Title: PSI 2015/2016 Condensed Matter - Oleg Tchernyshyov - Lecture 7

Date: Nov 17, 2015 10:45 AM

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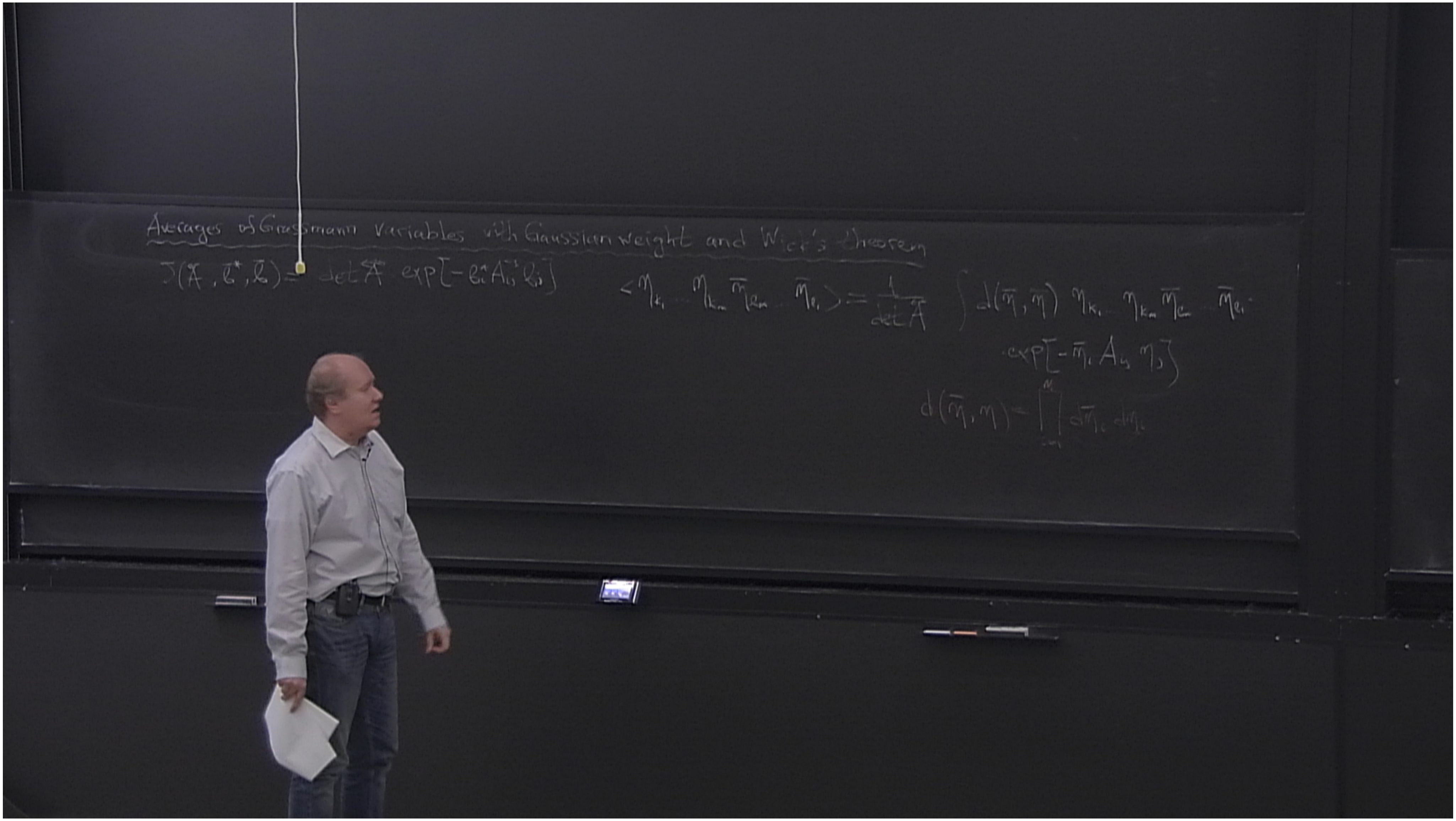
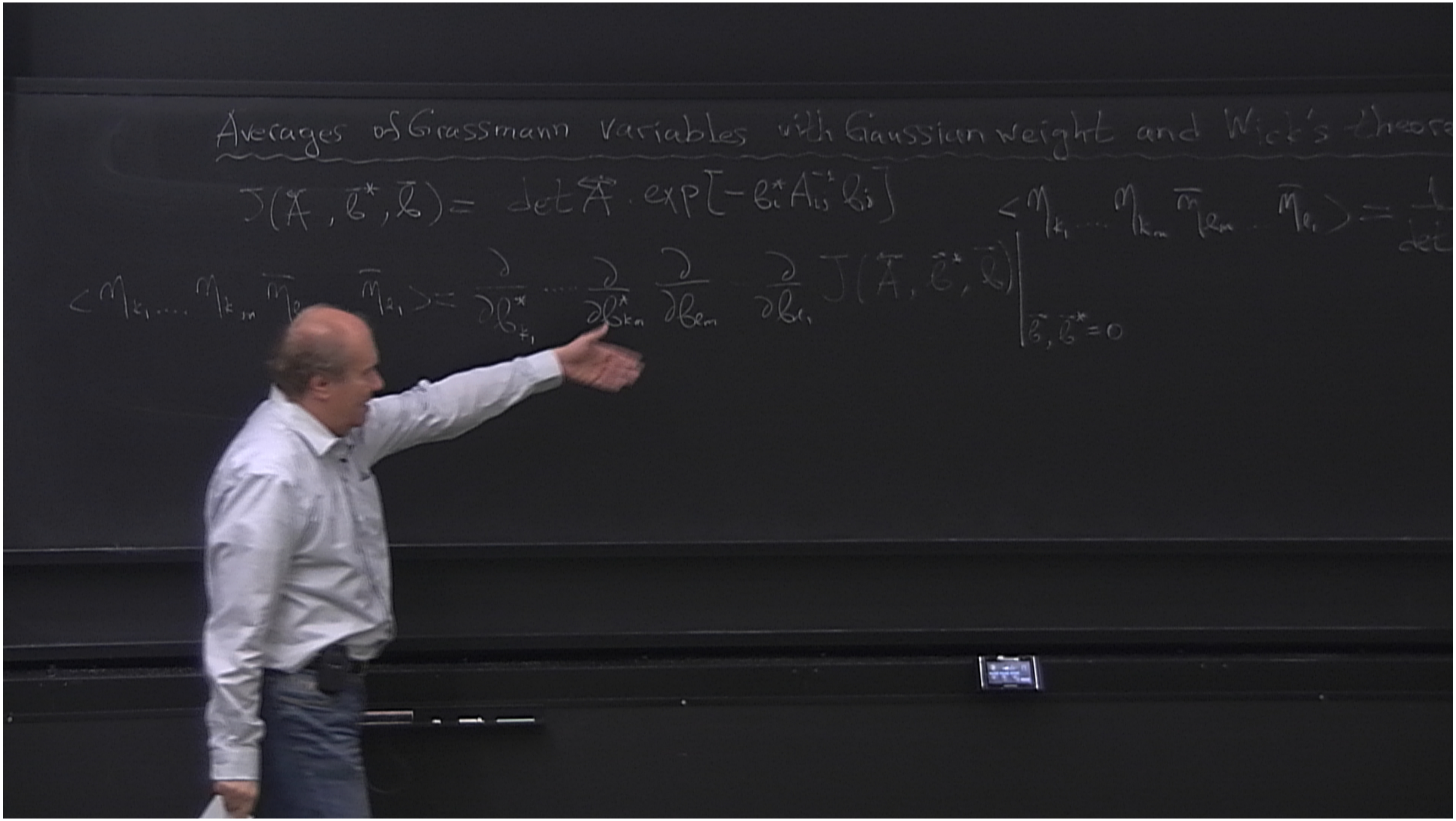
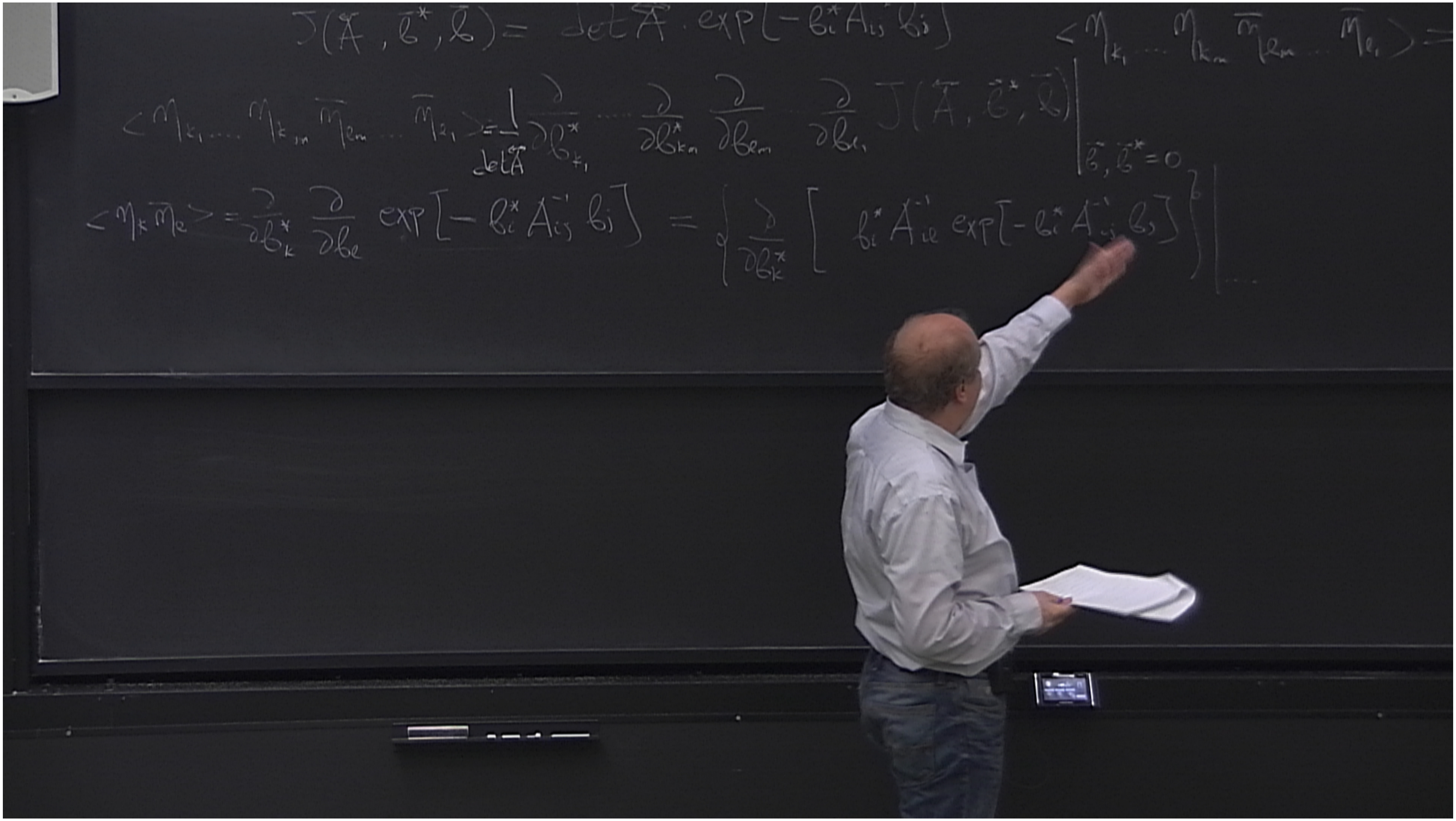
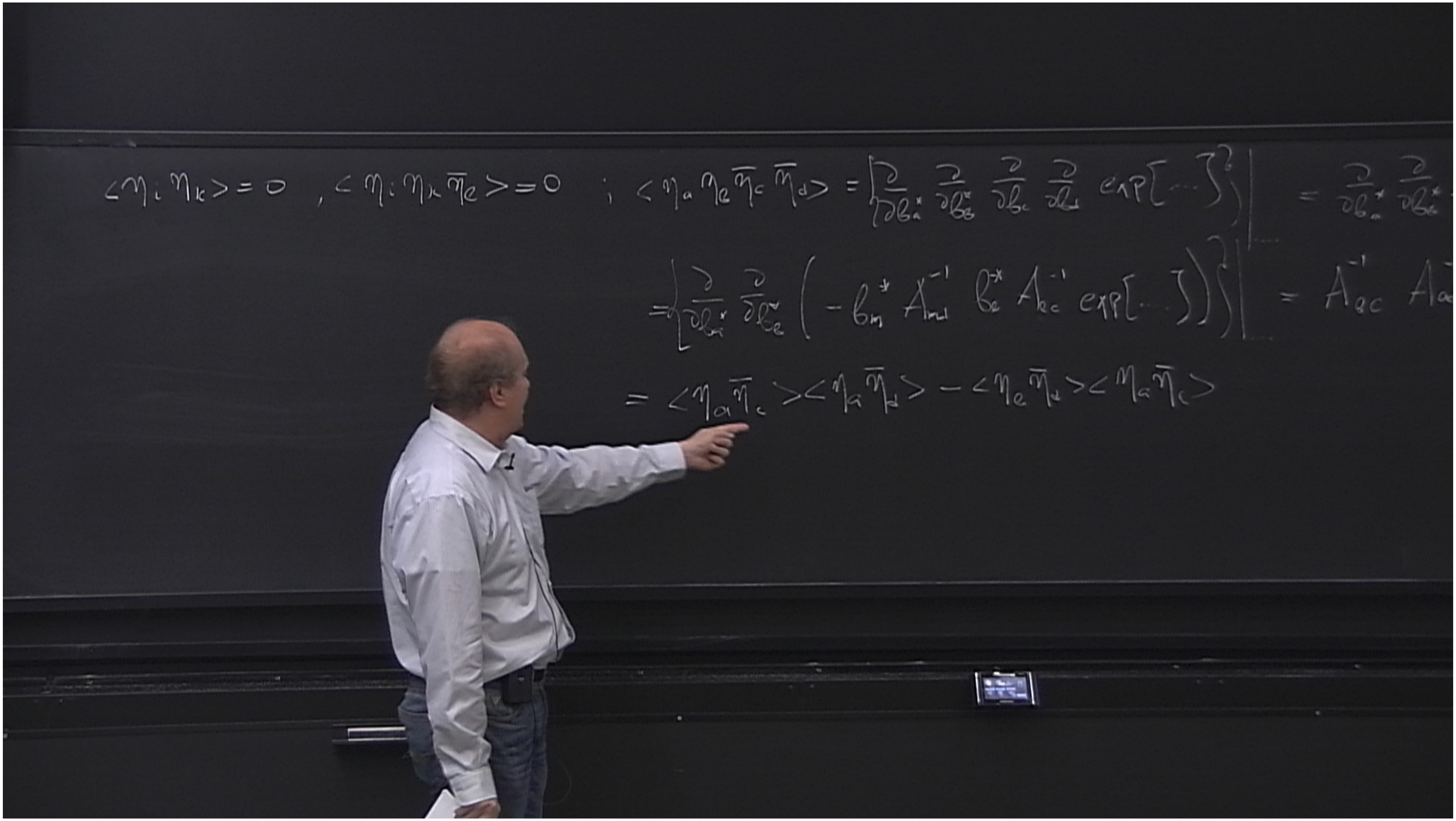
Abstract:


Averages of Grassmann variables with Gaussian weight and Wick's theorem


Averages of Grassmann variables with Gaussian weight and Wick's theore
$J(\vec{A}, \vec{b}^*, \vec{b}) = \det \vec{A} \cdot \exp[-b_i^* A_{ij}^{-1} b_j]$
$\langle \eta_{k_1} \dots \eta_{k_m} \bar{\eta}_{l_m} \dots \bar{\eta}_{l_1} \rangle = \frac{\partial}{\partial b^*_{k_1}} \dots \frac{\partial}{\partial b^*_{k_m}} \frac{\partial}{\partial b_{l_m}} \dots \frac{\partial}{\partial b_{l_1}} J(\vec{A}, \vec{b}^*, \vec{b}) \Big|_{\vec{b}, \vec{b}^* = 0}$


$J(\overleftrightarrow{A}, \vec{b}^*, \vec{b}) = \det \overleftrightarrow{A} \cdot \exp[-b_i^* A_{ij}^{-1} b_j]$
$\langle \eta_{k_1} \ldots \eta_{k_m} \bar{\eta}_{l_m} \ldots \bar{\eta}_{l_1} \rangle =$
$\langle \eta_{k_1} \ldots \eta_{k_m} \bar{\eta}_{l_m} \ldots \bar{\eta}_{l_1} \rangle = \frac{1}{\det \overleftrightarrow{A}} \frac{\partial}{\partial b_{k_1}^*} \ldots \frac{\partial}{\partial b_{k_m}^*} \frac{\partial}{\partial b_{l_m}} \ldots \frac{\partial}{\partial b_{l_1}} J(\overleftrightarrow{A}, \vec{b}^*, \vec{b}) \Big|_{\vec{b}, \vec{b}^* = 0}$
$\langle \eta_k \bar{\eta}_l \rangle = \frac{\partial}{\partial b_k^*} \frac{\partial}{\partial b_l} \exp[-b_i^* A_{ij}^{-1} b_j] = \left\{ \frac{\partial}{\partial b_k^*} \left[b_i^* A_{il}^{-1} \exp[-b_i^* A_{ij}^{-1} b_j] \right] \right\} \Big|_{\ldots}$


$\langle \eta_i \eta_k \rangle = 0 \quad , \langle \eta_i \eta_k \bar{\eta}_l \rangle = 0 \quad ; \langle \eta_a \eta_b \bar{\eta}_c \bar{\eta}_d \rangle = \left[\frac{\partial}{\partial b_a^*} \frac{\partial}{\partial b_b^*} \frac{\partial}{\partial b_c} \frac{\partial}{\partial b_d} \exp[\ldots] \right\} \Big| = \frac{\partial}{\partial b_a^*} \frac{\partial}{\partial b_b^*}$
$= \left[\frac{\partial}{\partial b_a^*} \frac{\partial}{\partial b_b^*} \left(- b_m^* A^{-1}_{md} b_e^* A^{-1}_{ec} \exp[\ldots] \right) \right\} \Big| = A^{-1}_{ac} A^{-1}$
$= \langle \eta_a \bar{\eta}_c \rangle \langle \eta_b \bar{\eta}_d \rangle - \langle \eta_b \bar{\eta}_d \rangle \langle \eta_a \bar{\eta}_c \rangle$

$$= \underbrace{\langle \eta_a \bar{\eta}_c \rangle \langle \eta_b \bar{\eta}_d \rangle - \langle \eta_a \bar{\eta}_d \rangle \langle \eta_b \bar{\eta}_c \rangle}_{\text{Wick's theorem}}$$

Coherent states (fermions)

$\{a_i, a_j^\dagger\} = \delta_{ij}$, $\{a_i, a_j\} = 0$

. Set of $\{\eta_i\}$ where η_i - Grassmann variables

Impose $\{\eta_i, a_j\} = 0$

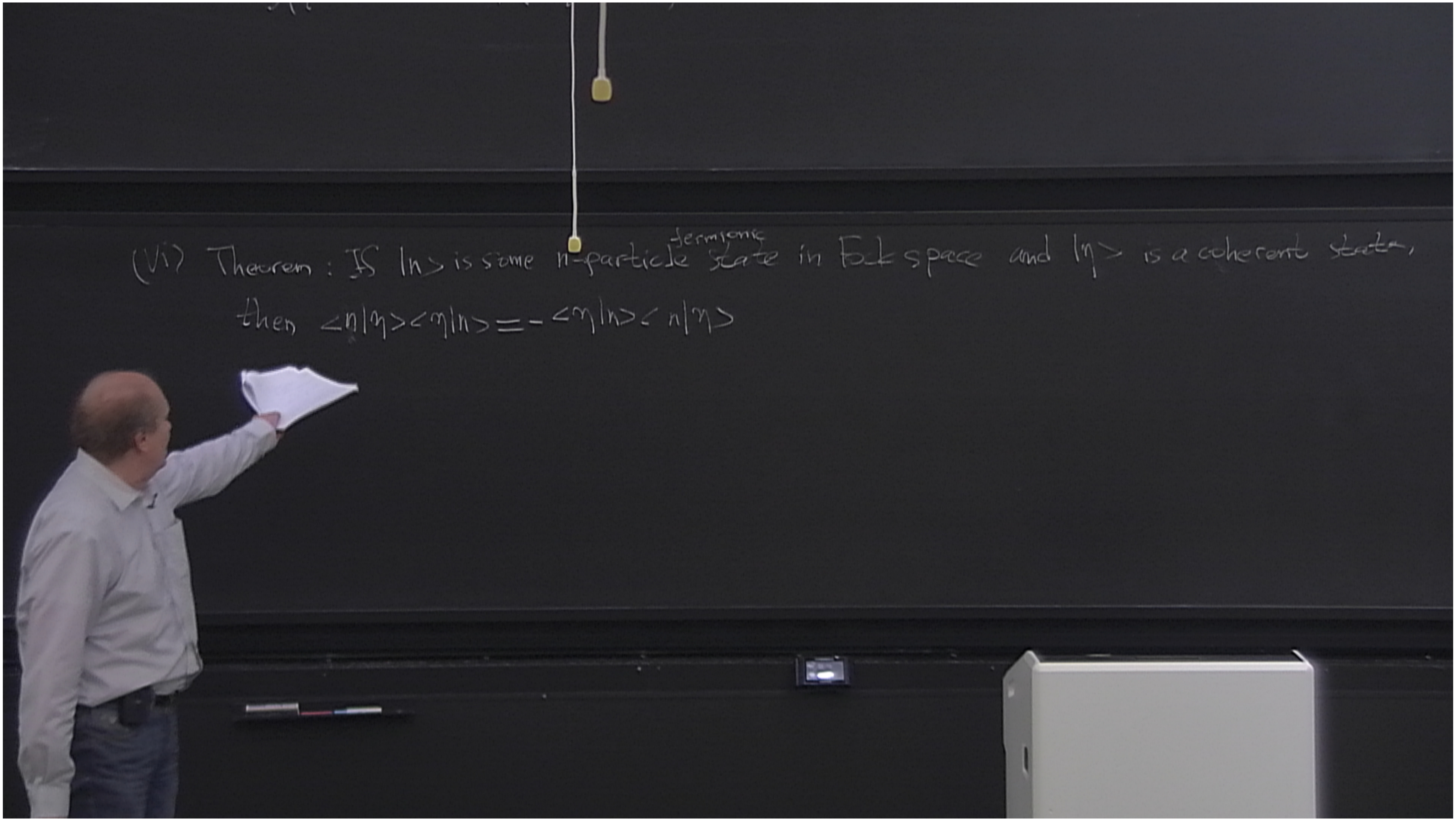
Definition. $|\eta\rangle = \exp\left(-\sum_i \eta_i a_i^\dagger\right)|0\rangle$, $\langle \eta| = \langle 0| \exp\left(-\sum_i a_i \bar{\eta}_i\right)$,

