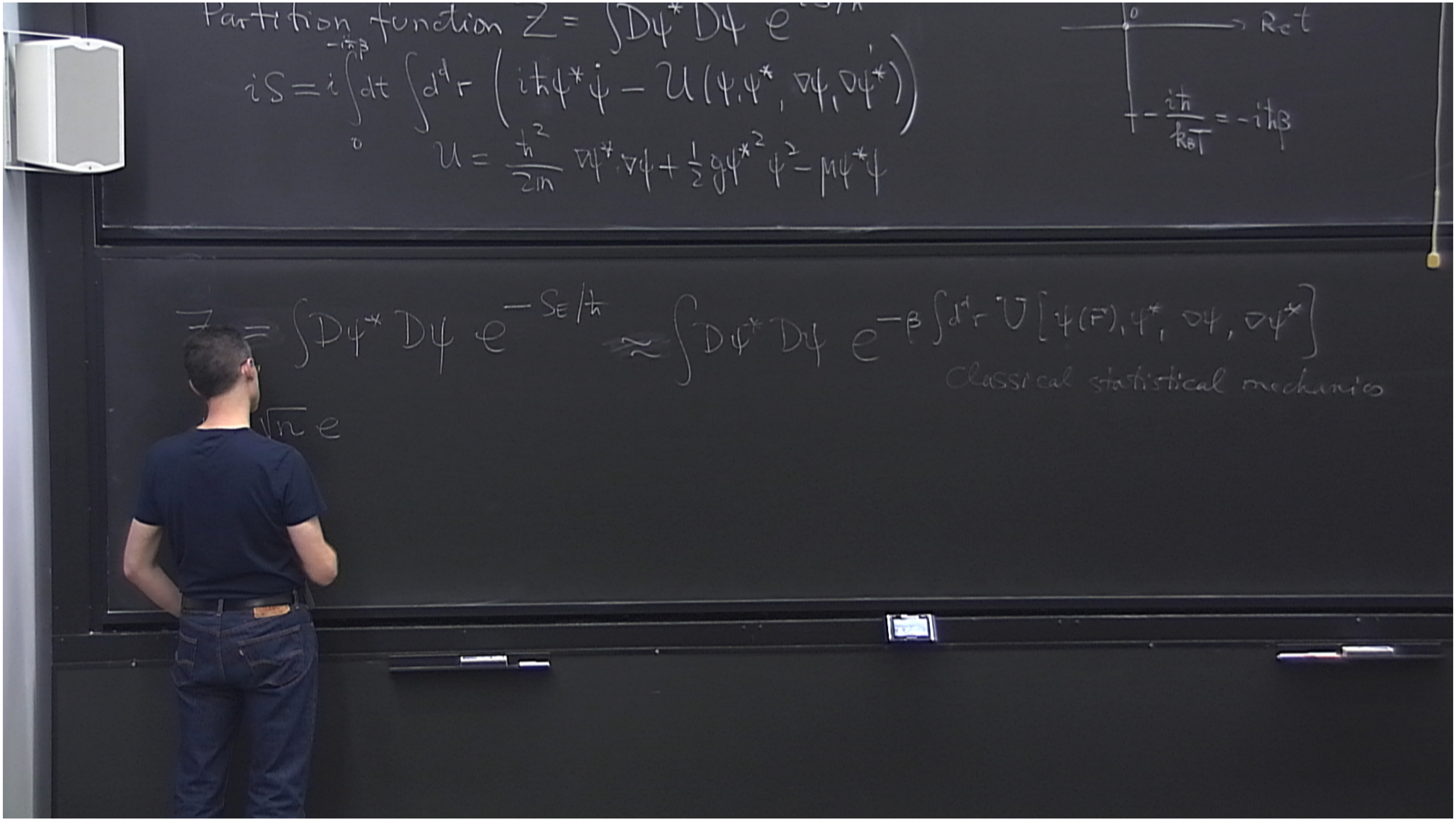


QFT in d+1 ⟶ classical one in d.

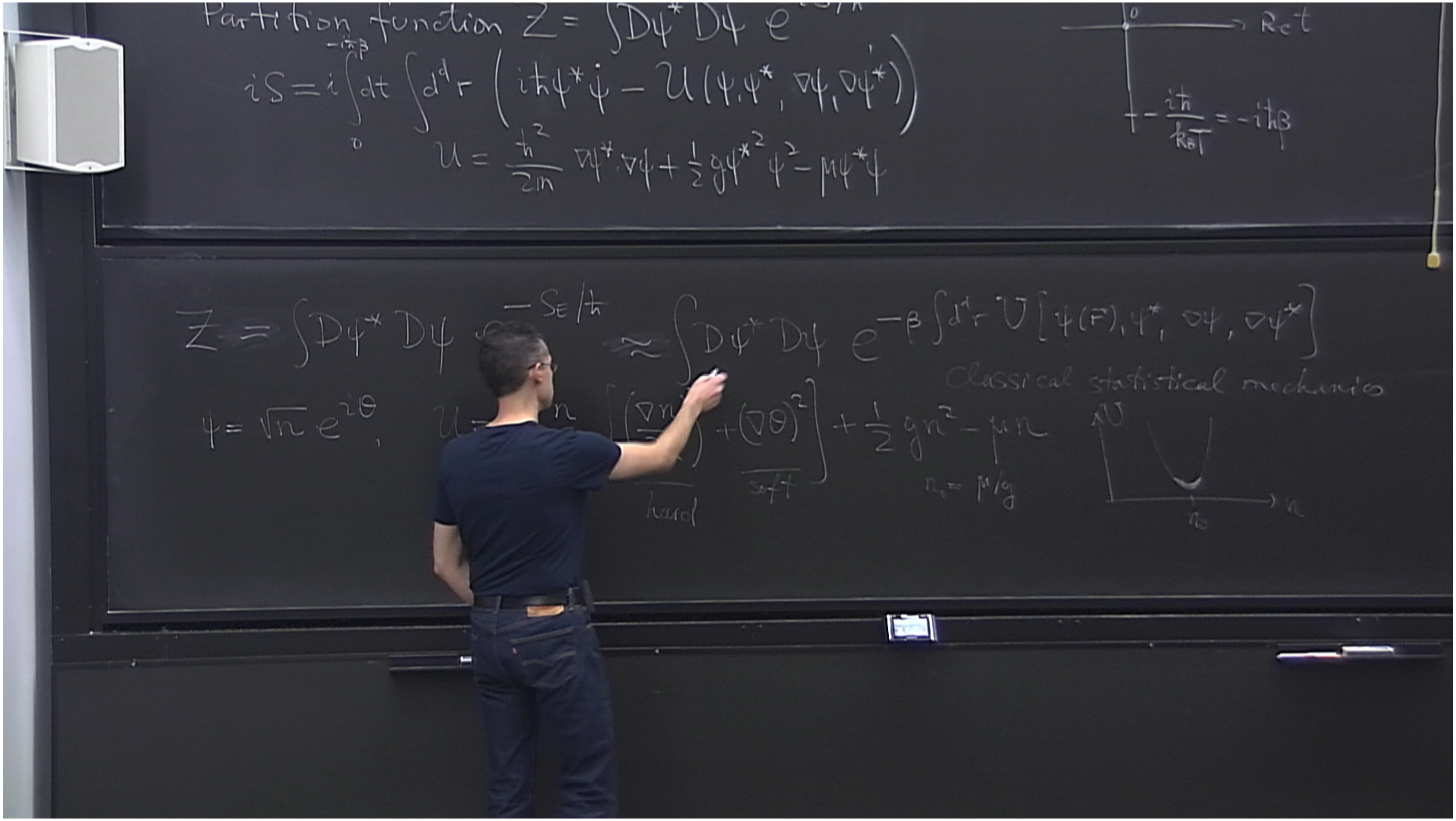
$t = -i\tau$, $0 \le \tau \le \hbar\beta$

$$iS = \int_0^{\hbar\beta} d\tau \int d^d r \left[\cancel{i}\hbar\psi^* \frac{\partial\psi}{\partial(-\cancel{i}\tau)} - \mathcal{U} \right] = -S_E, \quad S_E = \int_0^{\hbar\beta} d\tau \int d^d r \left[\hbar\psi^* \frac{\partial\psi}{\partial\tau} + \mathcal{U} \right]$$


Partition function Z = ∫Dψ* Dψ e
iS = i ∫dt ∫d^d r (iħψ* ψ̇ − U(ψ, ψ*, ∇ψ, ∇ψ*))
U = ħ²/2m ∇ψ*·∇ψ + ½ gψ*² ψ² − μψ*ψ
Re t
−iħ/k_B T = −iħβ
Z = ∫Dψ* Dψ e^(−S_E/ħ) = ∫Dψ* Dψ e^(−β∫d^d r U[ψ(r), ψ*, ∇ψ, ∇ψ*])



Partition function $Z = \int D\psi^* D\psi \, e$
$iS = i\int_0^{-i\hbar\beta} dt \int d^d r \left(i\hbar \psi^* \dot\psi - U(\psi, \psi^*, \nabla\psi, \nabla\psi^*) \right)$
$U = \frac{\hbar^2}{2m} \nabla\psi^* \cdot \nabla\psi + \frac{1}{2} g \psi^{*2}\psi^2 - \mu\psi^*\psi$
0
Re t
$-\frac{i\hbar}{k_B T} = -i\hbar\beta$
$Z = \int D\psi^* D\psi \, e^{-S_E/\hbar} \approx \int D\psi^* D\psi \, e^{-\beta \int d^d r \, U[\psi(\vec r), \psi^*, \nabla\psi, \nabla\psi^*]}$
Classical statistical mechanics
$\sqrt{n}\, e$


Partition function $Z = \int D\psi^* D\psi\, e^{\ldots}$
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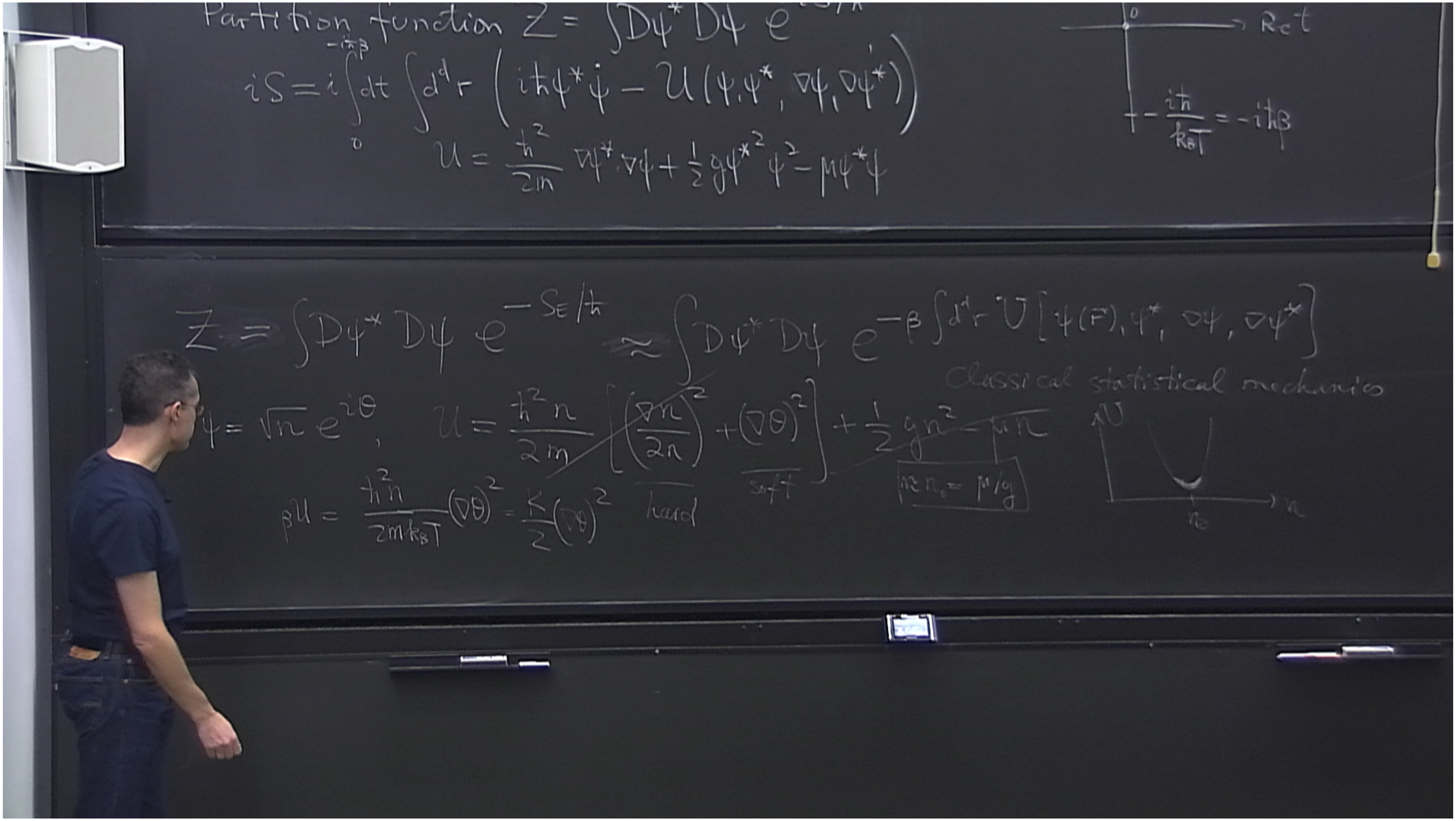
Classical statistical mechanics
$\psi = \sqrt{n}\, e^{i\theta}$
$+ (\nabla\theta)^2] + \frac{1}{2} g n^2 - \mu n$
hard
soft
$n_0 = \mu/g$
U
n
n_0

