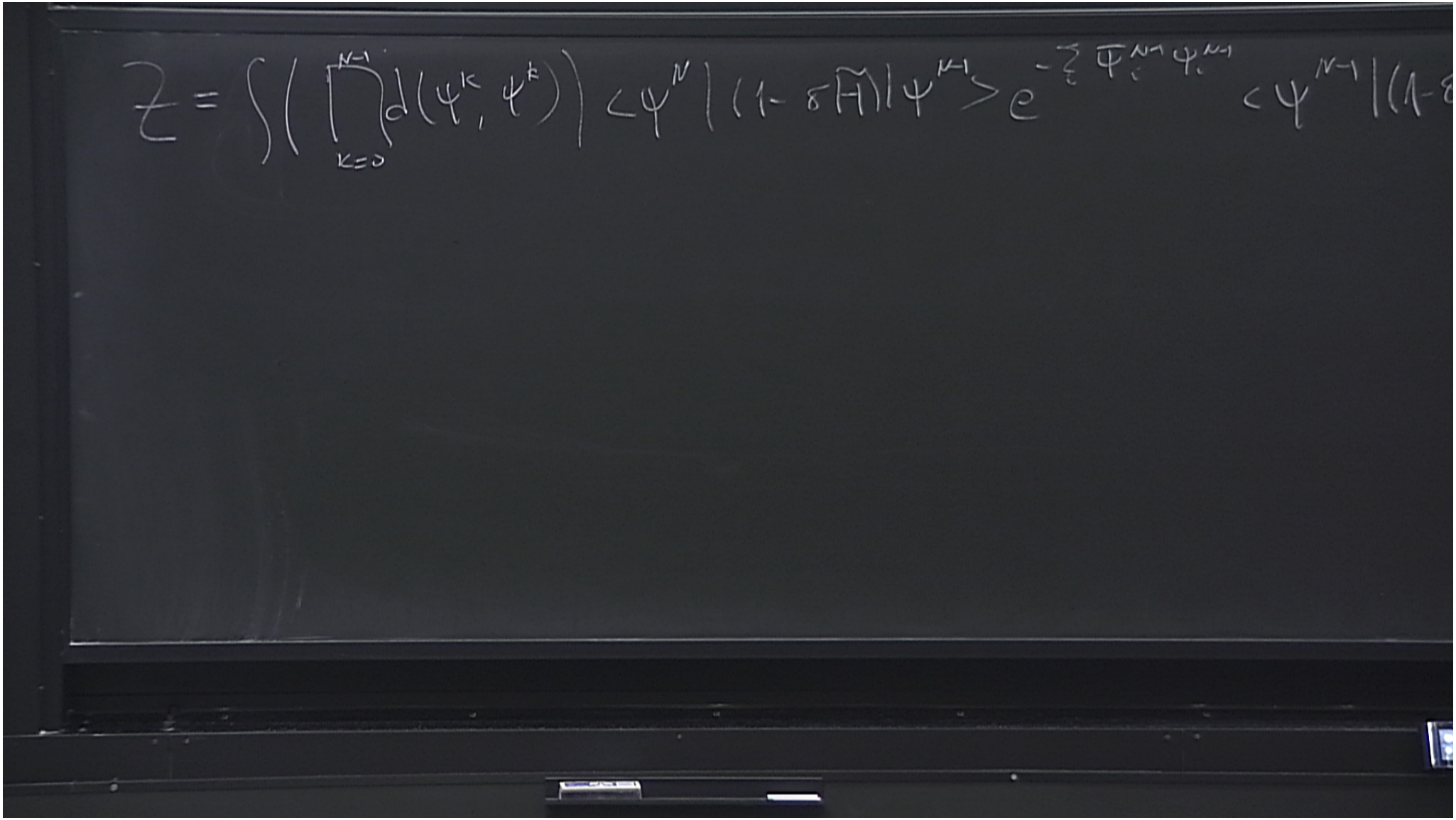
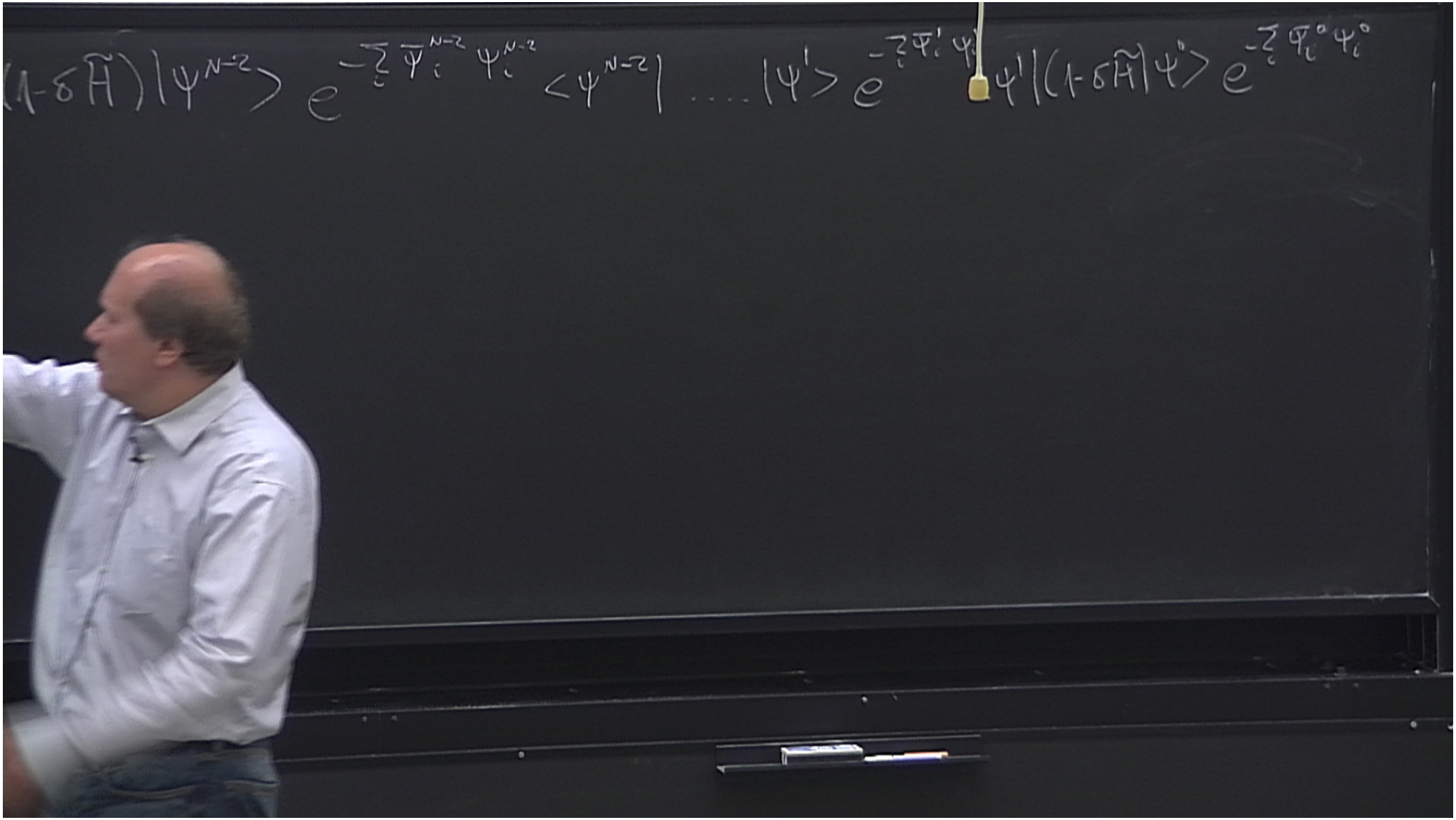
Title: PSI 2015/2016 Condensed Matter - Oleg Tchernyshyov - Lecture 8

Date: Nov 18, 2015 10:45 AM

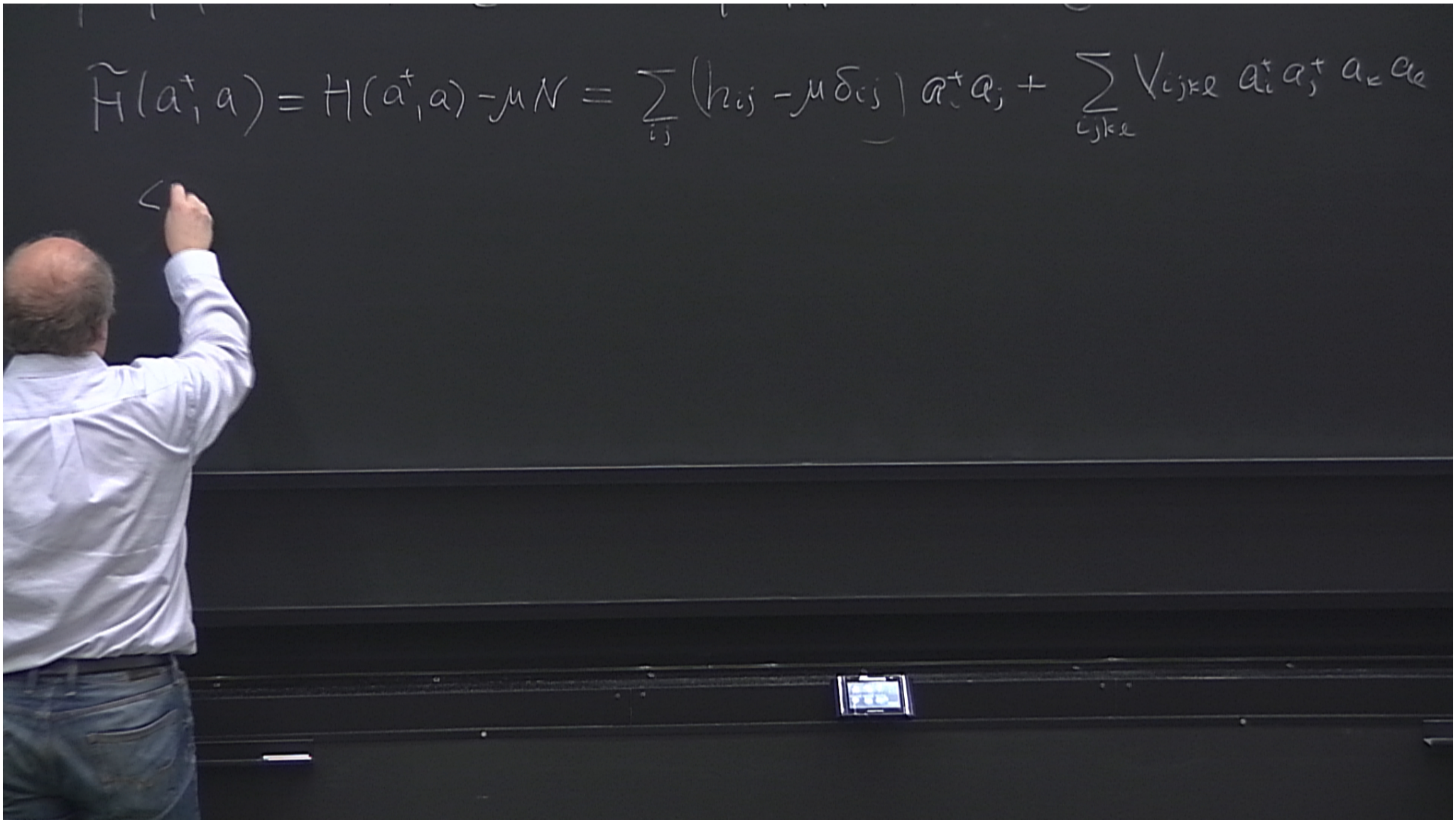
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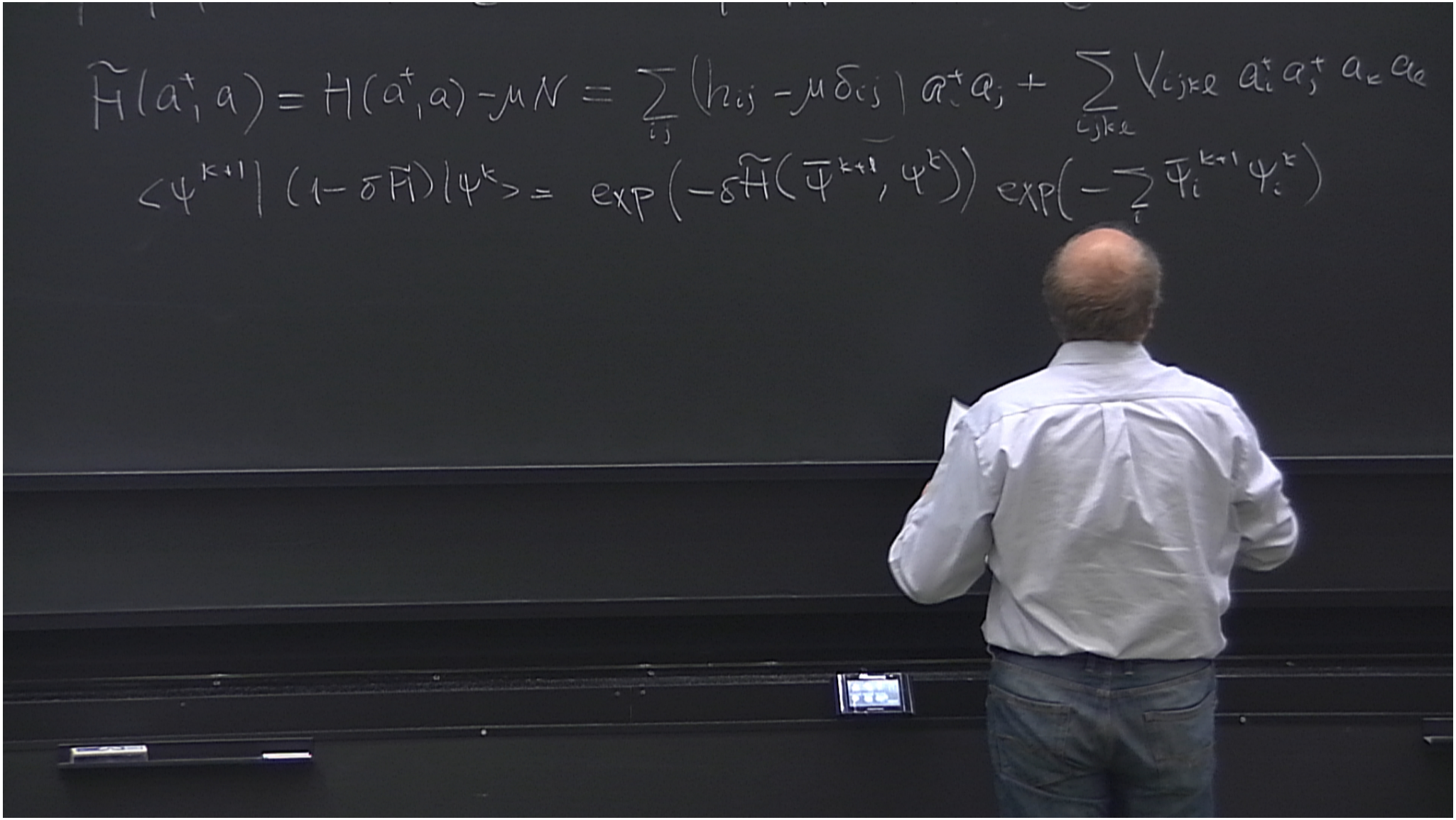
Abstract:


