Title: PSI 2015/2016 Condensed Matter - Oleg Tchernyshyov - Lecture 6

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Abstract:

Why path integrals in CM:

1) Efficient formulation of non-perturbative methods
2) Classical limit is more visible
3) New concepts are easily studied: topology. e.g

(Quant. statist. for bosons)

2) Classical limit is more visible

3) New concepts are easily studied: topology. eg

Coherent states for bosons

$a_i, a_i^\dagger$ that are bosonic, satisfy $[a_i, a_j^\dagger] = \delta_{ij}$

Can we find eigenstates of $a_i, a_i^\dagger$?

$$|n_1, n_2 \ldots\rangle = \frac{(a_1^\dagger)^{n_1}}{\sqrt{n_1!}} \frac{(a_2^\dagger)^{n_2}}{\sqrt{n_2!}} \cdots |0\rangle$$

Any state $|\phi\rangle = \sum_{n_1, n_2} C_{n_1, n_2 \ldots} |n_1, n_2, \ldots\rangle$,

$|\psi\rangle = \exp\left(\sum_i \phi_i a_i^\dagger\right)|0\rangle$, $\{\phi_i\}$ is a set of complex numbers

(i) $a_i |\phi\rangle = \phi_i |\phi\rangle$ $|\phi\rangle$ are coherent states for bosons

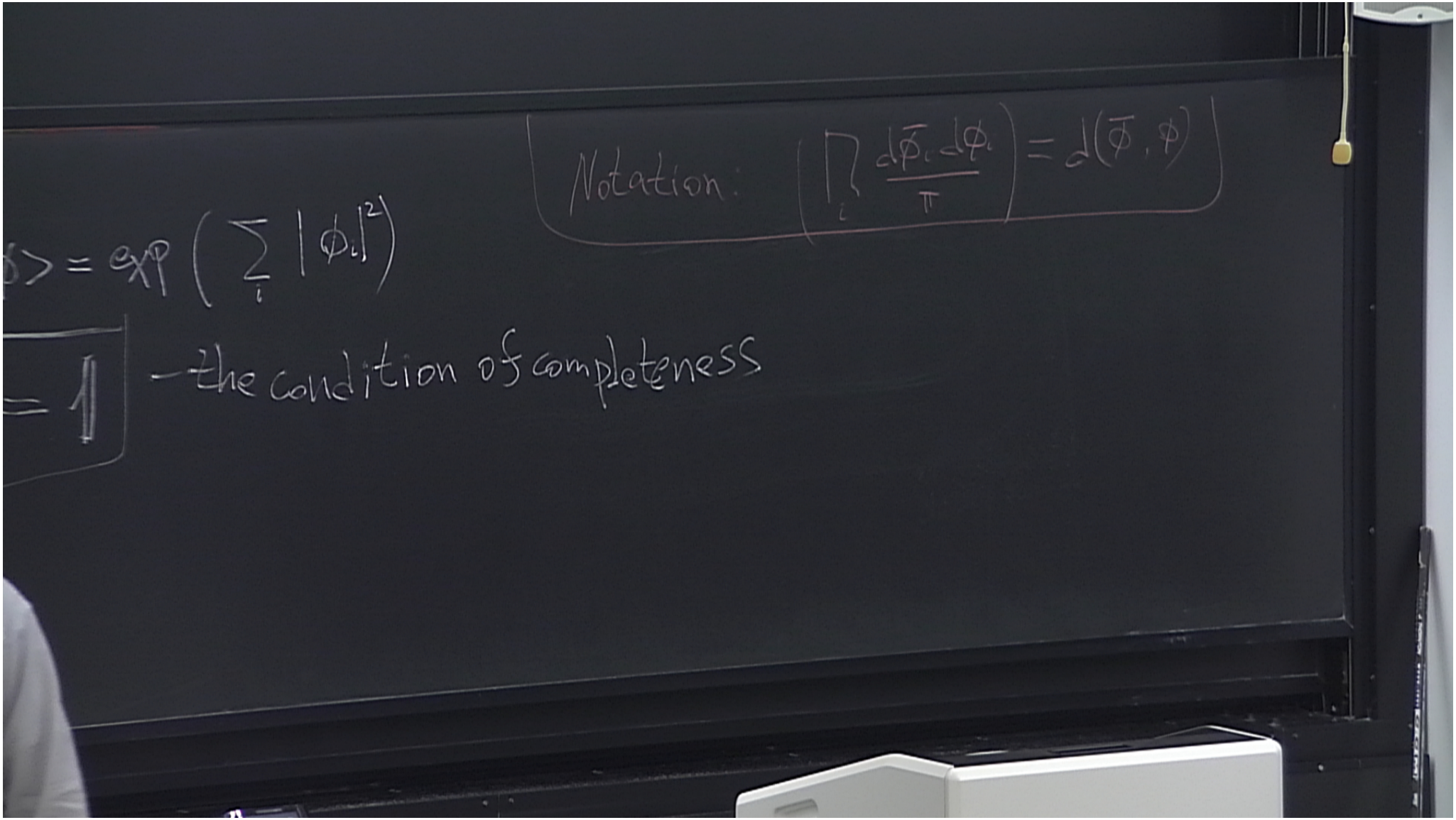
Construct $\langle\phi| = \langle 0| \exp\left(\sum_i \bar{\phi}_i a_i\right)$, where $\bar{\phi}_i$ are complex conjugates of ϕ_i