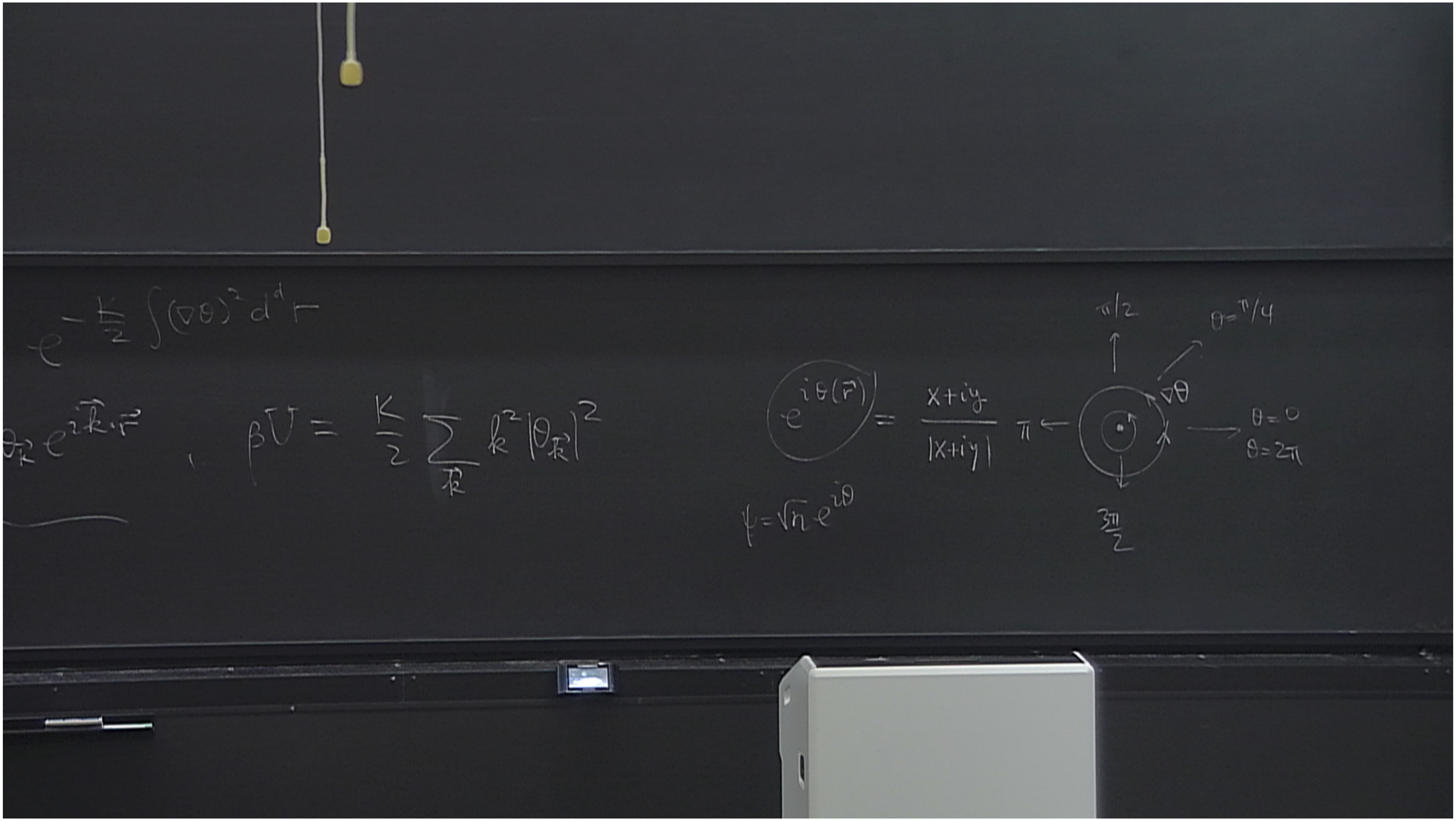


Partition function $Z = \int D\psi^* D\psi \, e^{iS/\hbar}$
$iS = i\int_0^{-i\hbar\beta} dt \int d^d r \left(i\hbar\psi^*\dot\psi - U(\psi,\psi^*,\nabla\psi,\nabla\psi^*) \right)$
$U = \frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi + \frac{1}{2}g\psi^{*2}\psi^2 - \mu\psi^*\psi$
Re t
$-\frac{i\hbar}{k_B T} = -i\hbar\beta$
$Z = \int D\psi^* D\psi \, e^{-S_E/\hbar} \approx \int D\psi^* D\psi \, e^{-\beta\int d^d r \, U[\psi(r),\psi^*,\nabla\psi,\nabla\psi^*]}$
Classical statistical mechanics
$\psi = \sqrt{n}\, e^{i\theta}$, $U = \frac{\hbar^2 n}{2m}\left[\left(\frac{\nabla n}{2n}\right)^2 + (\nabla\theta)^2\right] + \frac{1}{2}gn^2 - \mu n$
hard
soft
$n_0 = \mu/g$
$\beta U = \frac{\hbar^2 n}{2mk_BT}(\nabla\theta)^2 = \frac{K}{2}(\nabla\theta)^2$
U
n
n_0

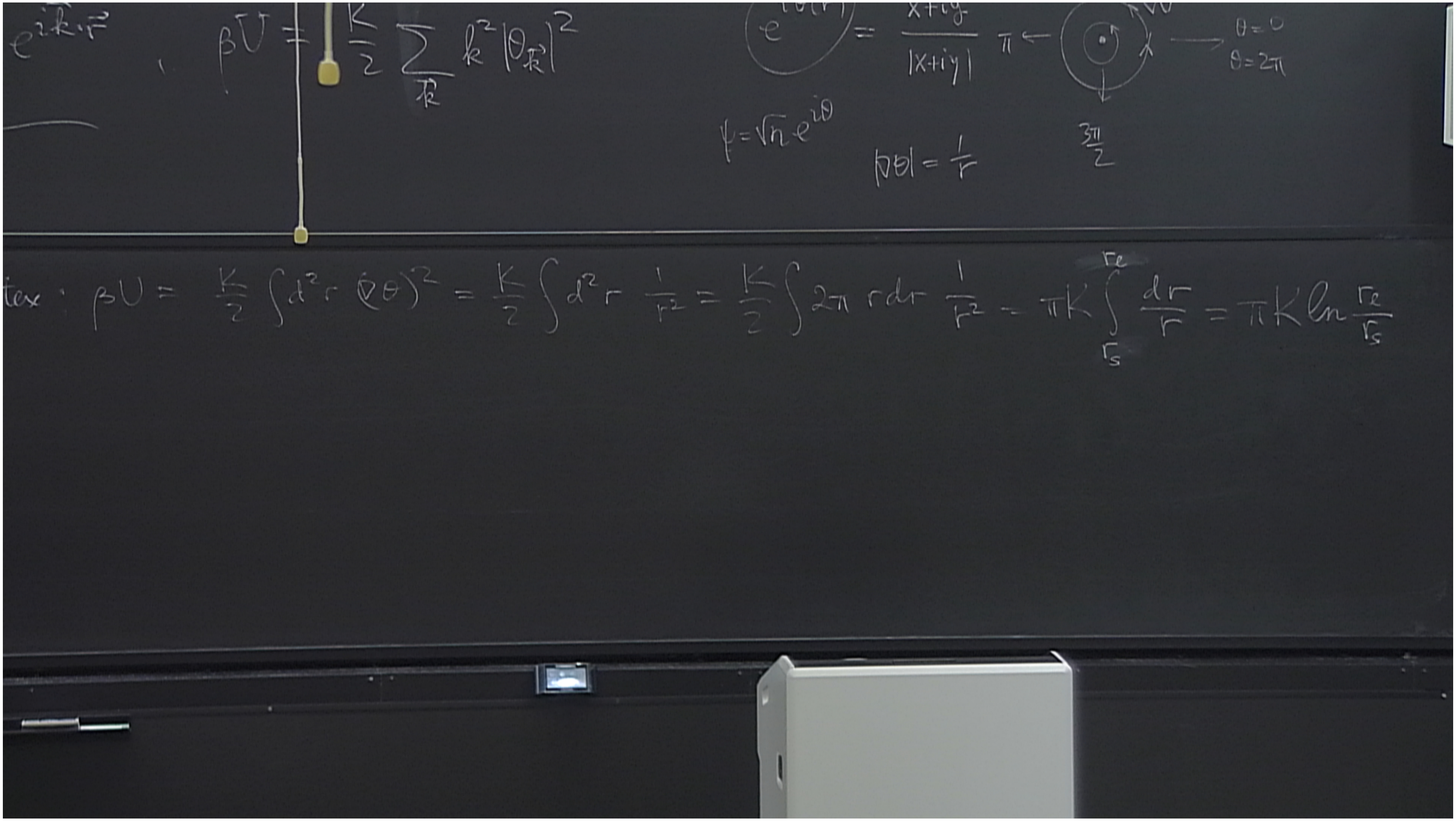
$$Z = \int D\psi^* D\psi \, e^{-\frac{K}{2}\int (\nabla\theta)^2 d^d r}$$