$Z = \int \left(\prod_{k=0}^{N-1} d(\psi^k, \psi^k) \right) \langle \psi^N | (1 - \delta \hat{H}) | \psi^{N-1} \rangle e^{-\sum \Phi^{N-1} \psi^{N-1}} \langle \psi^{N-1} | (1-$


(1-δH̃)|ψ^{N-2}> e^{-Σ ψ̄_i^{N-2} ψ_i^{N-2}} <ψ^{N-2}| ... |ψ'> e^{-Σ ψ̄_i' ψ_i'} ψ'|(1-δH̃|ψ'> e^{-Σ ψ̄_i° ψ_i°}

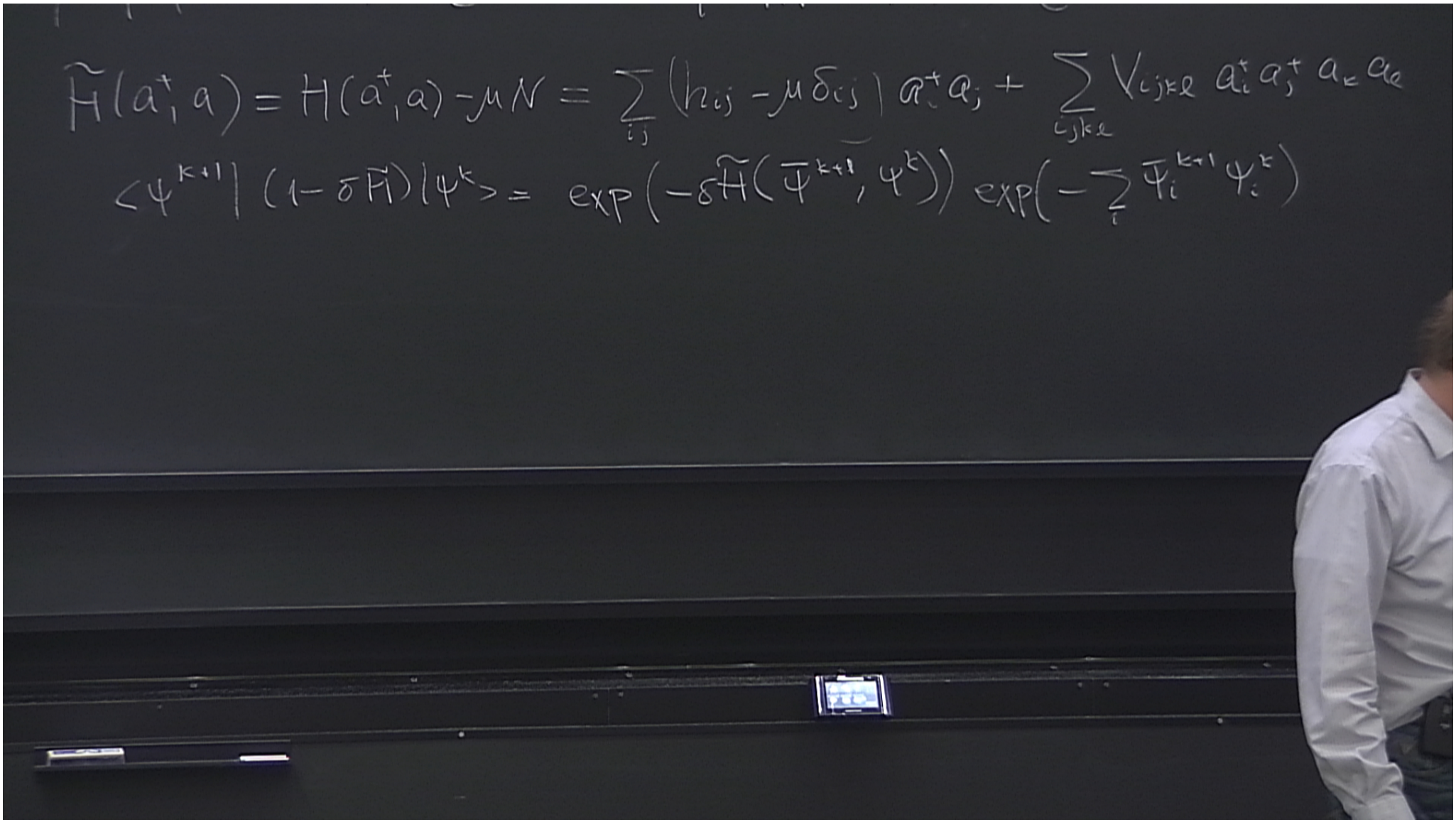
$$\tilde{H}(a^\dagger, a) = H(a^\dagger, a) - \mu N = \sum_{ij} (h_{ij} - \mu \delta_{ij})\, a_i^\dagger a_j + \sum_{ijkl} V_{ijkl}\, a_i^\dagger a_j^\dagger a_k a_l$$



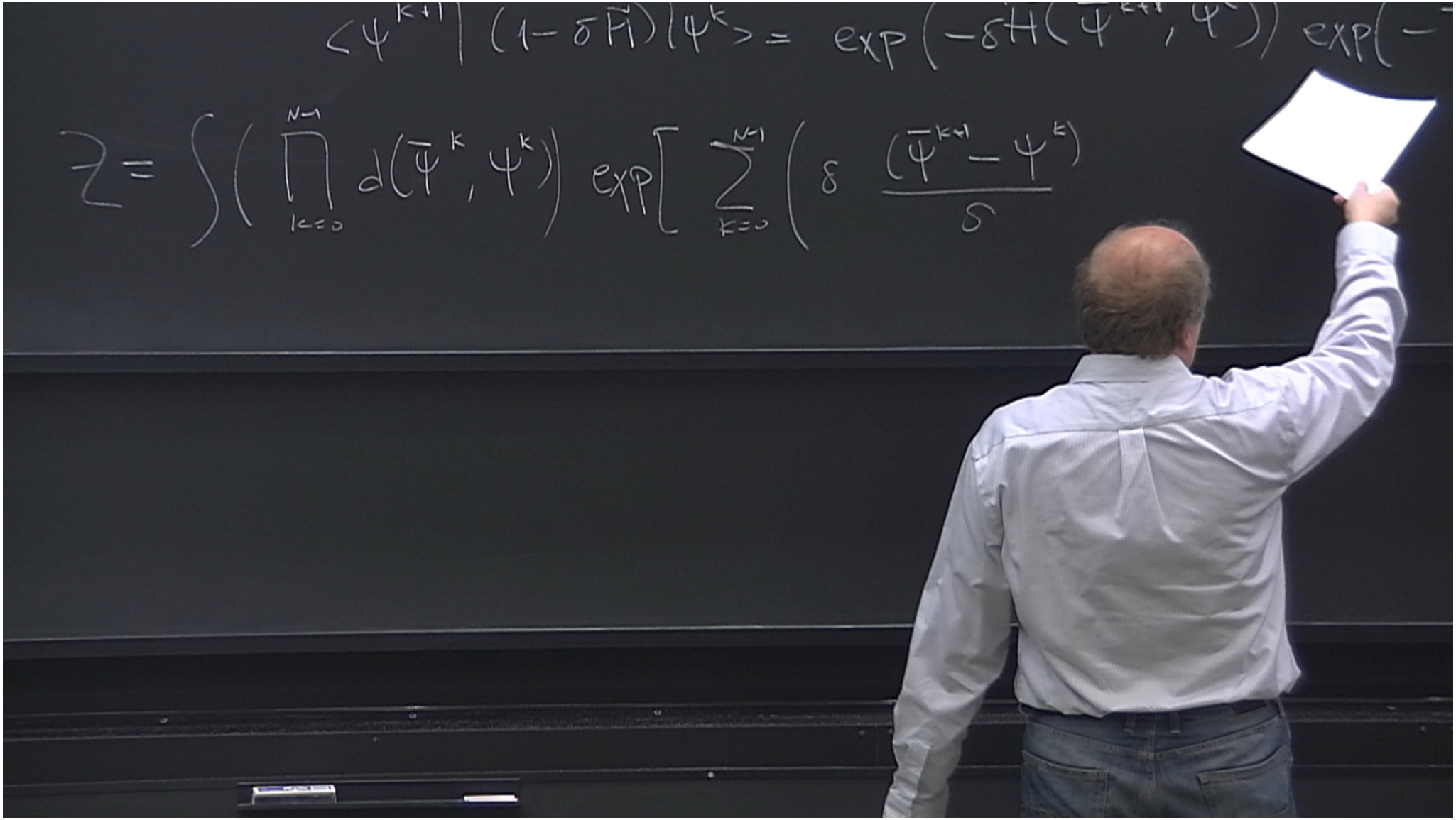

$\tilde{H}(a^\dagger, a) = H(a^\dagger, a) - \mu N = \sum_{ij} (h_{ij} - \mu \delta_{ij}) a_i^\dagger a_j + \sum_{ijkl} V_{ijkl} a_i^\dagger a_j^\dagger a_k a_l$
$\langle \psi^{k+1} | (1 - \delta \tilde{H}) | \psi^k \rangle = \exp(-\delta \tilde{H}(\Psi^{k+1}, \psi^k)) \exp(-\sum_i \Psi_i^{k+1} \psi_i^k)$

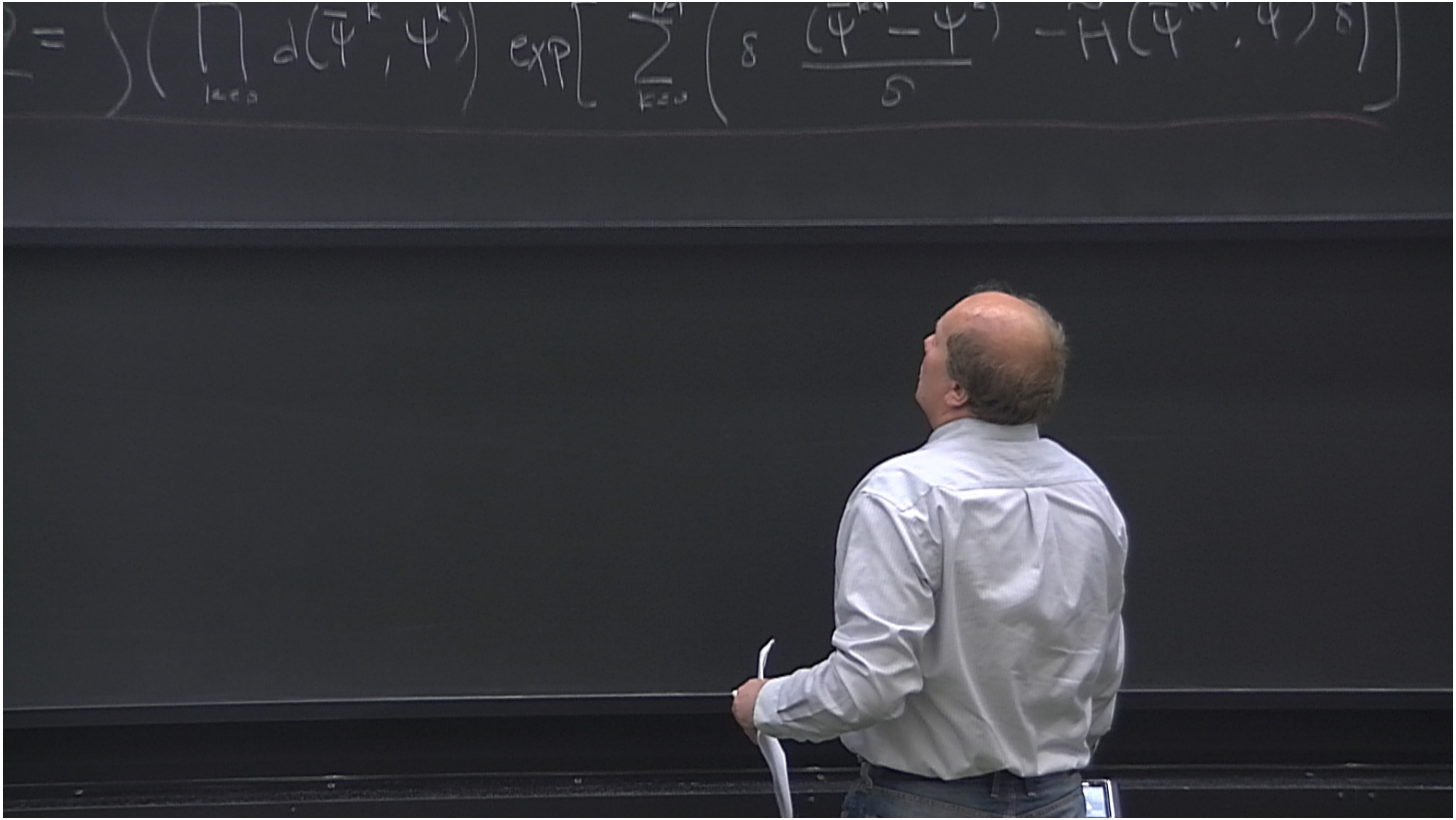
$$\tilde{H}(a^\dagger, a) = H(a^\dagger, a) - \mu N = \sum_{ij} (h_{ij} - \mu \delta_{ij})\, a_i^\dagger a_j + \sum_{ijkl} V_{ijkl}\, a_i^\dagger a_j^\dagger a_k a_l$$