(ii) $\langle\phi| a_i^\dagger = \langle\phi| \bar{\phi}_i$

(iii) Two coherent states $|\theta\rangle$ and $|\phi\rangle$

$$\langle\theta|\phi\rangle = \exp\left(\sum_i \bar{\theta}_i \phi_i\right) \Rightarrow \langle\phi|\phi\rangle = \exp\left(\sum_i |\phi_i|^2\right)$$

(iv) $$\int \left(\prod_i \frac{d\bar{\phi}_i\, d\phi_i}{\pi}\right) \exp\left(-\sum_i |\phi_i|^2\right) |\phi\rangle\langle\phi| = \mathbb{1}$$ – the condition


Notation: $\left(\prod_i \frac{d\bar{\phi}_i d\phi_i}{\pi}\right) = d(\bar{\phi}, \phi)$
$\rangle = \exp\left(\sum_i |\phi_i|^2\right)$
$= 1$ – the condition of completeness

Grassmann Variables

$\eta_i \eta_j + \eta_j \eta_i = 0 \quad , \quad \eta_i^2 = 0$

$f(\eta) = f(0) + \frac{df}{d\eta}\eta \quad , \quad f(\eta_1, \eta_2) = f(0,0) + \left.\frac{\partial f}{\partial \eta_1}\right|_{\eta_i=0} \eta_1 + \left.\frac{\partial f}{\partial \eta_2}\right|_{\eta_i=0} \eta_2 + \frac{1}{2}\left.\frac{\partial^2 f}{\partial \eta_1 \partial \eta_1}\right|_{\eta_i=0} \eta_1 \eta_2$