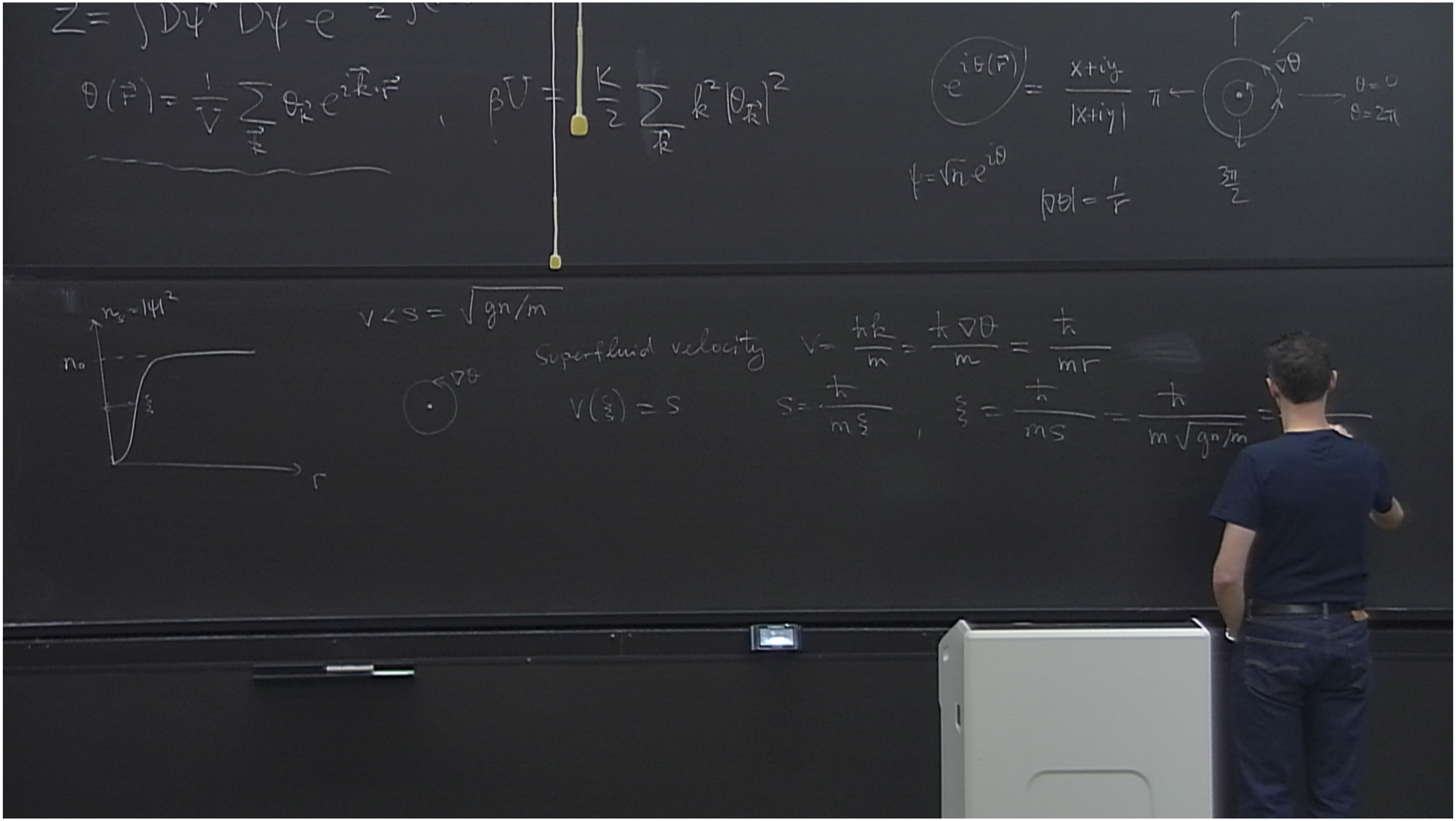
$$\theta(\vec{r}) = \frac{1}{V}\sum_{\vec{k}} \theta_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} \quad , \quad \beta U = \frac{K}{2}\sum_{\vec{k}} k^2 |\theta_{\vec{k}}|^2$$

$$e^{i\theta(\vec{r})} = \frac{x+iy}{|x+iy|}$$



$e^{-\frac{K}{2}\int(\nabla\theta)^2 d^d r}$
$\theta_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}$
$\beta U = \frac{K}{2}\sum_{\vec{k}} k^2 |\theta_{\vec{k}}|^2$
$e^{i\theta(\vec{r})} = \frac{x+iy}{|x+iy|}$
$\psi = \sqrt{n}\, e^{i\theta}$
$\pi/2$
$\theta = \pi/4$
$\nabla\theta$
$\theta = 0$
$\theta = 2\pi$
π
$\frac{3\pi}{2}$


$e^{i\vec{k}\cdot\vec{r}}$
$\beta U = \frac{K}{2}\sum_{\vec{k}} k^2 |\theta_{\vec{k}}|^2$
$e^{i\theta(\vec{r})} = \frac{x+iy}{|x+iy|}$
π
$\theta=0$
$\theta=2\pi$
$\psi=\sqrt{n}\,e^{i\theta}$
$|\nabla\theta| = \frac{1}{r}$
$\frac{3\pi}{2}$
vortex: $\beta U = \frac{K}{2}\int d^2r\,(\nabla\theta)^2 = \frac{K}{2}\int d^2r\,\frac{1}{r^2} = \frac{K}{2}\int 2\pi\,r\,dr\,\frac{1}{r^2} = \pi K\int_{r_s}^{r_e}\frac{dr}{r} = \pi K \ln\frac{r_e}{r_s}$



Z = ∫Dψ* Dψ e
θ(r) = 1/V Σ_k θ_k e^{ik·r}
βU = K/2 Σ_k k² |θ_k|²
e^{iθ(r)} = (x+iy)/|x+iy|
ψ = √n e^{iθ}
|∇θ| = 1/r
θ = 0
θ = 2π
π
3π/2
∇θ
n_s = |ψ|²
n₀
ξ
r
v < s = √(gn/m)
Superfluid velocity v = ħk/m = ħ∇θ/m = ħ/mr
v(ξ) = s
s = ħ/mξ , ξ = ħ/ms = ħ/(m√(gn/m))

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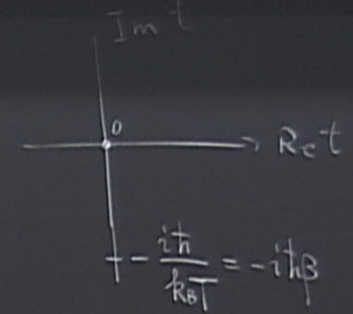
$$\mathcal{U} = \frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi + \frac{1}{2}g\psi^{*2}\psi^2 - \mu\psi^*\psi$$

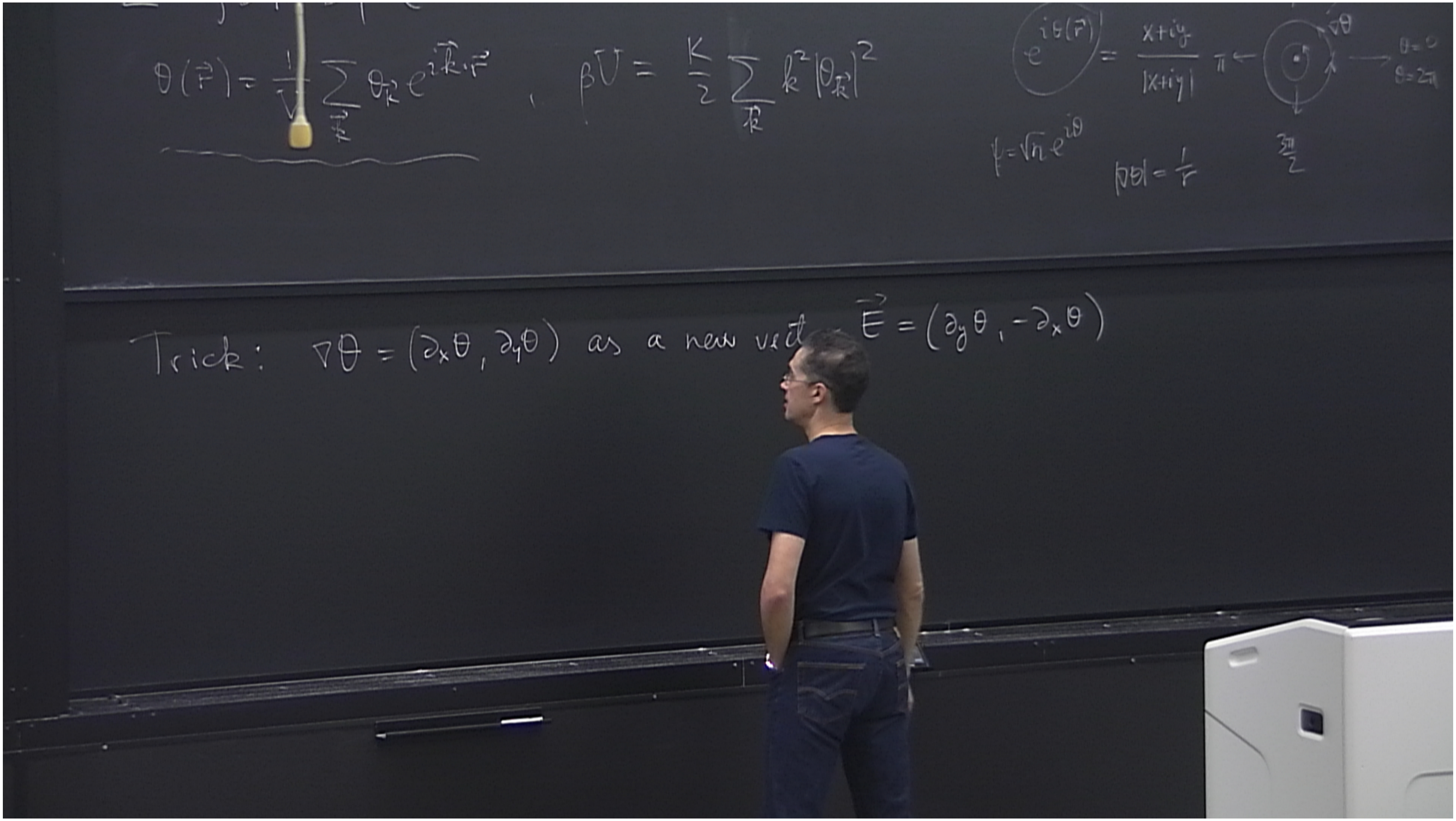


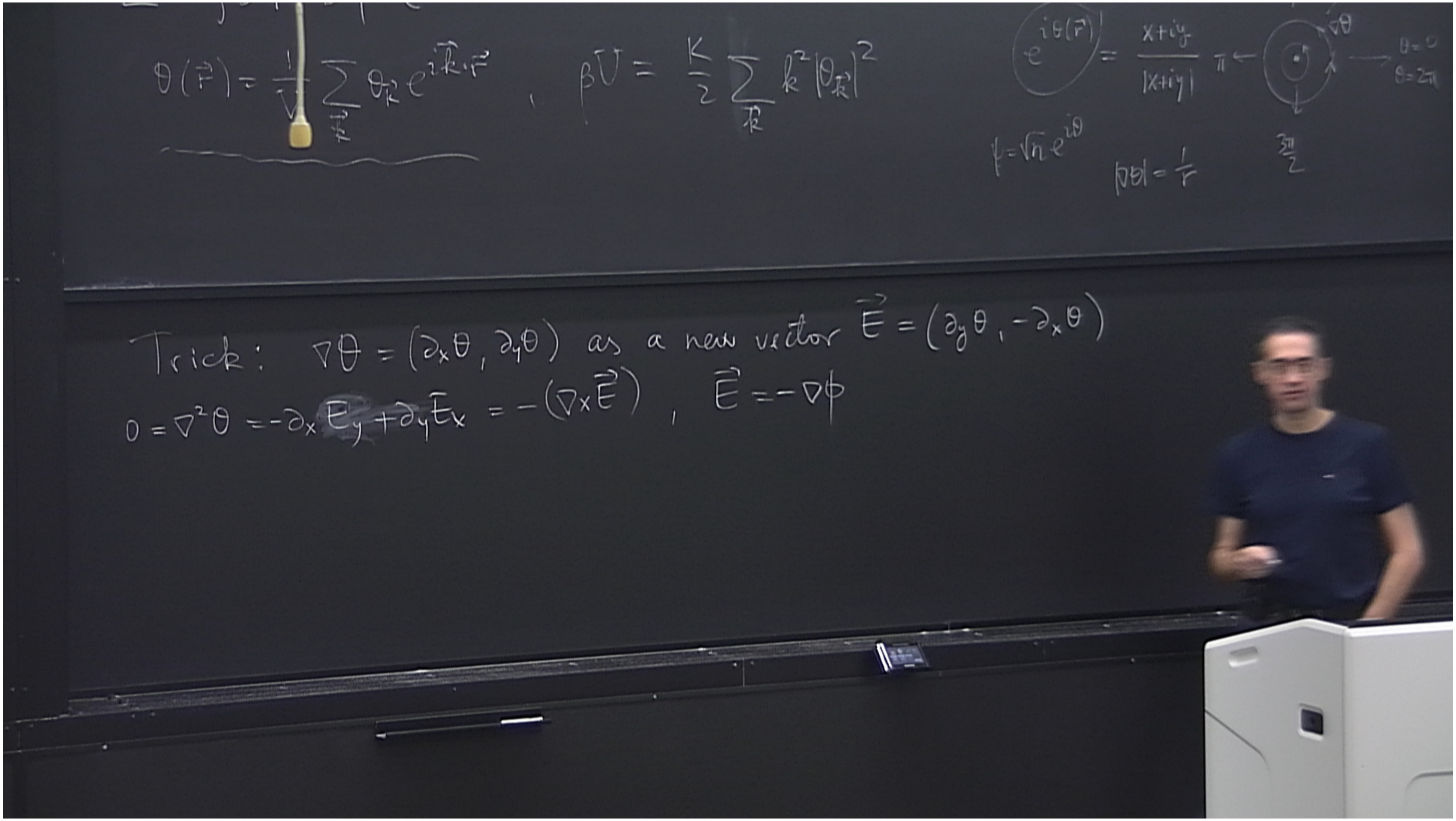
Next step: $\theta(\vec{r}) = \underset{\text{vortices}}{\theta_v(\vec{r})} + \underset{\text{waves (smooth)}}{\theta_w(\vec{r})} \longrightarrow \beta U = \underline{\underline{\beta U_v}} + \beta U_w$, $\beta U_w = \frac{K}{2}\int (\nabla\theta_w)^2 d^2 r$