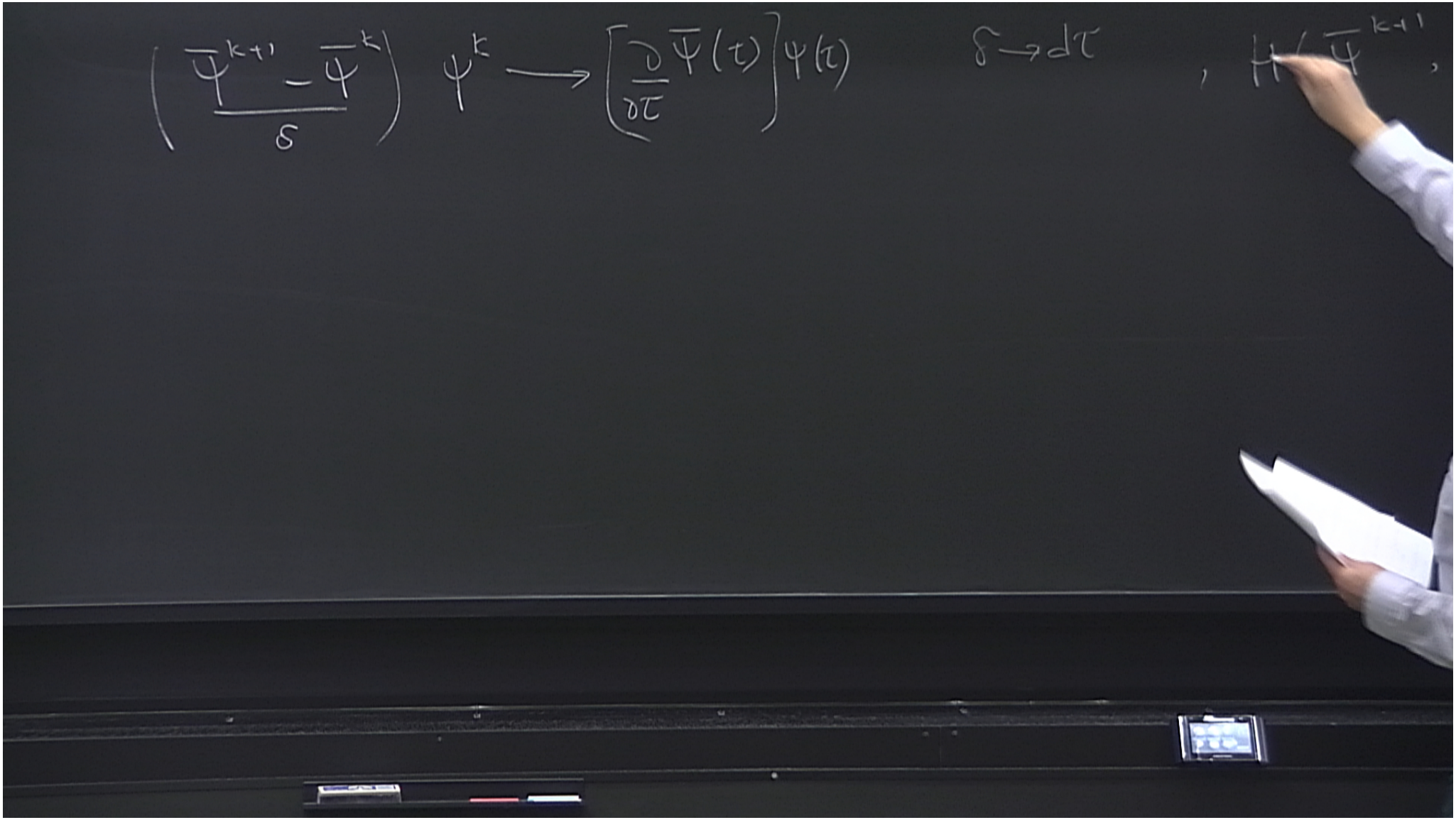
$$\langle \psi^{k+1} | (1 - \delta \tilde{H}) | \psi^k \rangle = \exp\left(-\delta \tilde{H}(\bar{\psi}^{k+1}, \psi^k)\right) \exp\left(-\sum_i \bar{\psi}_i^{k+1} \psi_i^k\right)$$

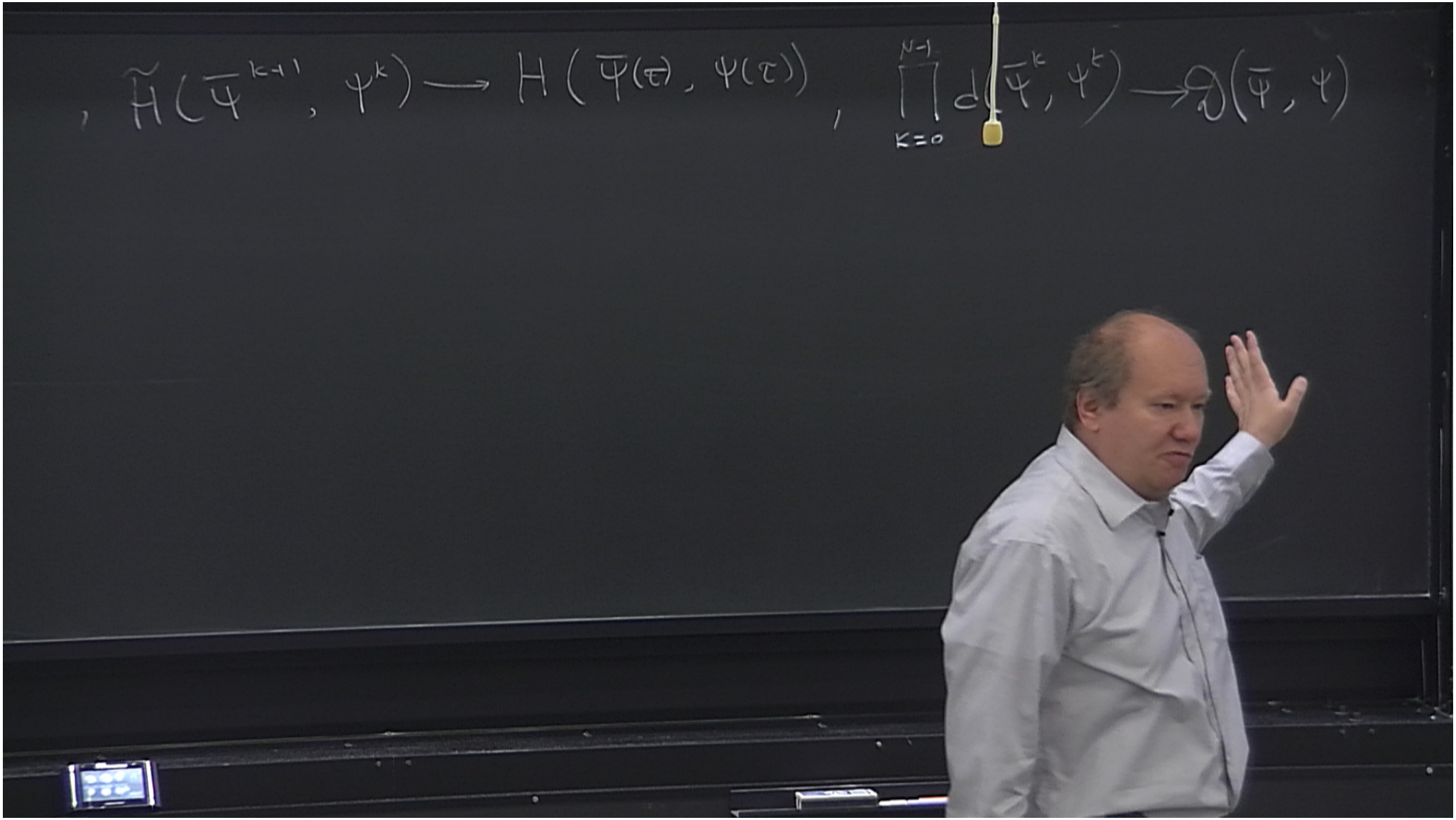


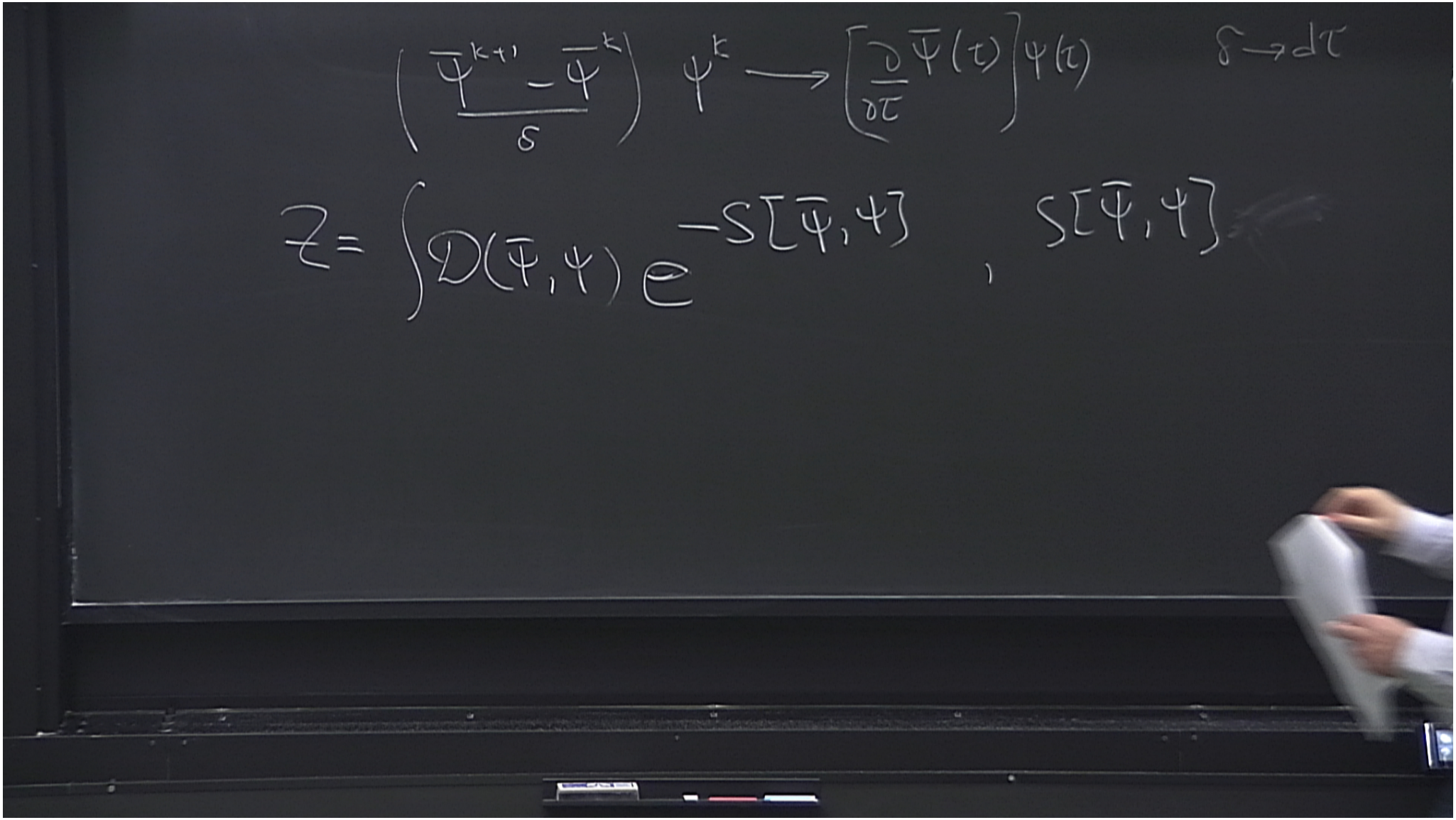
$$\langle \psi^{k+1} | (1 - \delta \hat{H}) | \psi^k \rangle = \exp\left(-\delta \tilde{H}(\bar{\psi}^{k+1}, \psi^k)\right) \exp\left(-\sum_i \bar{\psi}_i^{k+1} \psi_i^k\right)$$

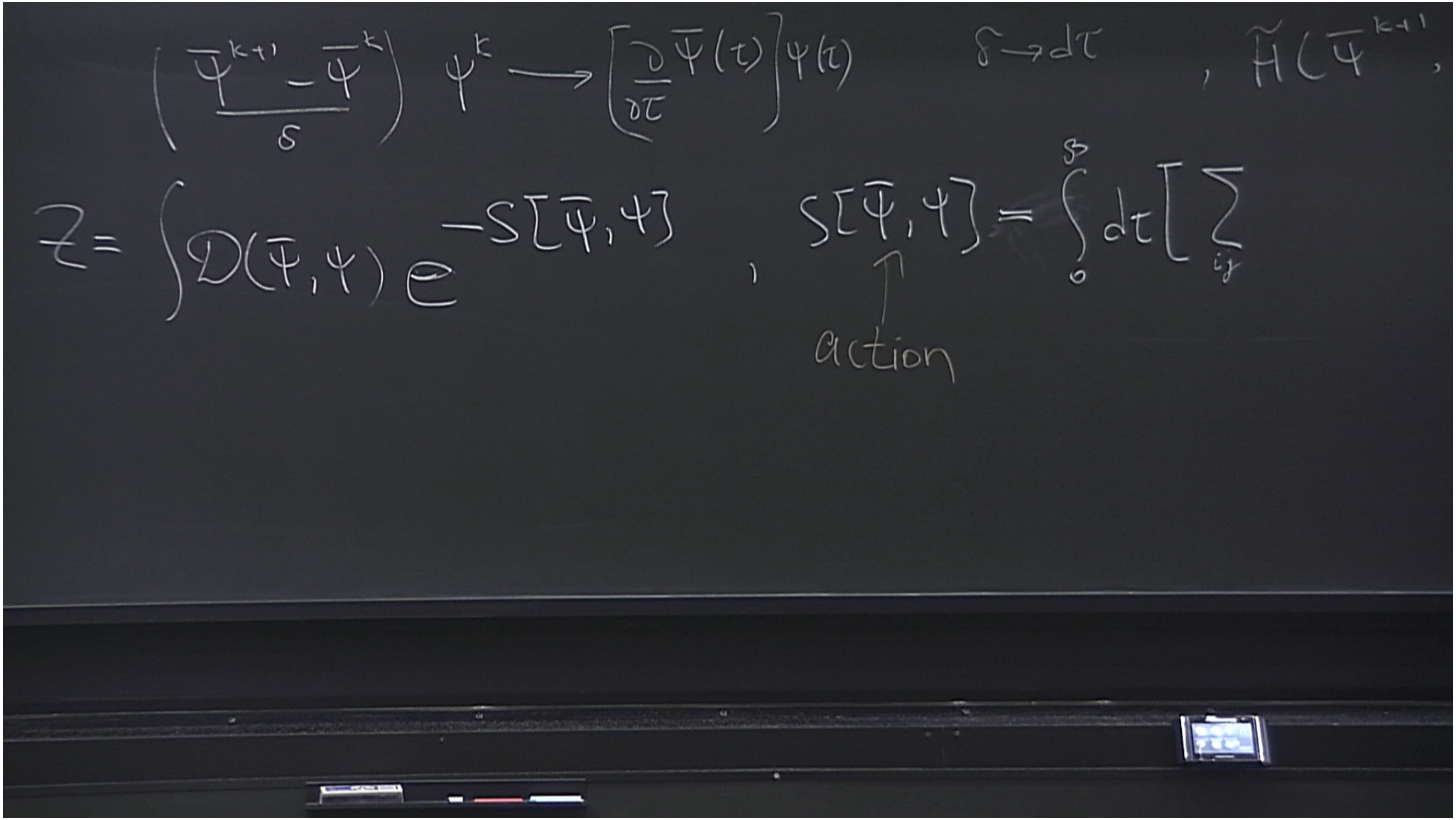

$\langle \psi^{k+1} | (1 - \delta \tilde{H}) | \psi^k \rangle = \exp(-\delta H(\bar{\psi}^{k+1}, \psi^k)) \exp($
$Z = \int \left(\prod_{k=0}^{N-1} d(\bar{\psi}^k, \psi^k) \right) \exp\left[\sum_{k=0}^{N-1} \left(\delta \frac{(\bar{\psi}^{k+1} - \psi^k)}{\delta} \right. \right.$




$\left(\frac{\bar{\Psi}^{k+1} - \bar{\Psi}^{k}}{\delta}\right) \psi^{k} \longrightarrow \left[\frac{\partial}{\partial \tau}\bar{\Psi}(\tau)\right]\psi(\tau)$
$\delta \to d\tau$
$H(\bar{\Psi}^{k+1})$


$\tilde{H}(\bar{\Psi}^{k+1}, \psi^k) \longrightarrow H(\bar{\Psi}(\tau), \psi(\tau))$
$\prod_{k=0}^{N-1} d(\bar{\Psi}^k, \psi^k) \longrightarrow \Omega(\bar{\Psi}, \psi)$