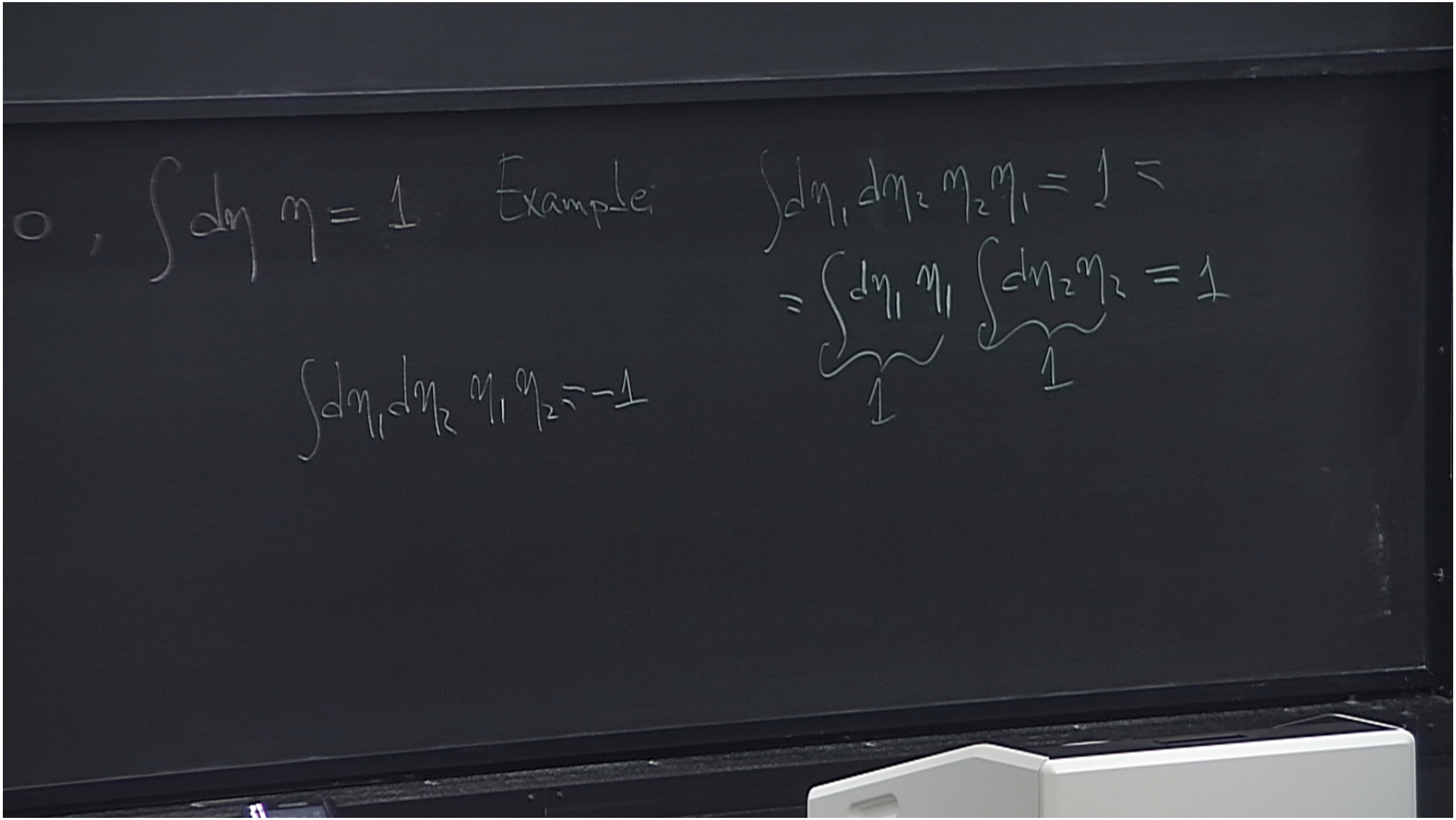
$\{\ldots\}$ ← anticommutator ⟶ $\left\{\frac{\partial}{\partial \eta_i}, \frac{\partial}{\partial \eta_j}\right\} = 0$, $\left\{\frac{\partial}{\partial \eta_i}, \eta_j\right\} = \delta_{ij}$

