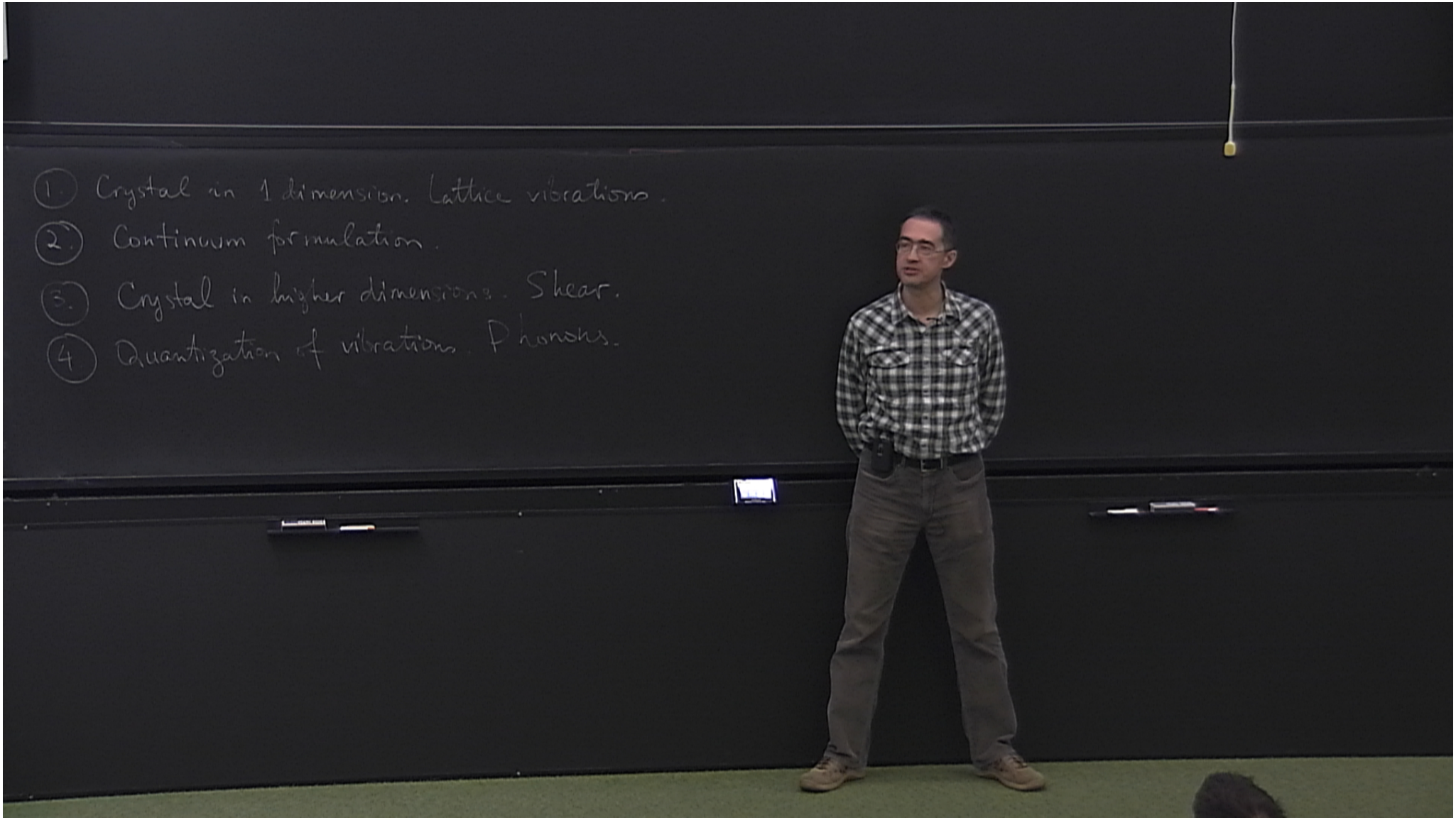
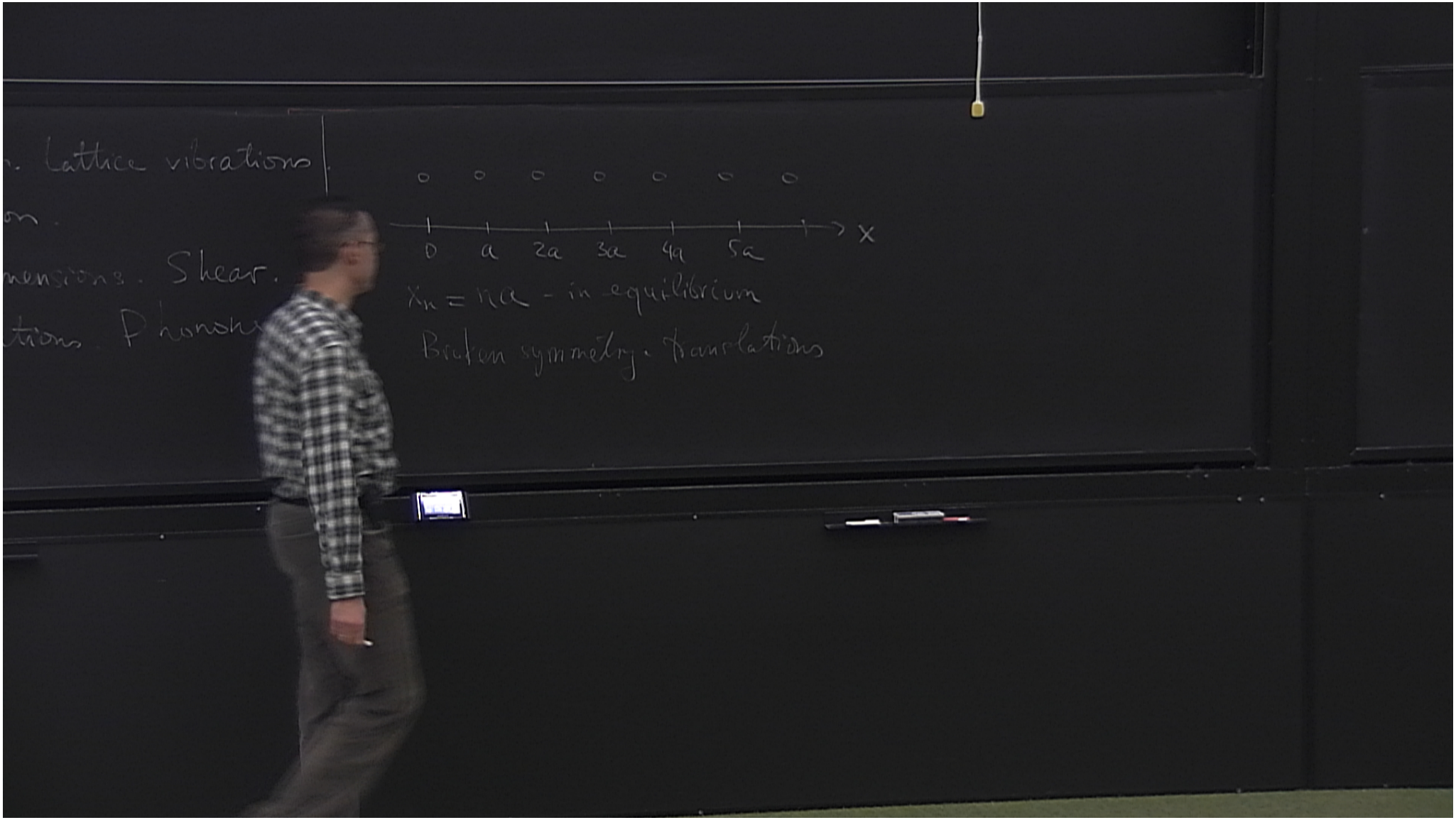
Title: PSI 2015/2016 Condensed Matter - Oleg Tchernyshyov - Lecture 1

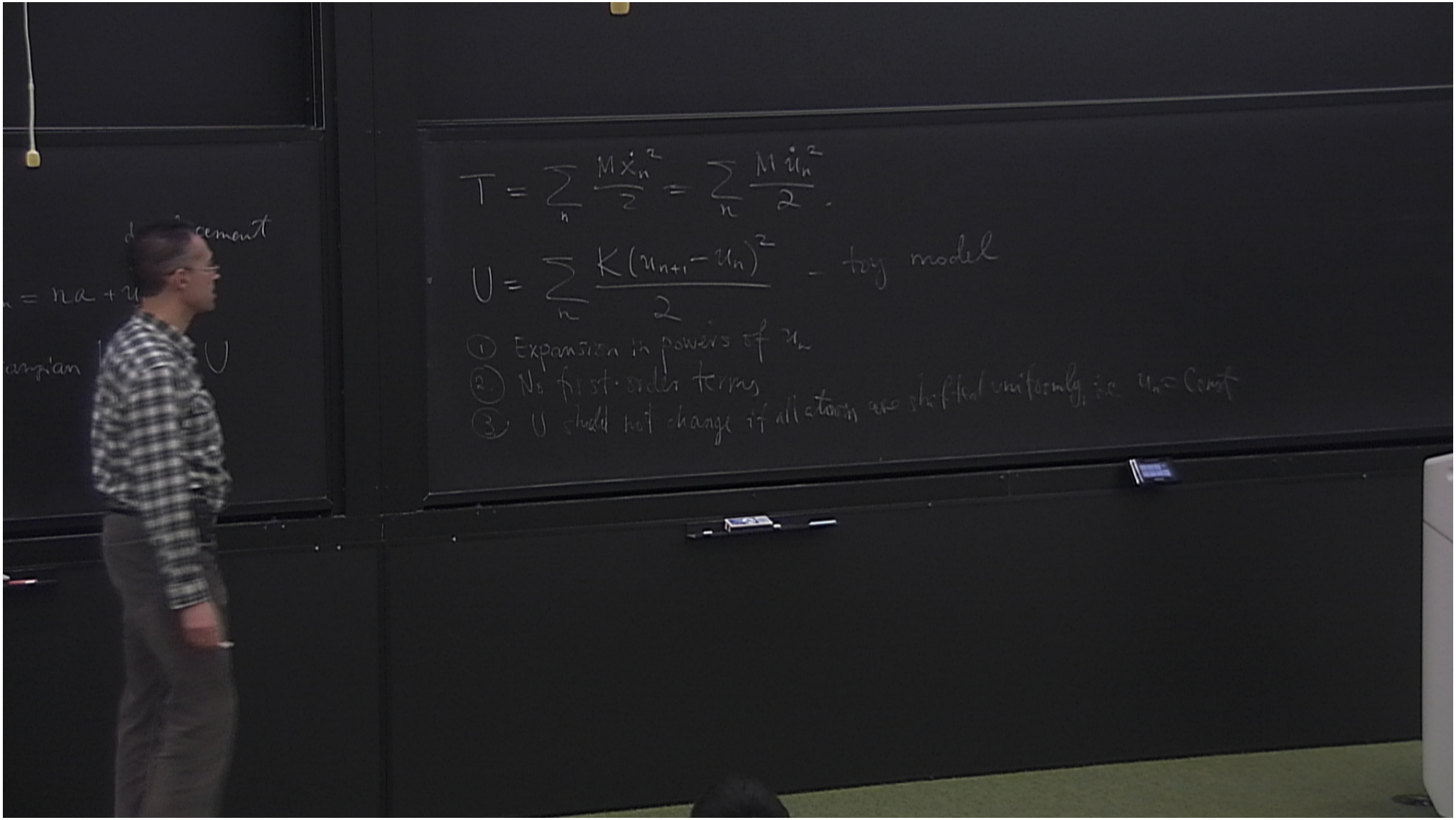
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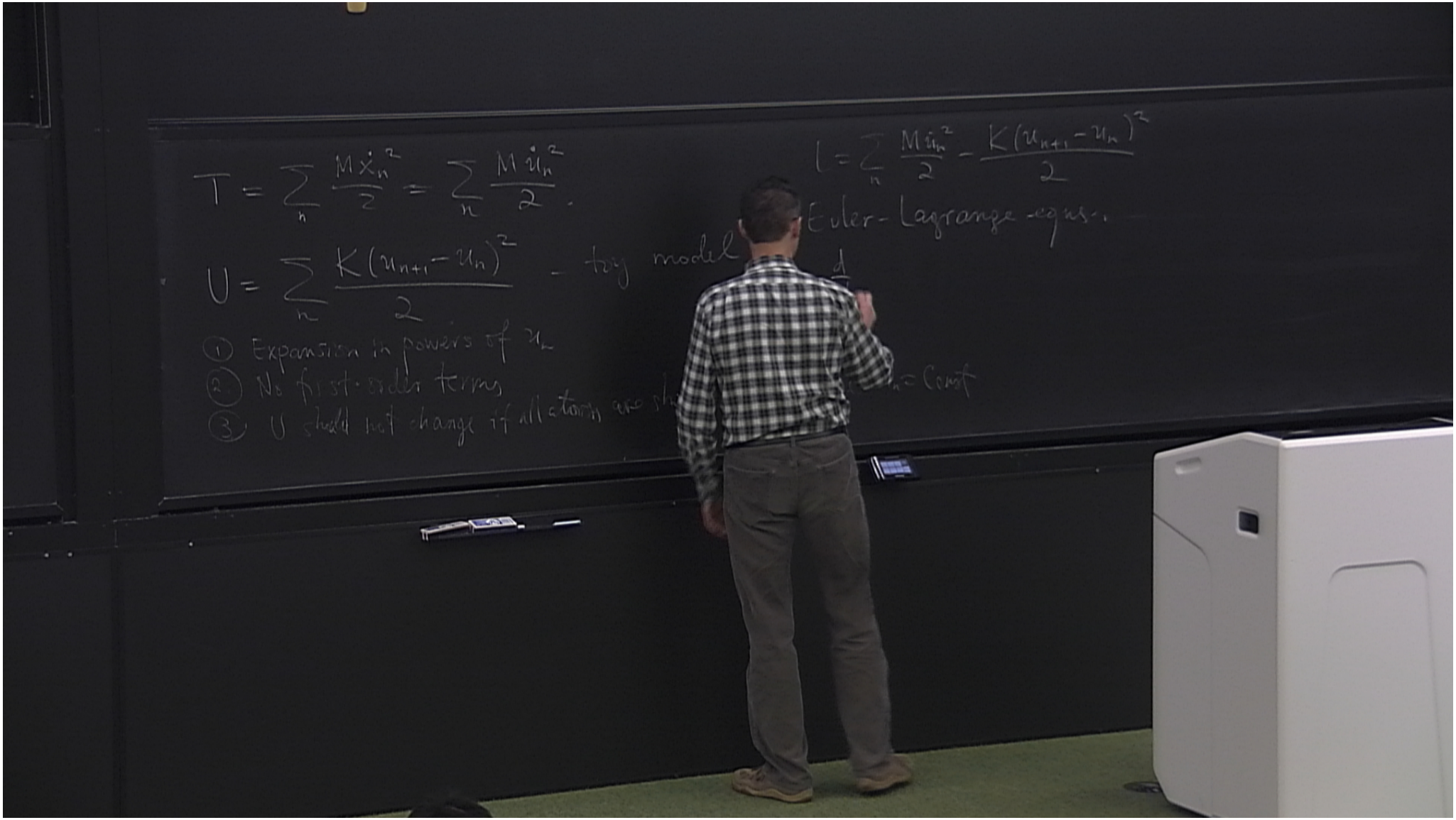
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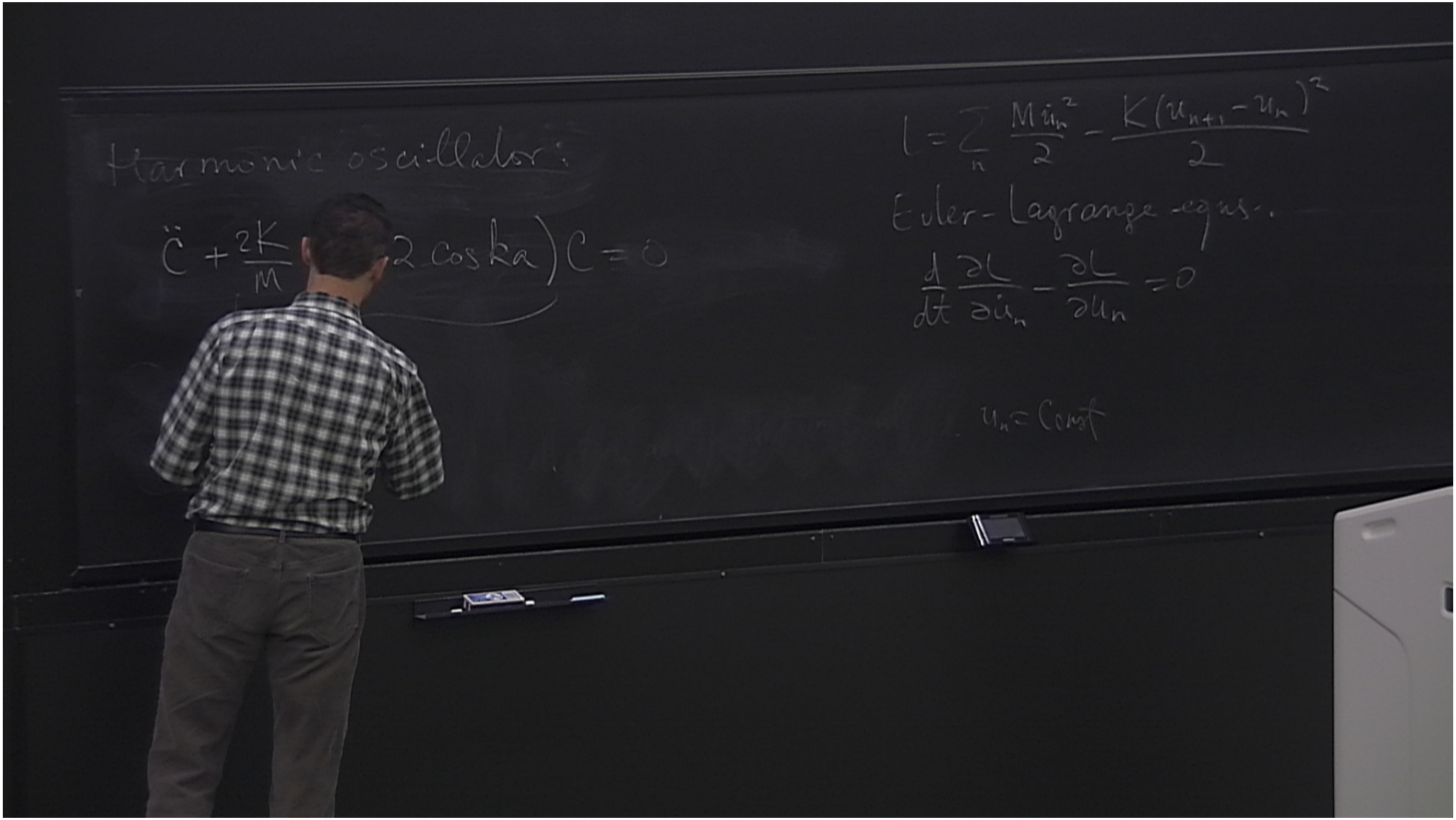
Abstract:

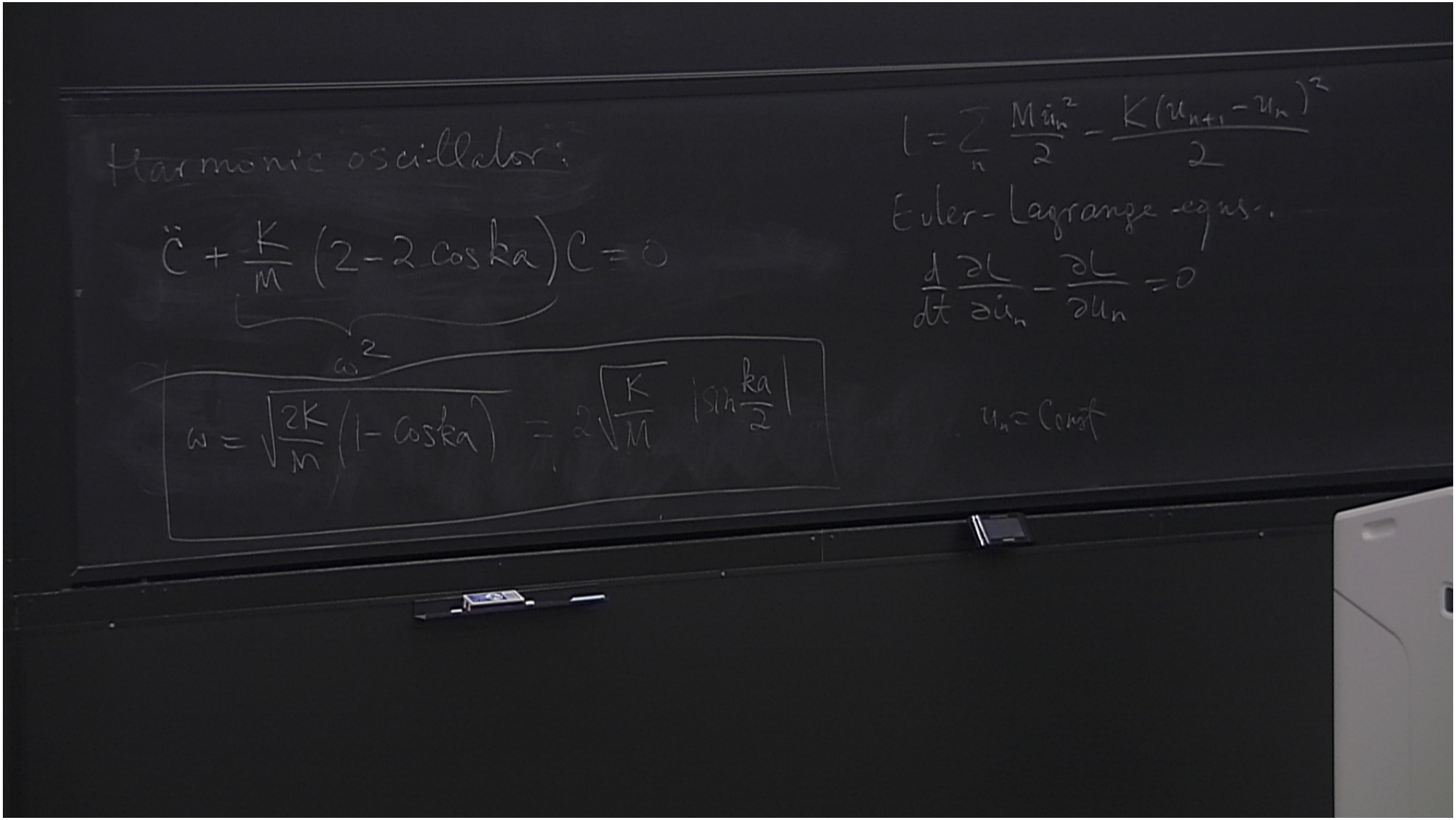

1. Crystal in 1 dimension. Lattice vibrations.
2. Continuum formulation.
3. Crystal in higher dimensions. Shear.
4. Quantization of vibrations. Phonons.

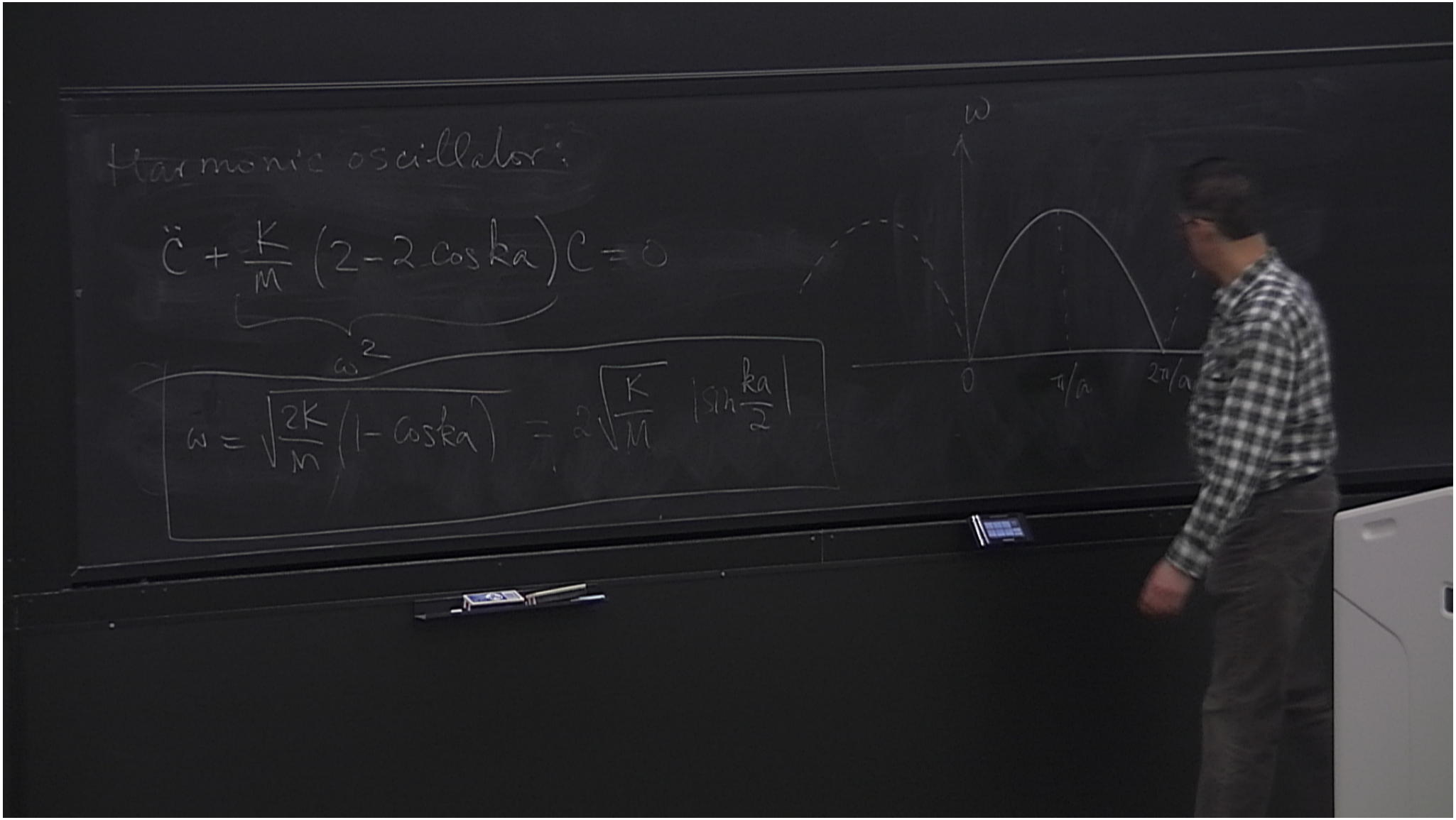

. Lattice vibrations
mensions. Shear.
tions. Phonons
0 a 2a 3a 4a 5a
X
$x_n = na$ - in equilibrium
Bravais symmetry. Translations

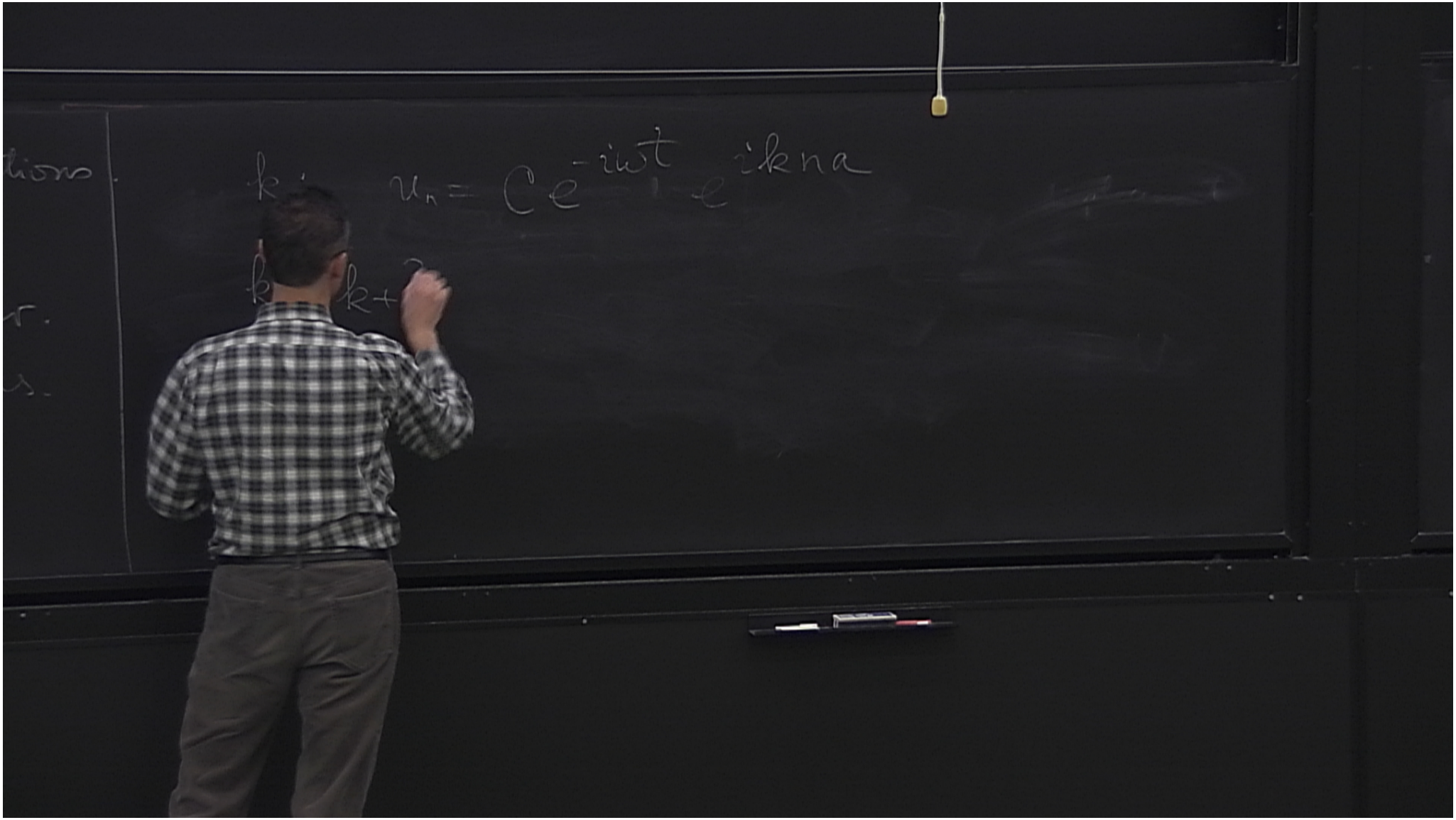

$T = \sum_n \frac{M\dot{x}_n^2}{2} = \sum_n \frac{M\dot{u}_n^2}{2}$
$U = \sum_n \frac{K(u_{n+1} - u_n)^2}{2}$ - toy model
1. Expansion in powers of u_n
2. No first-order terms
3. U should not change if all atoms are shifted uniformly, i.e. $u_n = \text{Const}$

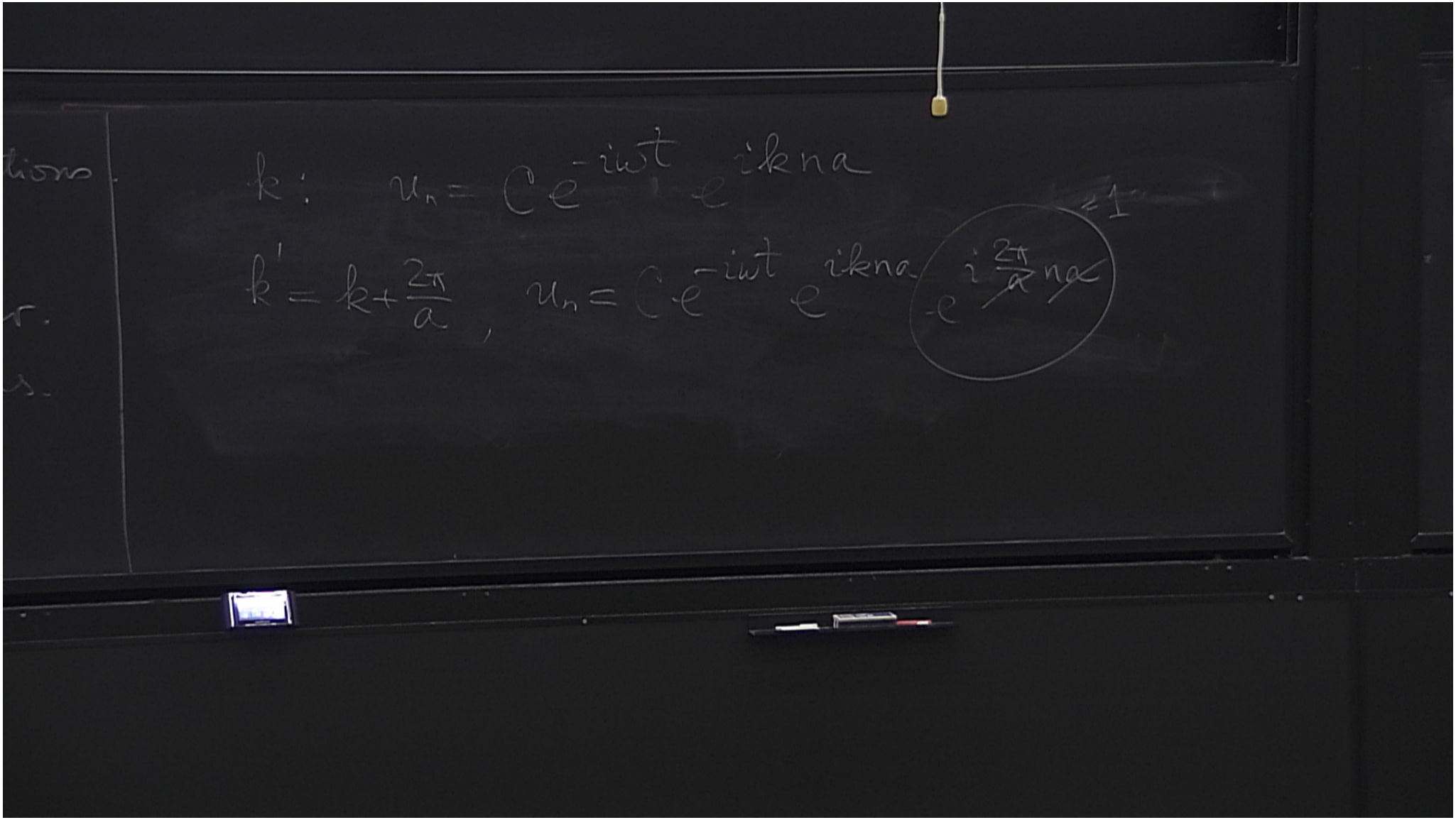

$T = \sum_n \frac{M\dot{x}_n^2}{2} = \sum_n \frac{M\dot{u}_n^2}{2}$
$U = \sum_n \frac{K(u_{n+1} - u_n)^2}{2}$ - toy model
$L = \sum_n \frac{M\dot{u}_n^2}{2} - \frac{K(u_{n+1} - u_n)^2}{2}$
Euler-Lagrange eqns.
$\frac{d}{}$
$= \text{Const}$
① Expansion in powers of u_n
② No first-order terms
③ U should not change if all atoms are

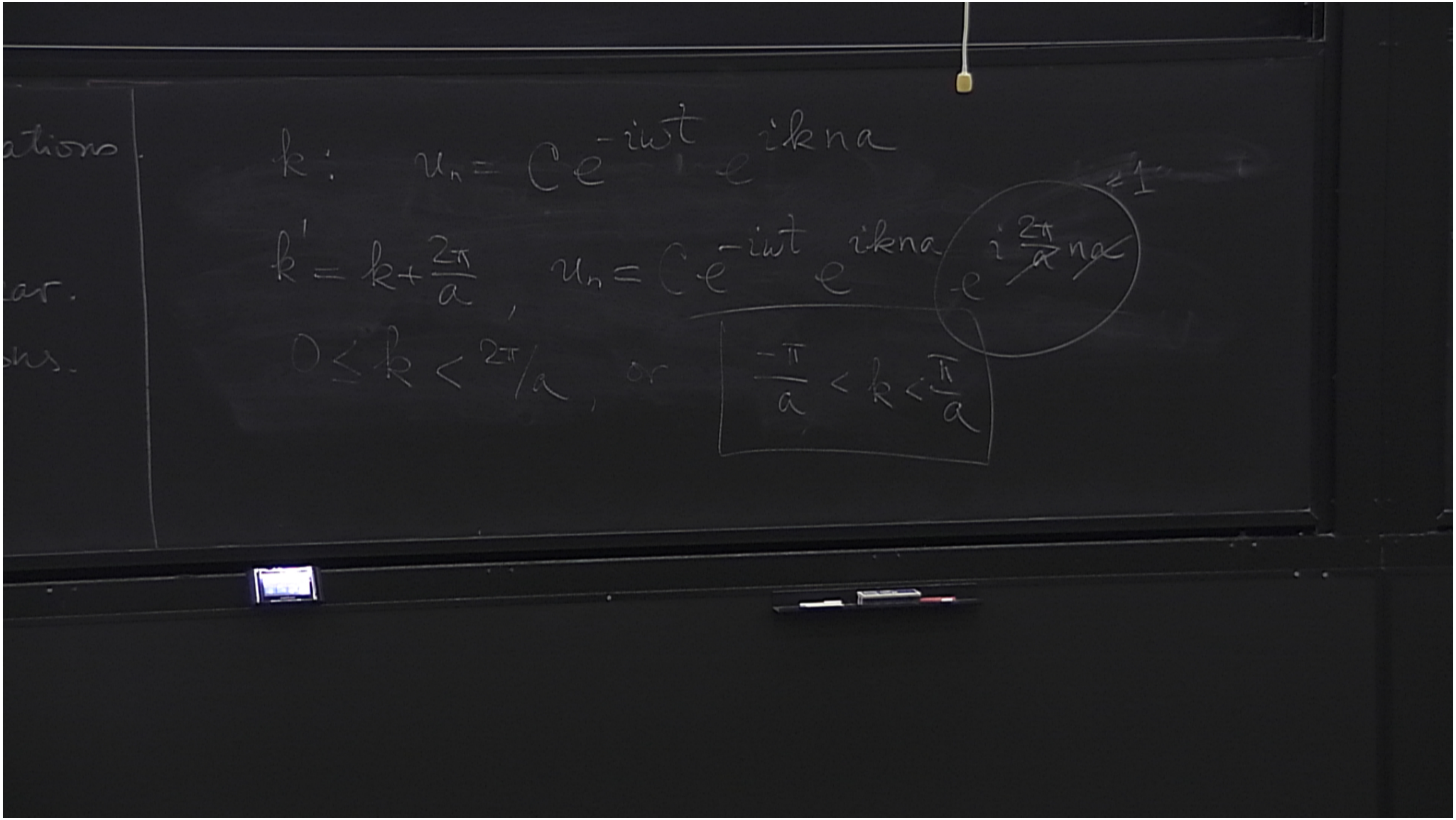

Harmonic oscillator:
$\ddot{C} + \frac{2K}{M}(\quad - 2\cos ka)C = 0$
$L = \sum_n \frac{M\dot{u}_n^2}{2} - \frac{K(u_{n+1} - u_n)^2}{2}$
Euler-Lagrange eqns.:
$\frac{d}{dt}\frac{\partial L}{\partial \dot{u}_n} - \frac{\partial L}{\partial u_n} = 0$
$u_n = \text{Const}$