$\left(\frac{\bar{\psi}^{k+1} - \bar{\psi}^{k}}{\delta}\right) \psi^{k} \longrightarrow \left[\frac{\partial}{\partial \tau}\bar{\psi}(\tau)\right]\psi(\tau)$
$\delta \to d\tau$
$Z = \int \mathcal{D}(\bar{\psi},\psi)\, e^{-S[\bar{\psi},\psi]}$
$S[\bar{\psi},\psi]$


$\left(\frac{\bar{\psi}^{k+1} - \bar{\psi}^{k}}{\delta}\right) \psi^{k} \longrightarrow \left[\frac{\partial}{\partial \tau}\bar{\psi}(\tau)\right]\psi(\tau)$
$\delta \to d\tau$
$, \tilde{H}(\bar{\psi}^{k+1},$
$Z = \int \mathcal{D}(\bar{\psi}, \psi)\, e^{-S[\bar{\psi}, \psi]}$
$, S[\bar{\psi}, \psi] = \int_0^{\beta} dt \left[\sum_{ij} \right.$
action

$(\tau)\,\psi(\tau)$ $\quad \delta \to d\tau$, $\tilde{H}(\bar{\psi}^{k-1}, \psi^k) \longrightarrow H(\bar{\psi}(\tau), \psi(\tau))$,

$$S[\bar{\psi}, \psi] = \int_0^\beta d\tau \Big[\sum_{ij} \bar{\psi}_i(\tau) \Big(\Big(\frac{\partial}{\partial \tau} - \mu \Big) \delta_{ij} + h_{ij} \Big) \psi_j(\tau) + \sum_{ijkl} V_{ijkl}$$

action

$$\tilde{H}(\bar{\psi}^{k+1}, \psi^k) \longrightarrow H(\bar{\psi}(\tau), \psi(\tau)) \quad , \quad \prod_{k=0}^{N-1} d(\bar{\psi}^k, \psi^k) \longrightarrow \mathcal{D}(\bar{\psi}, \psi)$$

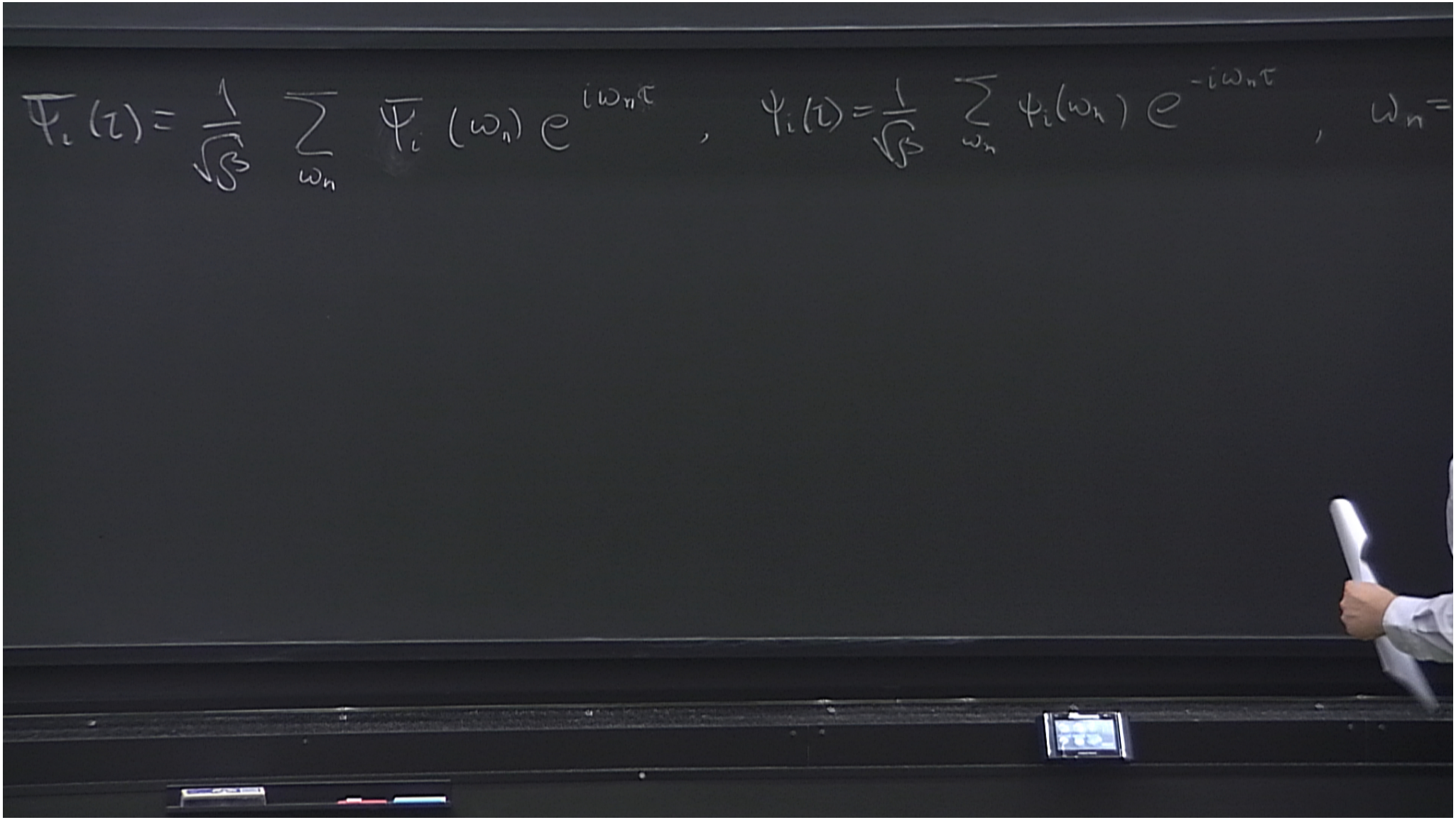
$$\sum_{ij} \bar{\psi}_i(\tau)\left(\left(\frac{\partial}{\partial \tau} - \mu\right)\delta_{ij} + h_{ij}\right)\psi_j(\tau) + \sum_{ijkl} V_{ijkl}\,\bar{\psi}_i(\tau)\bar{\psi}_j(\tau)\psi_k(\tau)\psi_l(\tau)\Big]$$

$$\left(\frac{\bar{\Psi}^{k+1} - \bar{\Psi}^{k}}{\delta}\right) \psi^{k} \longrightarrow \left[\frac{\partial}{\partial \tau}\bar{\Psi}(\tau)\right]\psi(\tau) \qquad \delta \to d\tau \qquad , \; \tilde{H}(\bar{\Psi}^{k+1}, \psi^{k}) \longrightarrow$$

$$\mathcal{D}(\bar{\Psi}, \psi)\, e^{-S[\bar{\Psi}, \psi]} \qquad , \qquad S[\bar{\Psi}, \psi] = \int_{0}^{\beta} dt \left[\sum_{ij} \bar{\Psi}_{i}(\tau) \left(\frac{\partial}{\partial \tau} - \mu \right) \delta_{ij} \right.$$

action

$$\Psi_i(\tau) = \frac{1}{\sqrt{\beta}} \sum_{\omega_n} \Psi_i(\omega_n) e^{i\omega_n \tau}, \quad \psi_i(\tau) = \frac{1}{\sqrt{\beta}} \sum_{\omega_n} \psi_i(\omega_n) e^{-i\omega_n \tau}, \quad \omega_n =$$



$$\psi_i(\tau) = \frac{1}{\sqrt{\beta}} \sum_{\omega_n} \psi_i(\omega_n)\, e^{-i\omega_n \tau}$$

$$\omega_n = 2\pi n T \quad \text{for bosons}$$

$$\omega_n = 2\pi\left(n + \tfrac{1}{2}\right) T \quad \text{for fermions}$$

$$\frac{1}{\sqrt{\beta}} \sum_{\omega_n} \psi_i(\omega_n) e^{-i\omega_n \tau}$$

$\omega_n = 2\pi n T$ for bosons

$\omega_n = 2\pi (n + \frac{1}{2}) T$ for fermions

← Matsubara frequencies

$$\frac{1}{\beta} \sum_{ijkl} \sum_{\omega_1, \omega_2, \omega_3, \omega_4} \delta_{\omega_1 + \omega_2 + \omega_3 + \omega_4, 0} V_{ijkl} \bar{\psi}_i(\omega_1) \bar{\psi}_j(\omega_2) \psi_k(\omega_3) \psi_l(\omega_4)$$

because $\int_0^\beta d\tau e^{i\omega_n \tau} = \beta \delta_{\omega_n, 0}$

$$\frac{1}{\beta}\sum_{\omega_n}\psi_i(\omega_n)\,e^{-i\omega_n\tau}$$

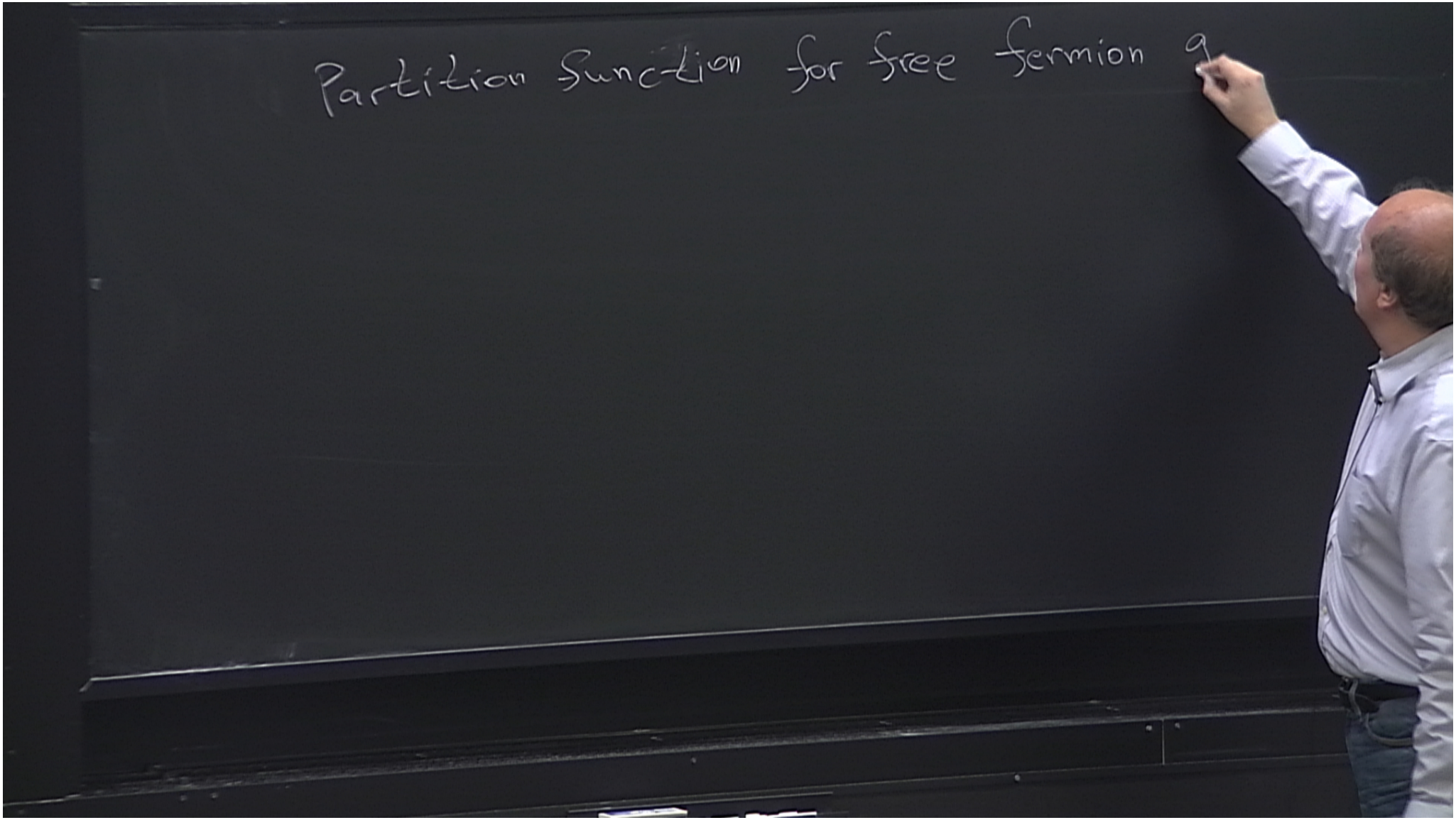
$\omega_n = 2\pi n T$ for bosons

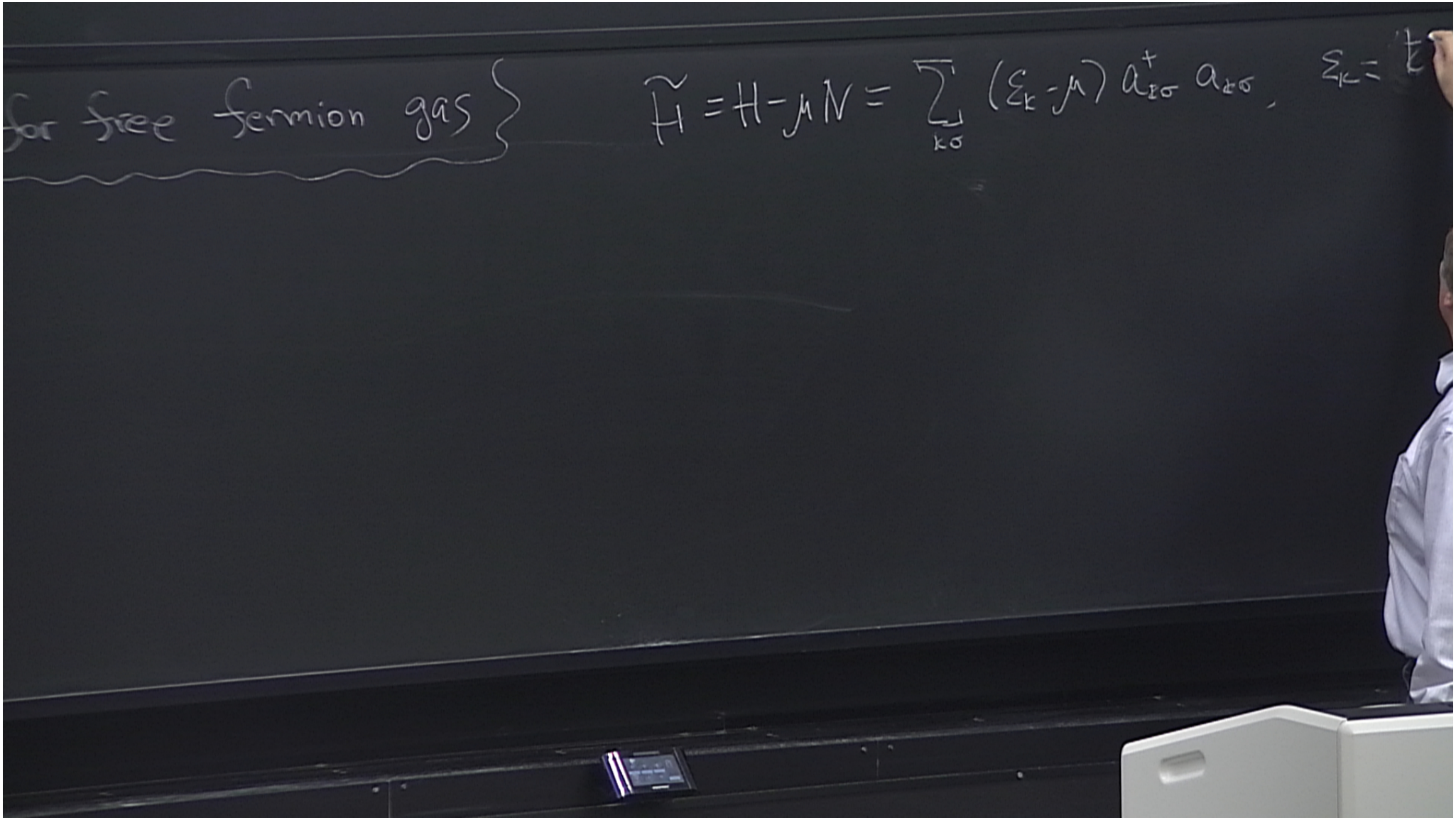
$\omega_n = 2\pi(n+\frac{1}{2})T$ for fermions