Define: $\int d\eta\, f(\eta) = \frac{df}{d\eta}$

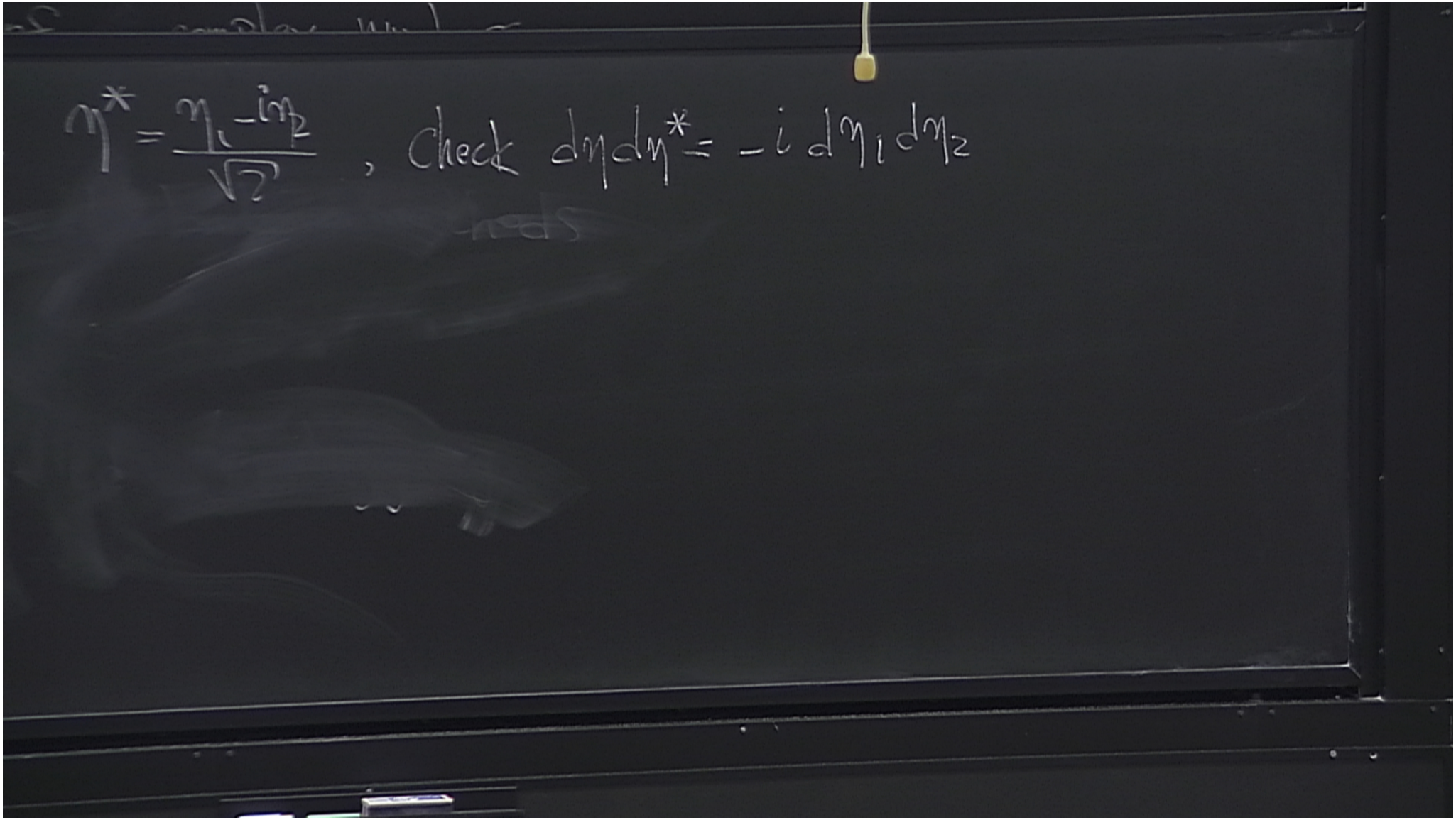
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Define: $\int d\eta\, f(\eta) = \frac{df}{d\eta} \Rightarrow \int d\eta = 0 \,,\; \int d\eta\, \eta = 1$

$$\int d\eta_1 d\eta_2 \dots d\eta_n\, \eta_n \dots \eta_2 \eta_1 = 1$$


0, $\int d\eta\, \eta = 1$ Example
$\int d\eta_1 d\eta_2\, \eta_1 \eta_2 = -1$
$\int d\eta_1 d\eta_2\, \eta_2 \eta_1 = 1 =$
$= \int d\eta_1\, \eta_1 \int d\eta_2\, \eta_2 = 1$
1
1

Linear combinations: - for η_1, η_2, $\eta = \frac{\eta_1 + i\eta_2}{\sqrt{2}}$, $\eta^* = \frac{\eta_1 - i\eta_2}{\sqrt{2}}$,


$\eta^* = \frac{\eta_1 - i\eta_2}{\sqrt{2}}$, Check $d\eta d\eta^* = -i\, d\eta_1 d\eta_2$

Linear combinations: - for η_1, η_2, $\eta = \frac{\eta_1 + i\eta_2}{\sqrt{2}}$, $\eta^* = \frac{\eta_1 -$