$$Z = \int D\psi^* D\psi \, e^{-\beta U[\psi,\psi^*]}$$

Configurations minimizing (locally) energy βU.

$\beta U = \frac{K}{2}\int d^2 r (\nabla\theta)^2$ $\qquad \theta = \text{const} \qquad 0 = \frac{\delta U}{\delta\theta} = -K\nabla^2\theta \qquad \nabla^2\theta = 0.$


$\theta(\vec{r}) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \theta_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}$
$\beta U = \frac{K}{2} \sum_{\vec{k}} k^2 |\theta_{\vec{k}}|^2$
$e^{i\theta(\vec{r})} = \frac{x+iy}{|x+iy|}$
$\psi = \sqrt{n} e^{i\theta}$
$|\nabla\theta| = \frac{1}{r}$
$\theta = 0$
$\theta = 2\pi$
Trick: $\nabla\theta = (\partial_x\theta, \partial_y\theta)$ as a new vect
$\vec{E} = (\partial_y\theta, -\partial_x\theta)$


$\theta(\vec r) = \frac{1}{\sqrt{V}} \sum_{\vec k} \theta_{\vec k} e^{i\vec k \cdot \vec r}$, $\beta U = \frac{K}{2} \sum_{\vec k} k^2 |\theta_{\vec k}|^2$
$e^{i\theta(\vec r)} = \frac{x+iy}{|x+iy|}$
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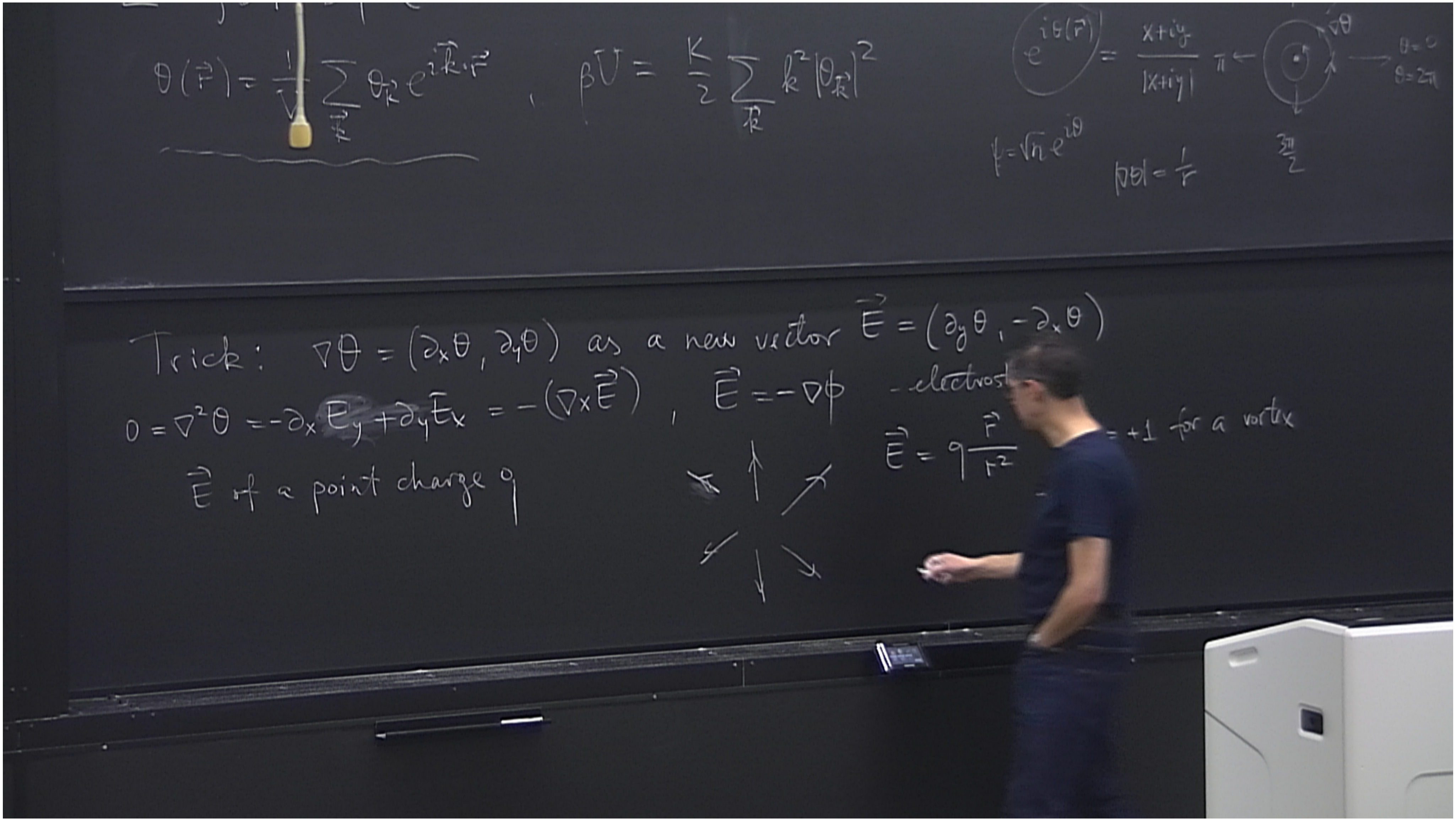
Trick: $\nabla\theta = (\partial_x\theta, \partial_y\theta)$ as a new vector $\vec E = (\partial_y\theta, -\partial_x\theta)$
$0 = \nabla^2\theta = -\partial_x E_y + \partial_y E_x = -(\nabla\times\vec E)$, $\vec E = -\nabla\phi$



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Trick: $\nabla\theta = (\partial_x\theta, \partial_y\theta)$ as a new vector $\vec E = (\partial_y\theta, -\partial_x\theta)$
$0 = \nabla^2\theta = -\partial_x E_y + \partial_y E_x = -(\nabla\times\vec E)$, $\vec E = -\nabla\phi$ - electrost
$\vec E = q\frac{\vec r}{r^2}$
$= +1$ for a vortex
$\vec E$ of a point charge q

Partition function $Z = \int D\psi \, D\psi^* \, e$

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$$\mathcal{U} = \frac{\hbar^2}{2m} \nabla\psi^* \cdot \nabla\psi + \frac{1}{2} g \psi^{*2} \psi^2 - \mu \psi^* \psi$$

$\mathrm{Re}\, t$

$-\frac{i\hbar}{k_B T} = -i\hbar\beta$

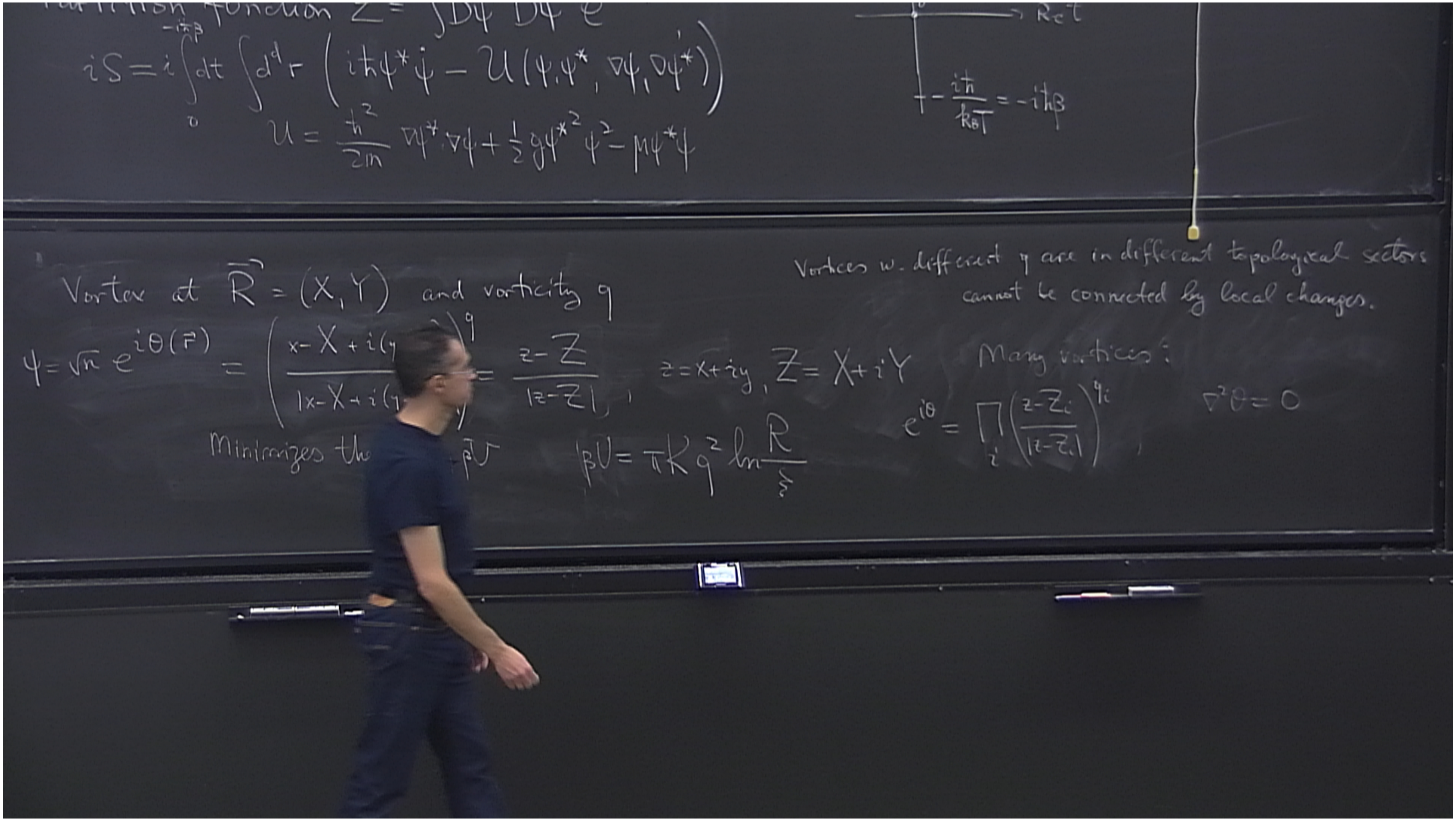
Vortex at $\vec{R} = (X, Y)$ and vorticity q

$$\psi = \sqrt{n}\, e^{i\theta(\vec{r})} = \left(\frac{x - X + i(y - Y)}{|x - X + i(y - Y)|} \right)^q = \frac{z - Z}{|z - Z|}, \quad z = x + iy, \; Z = X + iY$$