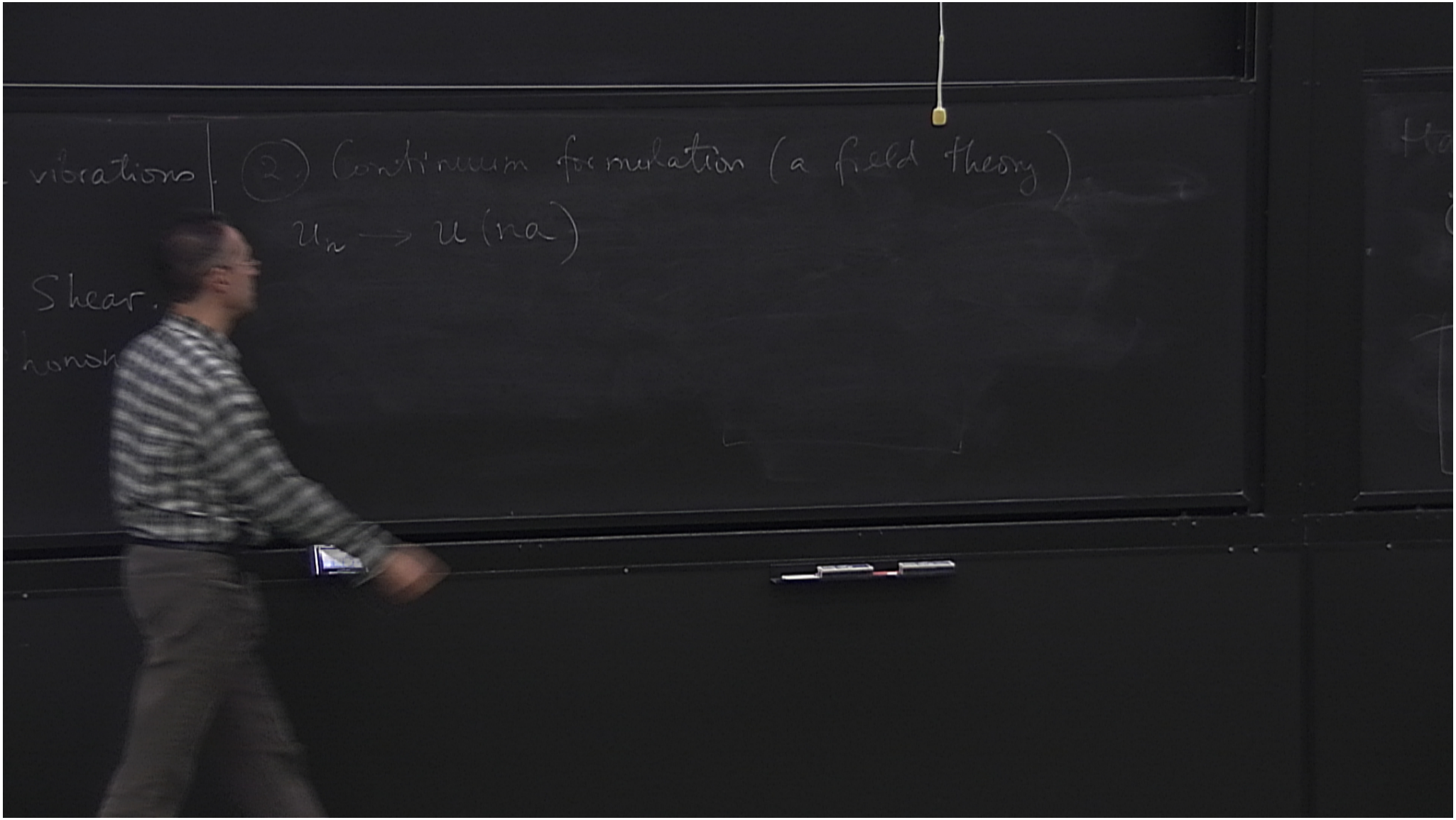

Harmonic oscillator:
$\ddot{C} + \frac{K}{M}(2 - 2\cos ka)C = 0$
a^2
$\omega = \sqrt{\frac{2K}{M}(1-\cos ka)} = 2\sqrt{\frac{K}{M}}\left|\sin\frac{ka}{2}\right|$
$L = \sum_n \frac{M\dot{u}_n^2}{2} - \frac{K(u_{n+1}-u_n)^2}{2}$
Euler-Lagrange eqns.
$\frac{d}{dt}\frac{\partial L}{\partial \dot{u}_n} - \frac{\partial L}{\partial u_n} = 0$
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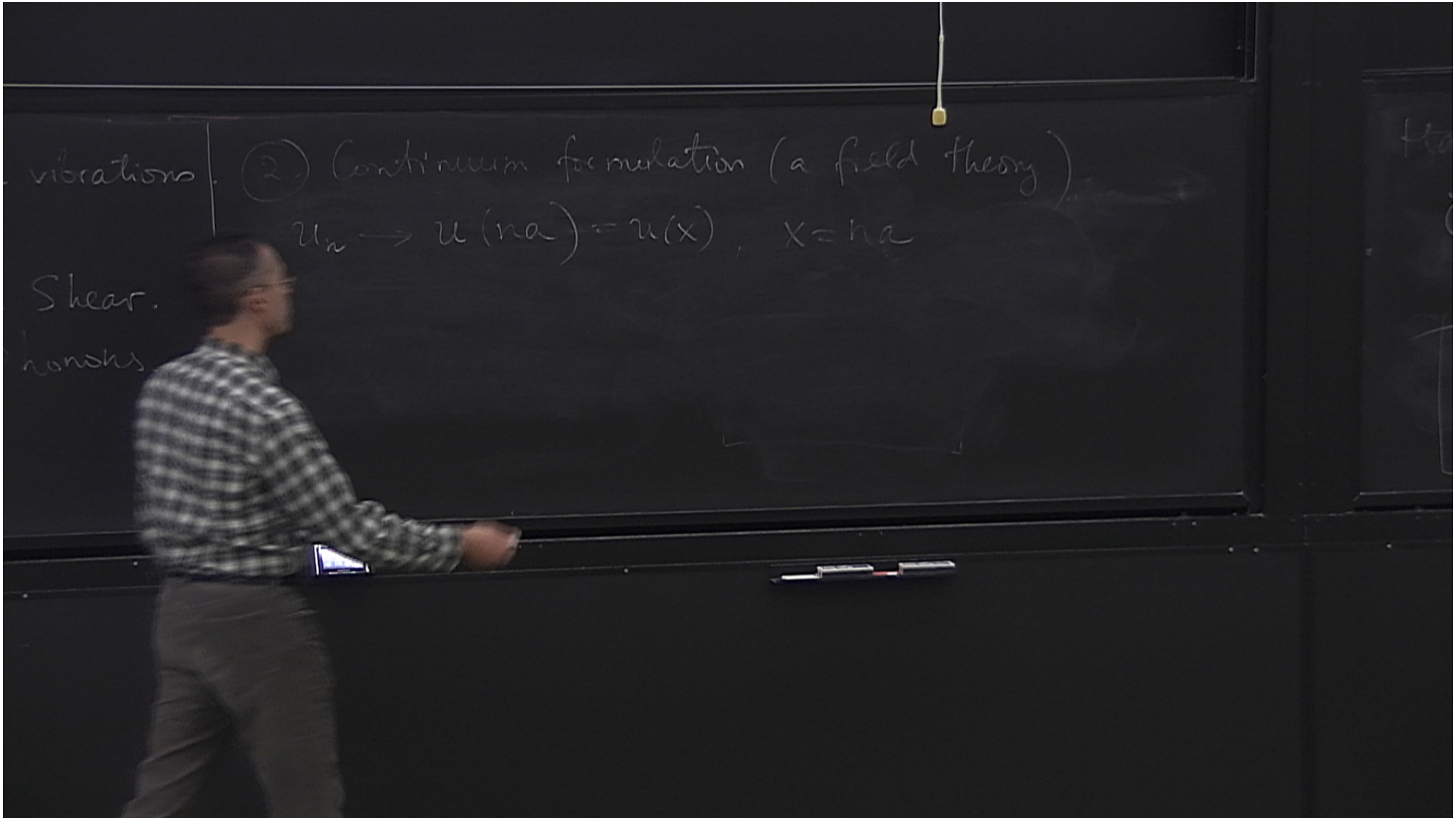

Harmonic oscillator:
$\ddot{C} + \frac{K}{M}(2 - 2\cos ka)C = 0$
ω^2
$\omega = \sqrt{\frac{2K}{M}(1 - \cos ka)} = 2\sqrt{\frac{K}{M}}\left|\sin\frac{ka}{2}\right|$
ω
0
π/a
$2\pi/a$

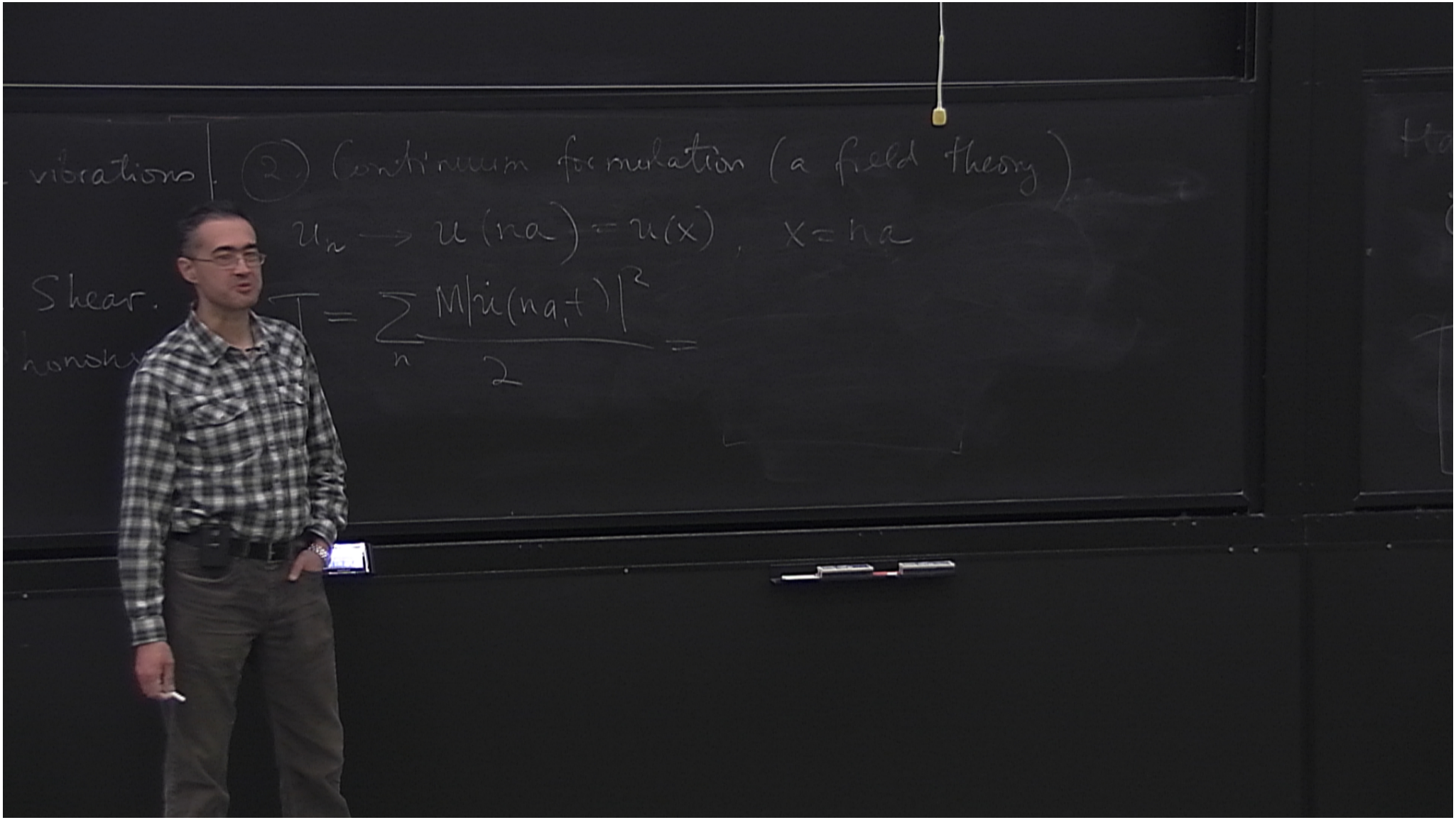

$u_n = Ce^{-i\omega t} e^{ikna}$

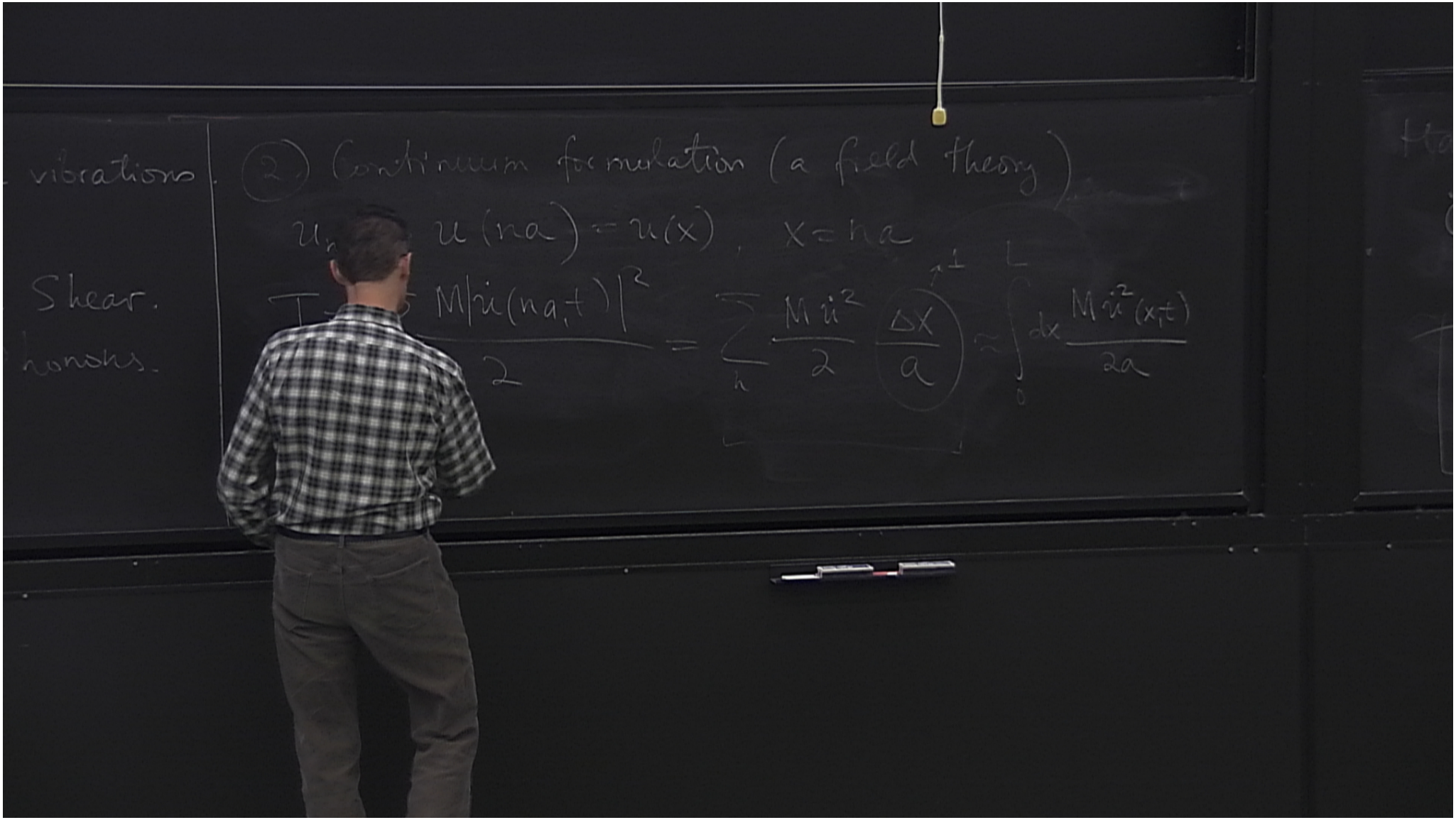

$k: \; u_n = C e^{-i\omega t} e^{ikna}$
$k' = k + \frac{2\pi}{a}, \; u_n = C e^{-i\omega t} e^{ikna} e^{i\frac{2\pi}{a}na}$
$= 1$


ations.
ar.
ns.
k: $u_n = Ce^{-i\omega t} e^{ikna}$
$k' = k + \frac{2\pi}{a}$, $u_n = Ce^{-i\omega t} e^{ikna} e^{i\frac{2\pi}{a}na} = 1$
$0 \le k < 2\pi/a$, or $\frac{-\pi}{a} < k \le \frac{\pi}{a}$


vibrations
② Continuum formulation (a field theory)
$u_n \longrightarrow u(na)$
Shear.


vibrations
(2) Continuum formulation (a field theory)
$u_n \rightarrow u(na) = u(x)$, $x = na$
Shear.


vibrations
② Continuum formulation (a field theory)
$u_n \to u(na) = u(x), \quad x = na$
Shear.
$T = \sum_n \frac{M|\dot{u}(na,t)|^2}{2} =$


vibrations
Shear.
② Continuum formulation (a field theory)
$u_n \quad u(na) = u(x), \quad x = na$
$T = \sum_n \frac{M|\dot{u}(na,t)|^2}{2} = \sum_n \frac{M\dot{u}^2}{2}\left(\frac{\Delta x}{a}\right) \simeq \int_0^L dx \frac{M\dot{u}^2(x,t)}{2a}$

vibrations

Shear.