Consider $\int d\bar{\eta}\, d\eta\, e^{-\bar{\eta} a \eta} = \int d\bar{\eta}\, d\eta\, (1 - \bar{\eta} a \eta) = -\int d\bar{\eta} d\eta\, \bar{\eta}$

a - complex number

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a - complex number

Note: $\int d\bar{\eta}\, d\eta\, \eta\bar{\eta}\, e^{-\bar{\eta} a \eta} = \int d\bar{\eta} d\eta\, \eta\bar{\eta} = 1 = \frac{1}{a} \cdot a$

a - complex number

Note: $\int d\bar{\eta}\, d\eta\; \eta\bar{\eta}\, e^{-\bar{\eta} a \eta} = \int d$

$$I = \int \left(\prod_i d\bar{\eta}_i\, d\eta_i \right) \exp\left[-\sum_i \bar{\eta}_i f_i \eta_i \right] = \int \left(\prod_i d\bar{\eta}_i\, d\eta_i \right) \left[\prod_i e^{-\bar{\eta}_i f_i \eta_i} \right] = \prod_i f_i$$

f_i - complex numbers

Gaussian integration over Grassmann variables

$$J(\vec{A}, \vec{B}, \vec{B}^*) = \int \left(\prod_{i=1}^{n} d\bar{\eta}_i\, d\eta_i \right) \exp\left[-\bar{\eta}_i A_{ij} \eta_j + B_i^* \eta_i + B_i \bar{\eta}_i \right]$$

Gaussian integration over Grassmann variables

$$\int(\vec{A},\vec{B},\vec{B}^*) = \int\left(\prod_{i=1}^{n} d\bar{\eta}_i\, d\eta_i\right) \exp\left[-\bar{\eta}_i A_{ij}\eta_j + B_i^*\eta_i + B_i\bar{\eta}_i\right]$$

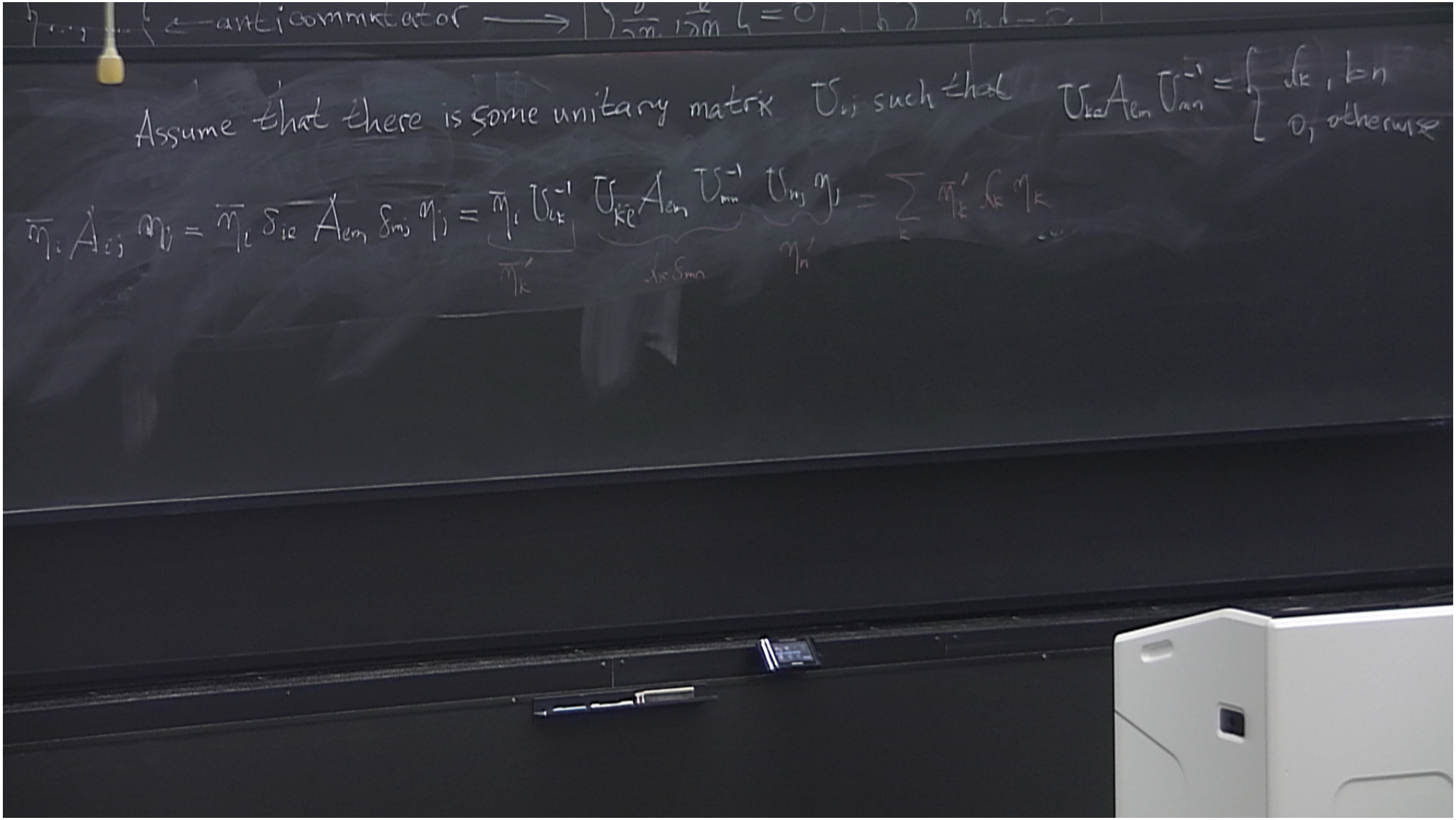
A_{ij} – Hermitean matrix, $\vec{B}=\{B_i\}$, $\vec{B}^*=\{B_i^*\}$. To proceed we make the linear transformations