Minimizes the energy βU $\qquad \beta U = \pi K q^2 \ln \frac{R}{\xi}$

Vortices w. different q are in different topological sect[ors]

cannot be connected by local changes.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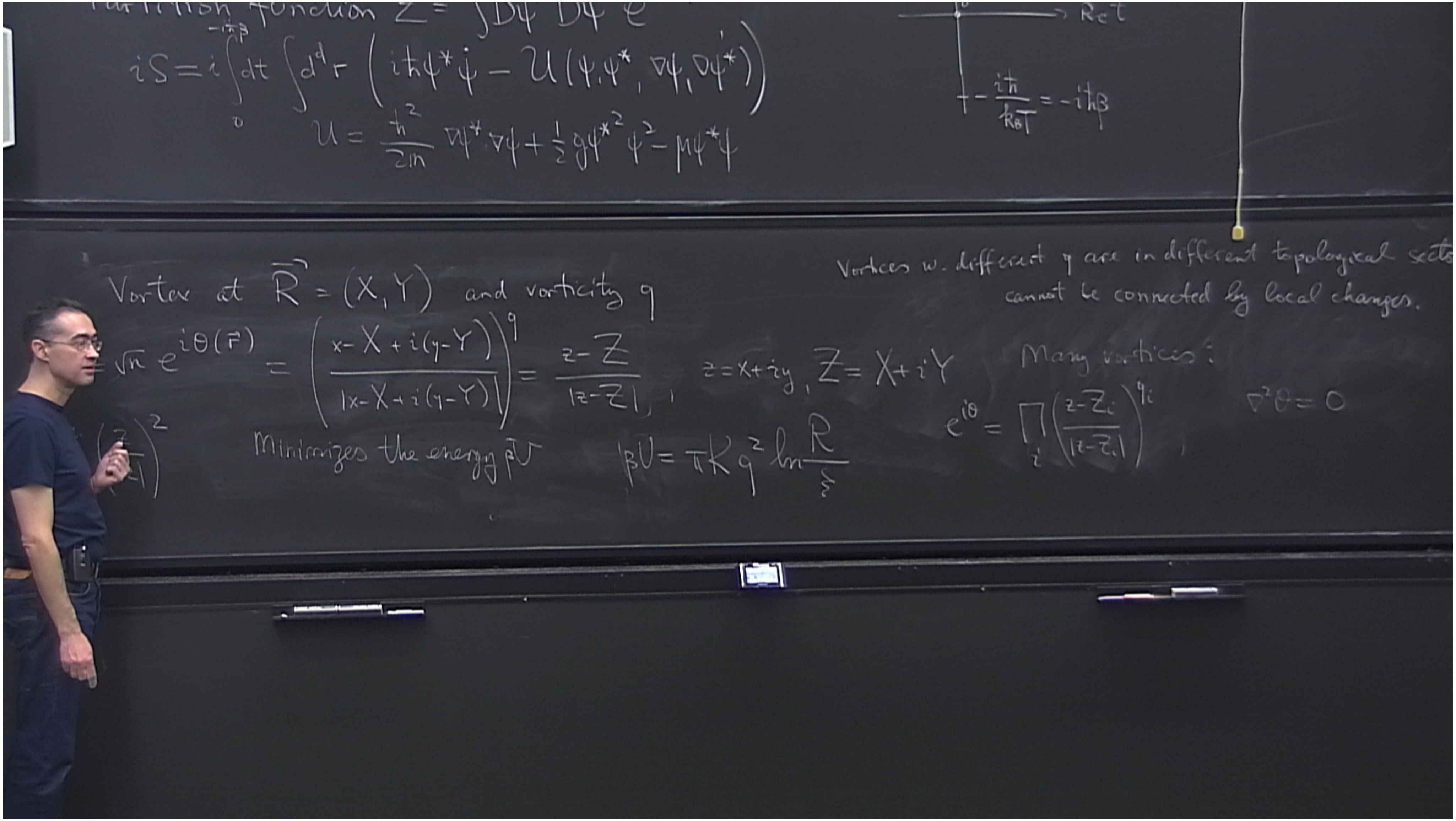
$-\frac{i\hbar}{k_B T} = -i\hbar\beta$
Vortex at $\vec{R} = (X, Y)$ and vorticity q
Vortices w. different q are in different topological sectors
cannot be connected by local changes.
Many vortices:
$z = x + iy$, $Z = X + iY$
$e^{i\theta} = \prod_i \left(\frac{z - Z_i}{|z - Z_i|}\right)^{q_i}$
$\nabla^2\theta = 0$
$\beta U = \pi K q^2 \ln\frac{R}{\xi}$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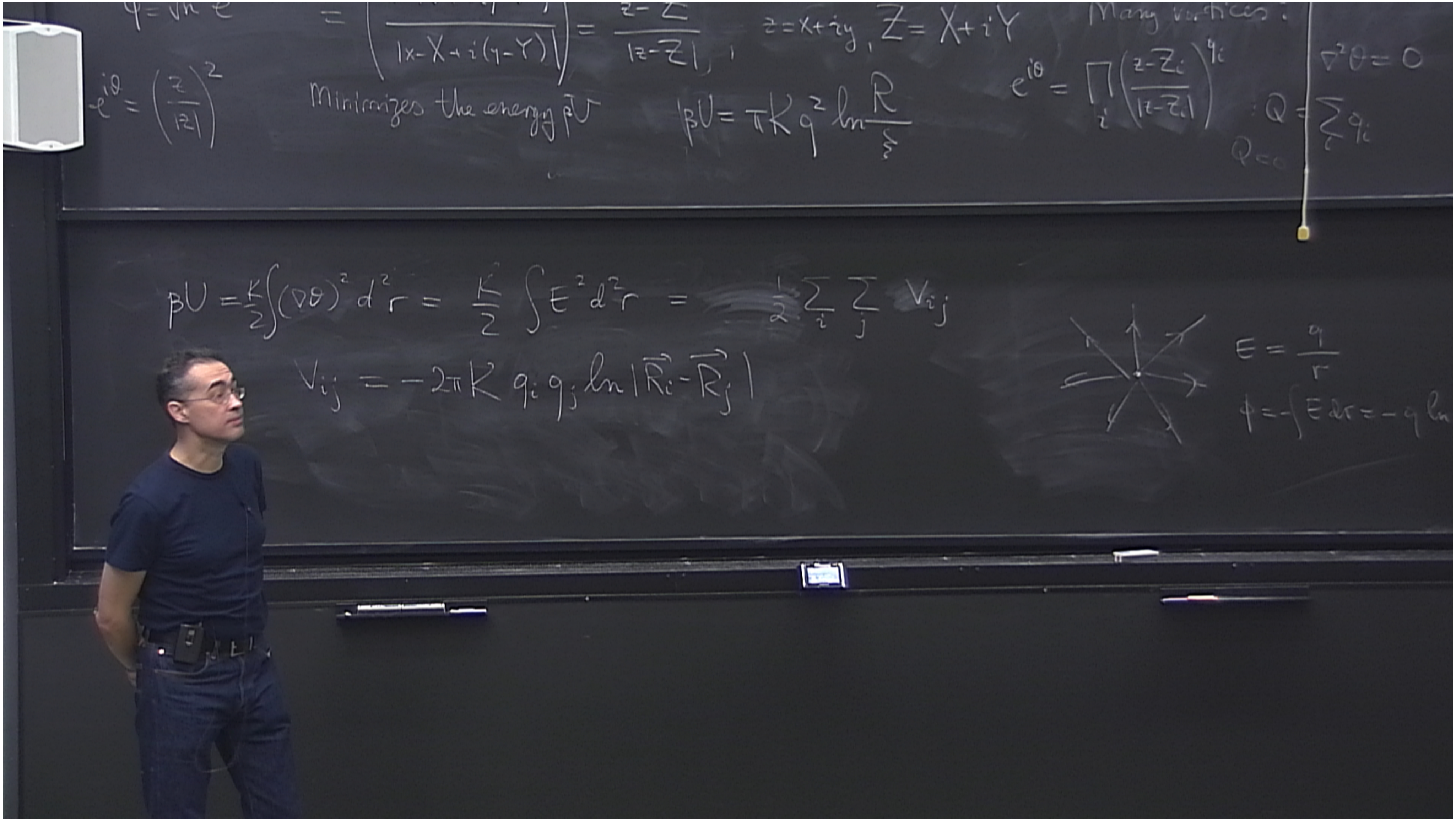
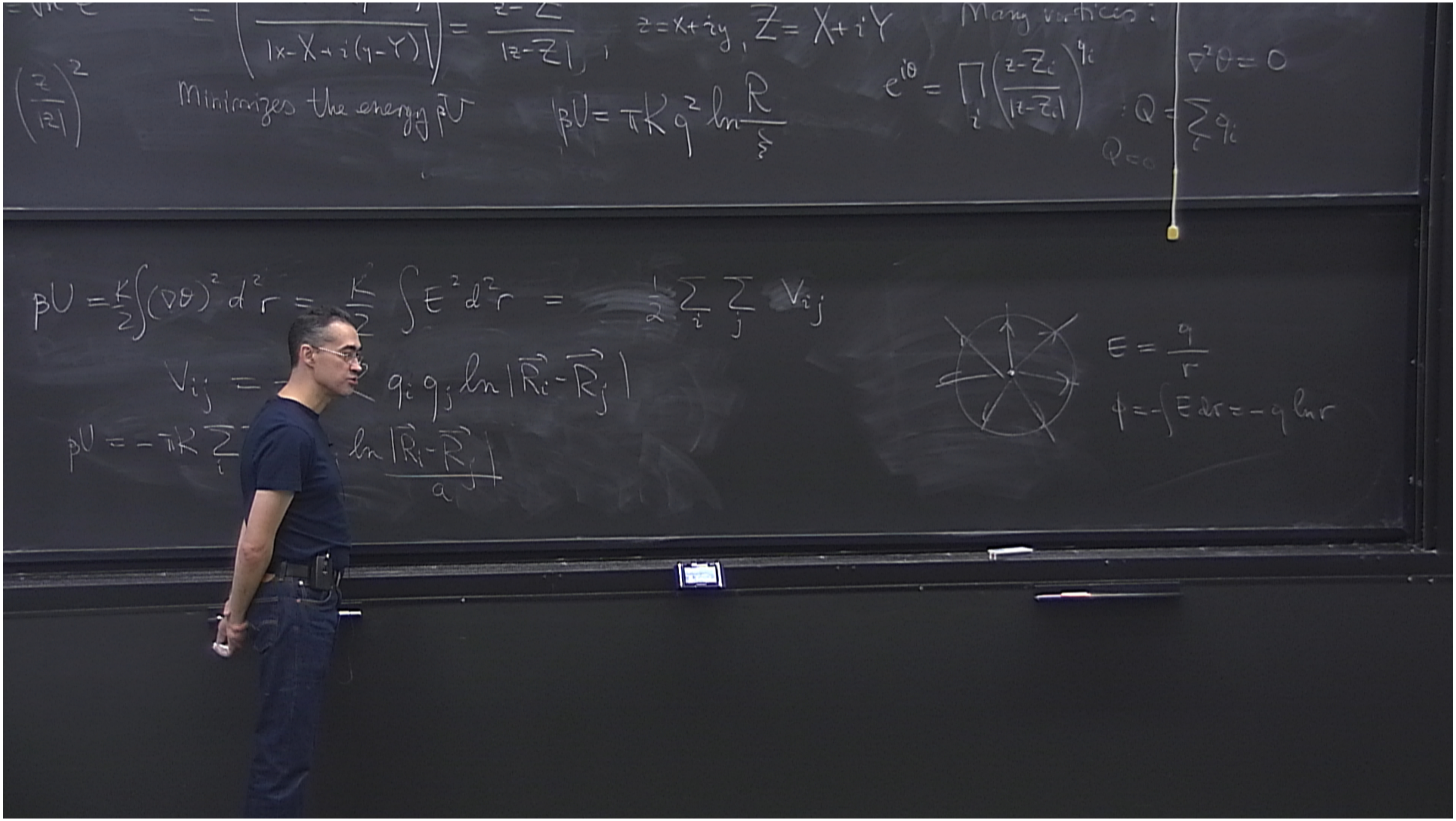
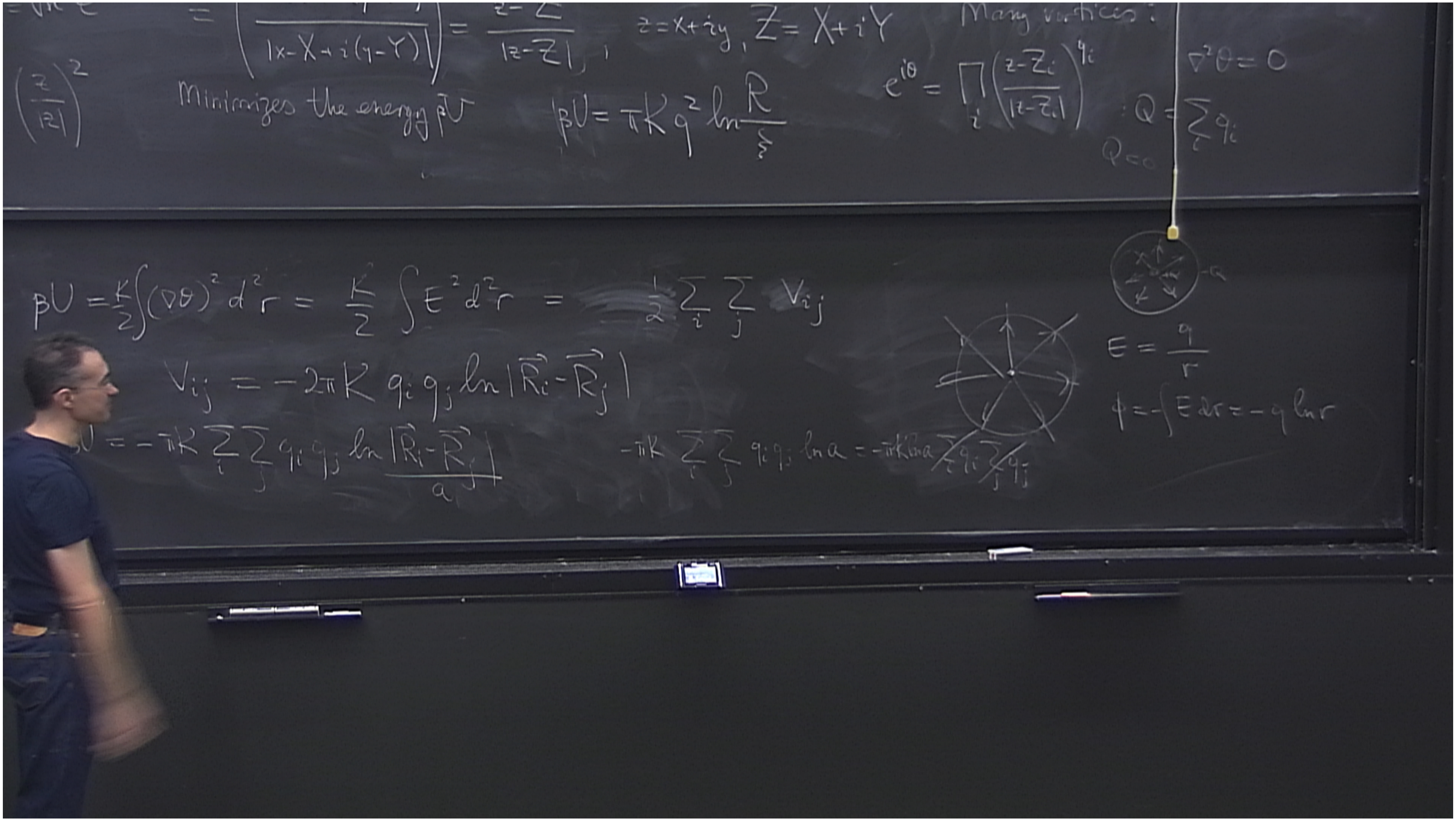
U = ħ²/2m ∇ψ*·∇ψ + ½ gψ*²ψ² − μψ*ψ
Re t
−iħ/k_BT = −iħβ
Vortex at R = (X, Y) and vorticity q
√n e^{iθ(r)} = ((x−X+i(y−Y)) / |x−X+i(y−Y)|)^q = (z−Z)/|z−Z|
z = x+iy, Z = X+iY
Minimizes the energy βU
βU = πKq² ln R/ξ
Vortices w. different q are in different topological sectors
cannot be connected by local changes.
Many vortices:
e^{iθ} = ∏_i ((z−Z_i)/|z−Z_i|)^{q_i}
∇²θ = 0