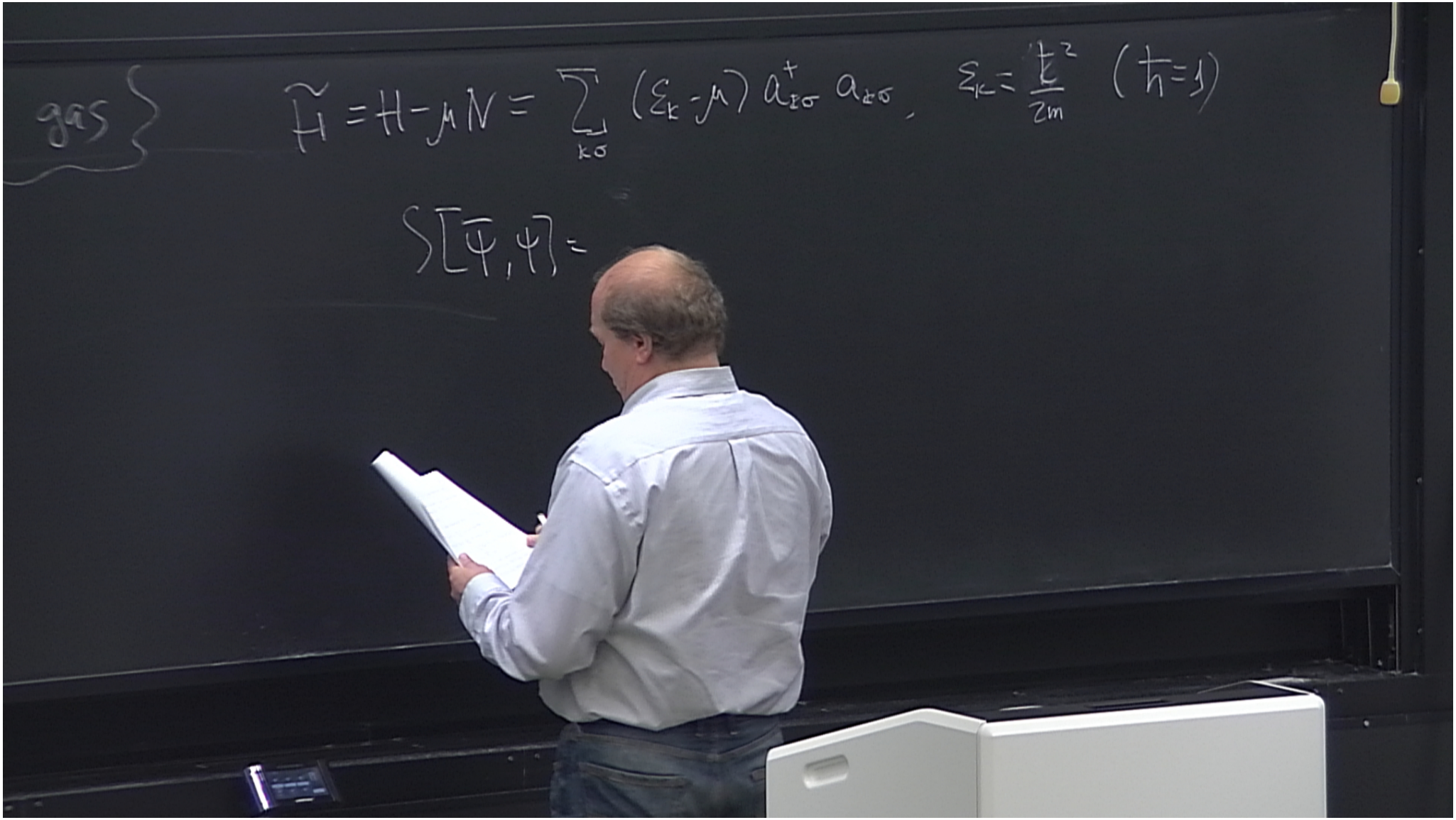
← Matsubara frequencies

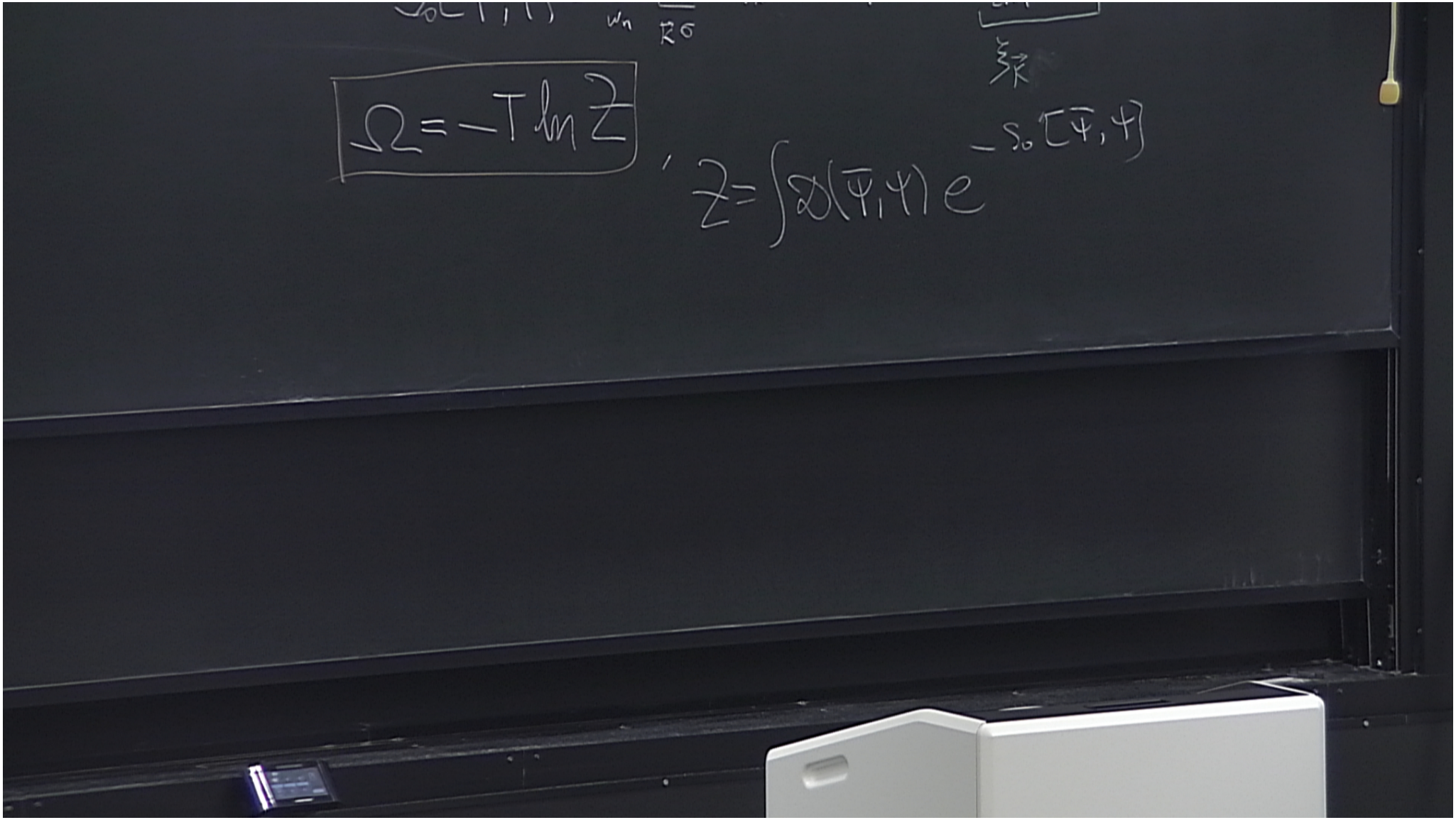
$$+\frac{1}{\beta}\sum_{ijkl}\sum_{\omega_1,\omega_2,\omega_3,\omega_4}\delta_{\omega_1+\omega_2+\omega_3+\omega_4,\,0}\,V_{ijkl}\,\bar{\psi}_i(\omega_1)\,\bar{\psi}_j(\omega_2)\,\psi_k(\omega_3)\,\psi_l(\omega_4)$$

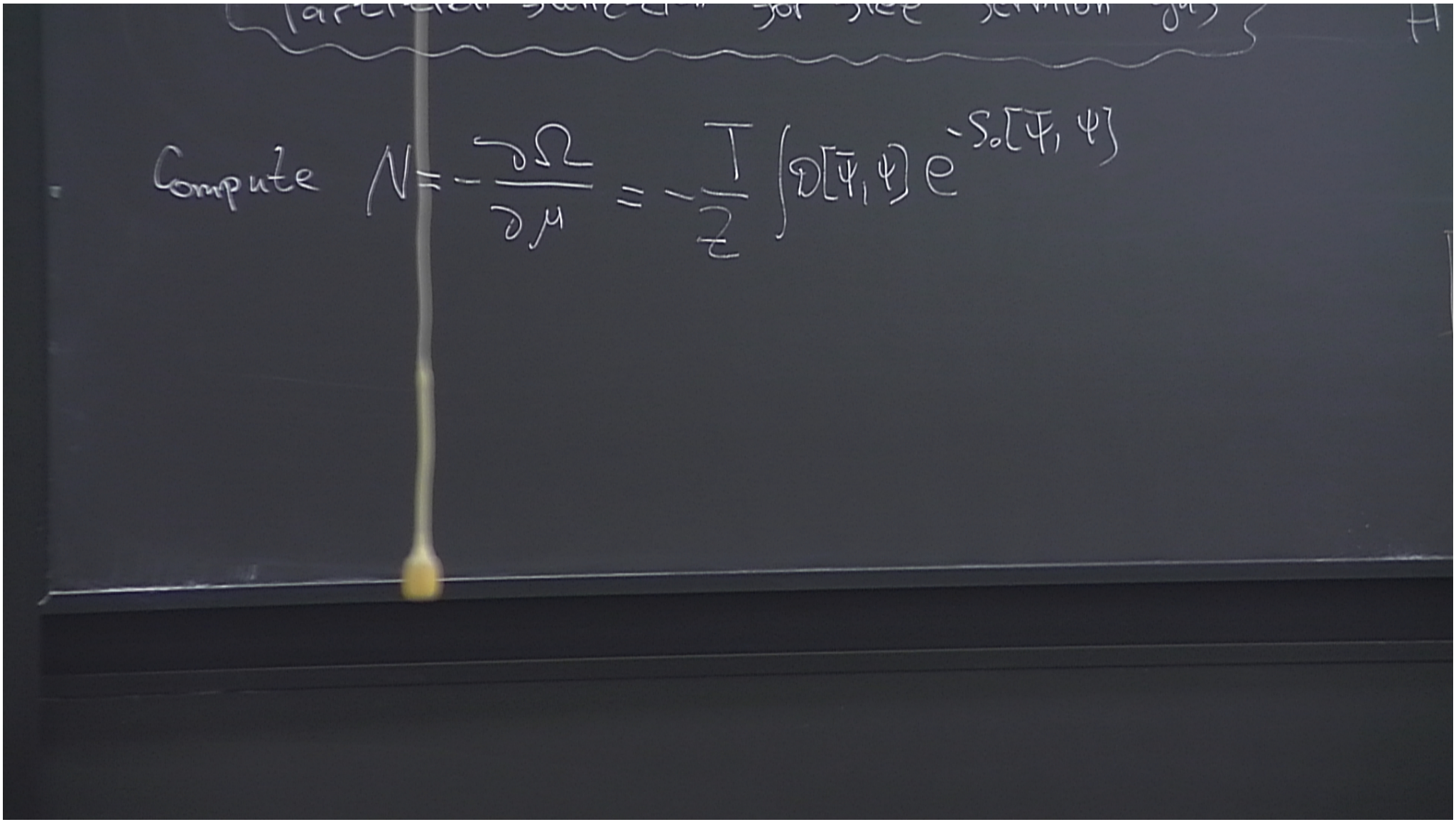
because $\int_0^\beta d\tau\, e^{i\omega_n\tau} = \beta\,\delta_{\omega_n,0}$

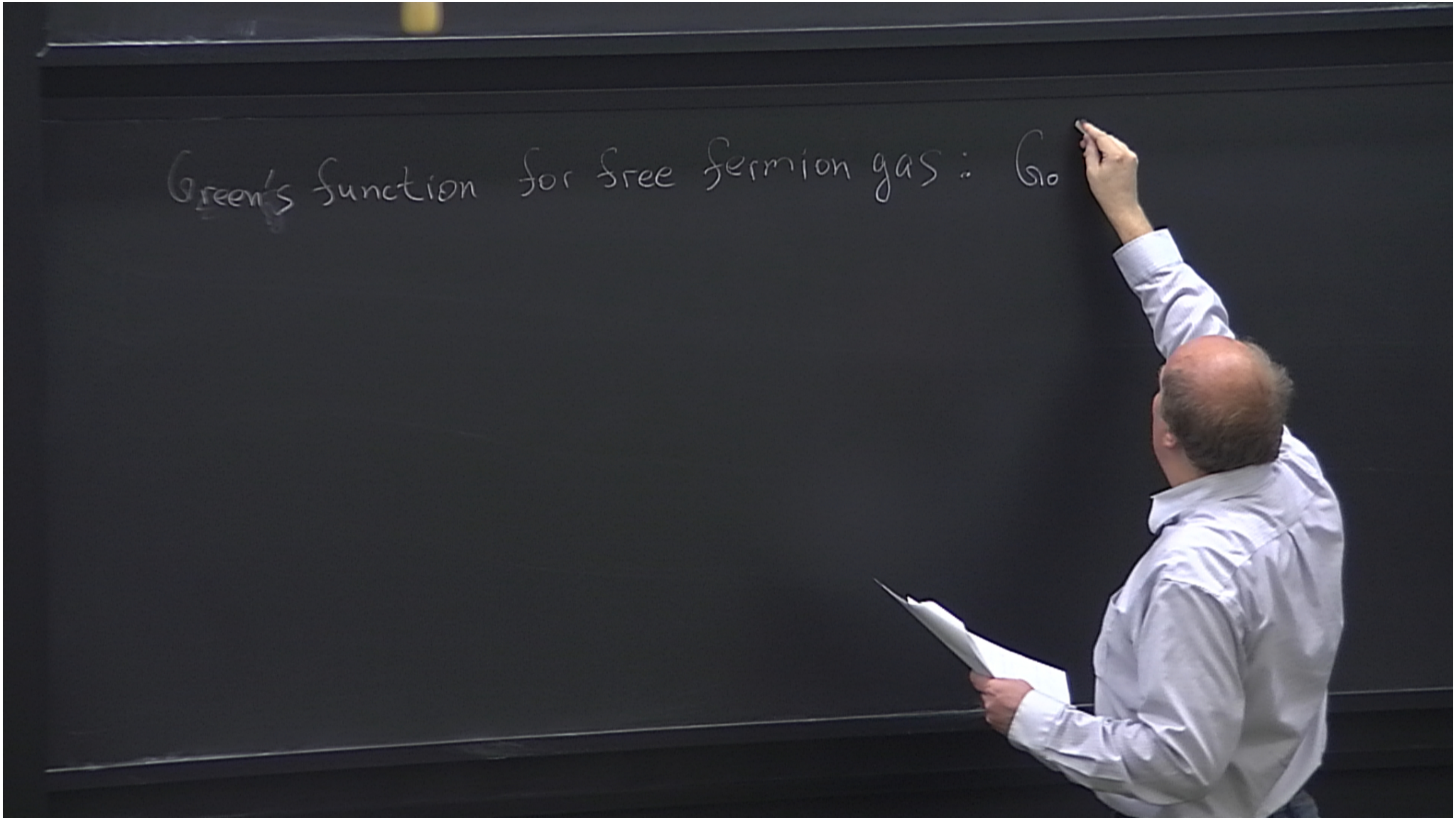

Partition function for free fermion


for free fermion gas
$\tilde{H} = H - \mu N = \sum_{k\sigma} (\varepsilon_k - \mu)\, a^\dagger_{k\sigma} a_{k\sigma}$, $\varepsilon_k =$