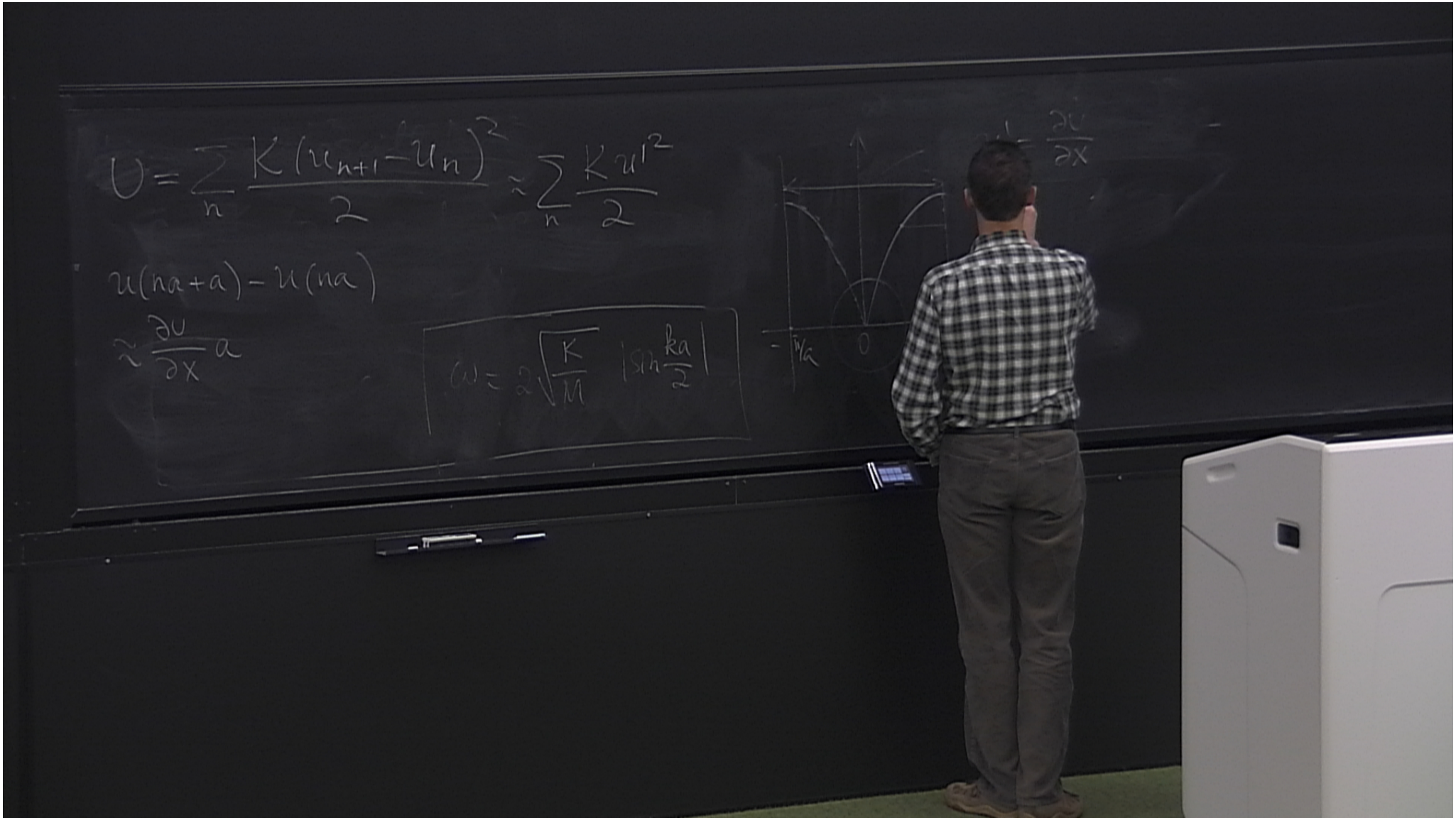
nonons.

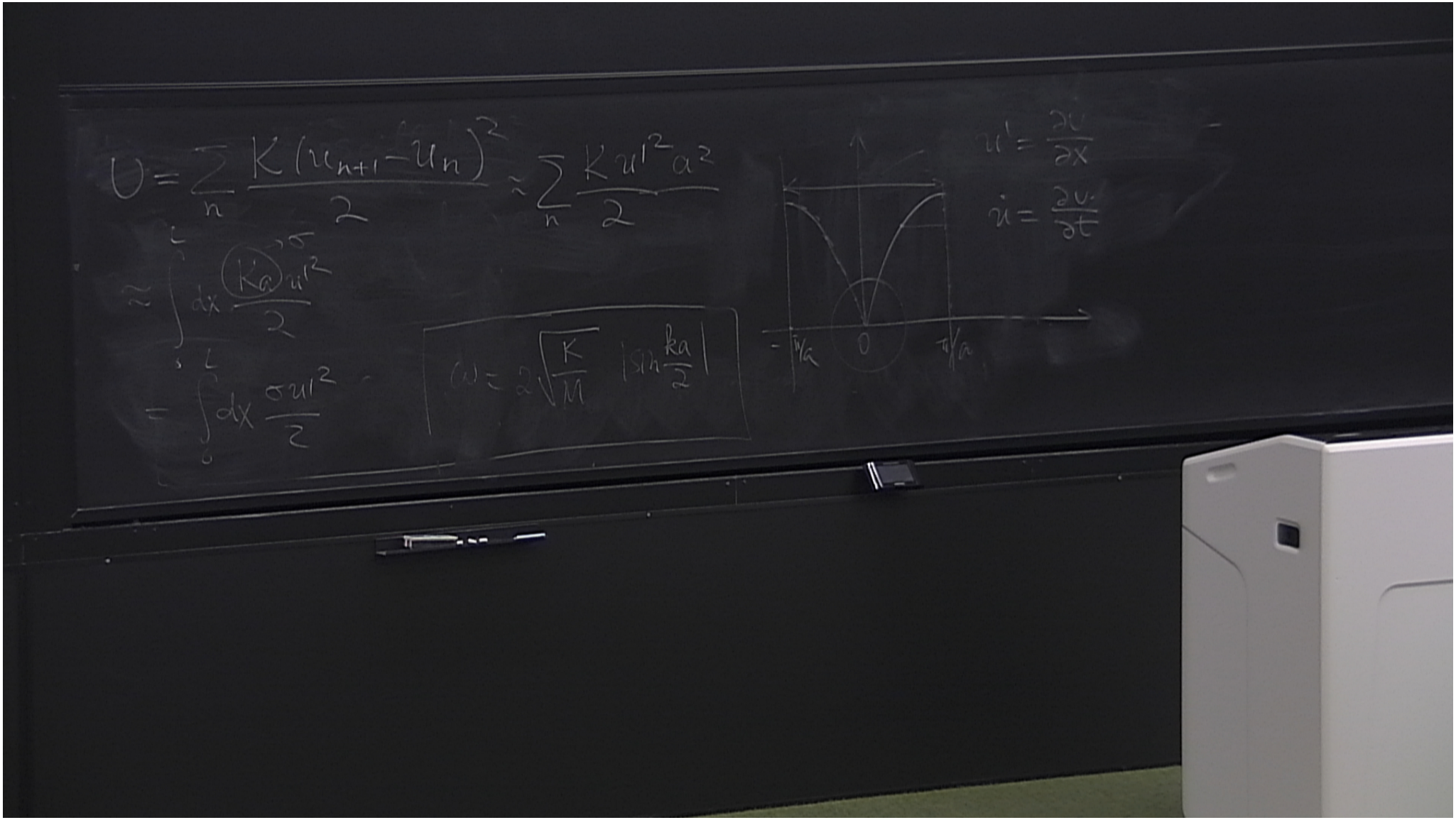
(2) Continuum formulation (a field theory)

$u_n \to u(na) = u(x), \quad x = na$

$$T = \sum_n \frac{M|\dot{u}(na,t)|^2}{2} = \sum_n \frac{M\dot{u}^2}{2}\left(\frac{\Delta x}{a}\right) \to \int_0^L dx \frac{M\dot{u}^2(x,t)}{2a}$$

$$= \int_0^L \rho \frac{\dot{u}^2}{2} dx, \quad \rho = M/a$$


$U = \sum_n \frac{K(u_{n+1} - u_n)^2}{2} \approx \sum_n \frac{K u'^2}{2}$
$u(na+a) - u(na)$
$\approx \frac{\partial u}{\partial x} a$
$\omega = 2\sqrt{\frac{K}{M}} \left| \sin \frac{ka}{2} \right|$
$\frac{\partial u}{\partial x}$
$-\pi/a$
0


$U = \sum_n \frac{K(u_{n+1}-u_n)^2}{2} \approx \sum_n \frac{K u'^2 a^2}{2}$
$\approx \int_0^L dx \frac{Ka}{2} u'^2$
$= \int_0^L dx \frac{\sigma u'^2}{2}$
$\omega = 2\sqrt{\frac{K}{M}} \left|\sin\frac{ka}{2}\right|$
$u' = \frac{\partial u}{\partial x}$
$\dot{u} = \frac{\partial u}{\partial t}$
$-\pi/a$
0
π/a

ce vibrations

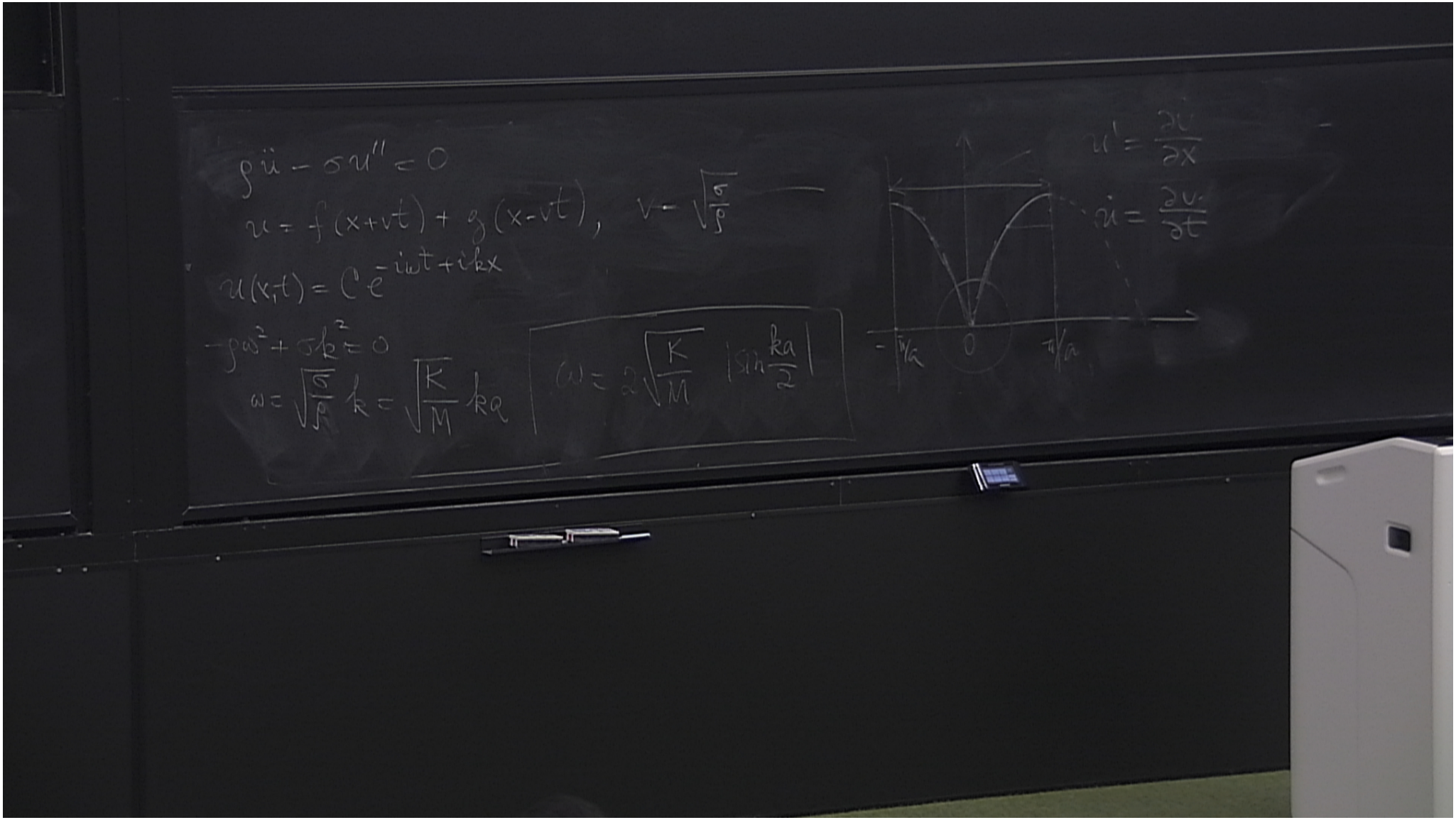
3. Shear.

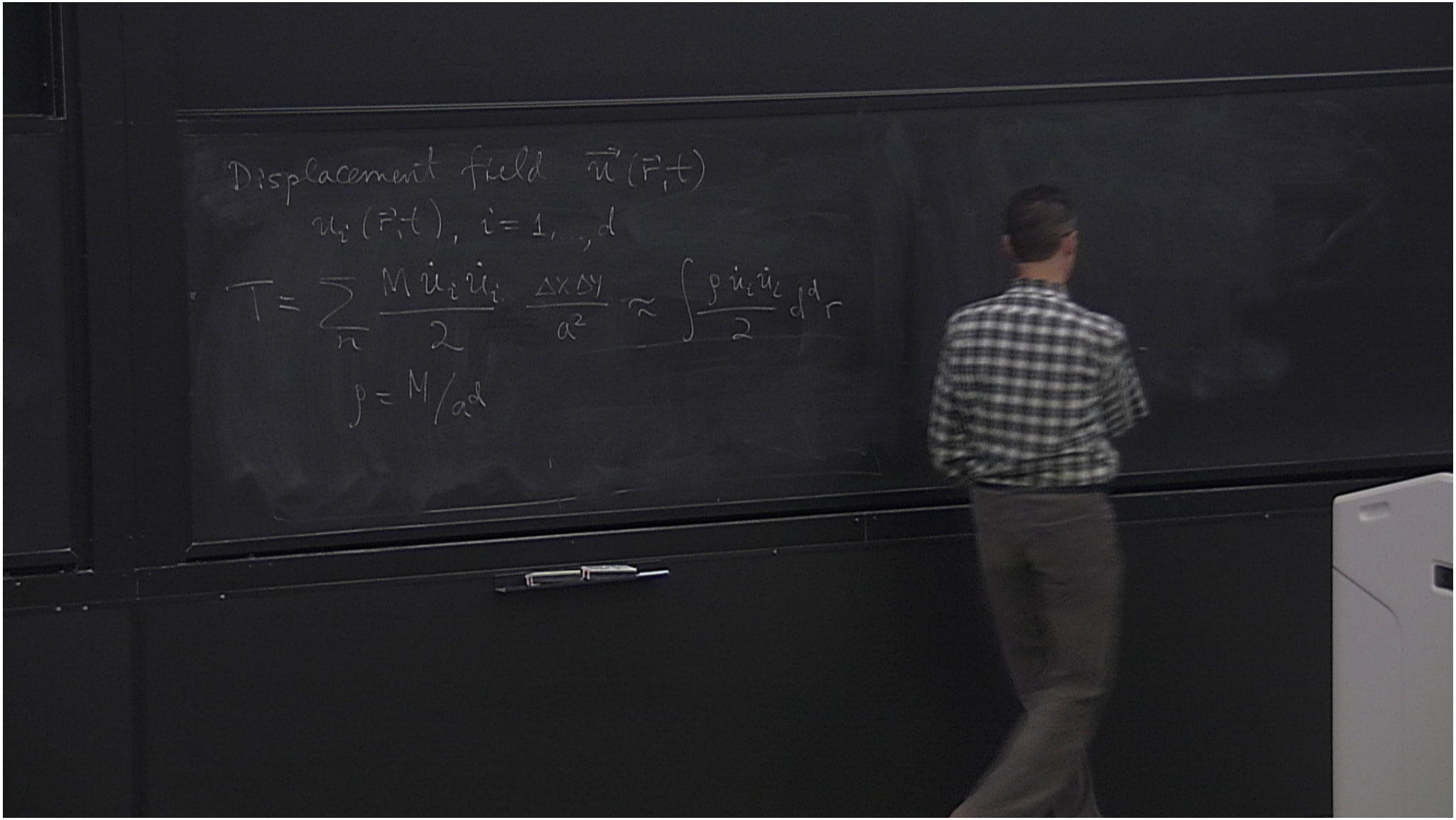
Phonons.

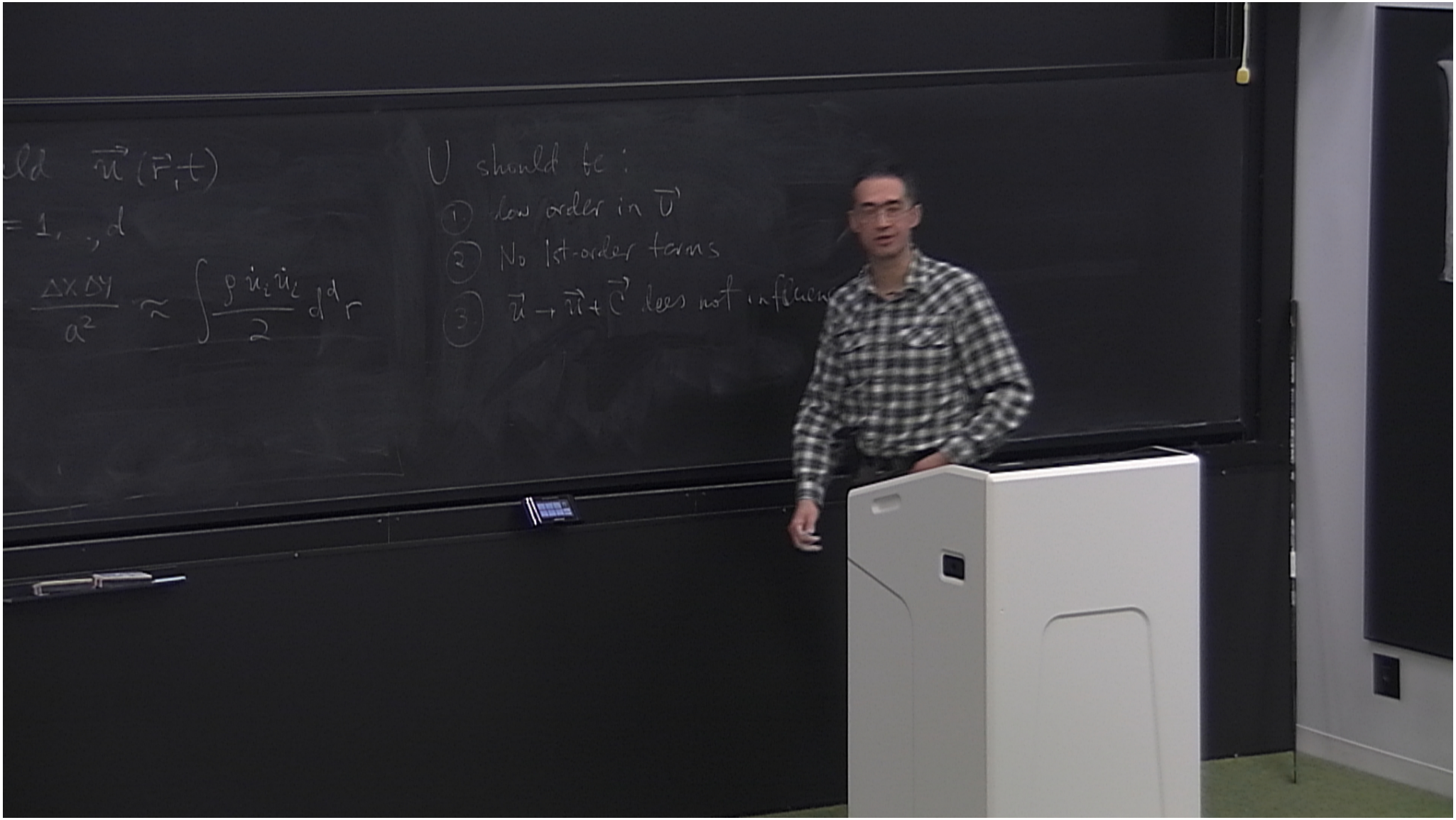
$$L = \int dx \, \mathcal{L}(u, \dot{u}, u')$$