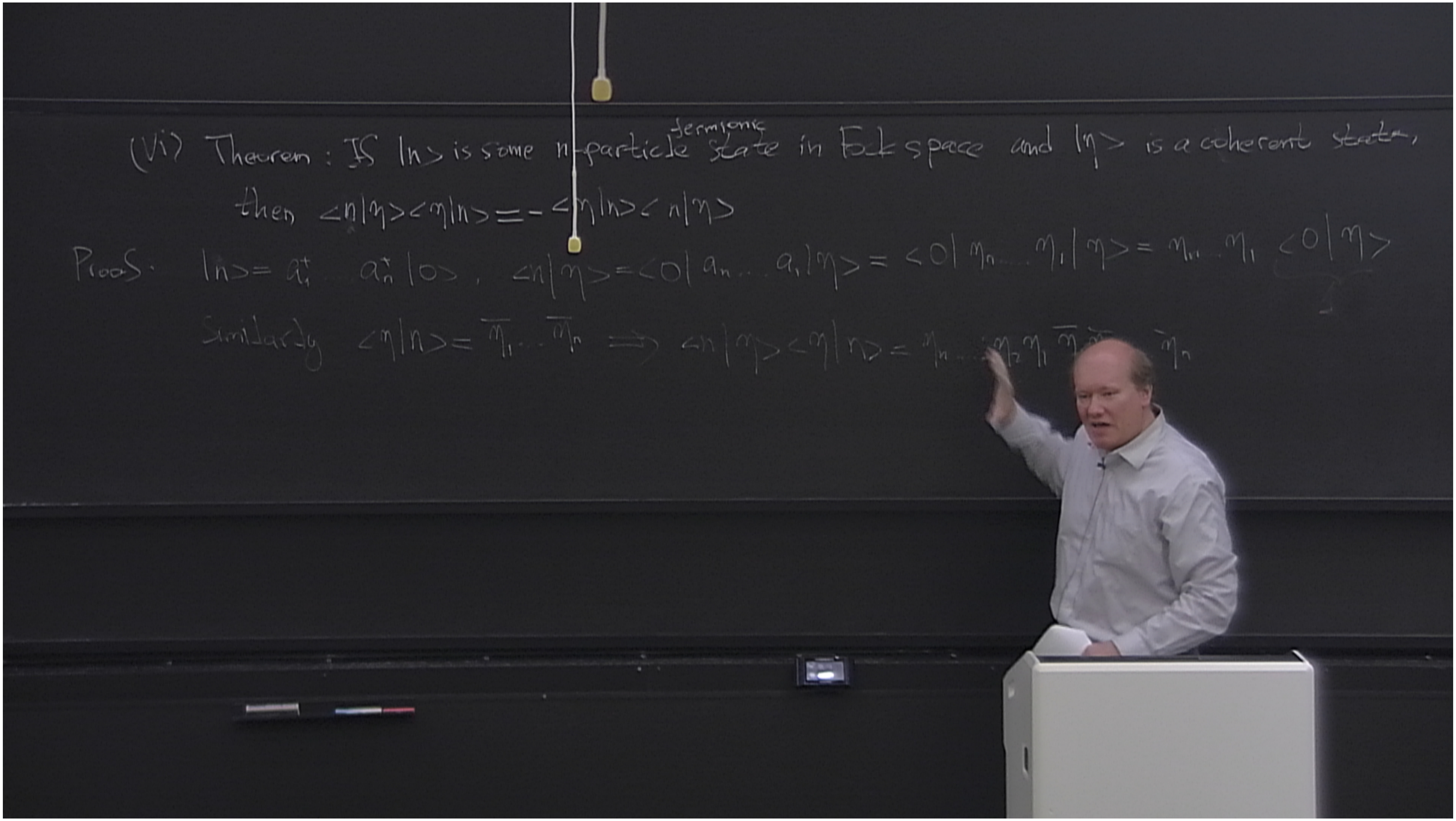
where $\{\bar{\eta}_i\}$ is independent of $\{\eta_i\}$

Properties: (i) $a_i|\eta\rangle = \eta_i|\eta\rangle$; (ii) $\langle\eta|a_i^\dagger = \langle\eta|\bar{\eta}_i$; (iii) $a_i^\dagger|\eta\rangle = -\frac{\partial}{\partial\eta_i}|\eta\rangle$

(iv) $\langle\xi|\eta\rangle = \exp\left(\sum_i \bar{\xi}_i\eta_i\right)$, $\langle\eta|\eta\rangle = \exp\left(\sum_i \bar{\eta}_i\eta_i\right)$

(v) $$\int\left(\prod_i d\bar{\eta}_i\, d\eta_i\right) \exp\left(-\sum_i \bar{\eta}_i\eta_i\right) |\eta\rangle\langle\eta| = \mathbb{1}$$


(vi) Theorem: If $|n\rangle$ is some n-particle fermionic state in Fock space and $|\eta\rangle$ is a coherent state,
then $\langle n|\eta\rangle\langle\eta|n\rangle = -\langle\eta|n\rangle\langle n|\eta\rangle$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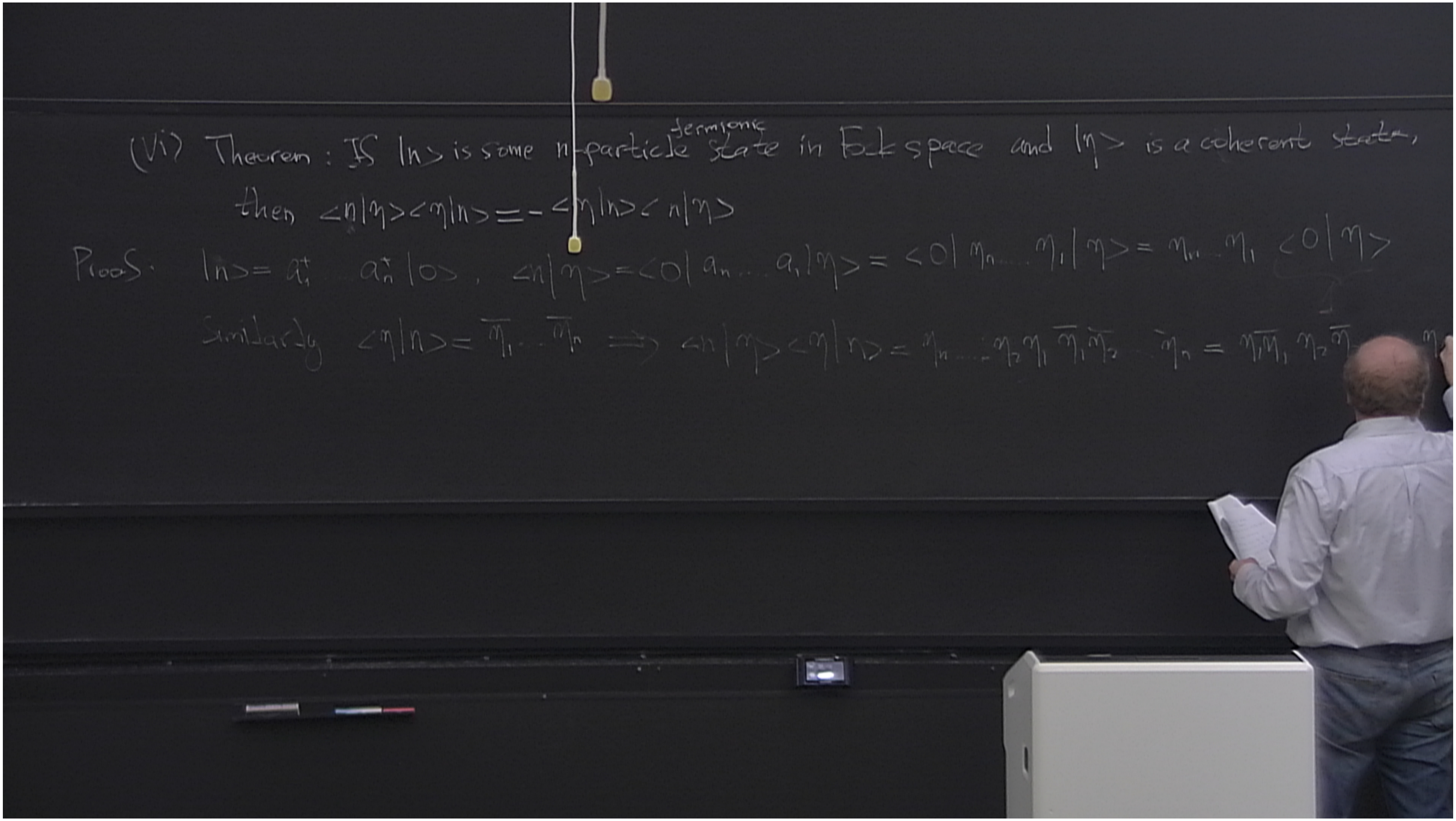
then $\langle n|\eta\rangle\langle\eta|n\rangle = -\langle\eta|n\rangle\langle n|\eta\rangle$
Proof. $|n\rangle = a_1^+ \cdots a_n^+|0\rangle$, $\langle n|\eta\rangle = \langle 0|a_n\ldots a_1|\eta\rangle = \langle 0|\eta_n\ldots\eta_1|\eta\rangle = \eta_n\ldots\eta_1\langle 0|\eta\rangle$
Similarly $\langle\eta|n\rangle = \bar\eta_1\ldots\bar\eta_n \Rightarrow \langle n|\eta\rangle\langle\eta|n\rangle = \eta_n\ldots\eta_2\eta_1\bar\eta_1 \ldots \bar\eta_n$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Proof. $|n\rangle = a_1^\dagger \ldots a_n^\dagger|0\rangle$, $\langle n|\eta\rangle = \langle 0|a_n \ldots a_1|\eta\rangle = \langle 0|\eta_n \ldots \eta_1|\eta\rangle = \eta_n \ldots \eta_1 \langle 0|\eta\rangle$
Similarly $\langle\eta|n\rangle = \bar\eta_1 \ldots \bar\eta_n \Rightarrow \langle n|\eta\rangle\langle\eta|n\rangle = \eta_n \ldots \eta_2\eta_1\bar\eta_1\bar\eta_2 \ldots \bar\eta_n = \eta_1\bar\eta_1\eta_2\bar\eta_2 \ldots \eta$