$z = x + iy$, $Z = X + iY$
Many vortices:
$\nabla^2 \theta = 0$
$e^{i\theta} = \left(\frac{z}{|z|}\right)^2$
Minimizes the energy βU
$\beta U = \pi K q^2 \ln \frac{R}{\xi}$
$e^{i\theta} = \prod_i \left(\frac{z - Z_i}{|z - Z_i|}\right)^{q_i}$
$Q = \sum_i q_i$
$\beta U = \frac{K}{2}\int (\nabla\theta)^2 d^2 r = \frac{K}{2}\int E^2 d^2 r = \frac{1}{2}\sum_i \sum_j V_{ij}$
$V_{ij} = -2\pi K q_i q_j \ln|\vec{R}_i - \vec{R}_j|$
$E = \frac{q}{r}$


Minimizes the energy βU
z = x + iy , Z = X + iY
Many vortices :
βU = πK q² ln R/ξ
∇²θ = 0
Q = Σ qᵢ
βU = K/2 ∫(∇θ)² d²r = K/2 ∫E² d²r = 1/2 Σᵢ Σⱼ Vᵢⱼ
Vᵢⱼ = qᵢ qⱼ ln |Rᵢ − Rⱼ|
E = q/r
φ = −∫E dr = −q ln r
βU = −πK Σᵢ ln |Rᵢ − Rⱼ|/a


z = x + iy, Z = X + iY
Many vortices:
∇²θ = 0
Minimizes the energy βU
βU = πKq² ln R/ξ
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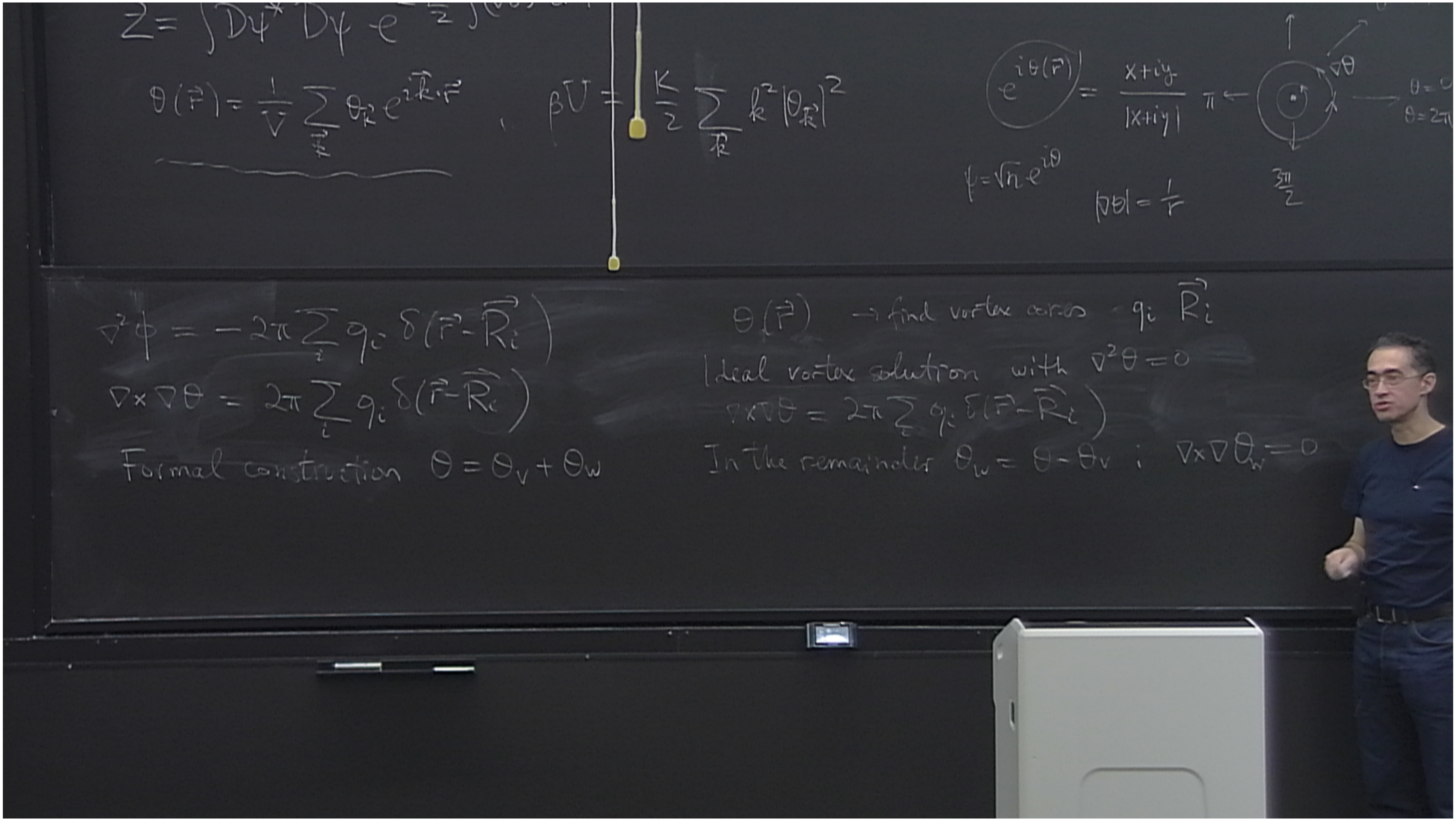
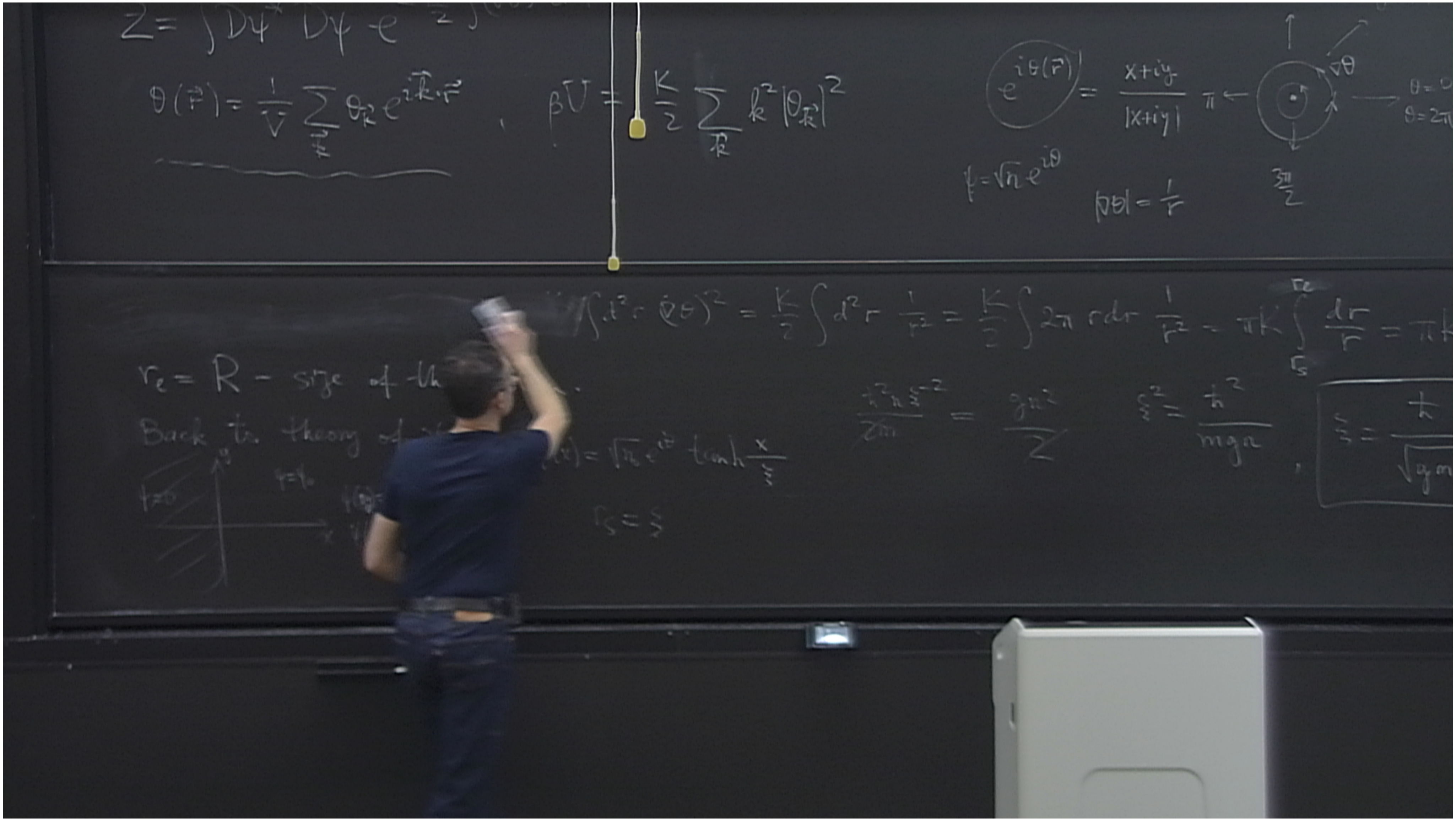
Vᵢⱼ = −2πK qᵢ qⱼ ln |Rᵢ − Rⱼ|
E = q/r
φ = −∫ E dr = −q ln r