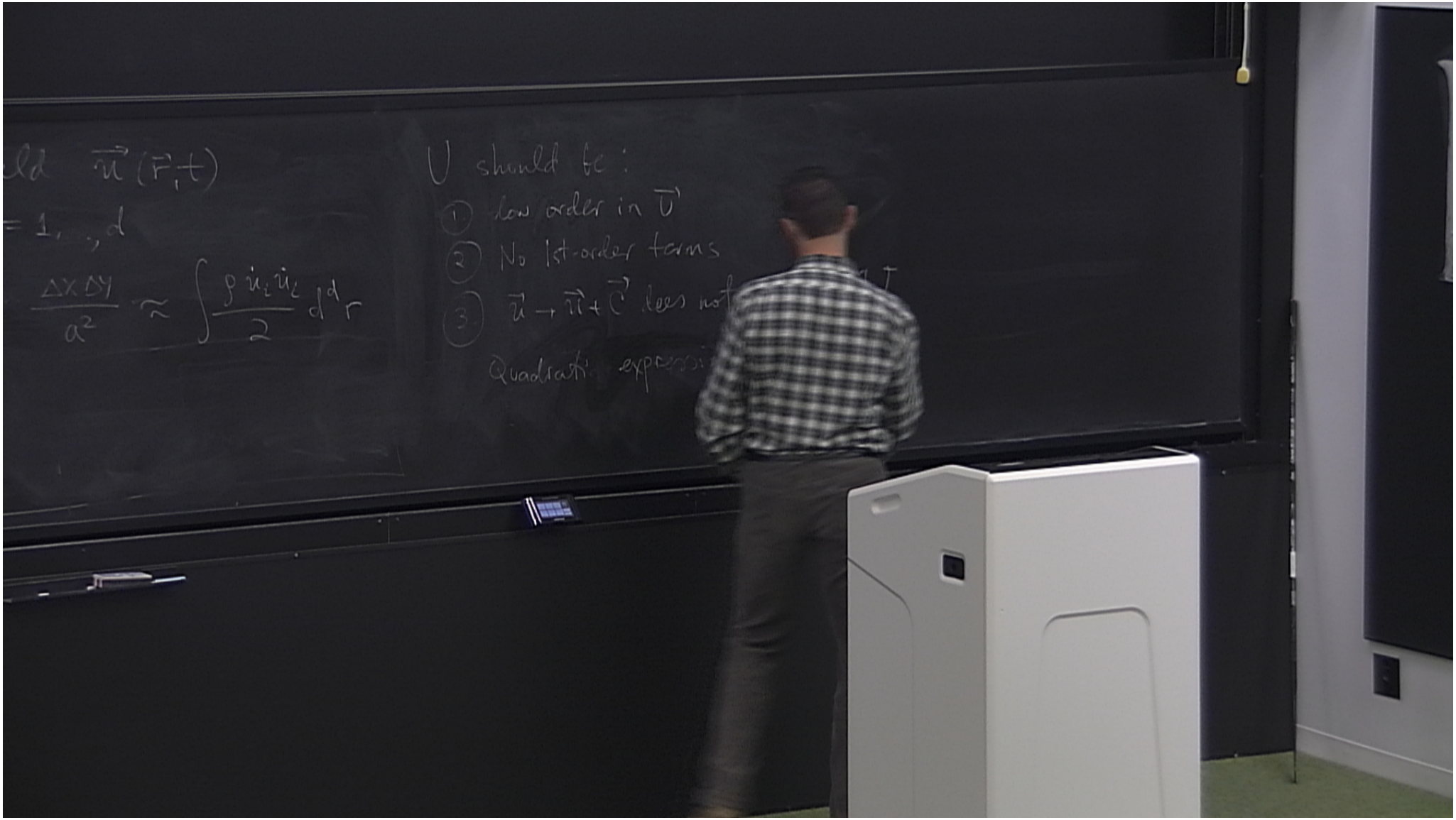
$$\mathcal{L} = \frac{\rho \dot{u}^2}{2} - \frac{\sigma u'^2}{2}$$

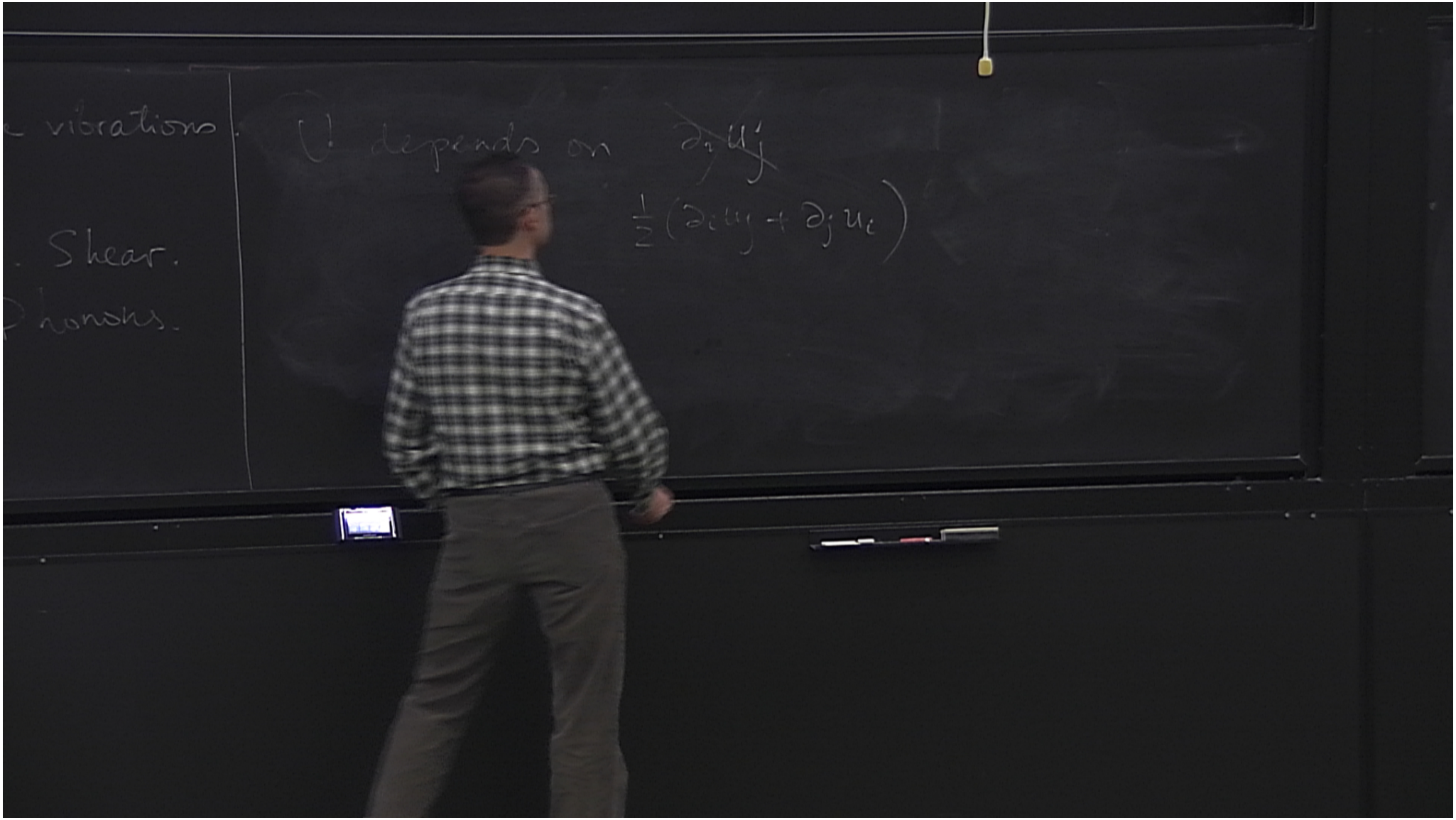
Euler-Lagrange eqns.: $\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{u}} + \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial u'} - \frac{\partial \mathcal{L}}{\partial u} = 0$

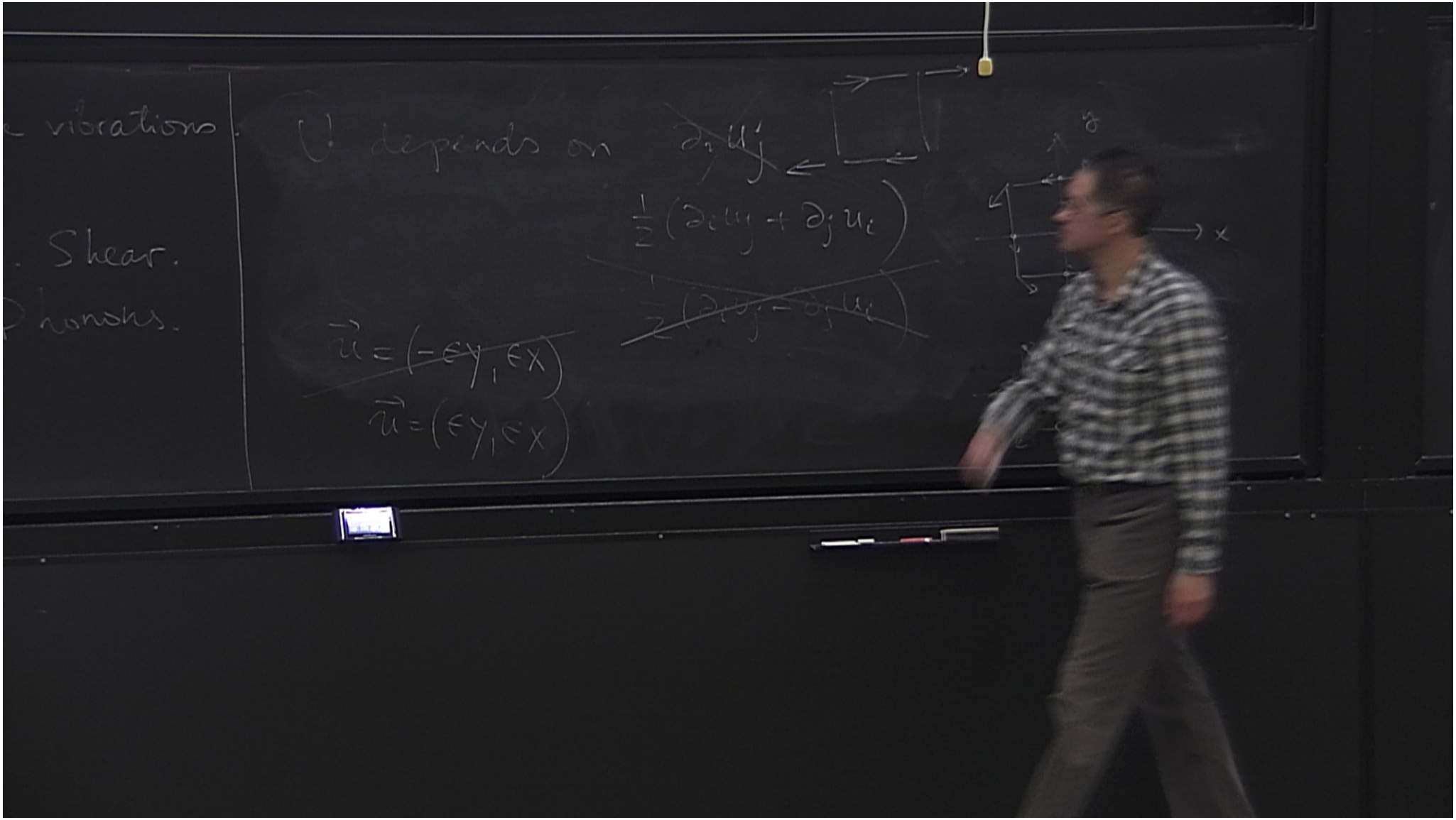

$\rho\ddot{u} - \sigma u'' = 0$
$u = f(x+vt) + g(x-vt), \quad v = \sqrt{\frac{\sigma}{\rho}}$
$u(x,t) = Ce^{-i\omega t + ikx}$
$-\rho\omega^2 + \sigma k^2 = 0$
$\omega = \sqrt{\frac{\sigma}{\rho}}\, k = \sqrt{\frac{K}{M}}\, ka$
$\omega = 2\sqrt{\frac{K}{M}}\left|\sin\frac{ka}{2}\right|$
$u' = \frac{\partial u}{\partial x}$
$\dot{u} = \frac{\partial u}{\partial t}$
$-\pi/a$
0
π/a

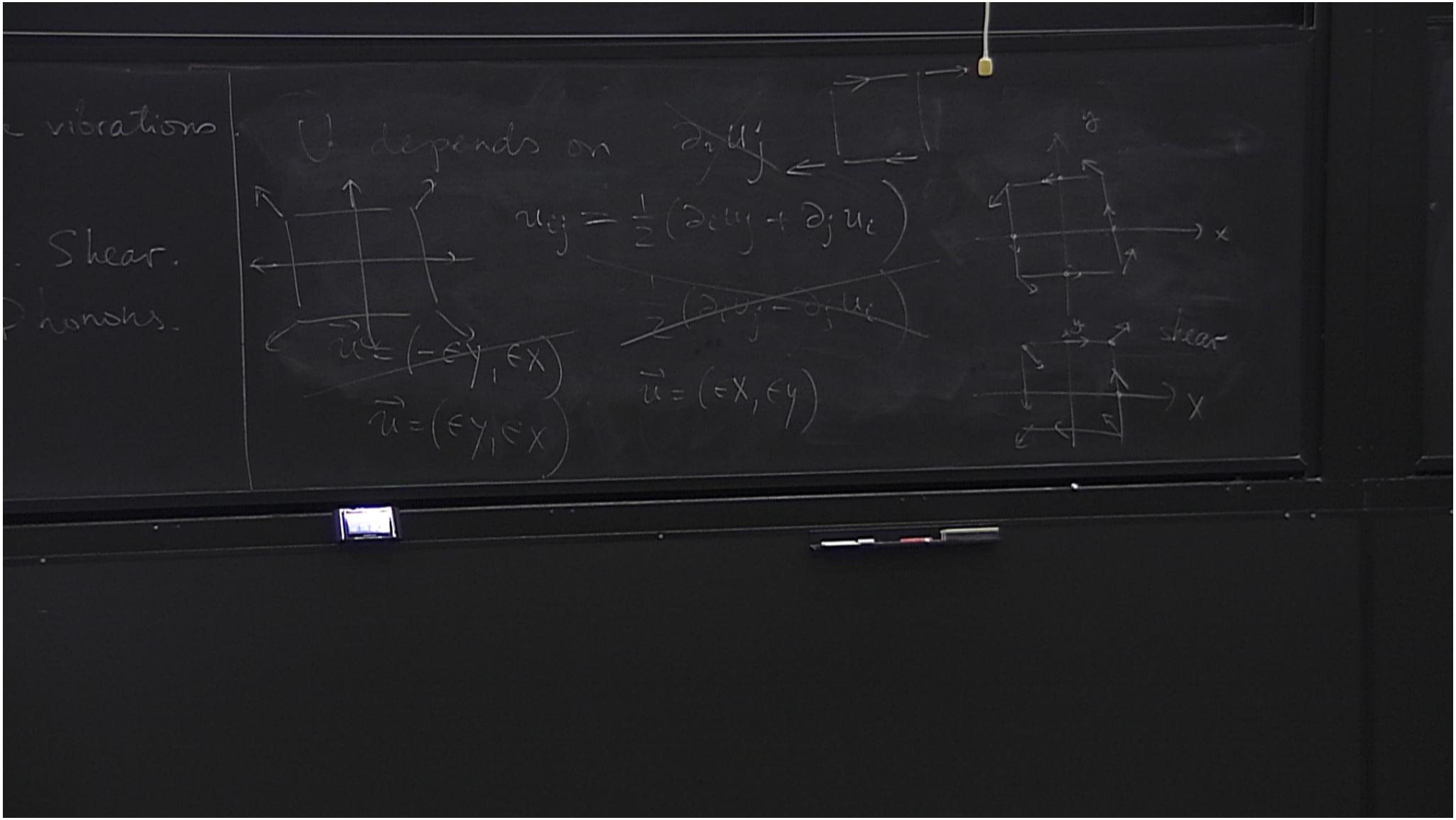

Displacement field $\vec{u}(\vec{r},t)$
$u_i(\vec{r},t)$, $i=1,\ldots,d$
$T=\sum_n \frac{M\dot{u}_i\dot{u}_i}{2}\frac{\Delta x\Delta y}{a^2}\approx\int\frac{\rho\dot{u}_i\dot{u}_i}{2}d^dr$
$\rho=M/a^d$

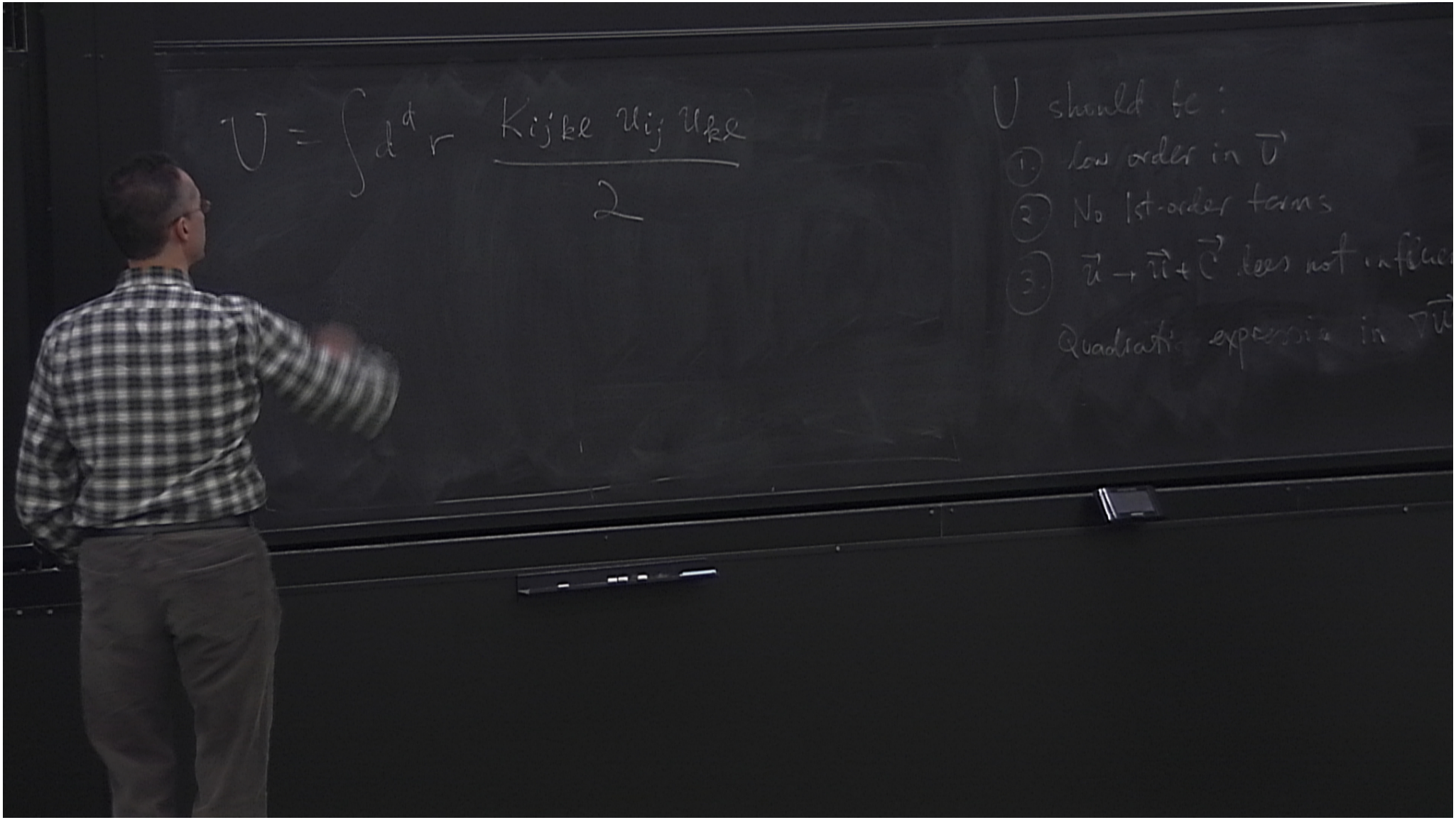

$\vec{u}(\vec{r},t)$
$= 1, \ldots, d$
$\frac{\Delta x \Delta y}{a^2} \approx \int \frac{\rho \dot{u}_i \dot{u}_i}{2} d^d r$
U should be:
1 low order in $\vec{v}$
2 No 1st-order terms
3 $\vec{u} \to \vec{u} + \vec{c}$ does not influen