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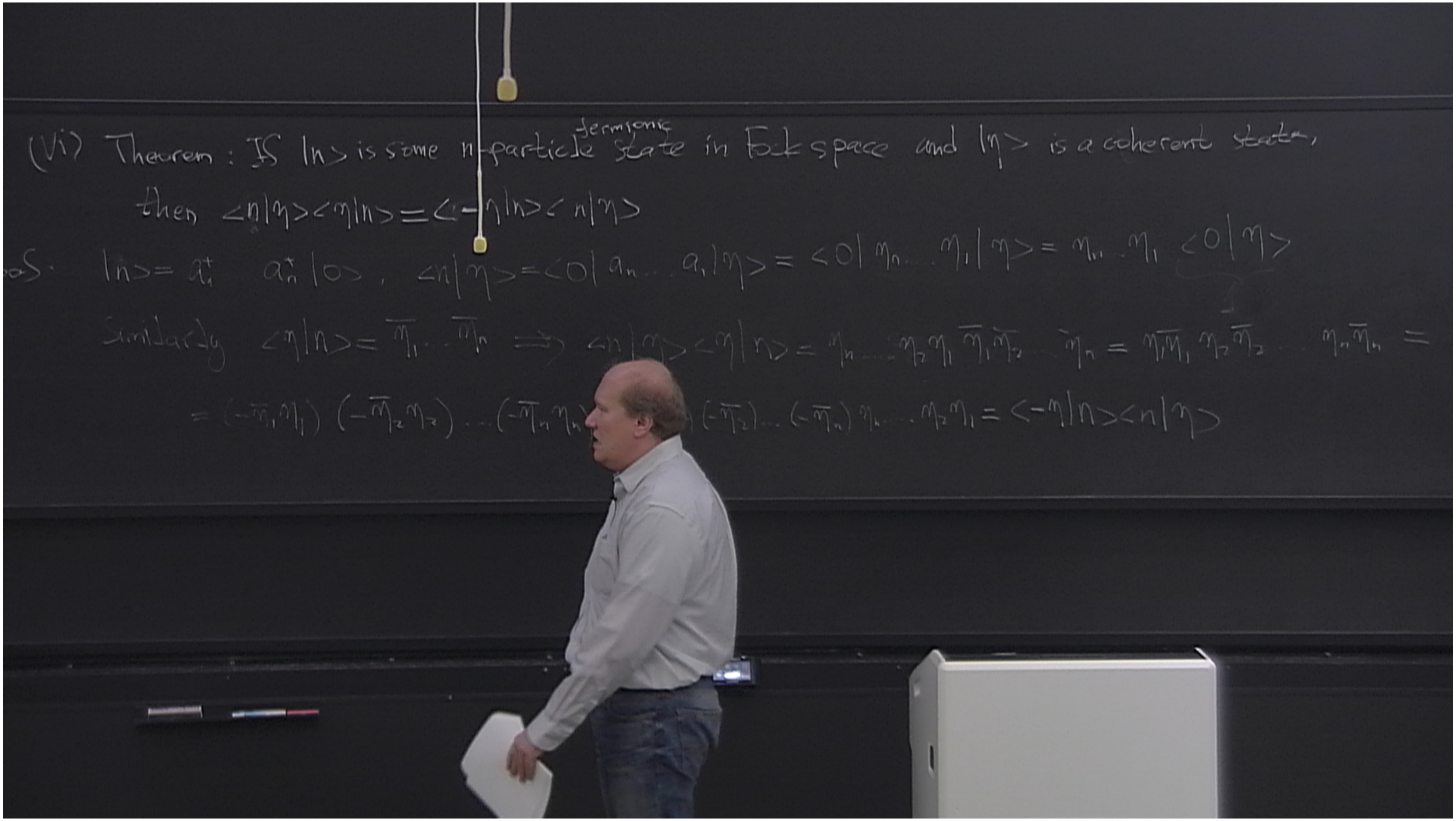
Similarly $\langle\eta|n\rangle = \bar\eta_1 \ldots \bar\eta_n \Rightarrow \langle n|\eta\rangle\langle\eta|n\rangle = \eta_n \ldots \eta_2\eta_1\bar\eta_1\bar\eta_2 \ldots \bar\eta_n = \eta_1\bar\eta_1\,\eta_2\bar\eta_2 \ldots \eta_n\bar\eta_n$