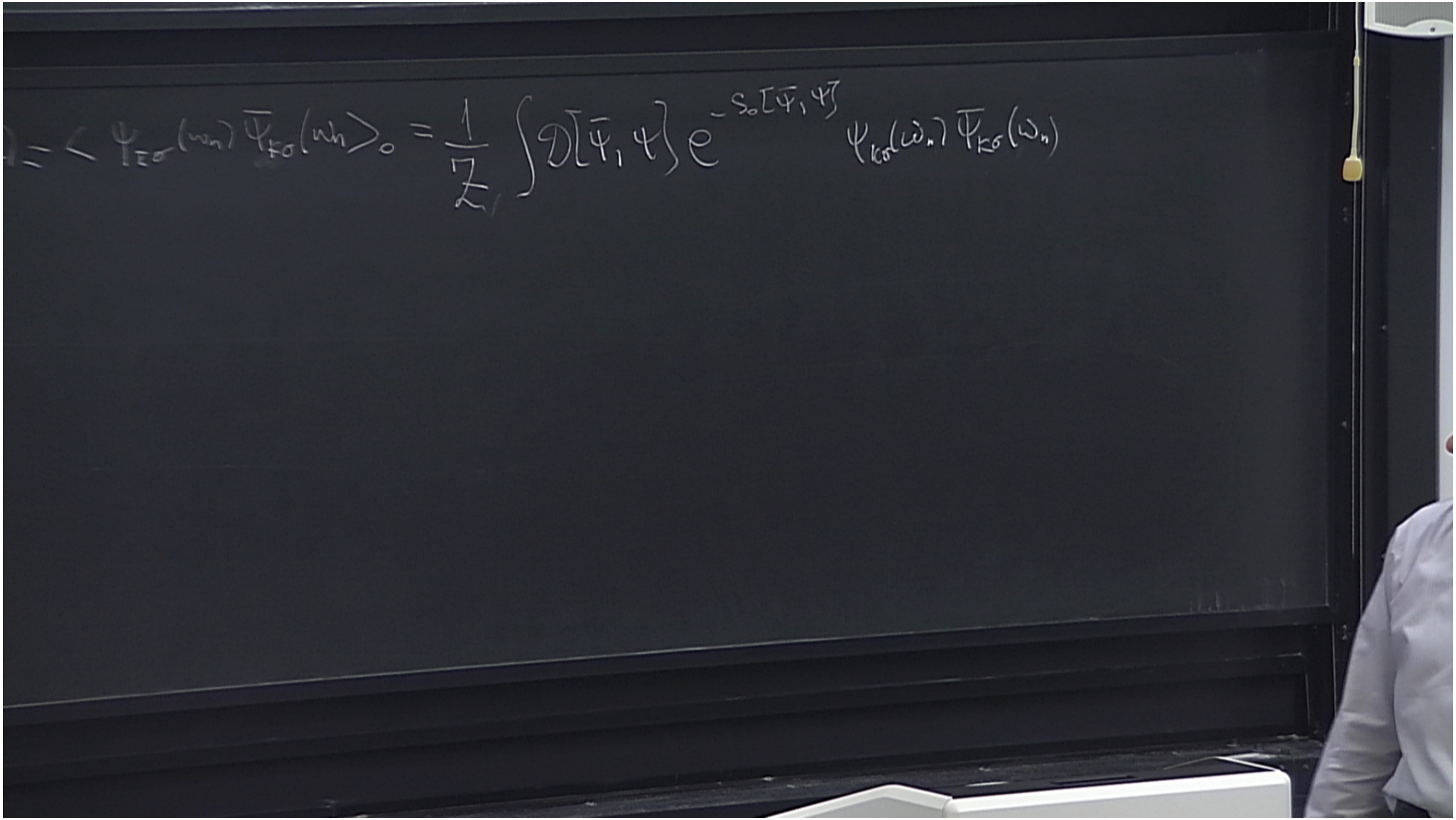

gas
$\tilde{H} = H - \mu N = \sum_{k\sigma} (\varepsilon_k - \mu)\, a^\dagger_{k\sigma}\, a_{k\sigma}$
$\varepsilon_k = \frac{\hbar^2 k^2}{2m}$ $(\hbar = 1)$
$S[\Psi, \psi] =$


$\Omega = -T \ln Z$
$Z = \int \mathcal{D}(\bar{\psi}, \psi)\, e^{-S_0[\bar{\psi}, \psi]}$


Compute $N = -\frac{\partial \Omega}{\partial \mu} = -\frac{I}{2} \int \mathcal{D}[\bar{\psi}, \psi] e^{-S_0[\bar{\psi}, \psi]}$


Green's function for free fermion gas: G0

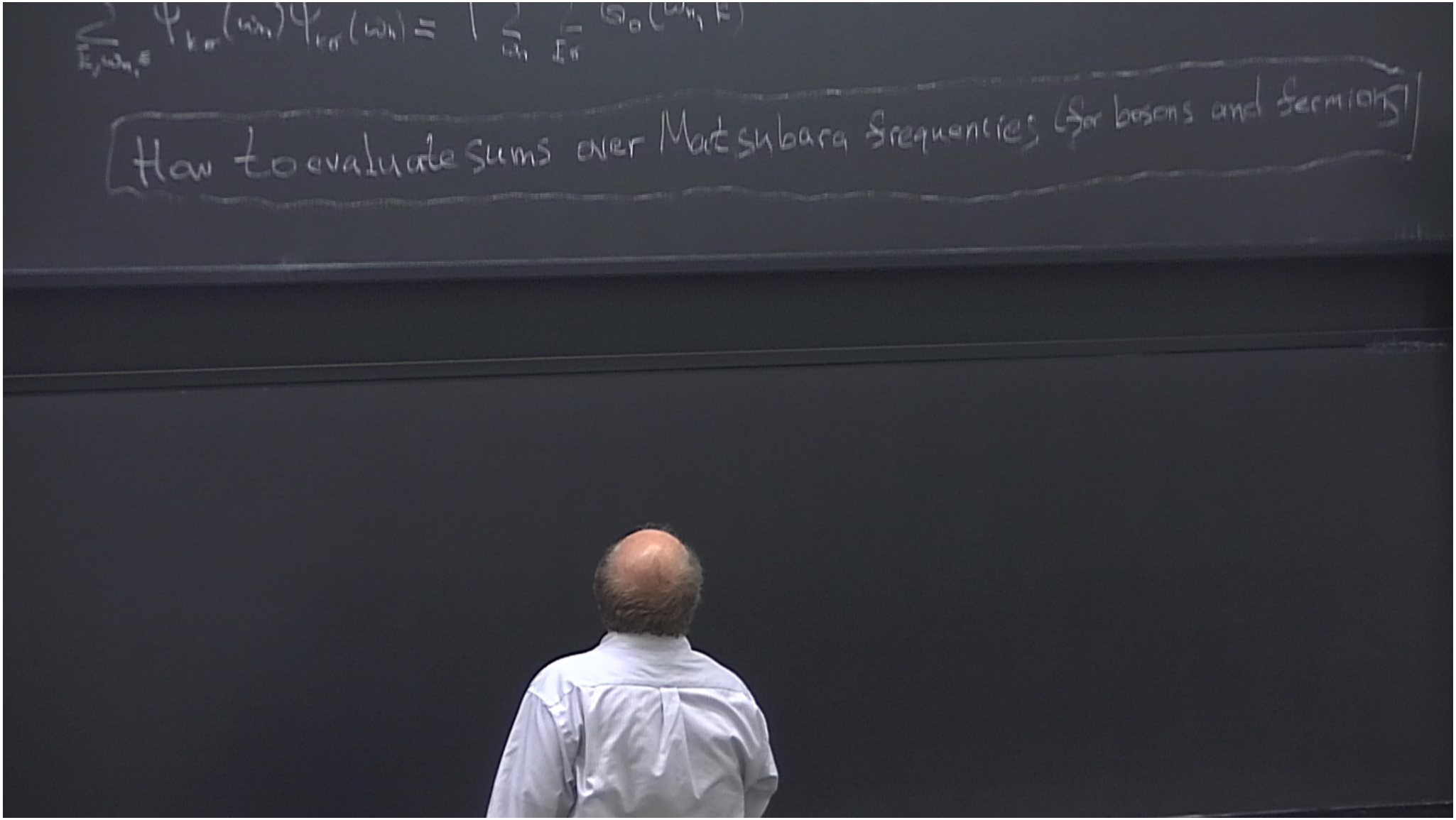
Green's function for free fermion gas: $G_0(\omega_n, \vec{k}) = \langle \psi_{\sigma 0}(\omega_n) \bar{\psi}_{\sigma 0}$

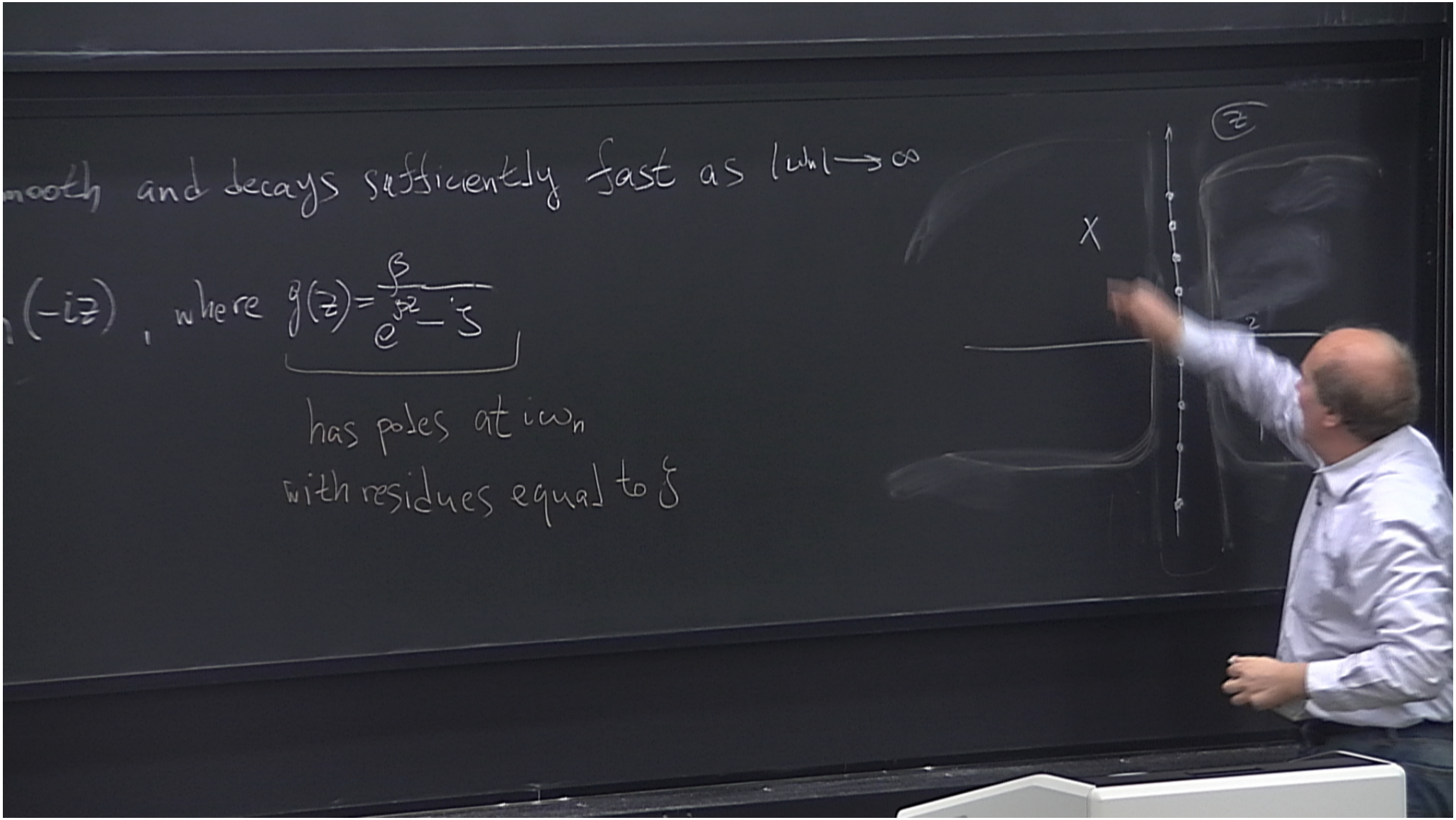

$= \langle \psi_{k\sigma}(\omega_n) \bar{\psi}_{k\sigma}(\omega_n) \rangle_0 = \frac{1}{Z} \int \mathcal{D}[\bar{\psi}, \psi] e^{-S_0[\bar{\psi}, \psi]} \psi_{k\sigma}(\omega_n) \bar{\psi}_{k\sigma}(\omega_n)$

Green's function for free fermion gas:

$$G_{k\sigma}(\omega_n, k) = \langle \psi_{k\sigma}(\omega_n)\, \bar{\psi}_{k\sigma}(\omega_n) \rangle = \frac{1}{-i\omega_n + \xi_k}$$

$$N = \frac{T}{2} \int \mathcal{D}[\bar{\psi}, \psi]\, e^{-S_0[\bar{\psi}, \psi]} \sum_{k, \omega_n, \sigma} \bar{\psi}_{k\sigma}(\omega_n) \psi_{k\sigma}(\omega_n)$$


How to evaluate sums over Matsubara frequencies (for bosons and fermions)


smooth and decays sufficiently fast as $|\omega| \to \infty$
$(-i\tau)$, where $g(\tau) = \frac{\beta}{e^{\beta\Omega} - \zeta}$
has poles at $i\omega_n$
with residues equal to ζ
X
τ
2