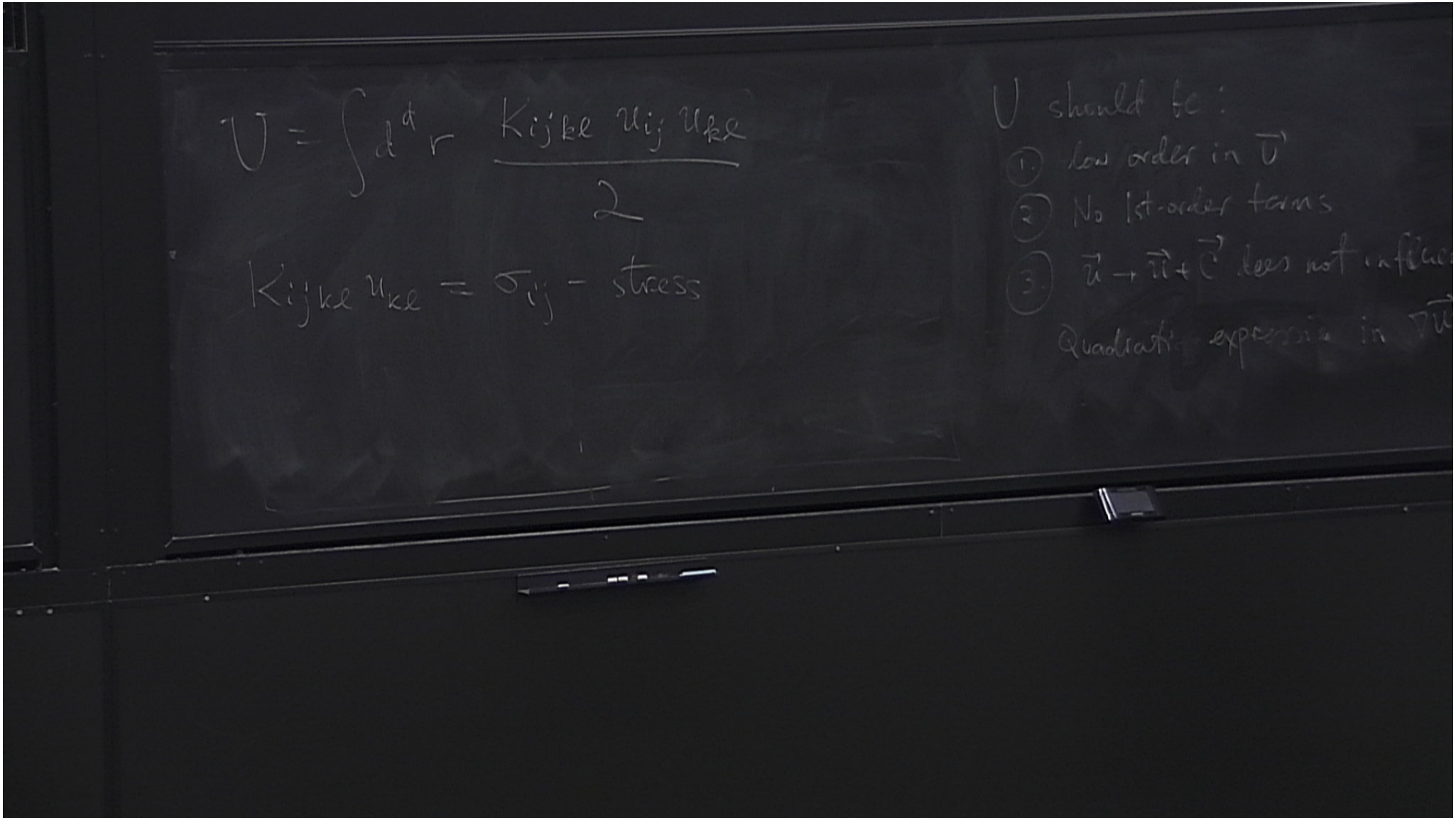

$\vec{u}(\vec{r},t)$
$= 1, \ldots, d$
$\frac{\Delta x \Delta y}{a^2} \approx \int \frac{\rho \dot{u}_i \dot{u}_i}{2} d^d r$
U should be:
1. low order in $\vec{U}$
2. No 1st-order terms
3. $\vec{u} \to \vec{u} + \vec{c}$ does not
Quadratic expres

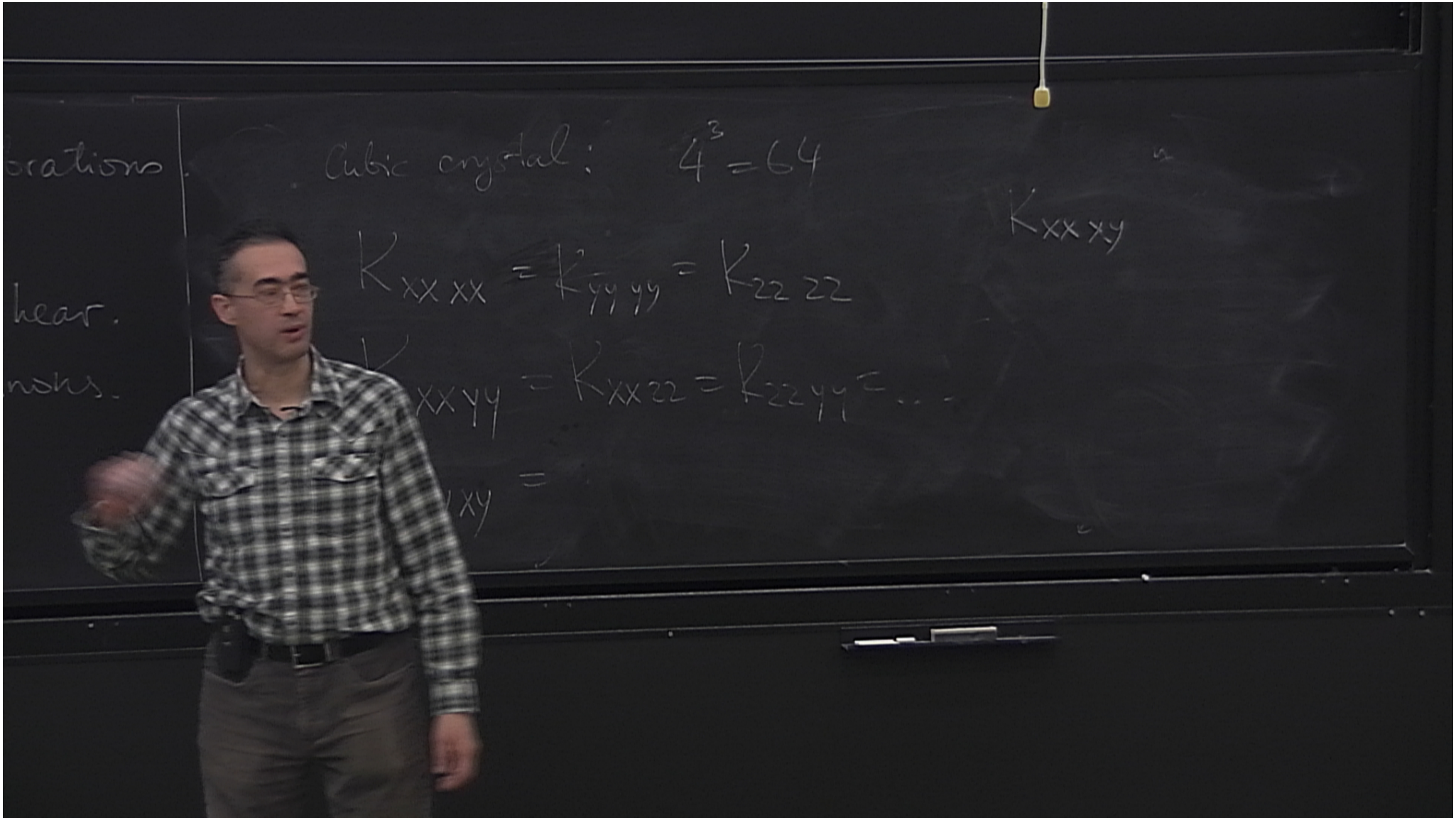

vibrations
Shear.
U depends on
$\frac{1}{2}(\partial_i u_j + \partial_j u_i)$

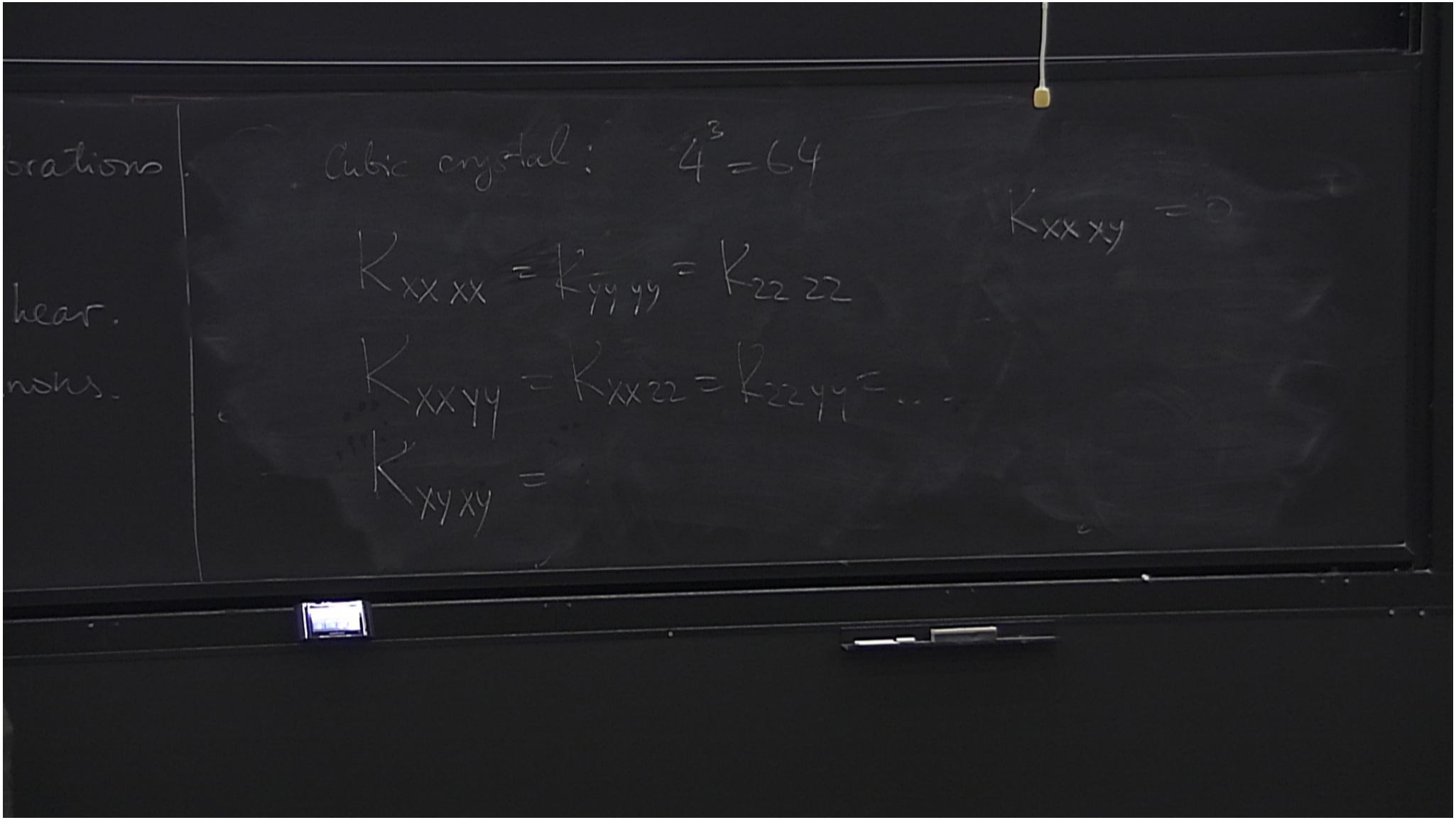

vibrations
Shear.
phonons.
U depends on
$\frac{1}{2}(\partial_i u_j + \partial_j u_i)$
$\vec{u} = (-\epsilon y, \epsilon x)$
$\vec{u} = (\epsilon y, \epsilon x)$
x

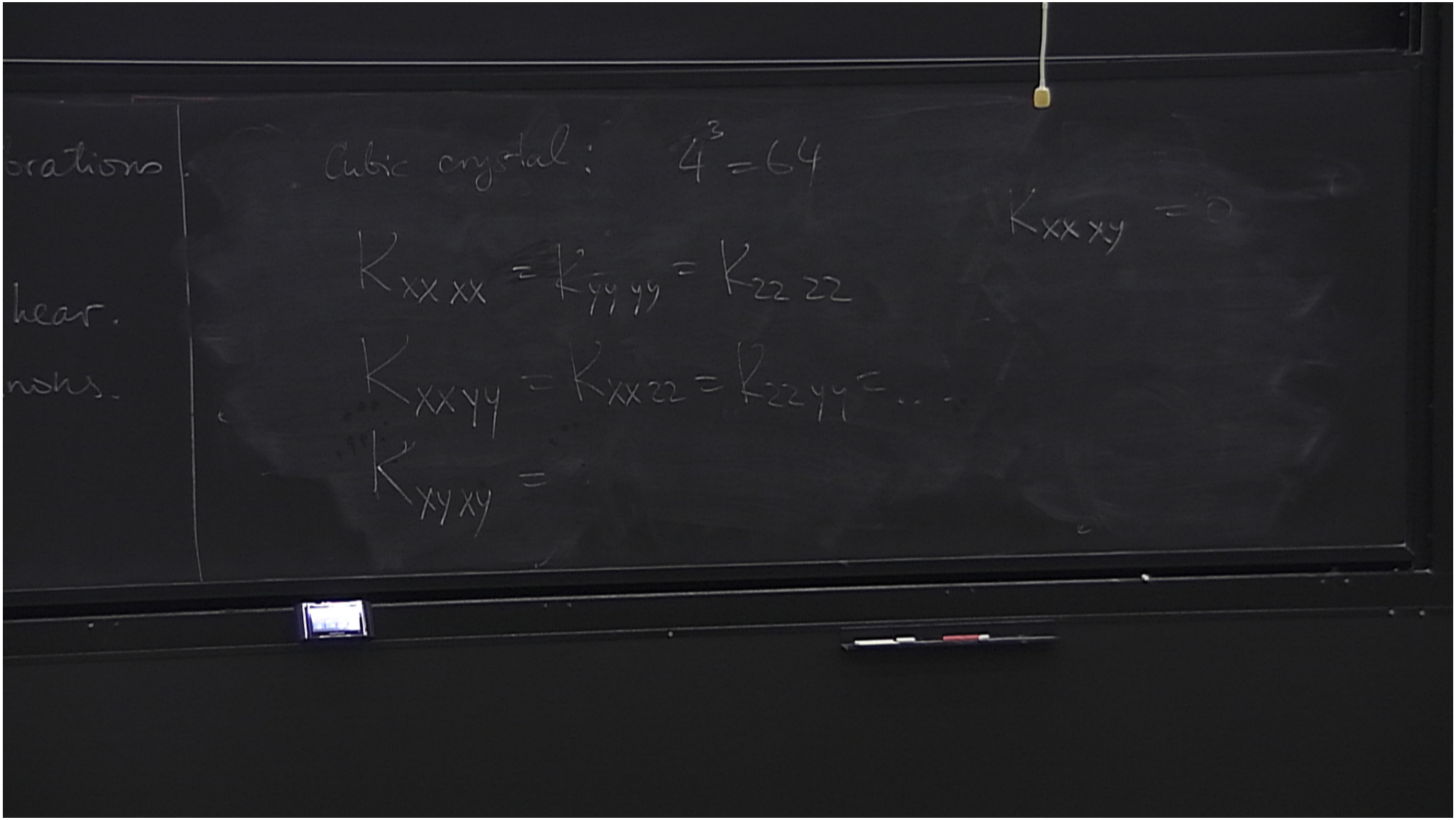

vibrations
Shear.
U depends on $\partial_j u_i$
$u_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$
$\vec{u} = (-\epsilon y, \epsilon x)$
$\vec{u} = (\epsilon y, \epsilon x)$
$\vec{u} = (\epsilon x, \epsilon y)$
y
x
shear
X