(vi) Theorem: If $|n\rangle$ is some n-particle fermionic state in Fock space and $|\eta\rangle$ is a coherent state,
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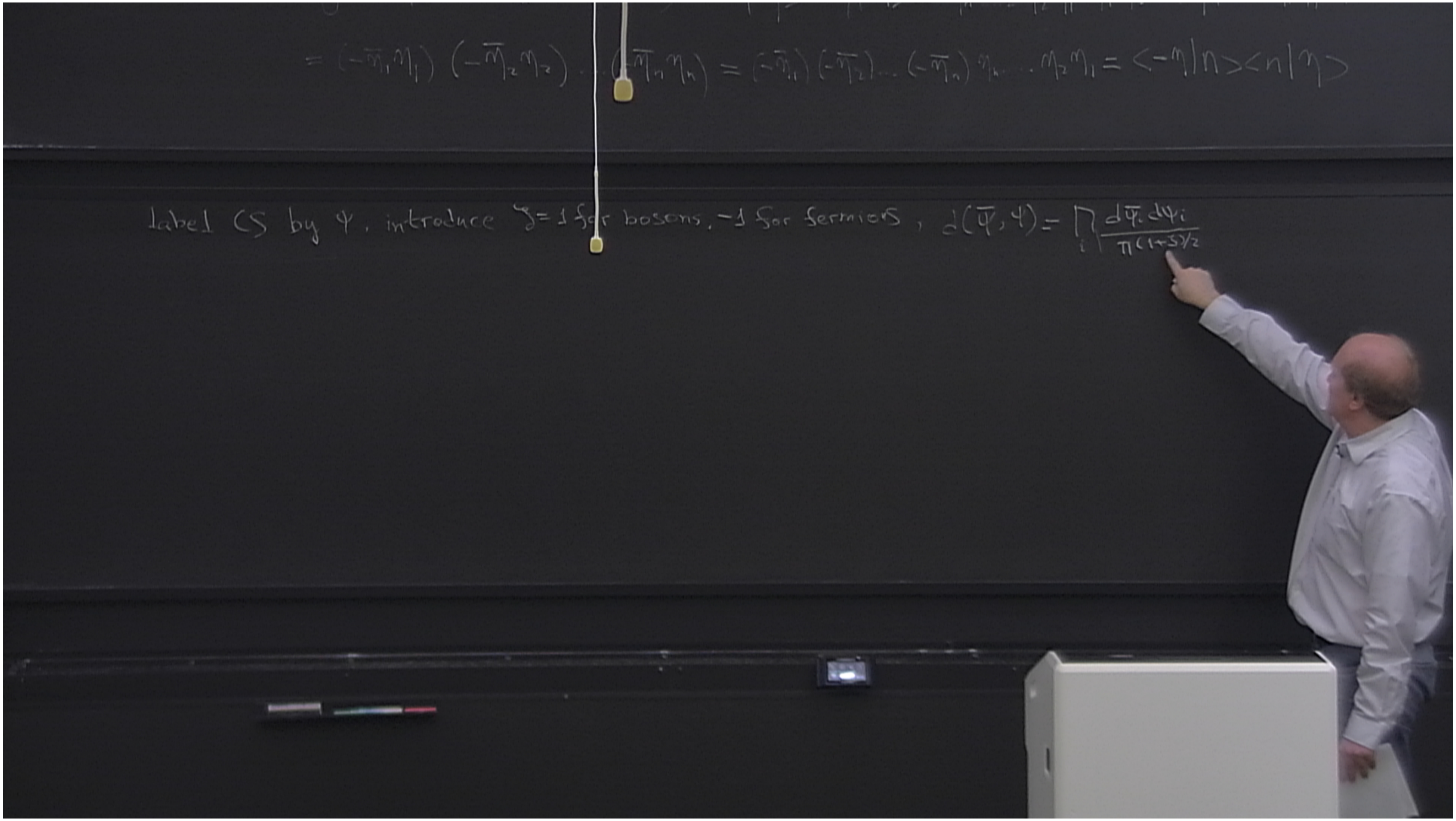
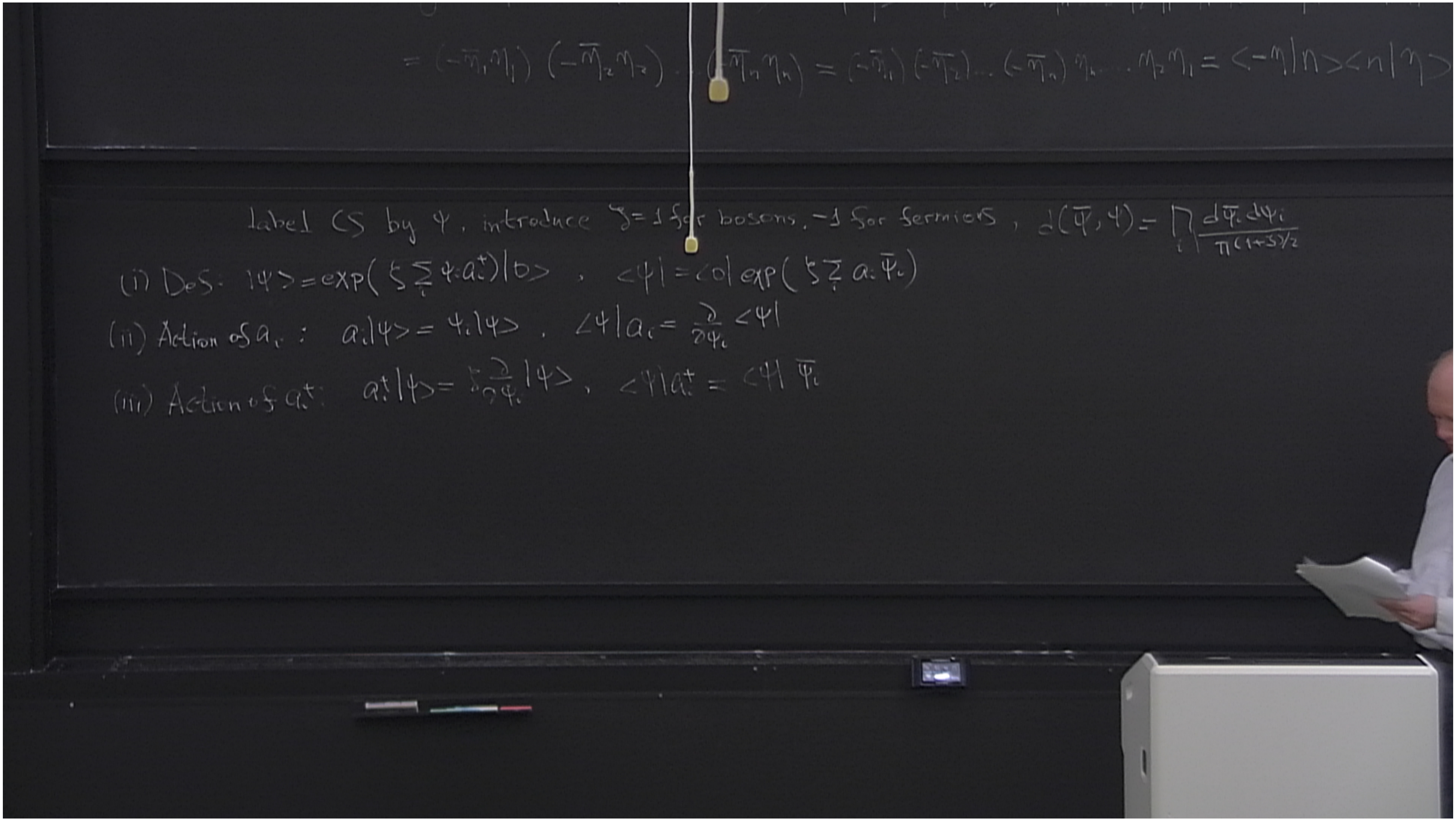
$= (-\bar\eta_1\eta_1)(-\bar\eta_2\eta_2)\ldots(-\bar\eta_n\eta_n)$
$(-\bar\eta_2)\ldots(-\bar\eta_n)\,\eta_n \ldots \eta_2\eta_1 = \langle -\eta|n\rangle\langle n|\eta\rangle$


$= (-\bar{\eta}_1\eta_1)(-\bar{\eta}_2\eta_2)\dots(-\bar{\eta}_n\eta_n) = (-\bar{\eta}_1)(-\bar{\eta}_2)\dots(-\bar{\eta}_n)\eta_n\dots\eta_2\eta_1 = \langle -\eta|n\rangle\langle n|\eta\rangle$
label CS by ψ, introduce $\zeta = 1$ for bosons, -1 for fermions, $d(\bar{\psi},\psi) = \prod_i \frac{d\bar{\psi}_i\, d\psi_i}{\pi^{(1+\zeta)/2}}$



$= (-\bar\eta_1\eta_1)(-\bar\eta_2\eta_2)\cdots(-\bar\eta_n\eta_n) = (-\bar\eta_1)(-\bar\eta_2)\cdots(-\bar\eta_n)\,\eta_n\cdots\eta_2\eta_1 = \langle -\eta|n\rangle\langle n|\eta\rangle$
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(i) Def: $|\psi\rangle = \exp(\zeta\sum_i \psi_i a_i^\dagger)|0\rangle$, $\langle\psi| = \langle 0|\exp(\zeta\sum_i a_i\bar\psi_i)$
(ii) Action of a_i: $a_i|\psi\rangle = \psi_i|\psi\rangle$, $\langle\psi|a_i = \frac{\partial}{\partial\bar\psi_i}\langle\psi|$
(iii) Action of $a_i^\dagger$: $a_i^\dagger|\psi\rangle = \zeta\frac{\partial}{\partial\psi_i}|\psi\rangle$, $\langle\psi|a_i^\dagger = \langle\psi|\bar\psi_i$

$$= (-\bar{\eta}_1\eta_1)(-\bar{\eta}_2\eta_2)\cdots(-\bar{\eta}_n\eta_n) = (-\bar{\eta}_1)(-\bar{\eta}_2)\cdots(-\bar{\eta}_n)\,\eta_n\cdots\eta_2\eta_1 = \langle -\eta | n\rangle\langle n|\eta\rangle$$