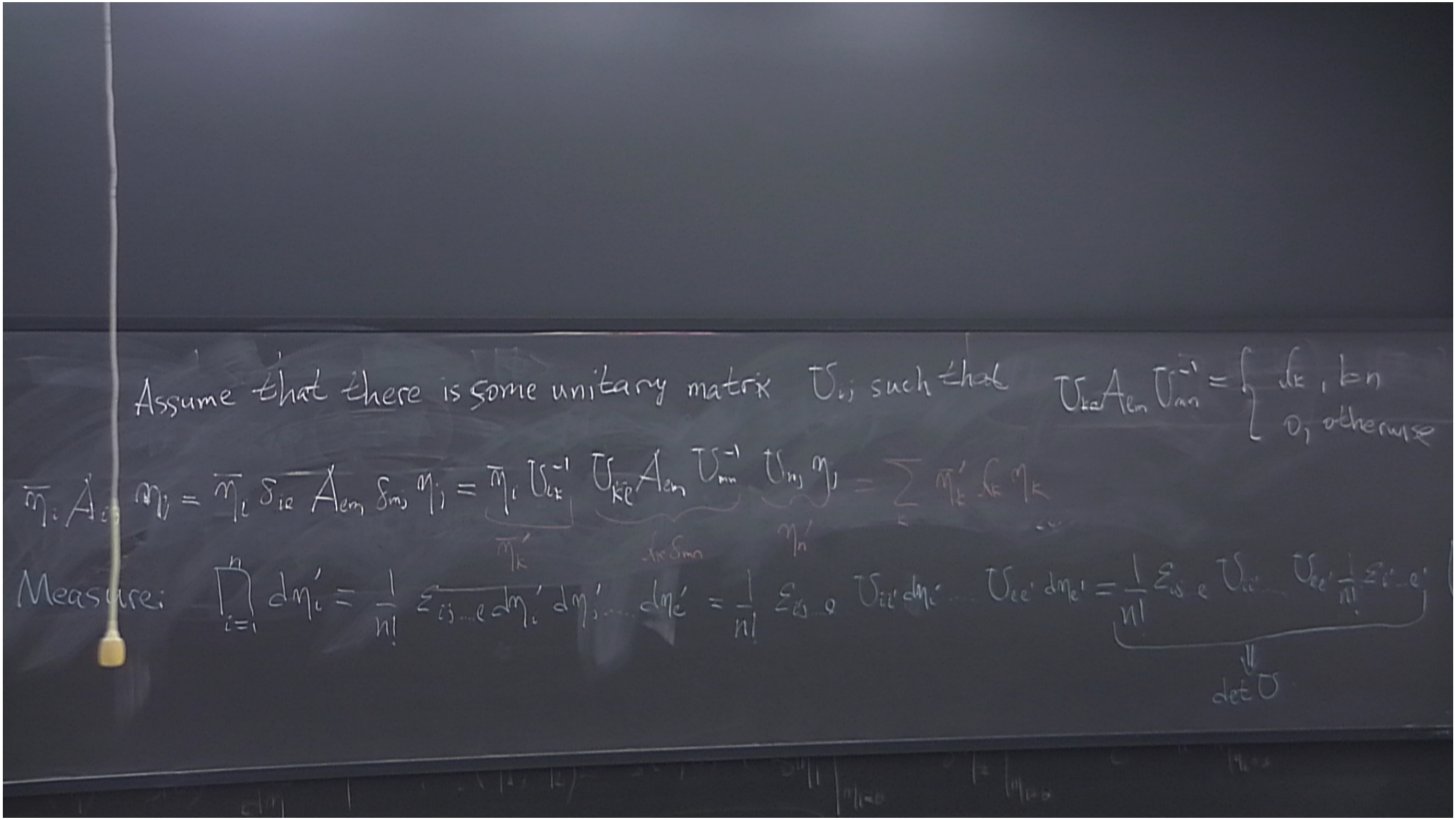
$$\eta_i \longrightarrow \eta_i - A^{-1}_{ij}B_j, \quad \bar{\eta}_i \longrightarrow \bar{\eta}_i + A^{-1}_{ji}B_j^*$$