Suppose we have $h(\omega_n)$ that is smooth and decays sufficient

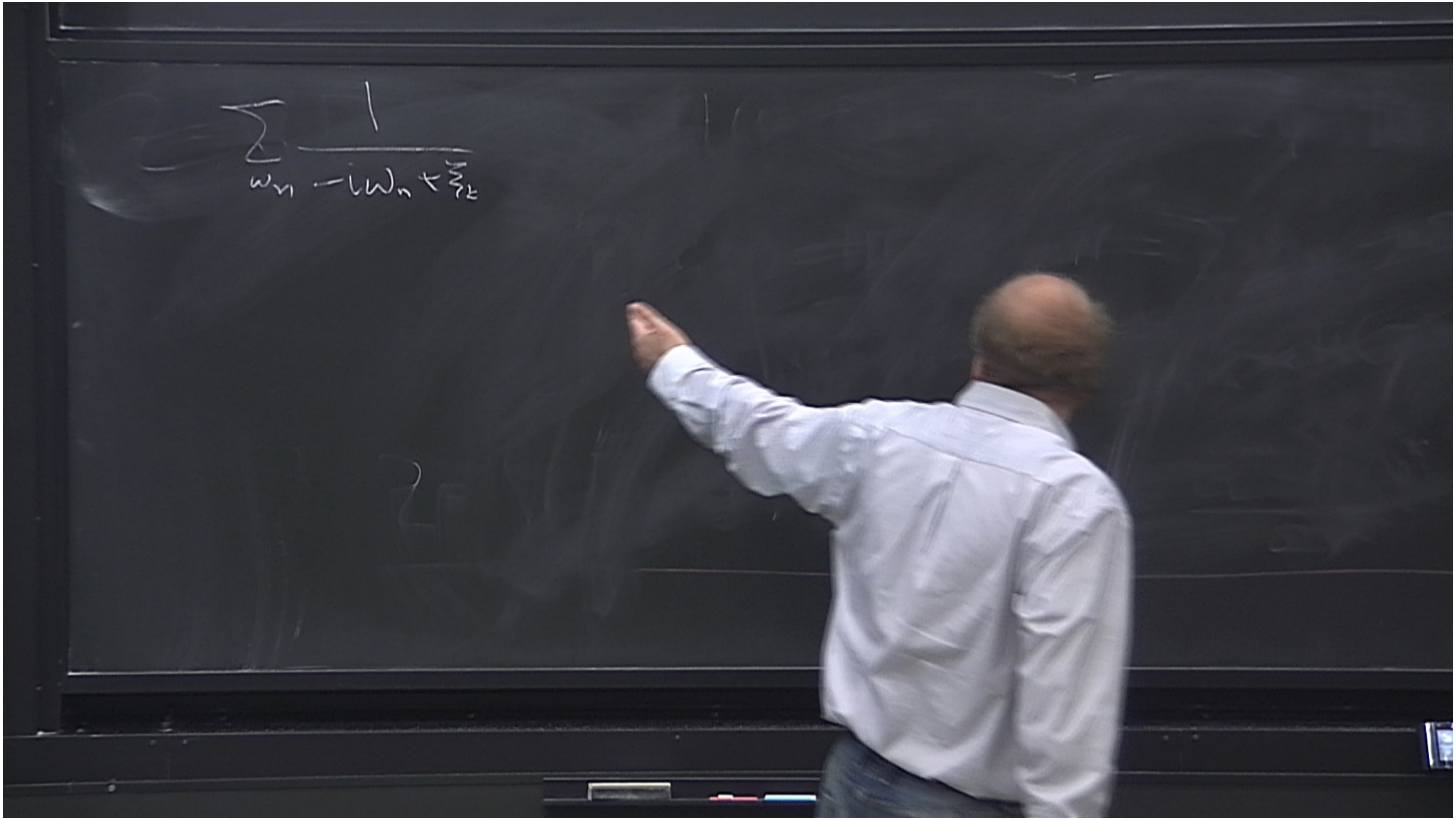
$$? \quad \sum_{\omega_n} h(\omega_n) = \frac{\beta}{2\pi i} \oint_C dz\, g(z)\, h(-iz)$$

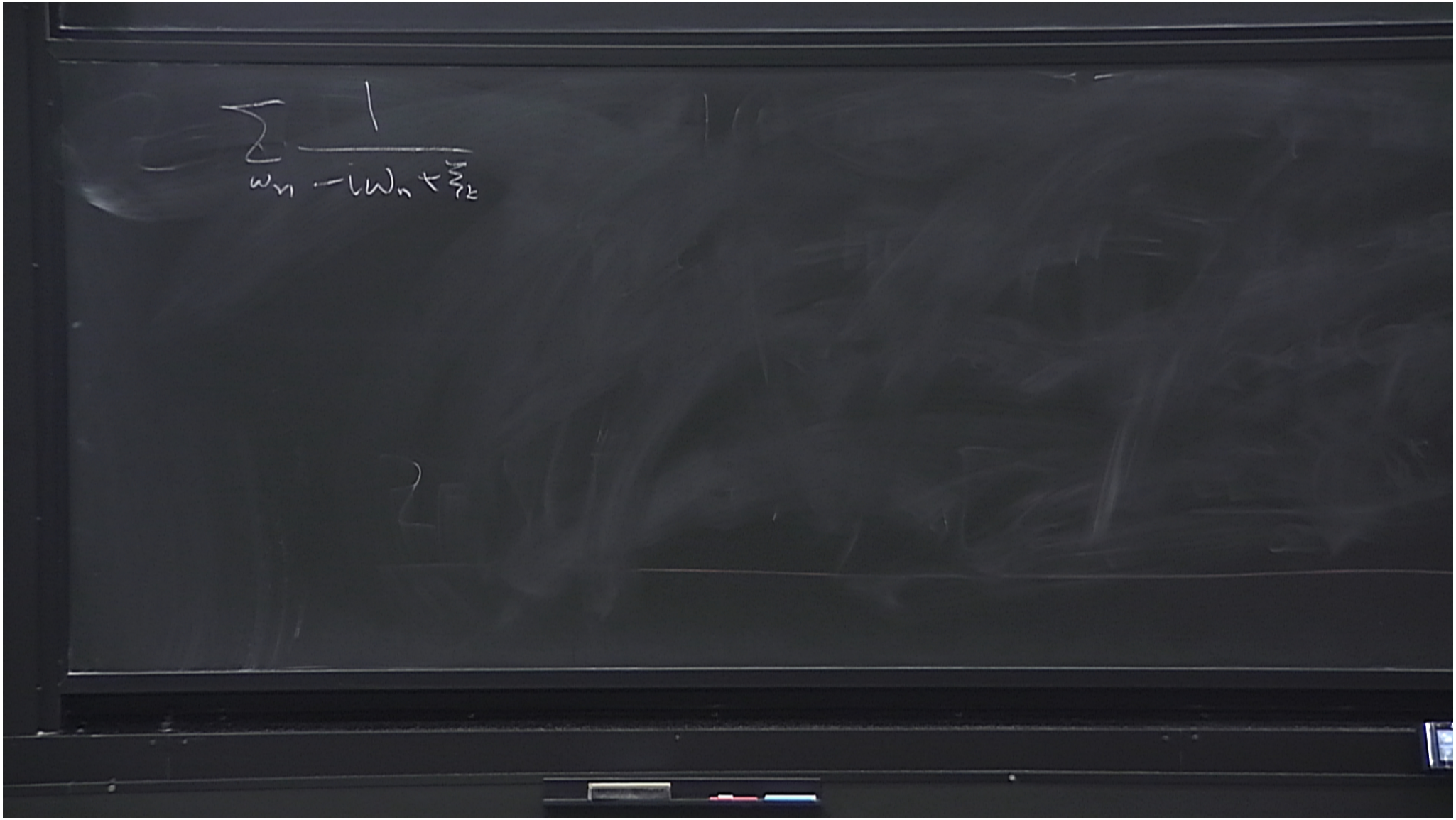
$$= -\beta \sum_{\text{poles } k} \left[\text{Res}\, h(-iz)\, g(z) \right]_{z=z_k}$$

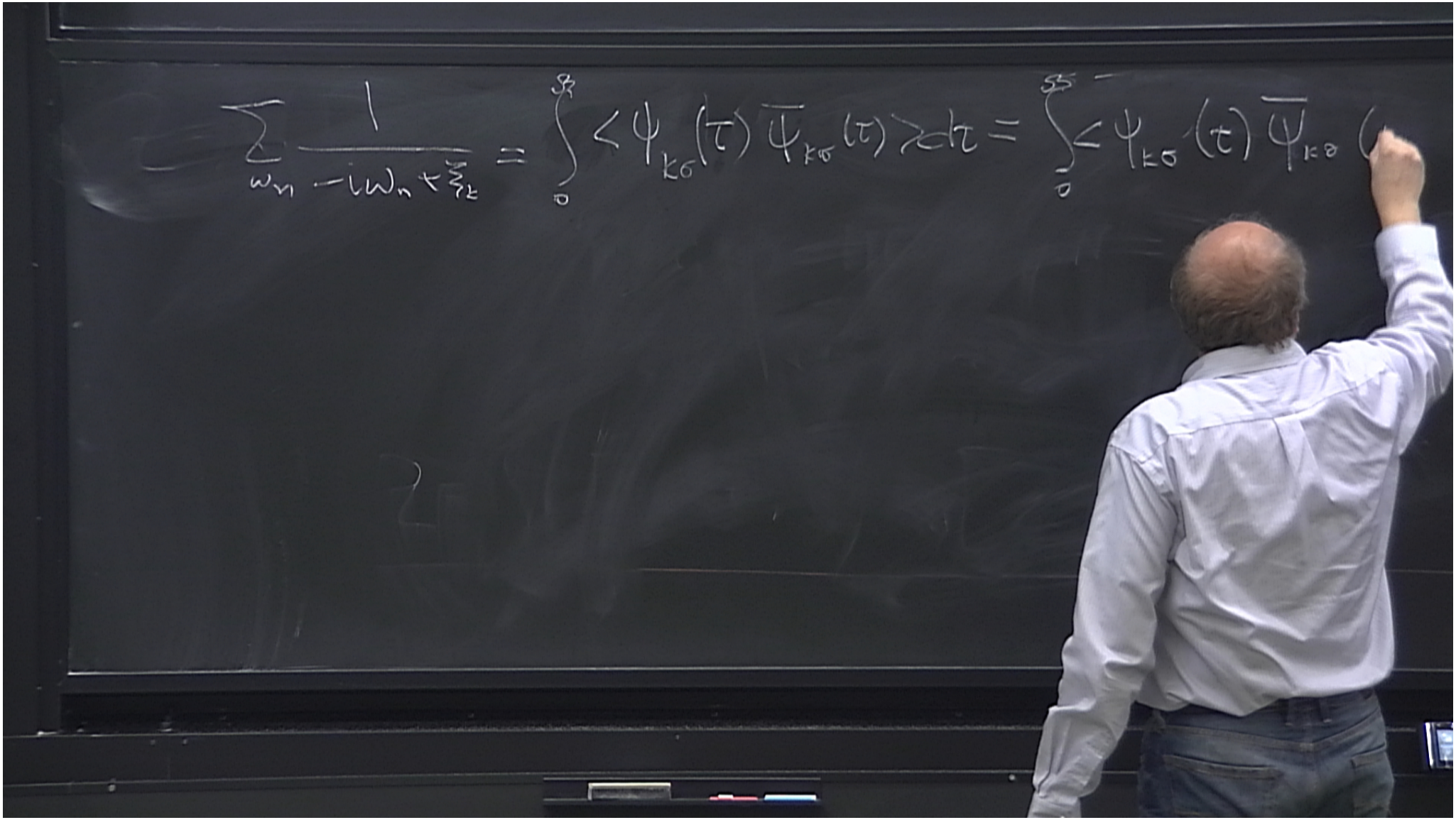
where $g(z) = \frac{\beta}{e^{\beta z} - 1}$