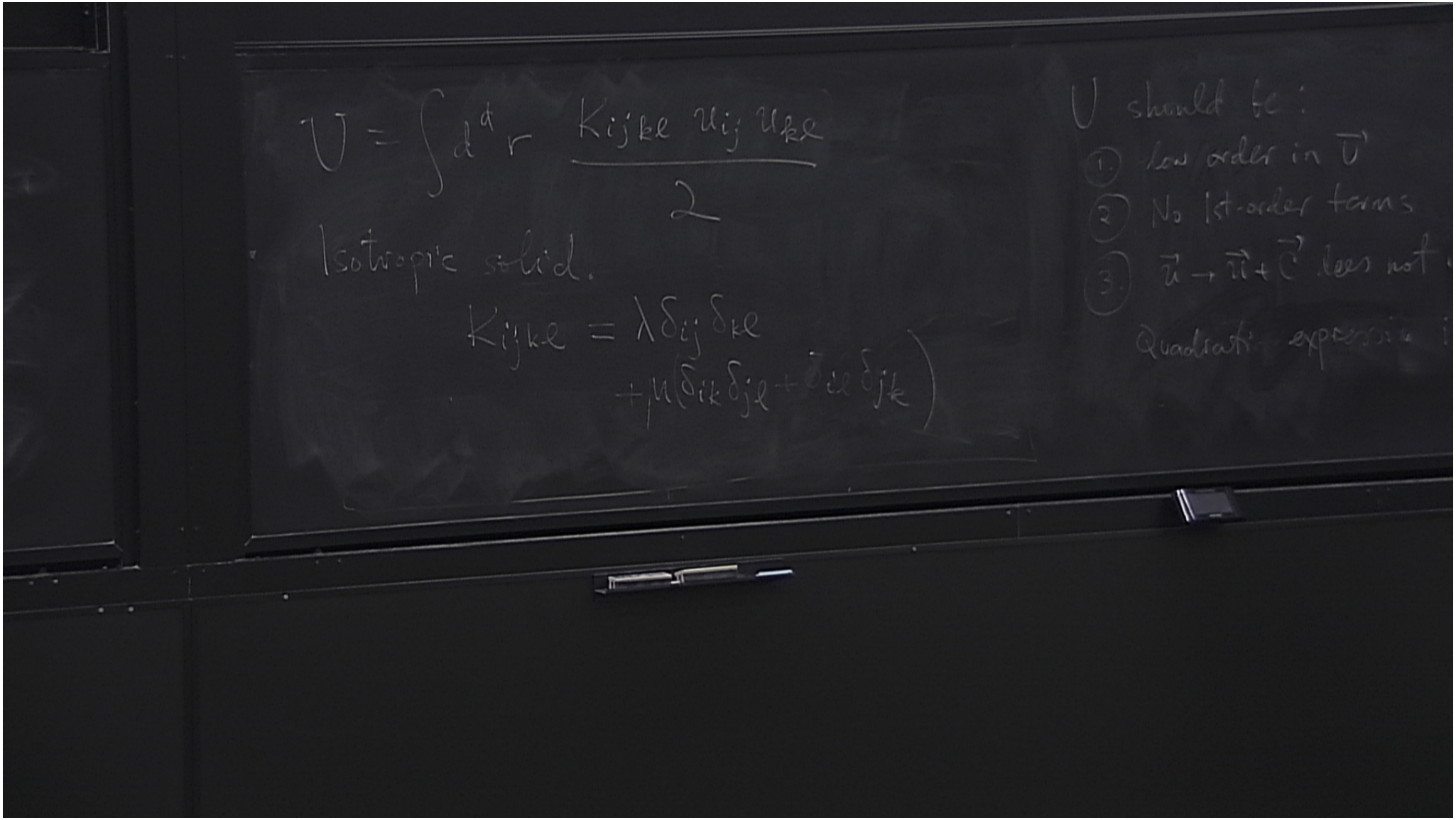

$U = \int d^q r \, \frac{K_{ijkl} u_{ij} u_{kl}}{2}$
U should be:
1. low order in $\vec{u}$
2. No 1st-order terms
3. $\vec{u} \to \vec{u} + \vec{c}$ leaves not inflec
Quadratic expressio in

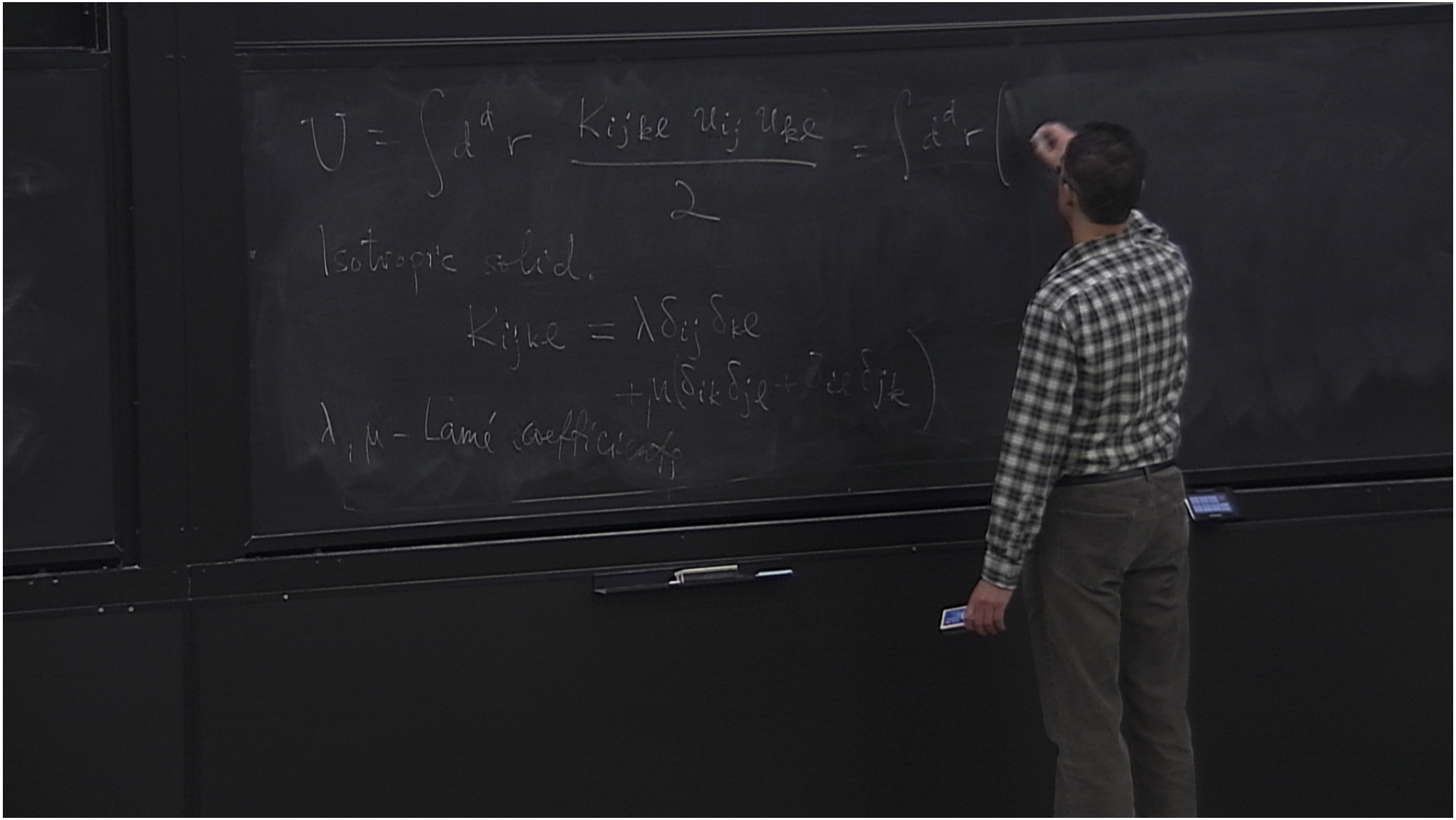

$U = \int d^q r \, \frac{K_{ijkl} u_{ij} u_{kl}}{2}$
$K_{ijkl} u_{kl} = \sigma_{ij}$ - stress
U should be:
1. low order in $\vec{u}$
2. No 1st-order terms
3. $\vec{u} \to \vec{u} + \vec{c}$ does not influe
Quadratic expressio in $\vec{u}$

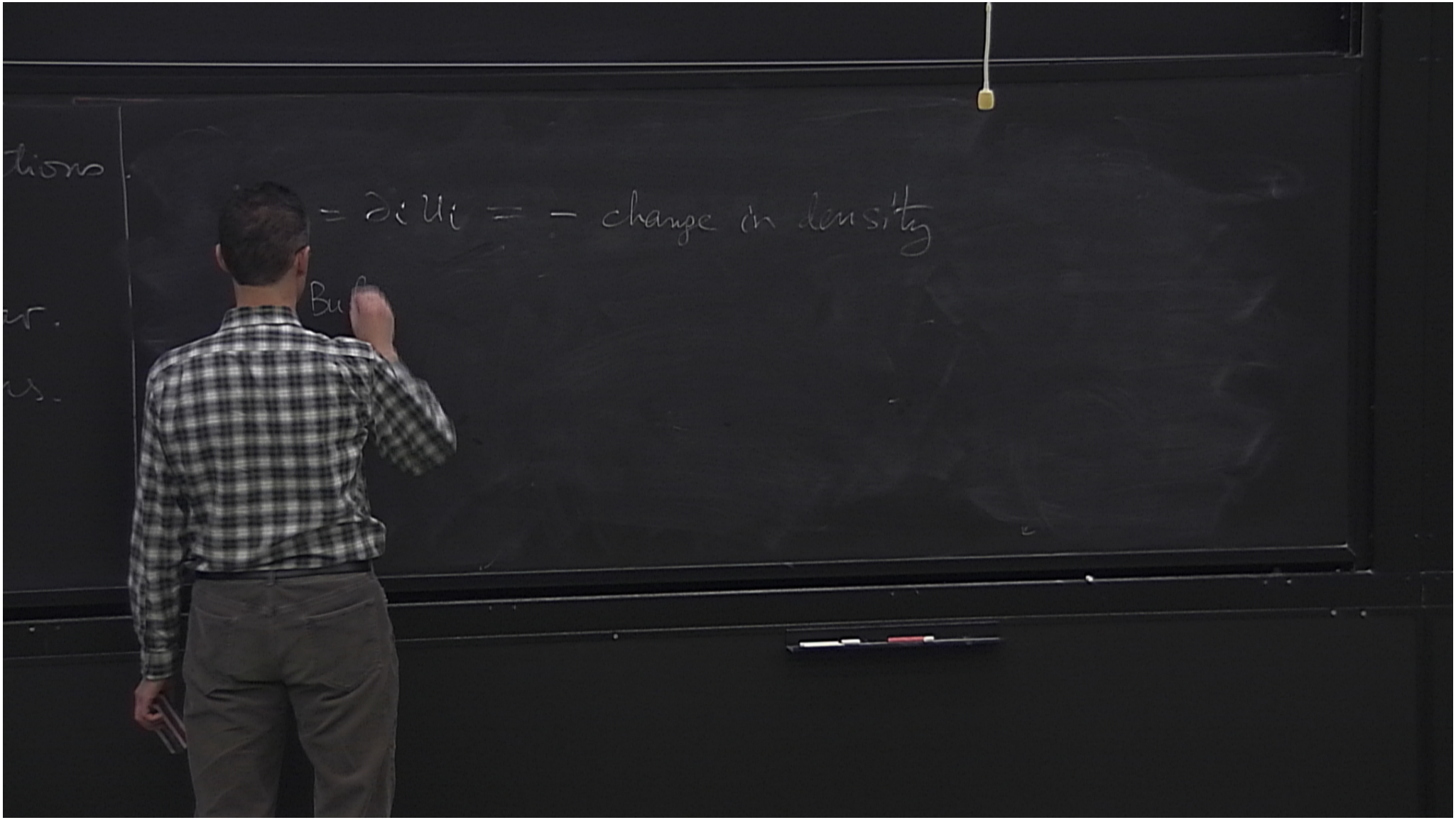

Cubic crystal: $4^3 = 64$
$K_{xxxx} = K_{yyyy} = K_{2222}$
K_{xxxy}
$K_{xxyy} = K_{xx22} = K_{22yy} = \ldots$

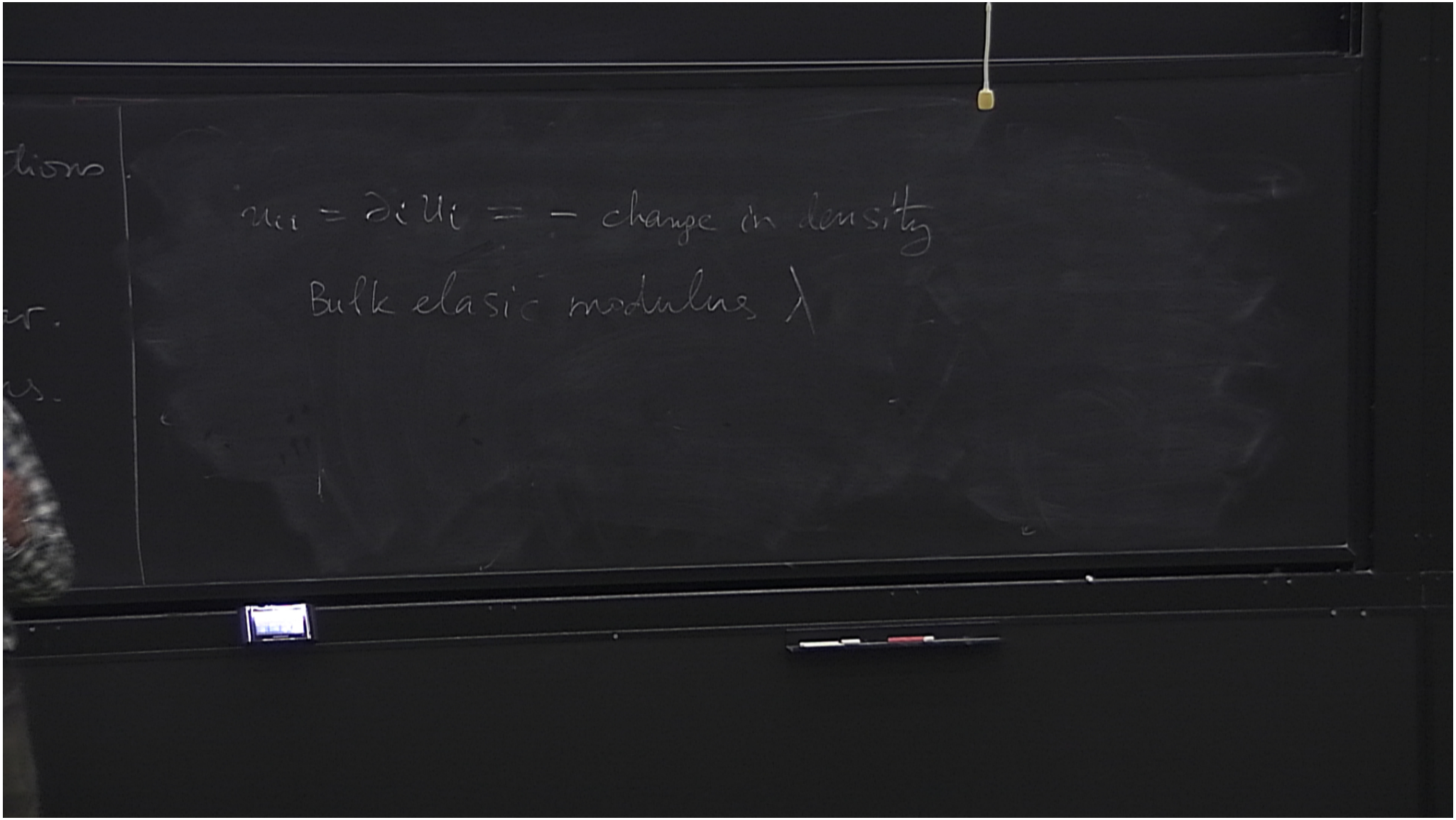

orations.
hear.
nons.
Cubic crystal: $4^3 = 64$
$K_{xxxy} = 0$
$K_{xxxx} = K_{yyyy} = K_{2222}$
$K_{xxyy} = K_{xx22} = K_{22yy} = \ldots$
$K_{xyxy} =$

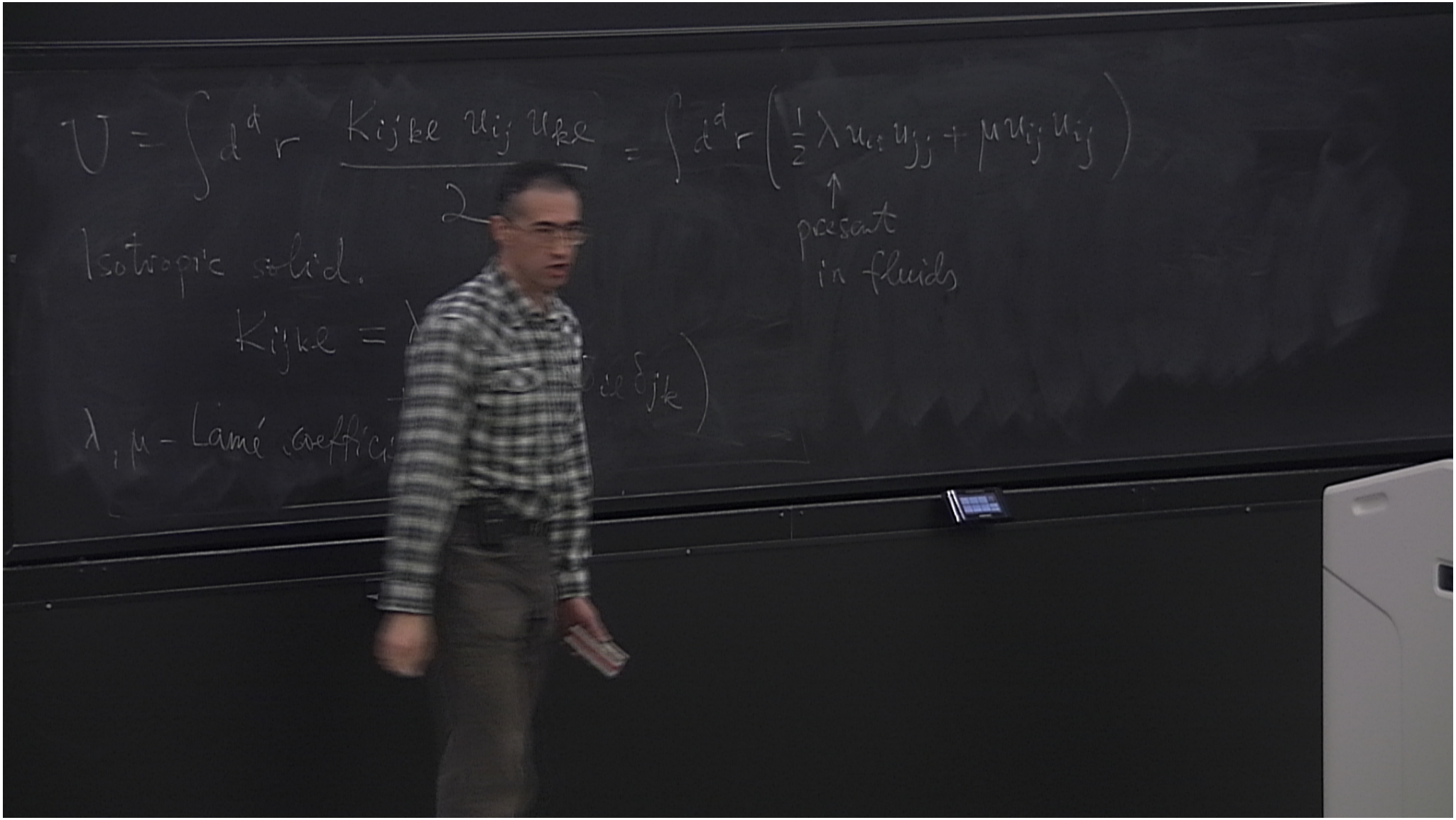

orations
hear.
nons.
Cubic crystal: $4^3 = 64$
$K_{xxxy} = 0$
$K_{xxxx} = K_{yyyy} = K_{zzzz}$
$K_{xxyy} = K_{xxzz} = K_{zzyy} = \ldots$
$K_{xyxy} =$