$\theta(\vec r) = \frac{1}{V}\sum_{\vec k} \theta_{\vec k} e^{i\vec k\cdot\vec r}$
$\beta U = \frac{K}{2}\sum_{\vec k} k^2 |\theta_{\vec k}|^2$
$e^{i\theta(\vec r)} = \frac{x+iy}{|x+iy|}$
$\psi = \sqrt{n}\, e^{i\theta}$
$|\nabla\theta| = \frac{1}{r}$
$\nabla\theta$
$\frac{3\pi}{2}$
$\nabla^2\phi = -2\pi\sum_i q_i\,\delta(\vec r - \vec R_i)$
$\nabla\times\nabla\theta = 2\pi\sum_i q_i\,\delta(\vec r - \vec R_i)$
Formal construction $\theta = \theta_v + \theta_w$
$\theta(\vec r)$ → find vortex cores q_i $\vec R_i$
Ideal vortex solution with $\nabla^2\theta = 0$
$\nabla\times\nabla\theta = 2\pi\sum q_i\,\delta(\vec r - \vec R_i)$
In the remainder $\theta_w = \theta - \theta_v$; $\nabla\times\nabla\theta_w = 0$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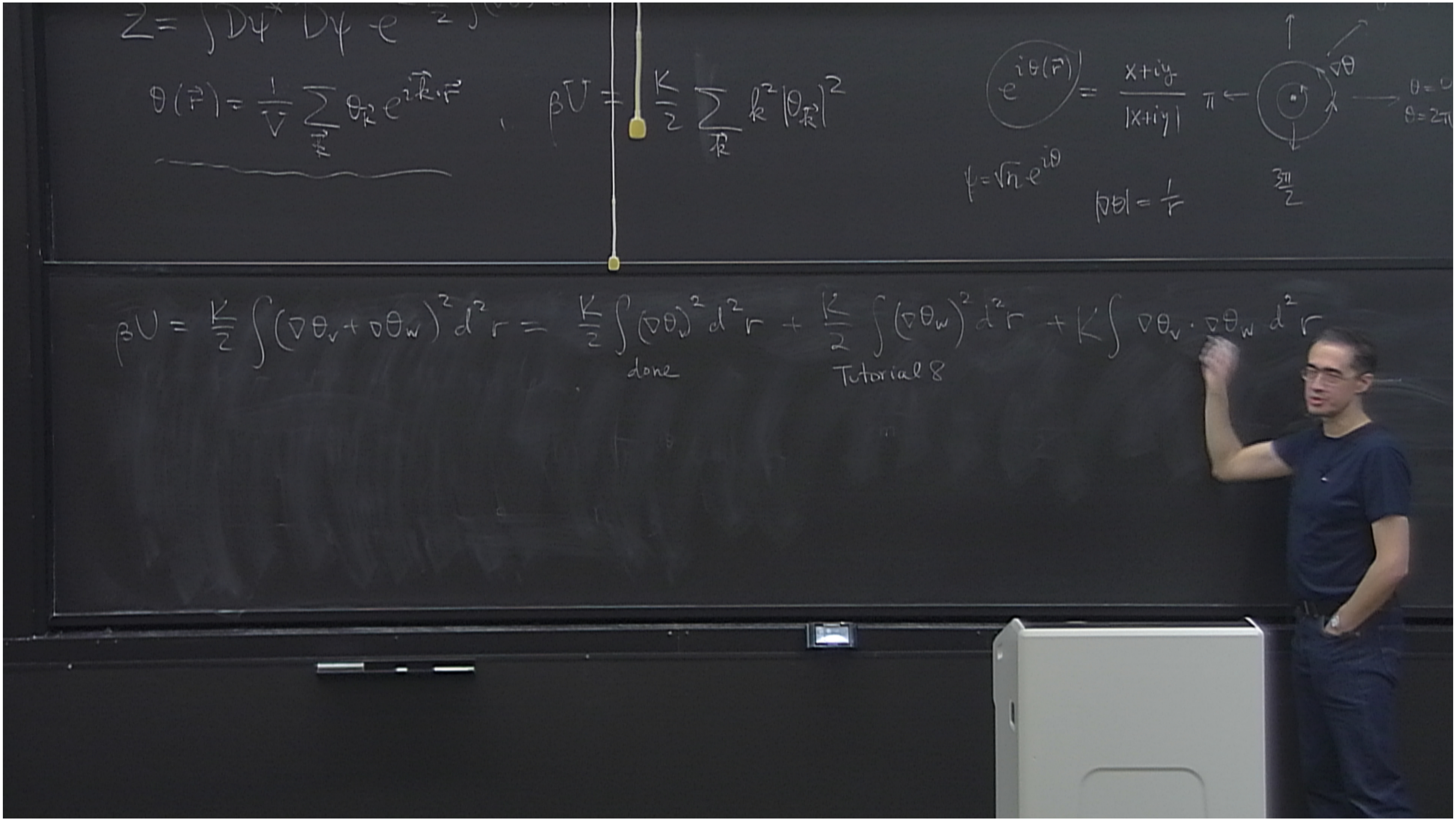
$e^{i\theta(\vec{r})} = \frac{x+iy}{|x+iy|}$
$\psi = \sqrt{n}e^{i\theta}$
$|\nabla\theta| = \frac{1}{r}$
$r_e = R$ - size of th
Back to theory of
$\xi = \frac{\hbar}{\sqrt{gm}}$
$r_\xi = \xi$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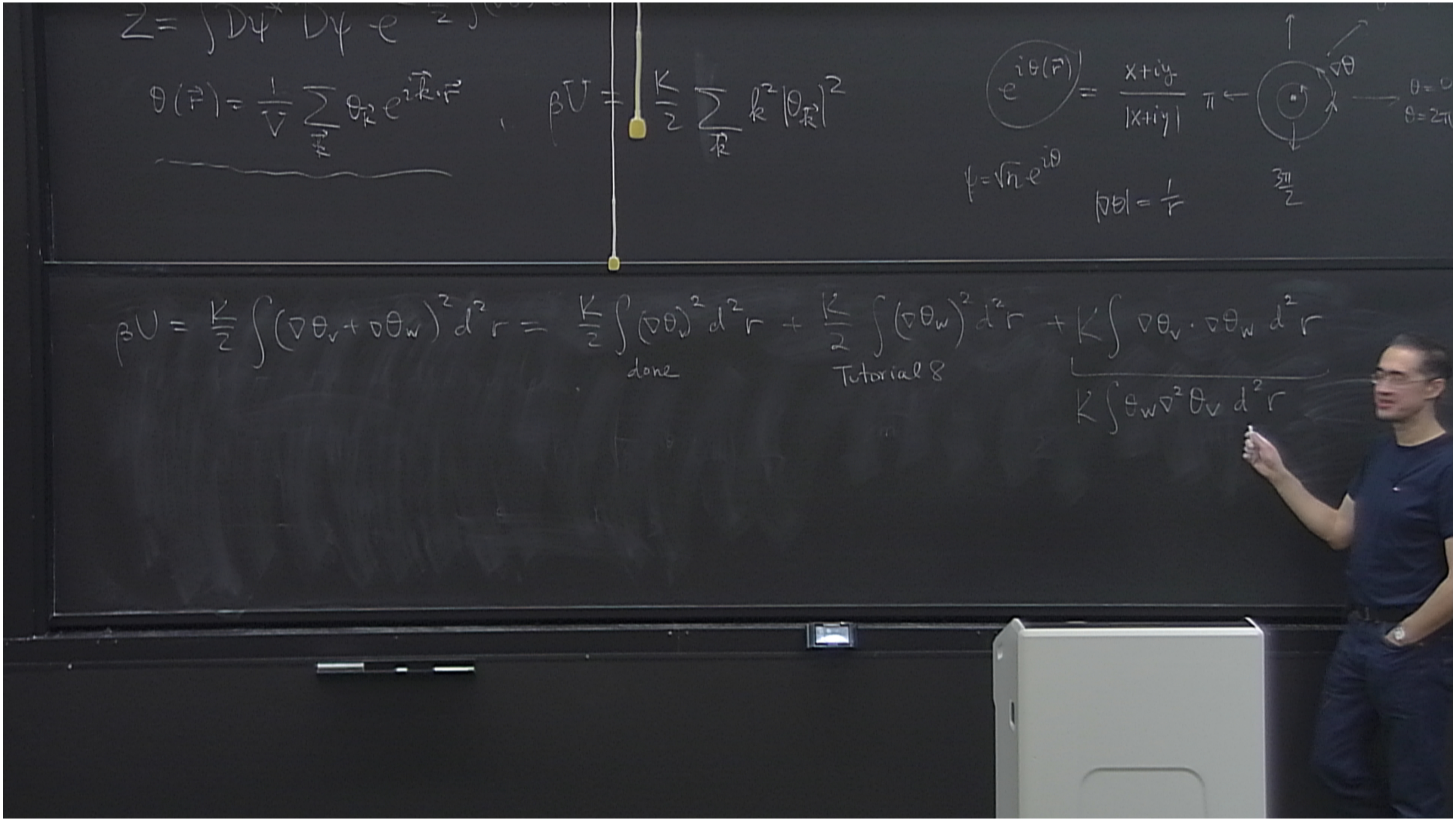
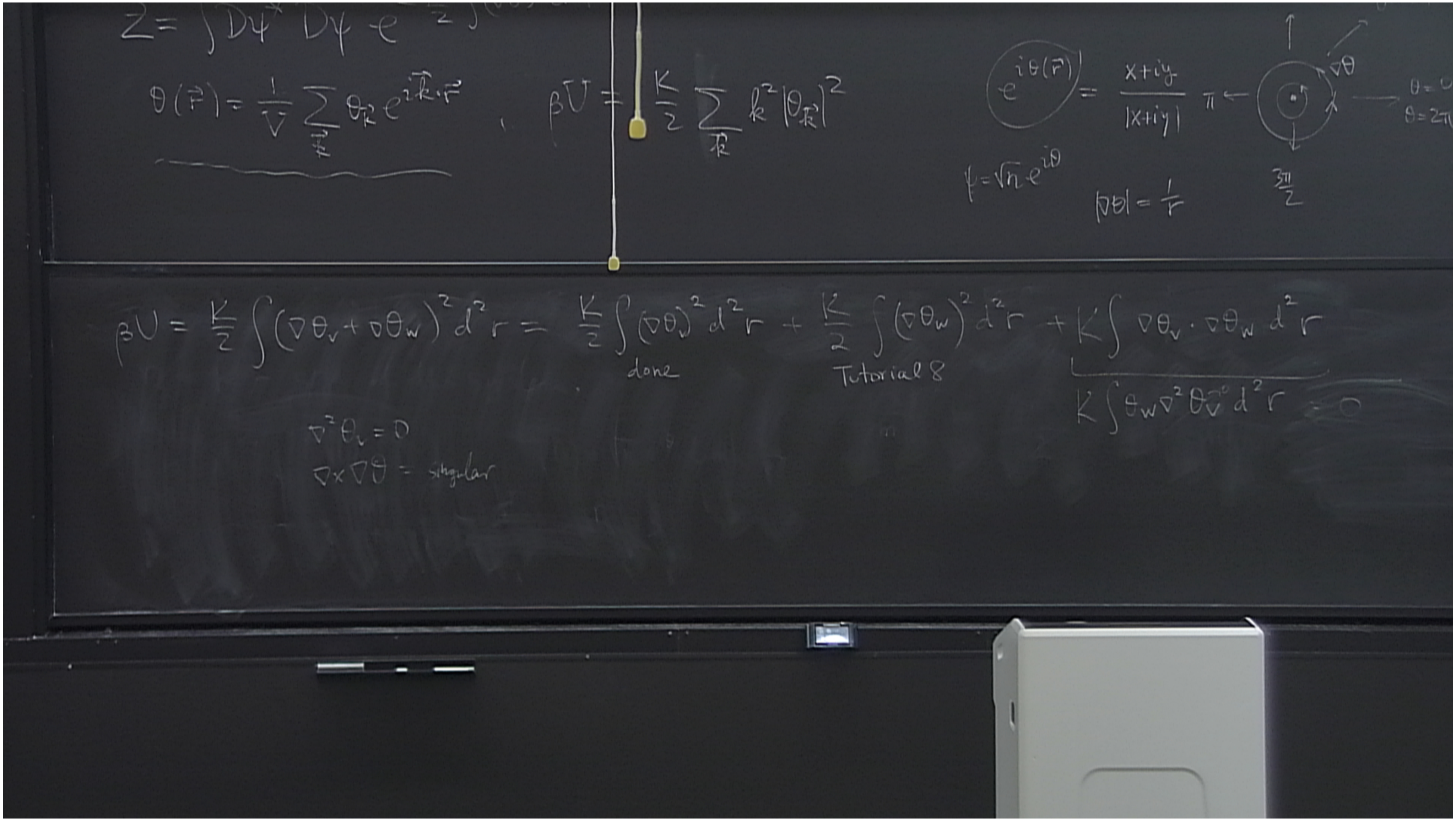
$\beta U = \frac{K}{2}\sum_{\vec{k}} k^2 |\theta_{\vec{k}}|^2$
$e^{i\theta(\vec{r})} = \frac{x+iy}{|x+iy|}$
$\psi = \sqrt{n} e^{i\theta}$
$|\nabla\theta| = \frac{1}{r}$
$\nabla\theta$
π
$\frac{3\pi}{2}$
$\beta U = \frac{K}{2}\int(\nabla\theta_v + \nabla\theta_w)^2 d^2r = \frac{K}{2}\int(\nabla\theta_v)^2 d^2r + \frac{K}{2}\int(\nabla\theta_w)^2 d^2r + K\int \nabla\theta_v\cdot\nabla\theta_w \, d^2r$
done
Tutorial 8