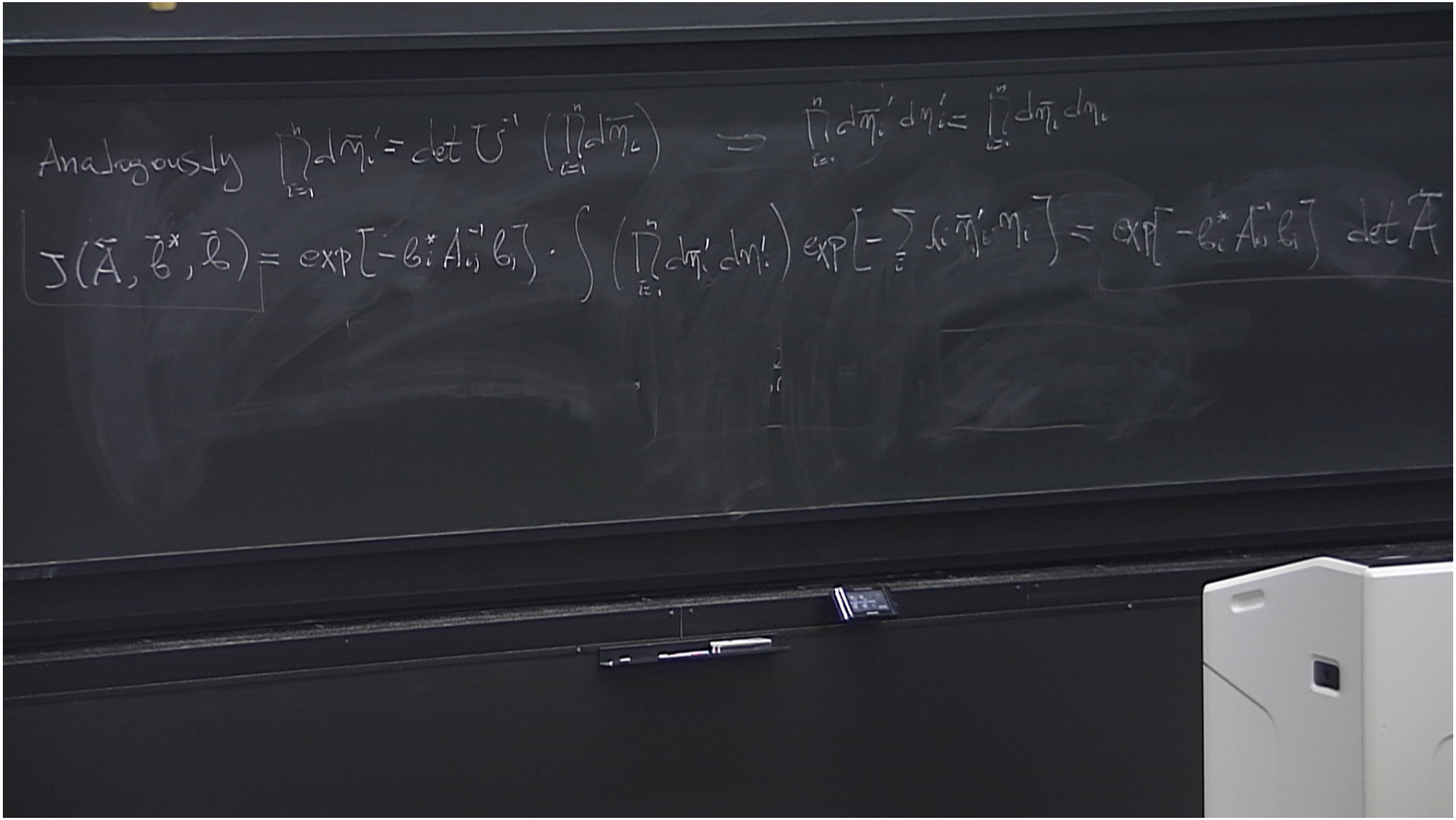
$$A_{ij}\eta_j + B_i^*\eta_i + B_i\bar{\eta}_i \longrightarrow -(\bar{\eta}_i + A^{-1}_{ki}B_k^*)A_{ij}(\eta_j - A^{-1}_{je}B_e) + B_i^*(\eta_i - A^{-1}_{ij}B_j) + B_i(\bar{\eta}_i + A^{-1}_{ji}B_j^*) =$$