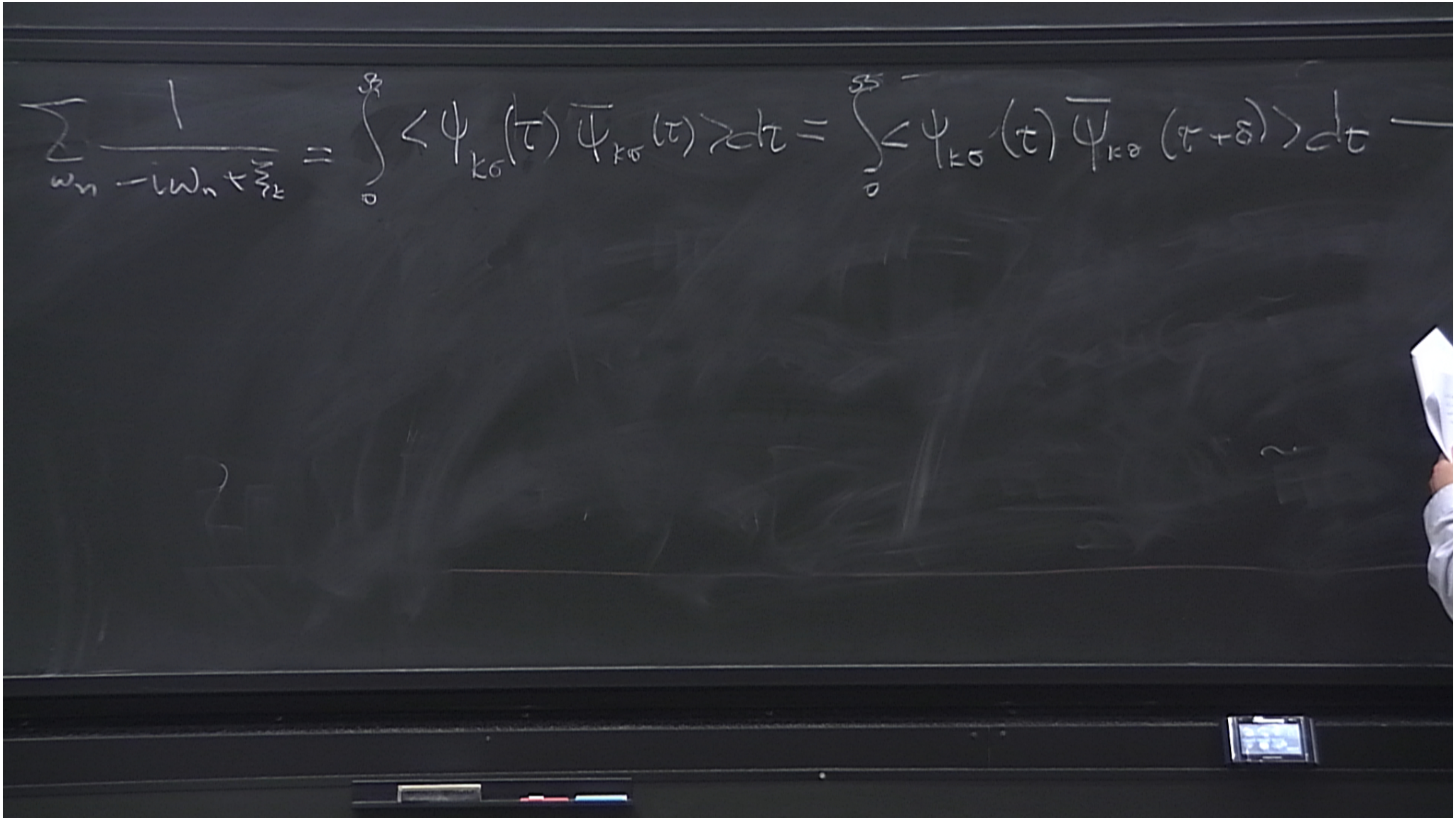
has poles at

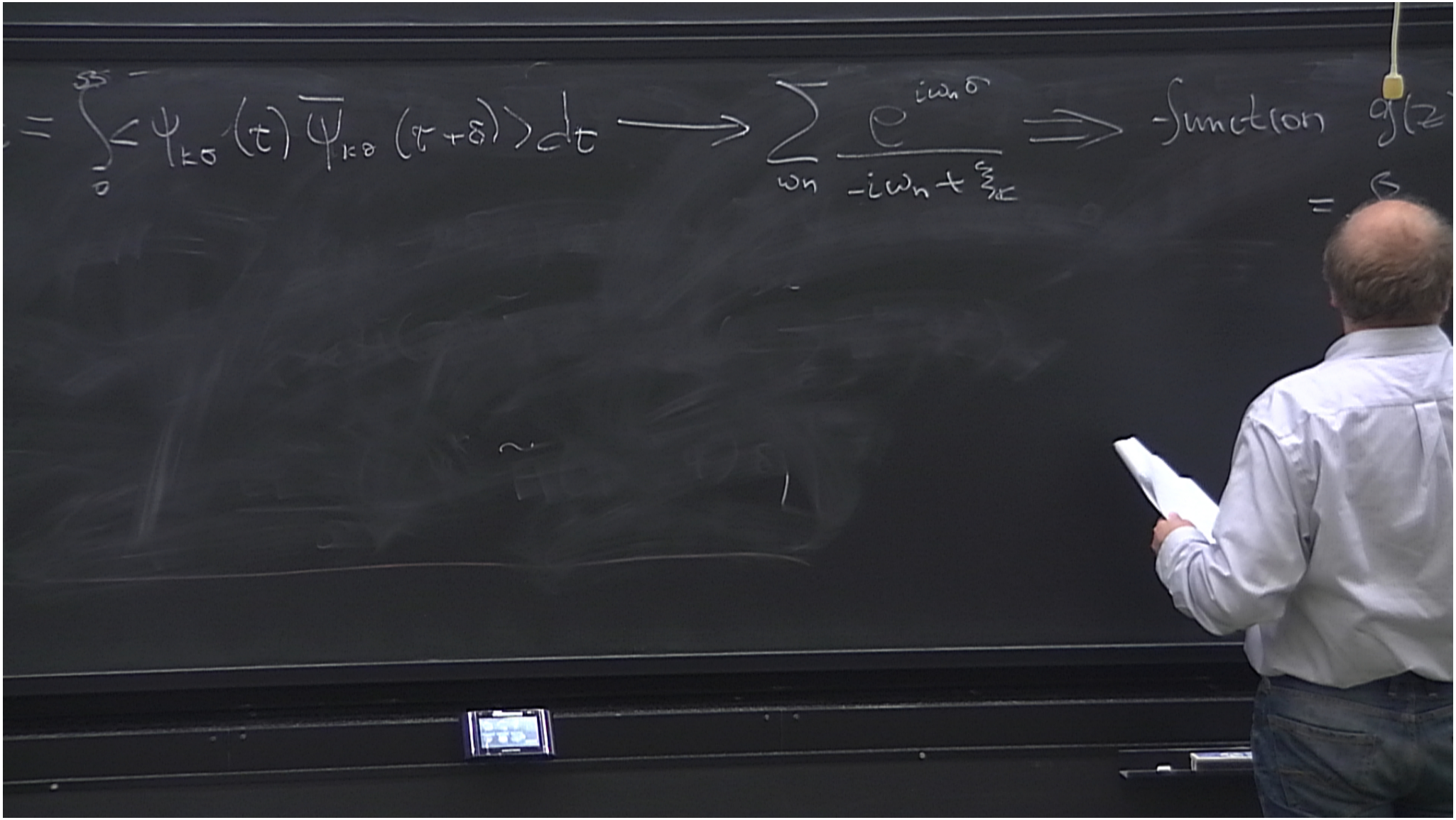
with residues

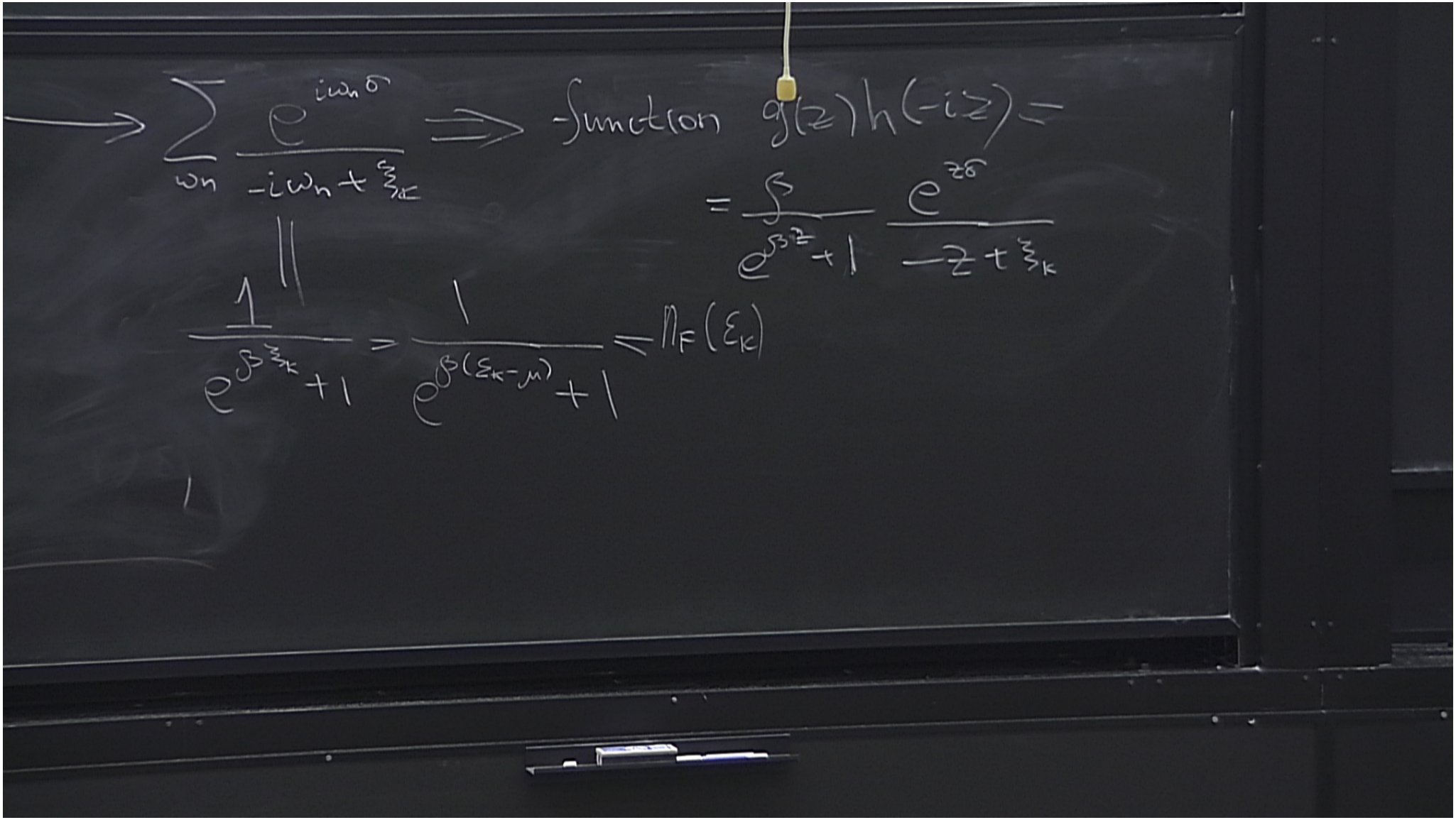

$\sum \frac{1}{\omega_n - i\omega_n + \xi_k}$


$\sum \frac{1}{\omega_n - i\omega_n + \xi_k}$


$\sum_{\omega_n} \frac{1}{\omega_n - i\omega_n + \xi_k} = \int_0^\beta \langle \psi_{k\sigma}(\tau)\, \bar{\psi}_{k\sigma}(\tau) \rangle d\tau = \int_0^\beta \langle \psi_{k\sigma}(\tau)\, \bar{\psi}_{k\sigma}$


$\sum_{\omega_n} \frac{1}{-i\omega_n + \xi_k} = \int_0^\beta \langle \psi_{k\sigma}(\tau)\, \bar{\psi}_{k\sigma}(\tau) \rangle d\tau = \int_0^\beta \langle \psi_{k\sigma}(t)\, \bar{\psi}_{k\sigma}(\tau+\delta) \rangle dt$


$= \int_0^{\infty} \langle \psi_{k\sigma}(t)\, \bar{\psi}_{k\sigma}(\tau+\delta) \rangle dt \longrightarrow \sum_{\omega_n} \frac{e^{i\omega_n \delta}}{-i\omega_n + \xi_k} \Rightarrow$ -function $g(z)$
$= \delta$


$\sum_{\omega_n} \frac{e^{i\omega_n 0^+}}{-i\omega_n + \xi_k} \Rightarrow$ -function $g(z)h(-iz) =$
$= \frac{\beta}{e^{\beta z}+1} \frac{e^{z0^+}}{-z+\xi_k}$
$\frac{1}{e^{\beta\xi_k}+1} = \frac{1}{e^{\beta(\xi_k-\mu)}+1} = n_F(\xi_k)$

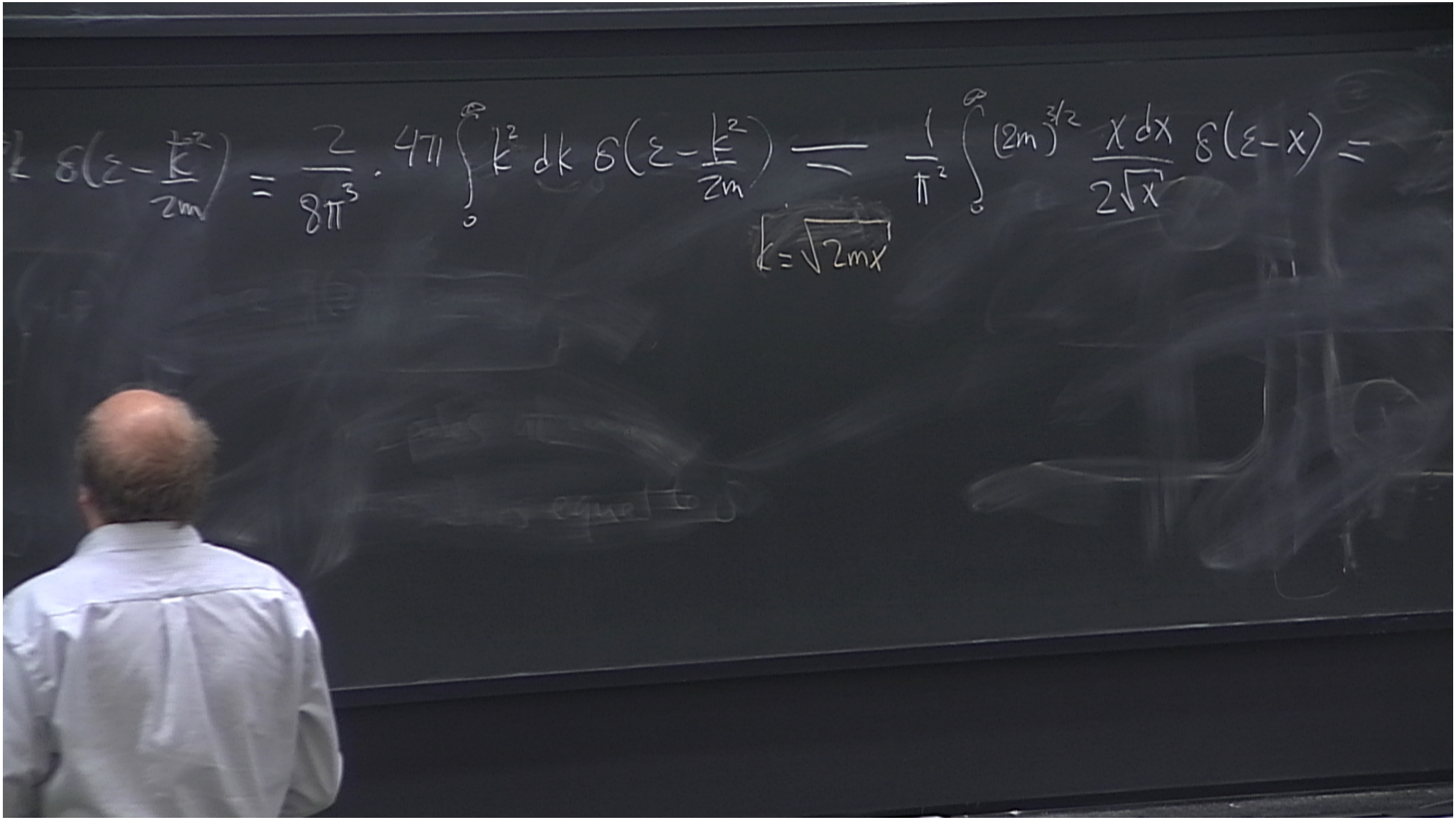
$$N = -T \sum_{\omega_n, k\sigma} G_0(\omega_n, k) = \sum_{k\sigma} n_F(\varepsilon_k) = \sum_{k\sigma} \frac{1}{e^{\beta(\varepsilon_k - \mu)} + 1}$$

$$N = -\frac{\partial \Omega}{\partial \mu} \Rightarrow \Omega = -T \sum_{k\sigma} \ln\left(1 + e^{-\beta(\varepsilon_k - \mu)}\right)$$

$$\sum_{k\sigma} F(\varepsilon_k) = \underbrace{2}_{\text{from spins}} \frac{V}{(2\pi)^3} \int d^3k\, F(\varepsilon_k) = \frac{2V}{(2\pi)^3} \int d^3k \int d\varepsilon\, F(\varepsilon)\, \delta(\varepsilon$$

$$= V \int d\varepsilon\, \nu(\varepsilon) F(\varepsilon) \quad \text{with} \quad \boxed{\nu(\varepsilon) = \frac{2}{(2\pi)^3} \int d^3k\, \delta(\varepsilon - \varepsilon_k)}$$

density of states


$\delta\left(\varepsilon - \frac{k^2}{2m}\right) = \frac{2}{8\pi^3} \cdot 4\pi \int_0^\infty k^2\, dk\, \delta\left(\varepsilon - \frac{k^2}{2m}\right) = \frac{1}{\pi^2} \int_0^\infty (2m)^{3/2} \frac{x\, dx}{2\sqrt{x}}\, \delta(\varepsilon - x) =$
$k = \sqrt{2mx}$