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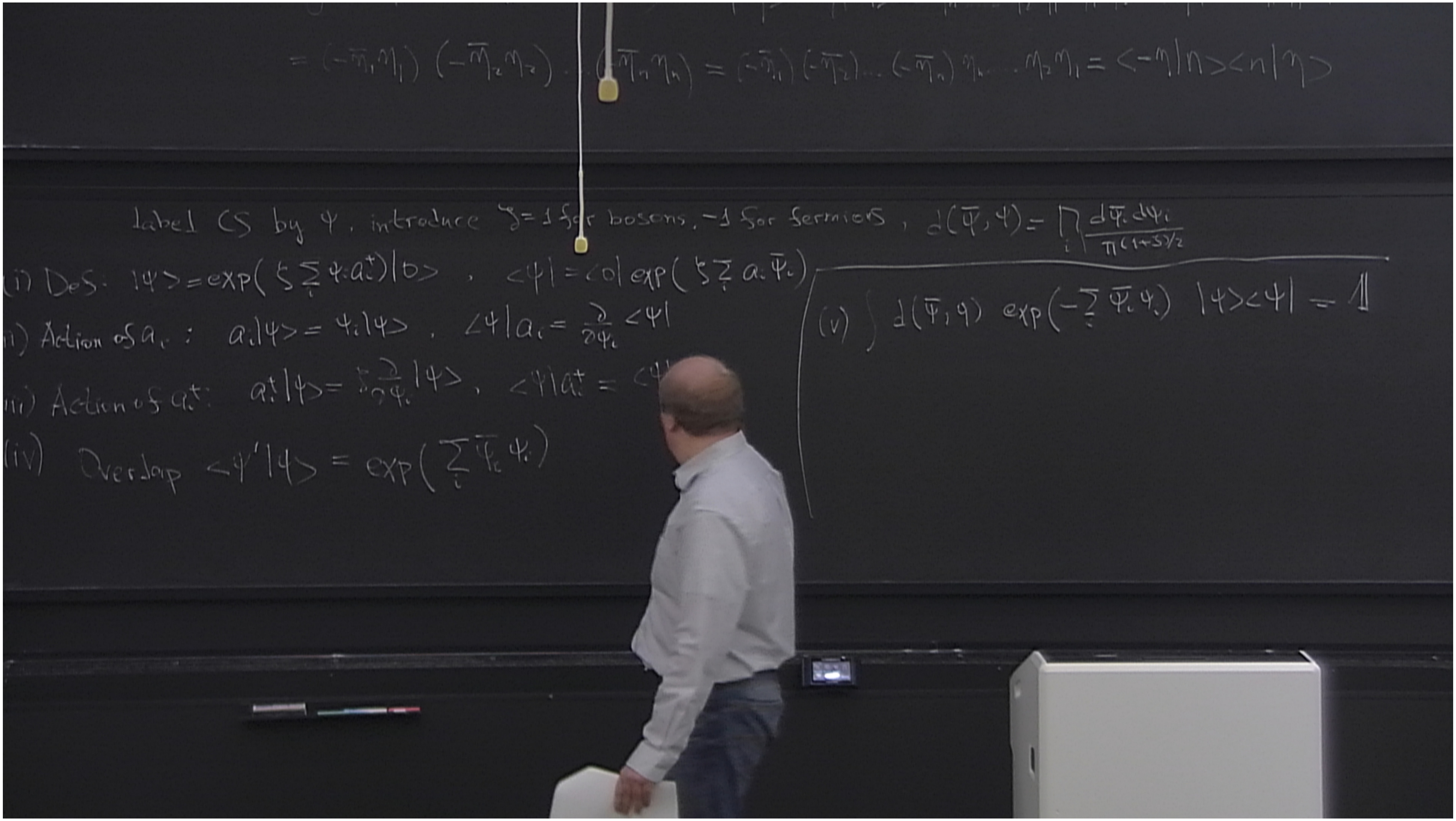

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(iv) Overlap $\langle\psi'|\psi\rangle = \exp(\sum_i \bar\psi_i \psi_i)$
(v) $\int d(\bar\psi,\psi)\, \exp(-\sum_i \bar\psi_i\psi_i)\, |\psi\rangle\langle\psi| = \mathbb{1}$

$= (-\bar\eta_1\eta_1)(-\bar\eta_2\eta_2)\cdots(-\bar\eta_n\eta_n) = (-\bar\eta_1)(-\bar\eta_2)\cdots(-\bar\eta_n)\,\eta_n\cdots\eta_2\eta_1 = \langle -\eta|n\rangle\langle n|\eta\rangle$

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a_i : $a_i|\psi\rangle = \psi_i|\psi\rangle$, $\langle\psi|a_i = \frac{\partial}{\partial\psi_i}\langle\psi|$

of $a_i^\dagger$: $a_i^\dagger|\psi\rangle = \zeta\frac{\partial}{\partial\psi_i}|\psi\rangle$, $\langle\psi|a_i^\dagger = \langle\psi|\bar\psi_i$

lap $\langle\psi'|\psi\rangle = \exp(\sum_i \bar\psi'_i\psi_i)$

(v) $\int d(\bar\psi,\psi)\exp(-\sum_i \bar\psi_i\psi_i)\,|\psi\rangle\langle\psi| = \mathbb{1}$

(vi) $\langle n|\psi\rangle\langle\psi|n\rangle = \langle\zeta\psi|n\rangle\langle n|\psi\rangle$

$$= (-\bar\eta_1\eta_1)(-\bar\eta_2\eta_2)\ldots(-\bar\eta_n\eta_n) = (-\bar\eta_1)(-\bar\eta_2)\ldots(-\bar\eta_n)\,\eta_n\ldots\eta_2\eta_1 = \langle -\eta|n\rangle\langle n|\eta\rangle$$