$$= -\bar{\eta}_i A_{ij}\eta_j + \cancel{\bar{\eta}_i B_i} - \cancel{B_k^*\eta_k} + B_k^* A^{-1}_{ke} B_e + \cancel{B_i^*\eta_i} + \cancel{B_i\bar{\eta}_i} - B_i^* A^{-1}_{ij} B_j - B_j^* A^{-1}_{ji} B_i \qquad A^{-1}_{ki}A_{ij} = \delta_{kj}$$

$$= -\bar{\eta}_i A_{ij}\eta_j - B_i^* A^{-1}_{ij} B_j$$


anticommutator
Assume that there is some unitary matrix U_{ij} such that $U_{ki} A_{em} U^{-1}_{mn} = \begin{cases} \delta_{ik}, & k=n \\ 0, & \text{otherwise} \end{cases}$
$\bar{\eta}_i A_{ij} \eta_j = \bar{\eta}_i \delta_{ik} \bar{A}_{em} \delta_{mj} \eta_j = \bar{\eta}_i U^{-1}_{ik} U_{ke} A_{em} U^{-1}_{mn} U_{nj} \eta_j = \sum_k \bar{\eta}'_k \delta_{kk} \eta'_k$
$\bar{\eta}'_k$
$\delta_{ik}\delta_{mn}$
η'_n


Assume that there is some unitary matrix U_{ij} such that $U_{ke} A_{em} U^{-1}_{mn} = \begin{cases} d_k, k=n \\ 0, \text{otherwise} \end{cases}$
$\bar{\eta}_i A_{ij} \eta_j = \bar{\eta}_i \delta_{ik} A_{em} \delta_{mj} \eta_j = \bar{\eta}_i U_{ik} U^{-1}_{ke} A_{em} U^{-1}_{mn} U_{nj} \eta_j = \sum_k \bar{\eta}'_k d_k \eta'_k$
$\bar{\eta}'_k$
$\delta_{ke} \delta_{mn}$
η'_n
Measure: $\prod_{i=1}^{n} d\eta'_i = \frac{1}{n!} \varepsilon_{ij\ldots e} d\eta'_i d\eta'_j \ldots d\eta'_e = \frac{1}{n!} \varepsilon_{ij\ldots e} U_{ii'} d\eta'_{i'} \ldots U_{ee'} d\eta'_{e'} = \frac{1}{n!} \varepsilon_{i'j'\ldots e'} \det U$


Analogously $\prod_{i=1}^{n} d\bar{\eta}_i' = \det U^{-1} \left(\prod_{i=1}^{n} d\bar{\eta}_i\right) \Rightarrow \prod_{i=1}^{n} d\bar{\eta}_i' d\eta_i' = \prod_{i=1}^{n} d\bar{\eta}_i d\eta_i$
$J(\bar{A}, \bar{b}^*, \bar{b}) = \exp\left[-b_i^* A_{ij}^{-1} b_j\right] \cdot \int \left(\prod_{i=1}^{n} d\eta_i' d\eta_i'\right) \exp\left[-\sum_i \bar{\eta}_i' \eta_i\right] = \exp\left[-b_i^* A_{ij}^{-1} b_j\right] \det \bar{A}$