$\theta(\vec r) = \frac{1}{V}\sum_{\vec k} \theta_{\vec k} e^{i\vec k\cdot\vec r}$
$\beta U = \frac{K}{2}\sum_{\vec k} k^2 |\theta_{\vec k}|^2$
$e^{i\theta(\vec r)} = \frac{x+iy}{|x+iy|}$
$\nabla\theta$
$\psi = \sqrt{n}\, e^{i\theta}$
$|\nabla\theta| = \frac{1}{r}$
$\beta U = \frac{K}{2}\int(\nabla\theta_v + \nabla\theta_w)^2 d^2r = \frac{K}{2}\int(\nabla\theta_v)^2 d^2r + \frac{K}{2}\int(\nabla\theta_w)^2 d^2r + K\int \nabla\theta_v\cdot\nabla\theta_w\, d^2r$
done
Tutorial 8
$K\int \theta_w \nabla^2\theta_v\, d^2r$



$\theta(\vec{r}) = \frac{1}{V}\sum_{\vec{k}} \theta_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}$
$\beta U = \frac{K}{2}\sum_{\vec{k}} k^2 |\theta_{\vec{k}}|^2$
$e^{i\theta(\vec{r})} = \frac{x+iy}{|x+iy|}$
$\psi = \sqrt{n}\, e^{i\theta}$
$|\nabla\theta| = \frac{1}{r}$
$\nabla\theta$
π
$\frac{3\pi}{2}$
$\beta U = \frac{K}{2}\int(\nabla\theta_v + \nabla\theta_w)^2 d^2r = \frac{K}{2}\int(\nabla\theta_v)^2 d^2r + \frac{K}{2}\int(\nabla\theta_w)^2 d^2r + K\int \nabla\theta_v\cdot\nabla\theta_w\, d^2r$
done
Tutorial 8
$K\int \theta_w \nabla^2\theta_v\, d^2r = 0$
$\nabla^2\theta_v = 0$
$\nabla\times\nabla\theta$ = singular

$$\psi = \sqrt{n}\, e^{i\theta(r)} \quad = \left(\frac{x-X+i(y-Y)}{|x-X+i(y-Y)|}\right) = \frac{z-Z}{|z-Z|}, \quad z = x+iy,\ Z = X+iY$$

$$e^{i\theta} = \left(\frac{z}{|z|}\right)^2$$

Minimizes the energy βU

$$\beta U = \pi K q^2 \ln\frac{R}{\xi}$$

Many vortices:

$$e^{i\theta} = \prod_i \left(\frac{z-Z_i}{|z-Z_i|}\right)^{q_i}$$

$$Q = \sum_i q_i \qquad Q=0 \qquad \nabla^2\theta =$$

Pair of vortices, charges q and $-q$

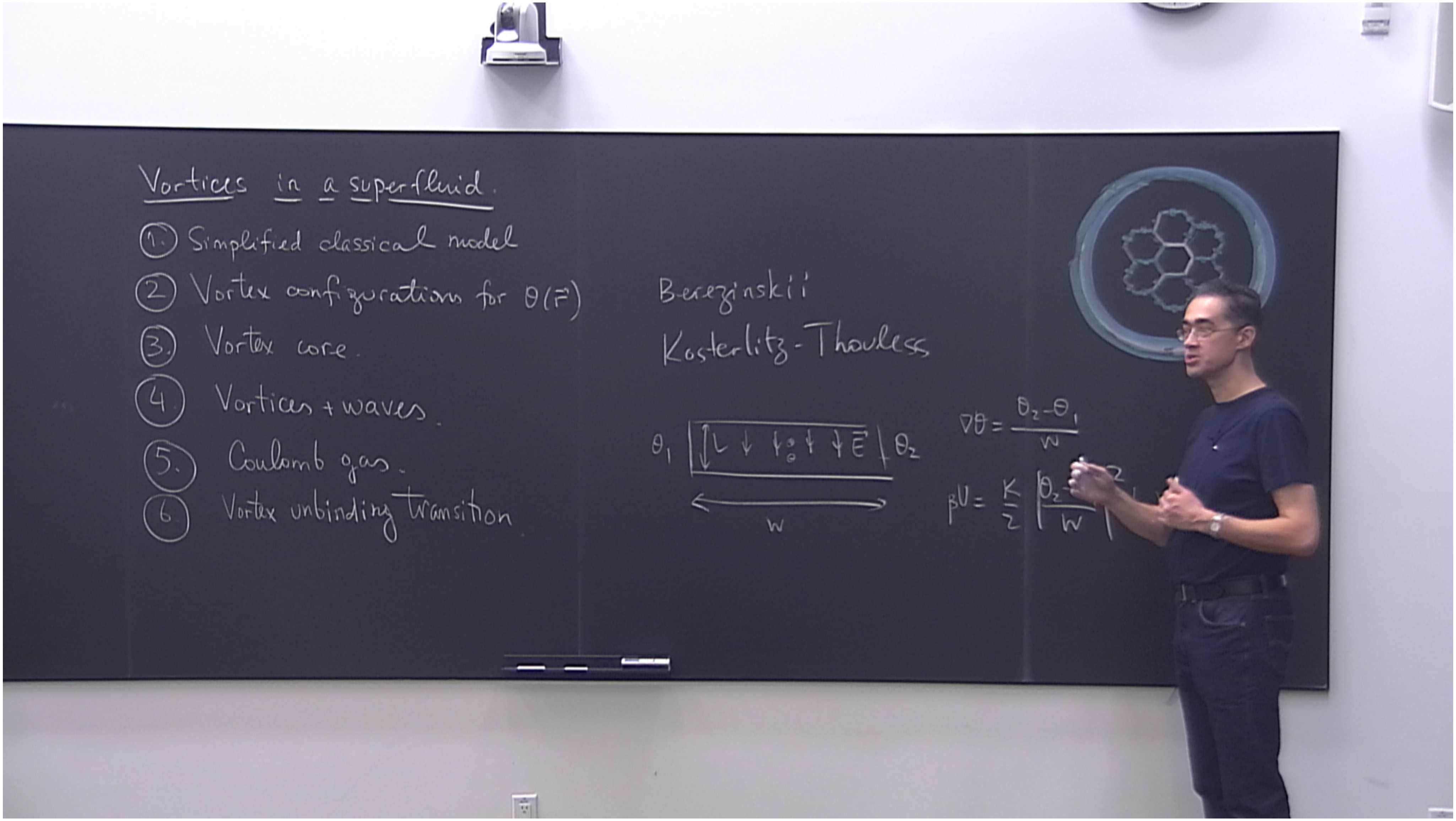
$$\beta U = -2\pi K\, q(-q) \ln \underbrace{|\vec{R}_1 - \vec{R}_2|}_{R}$$

$$\text{prob.} = e^{-\beta U} = e^{-2\pi K q^2 \ln r} = \frac{1}{r^{2\pi q^2 K}}$$

$$\int p\, d^2 r = \int d^2 r \frac{1}{r^{2\pi q^2 K}} = \int_\xi \frac{dr}{r^{2\pi q^2 K - 1}}$$

change from short-distance physics $(r \sim \xi)$ to long-distance physics $(r \sim R)$ happens at $2\pi q^2 K - 1 = 1$

$$K = \frac{1}{\pi q^2}$$


Vortices in a superfluid.
1. Simplified classical model
2 Vortex configurations for θ(r)
3. Vortex core.
4 Vortices + waves.
5. Coulomb gas.
6. Vortex unbinding transition
Berezinskii
Kosterlitz-Thouless
θ1
θ2
w
∇θ = (θ2-θ1)/w
βU = K/2