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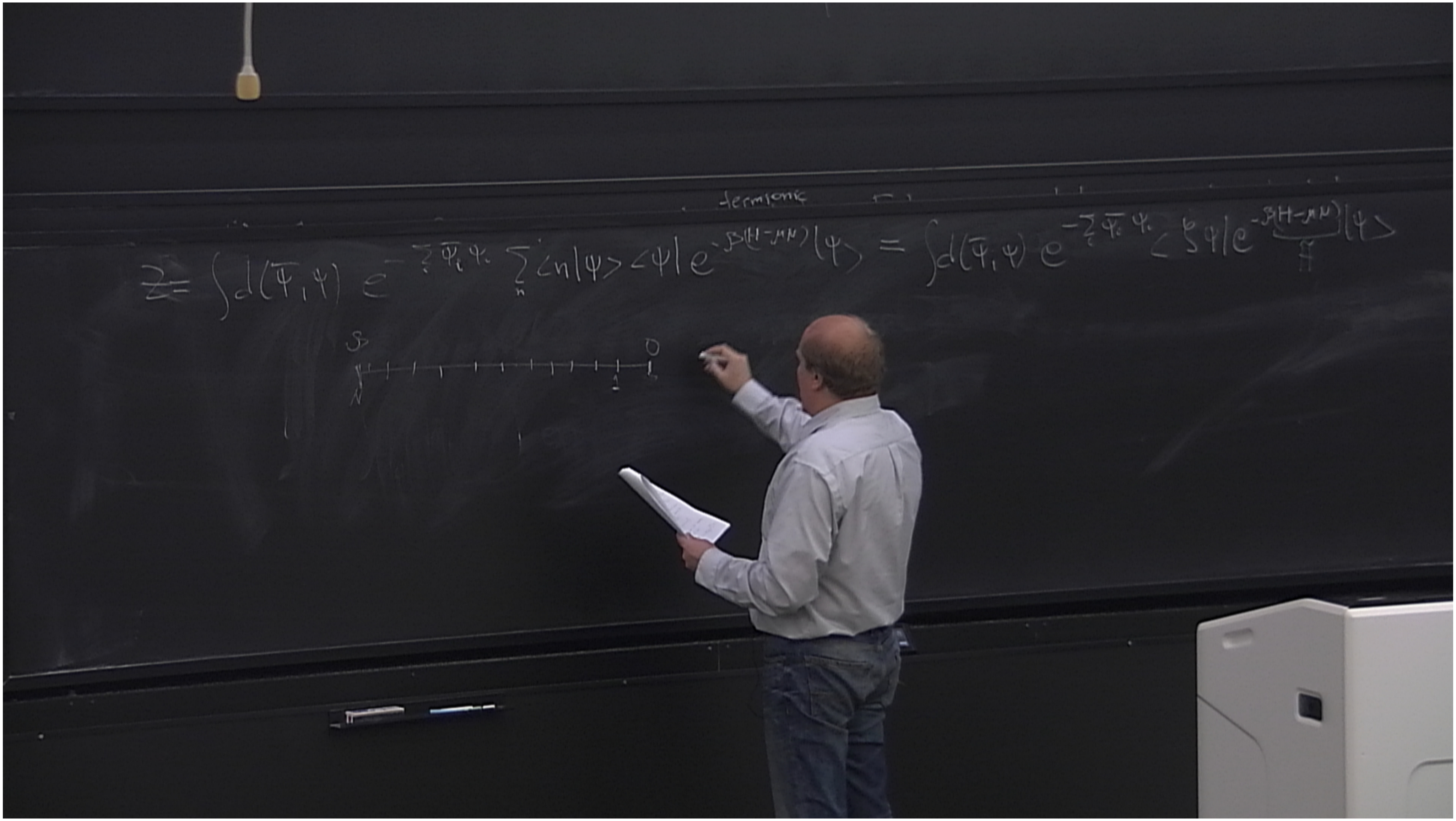
lap $\langle\psi'|\psi\rangle = \exp(\sum_i \bar\psi_i'\psi_i)$

(v) $\int d(\bar\psi,\psi)\exp(-\sum_i \bar\psi_i\psi_i)\,|\psi\rangle\langle\psi| = \mathbb{1}$

(vi) $\langle n|\psi\rangle\langle\psi|n\rangle = \langle\zeta\psi|n\rangle\langle n|\psi\rangle$


fermionic

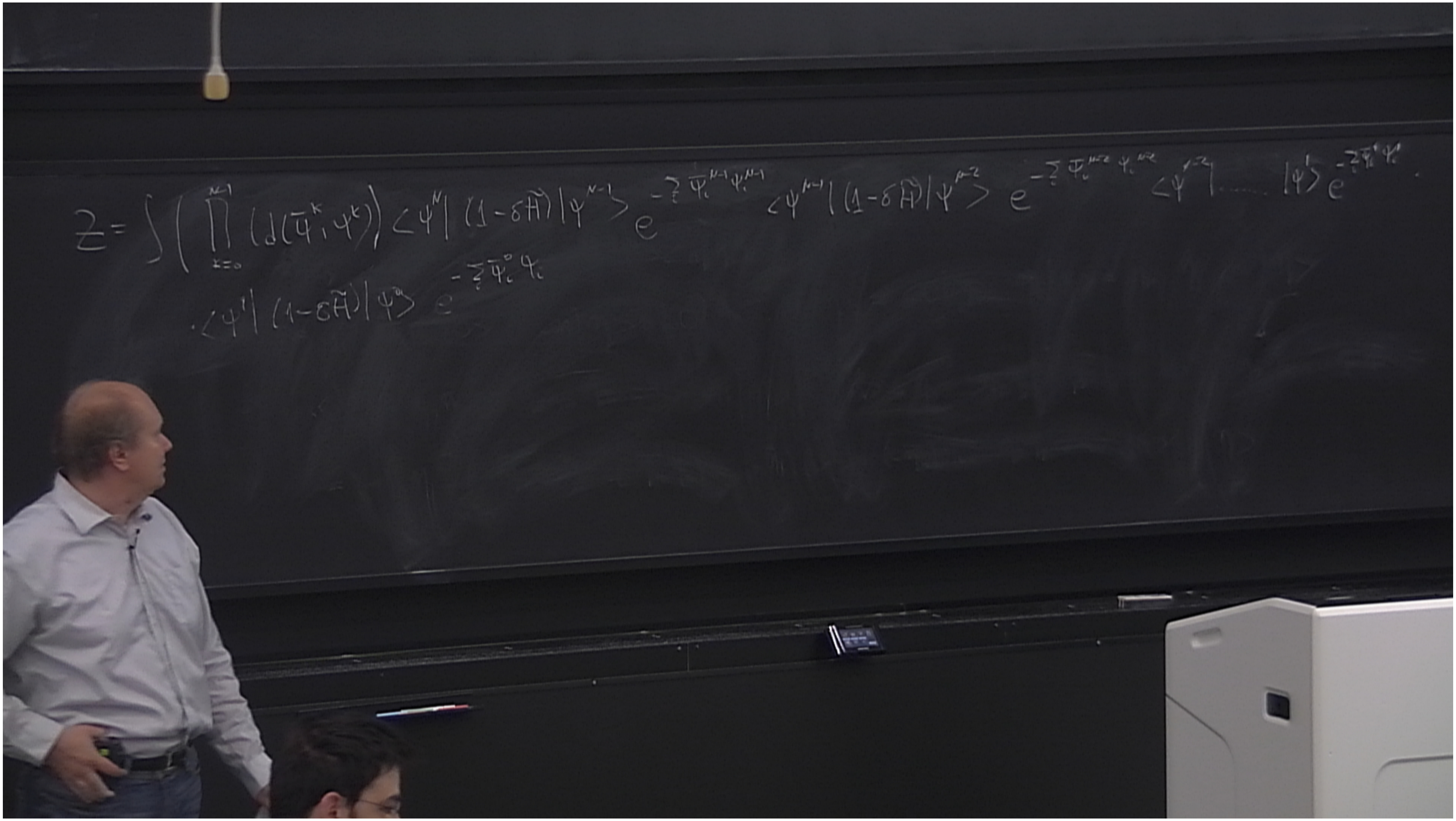
fermionic

$$Z = \int d(\bar\psi, \psi)\, e^{-\sum_i \bar\psi_i \psi_i} \sum_n \langle n|\psi\rangle\langle\psi| e^{-\beta(H-\mu N)}|\psi\rangle = \int d(\bar\psi,\psi)\, e^{-\sum_i \bar\psi_i \psi_i} \langle \zeta\psi| e^{-\beta\underbrace{(H-\mu N)}_{\hat H}}|\psi\rangle$$

β $\quad$ 0

N $\quad$ 1 $\quad$ $\delta = \frac{\beta}{N}$

$$\langle \zeta\psi| e^{-\beta \hat H}|\psi\rangle = \langle \zeta\psi| \lim_{N\to\infty} (e$$


$Z = \int \left(\prod_{k=0}^{N-1} d(\bar{\psi}^k, \psi^k) \right) \langle \psi^N | (1 - \delta \tilde{H}) | \psi^{N-1} \rangle e^{-\sum_i \bar{\psi}_i^{N-1} \psi_i^{N-1}} \langle \psi^{N-1} | (1 - \delta \tilde{H}) | \psi^{N-2} \rangle e^{-\sum_i \bar{\psi}_i^{N-2} \psi_i^{N-2}} \langle \psi^{N-2} | \ldots\ldots | \psi^1 \rangle e^{-\sum_i \bar{\psi}_i^1 \psi_i^1}$
$\cdot \langle \psi^1 | (1 - \delta \tilde{H}) | \psi^0 \rangle e^{-\sum_i \bar{\psi}_i^0 \psi_i}$