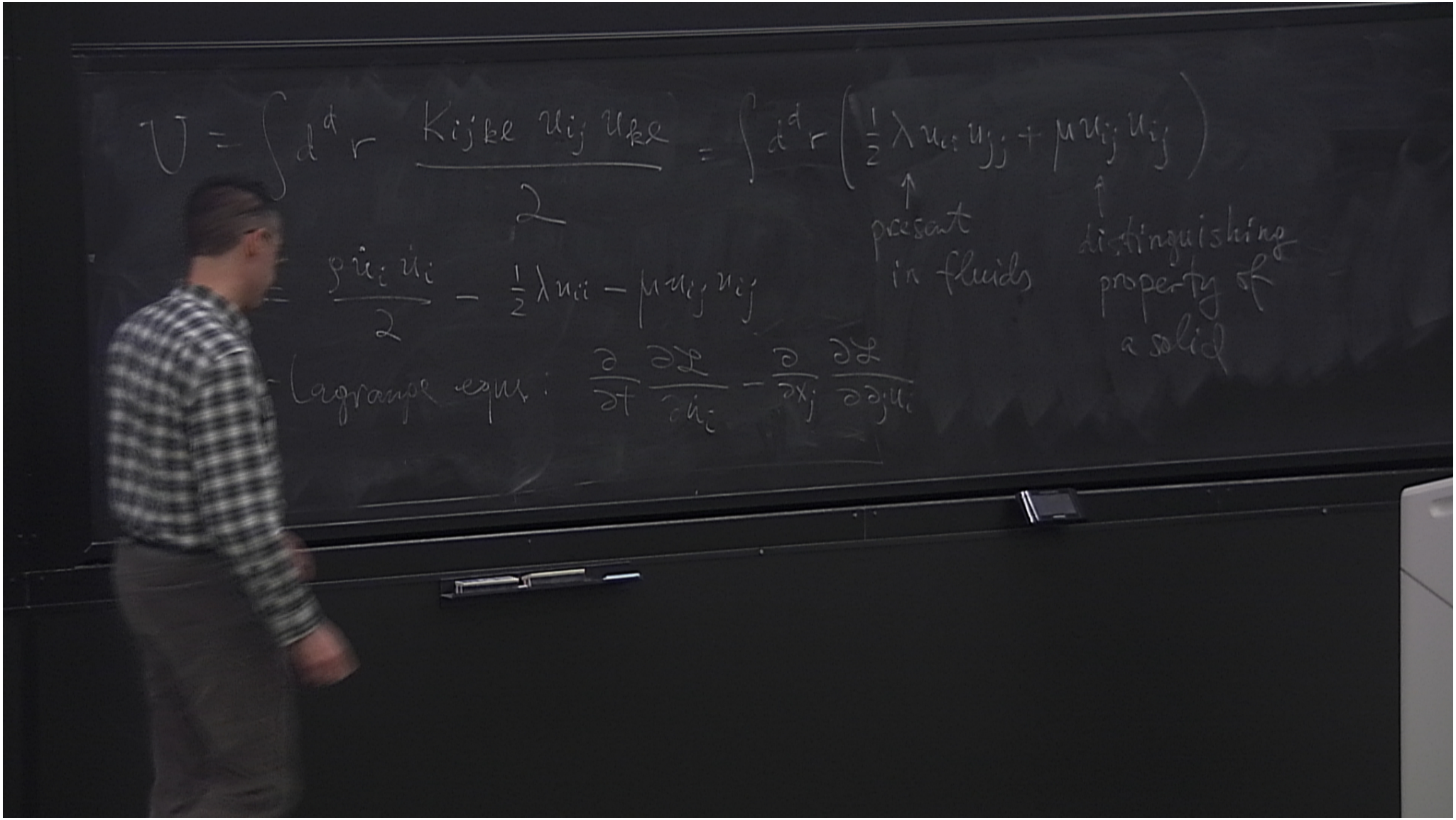

$U = \int d^d r \, \frac{K_{ijkl} u_{ij} u_{kl}}{2}$
Isotropic solid.
$K_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$
U should be:
1. low order in $\vec{u}$
2. No 1st-order terms
3. $\vec{u} \to \vec{u} + \vec{c}$ leaves not
Quadratic expressio

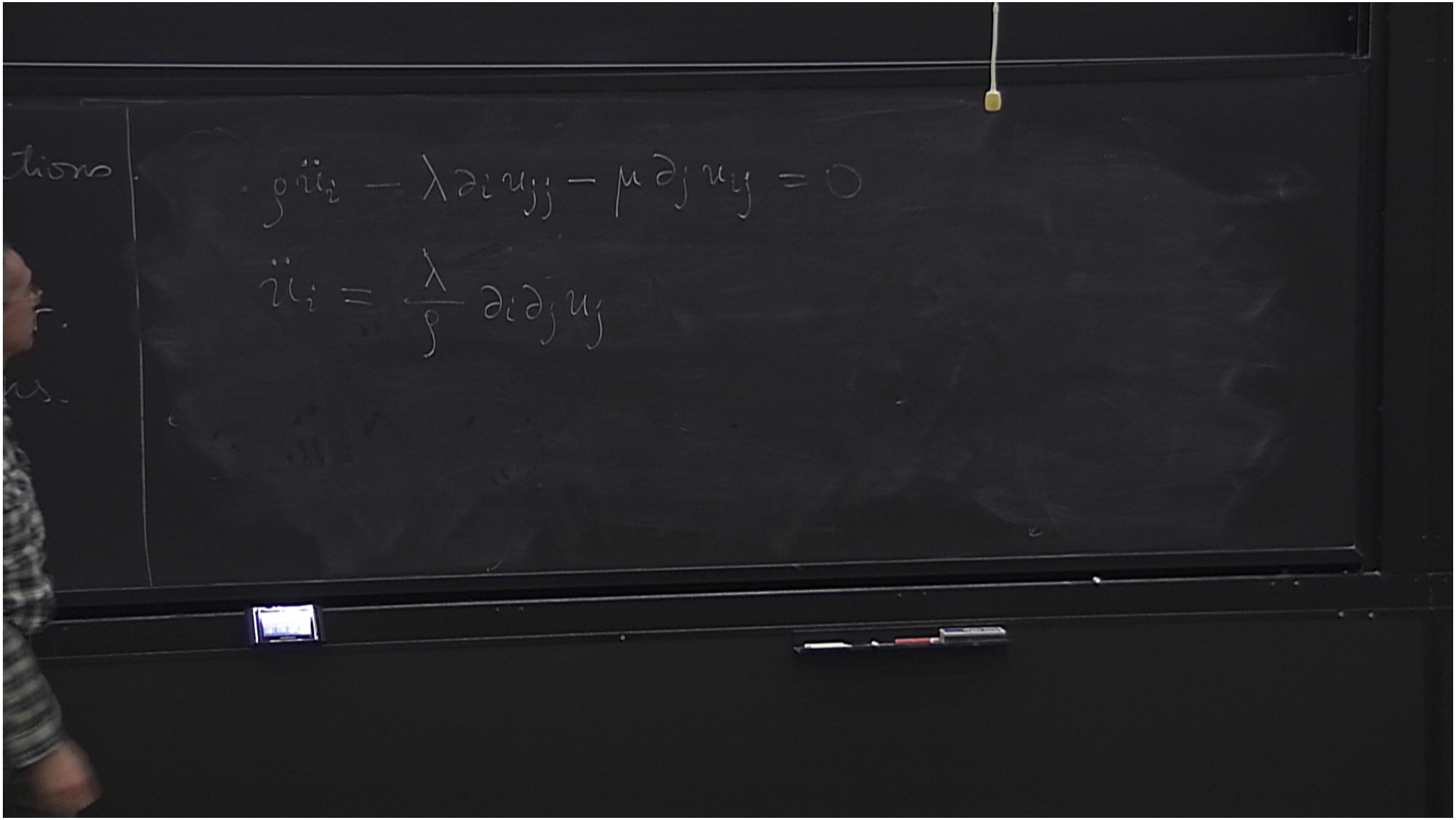

$U = \int d^d r \frac{K_{ijkl} u_{ij} u_{kl}}{2} = \int d^d r ($
Isotropic solid.
$K_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$
λ, μ - Lamé coefficients

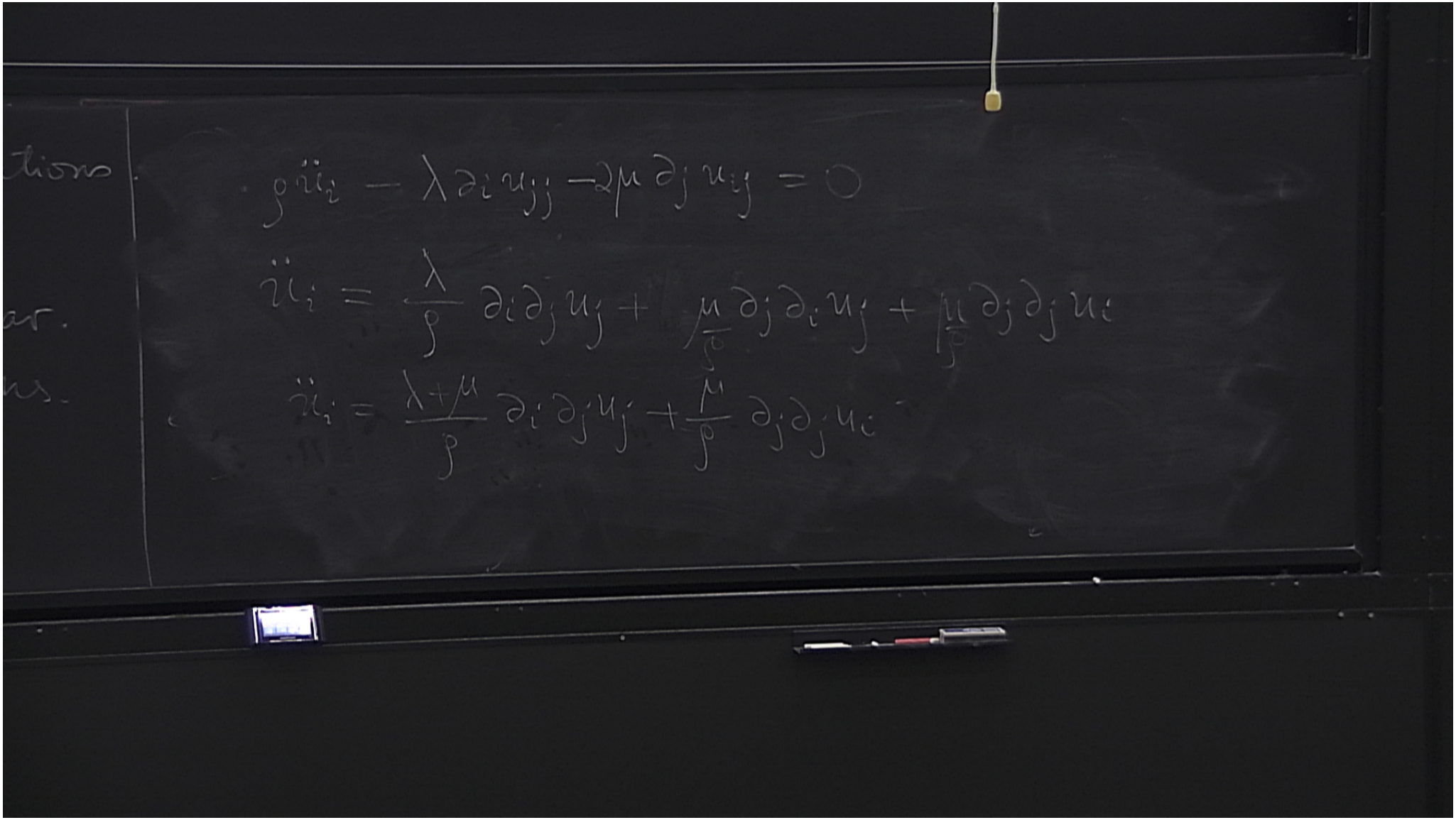

= ∂i ui = − change in density

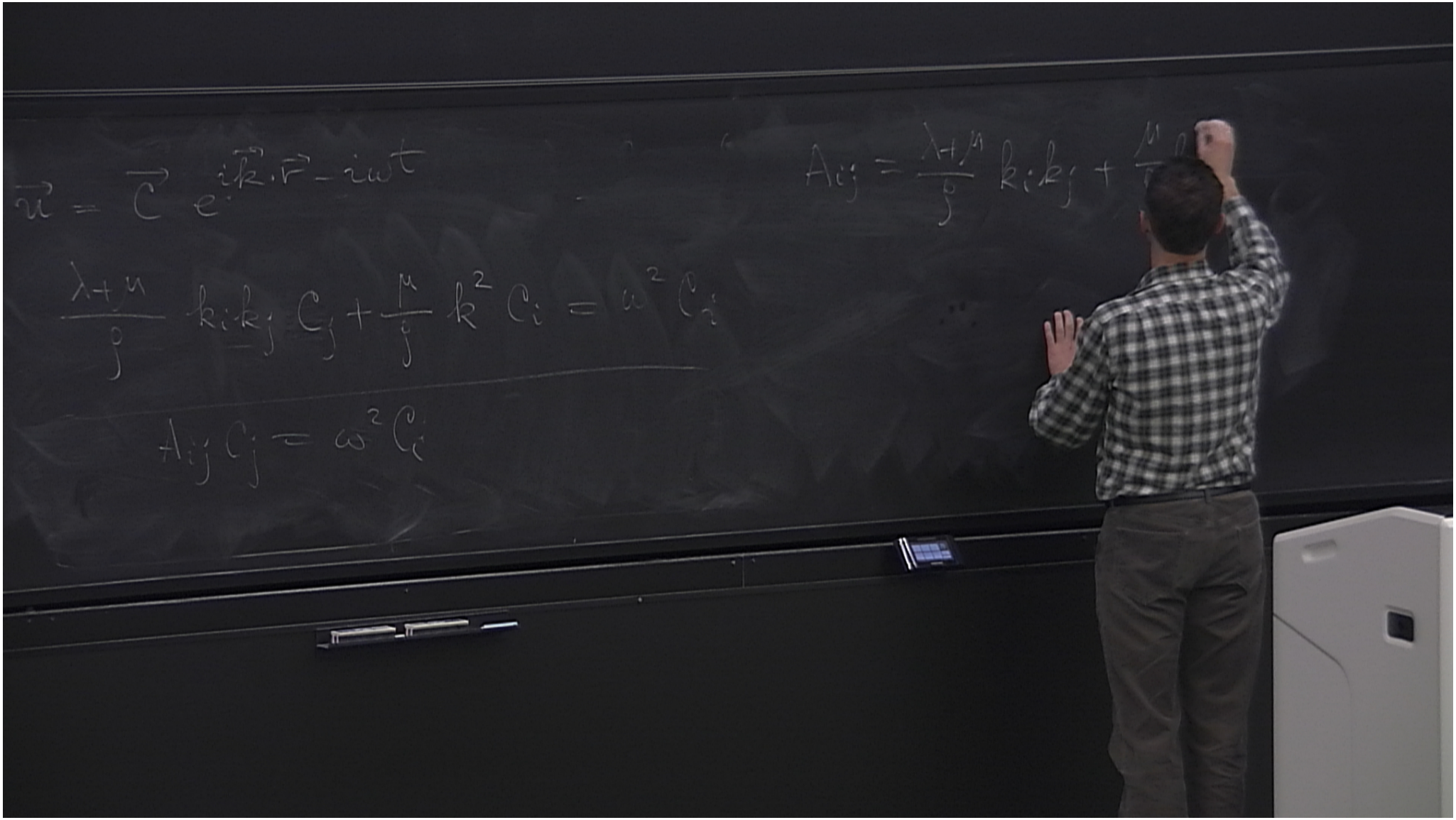

$\partial_i u_i = -$ change in density
Bulk elasic modulus λ

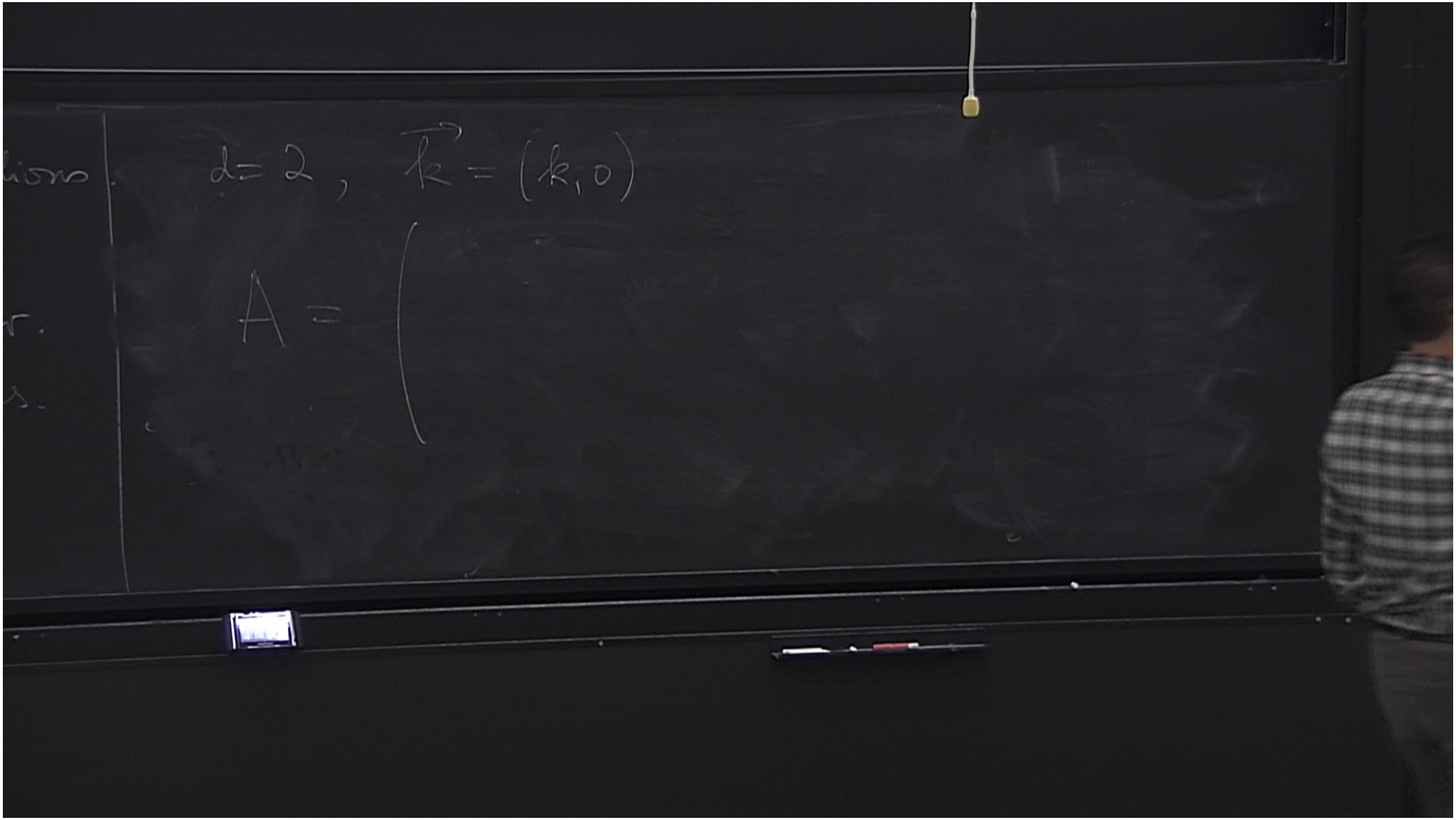

$U = \int d^d r \frac{K_{ijkl} u_{ij} u_{kl}}{2} = \int d^d r \left(\frac{1}{2} \lambda u_{ii} u_{jj} + \mu u_{ij} u_{ij} \right)$
present in fluids
Isotropic solid.
$K_{ijkl} =$
λ, μ - Lamé coefficients

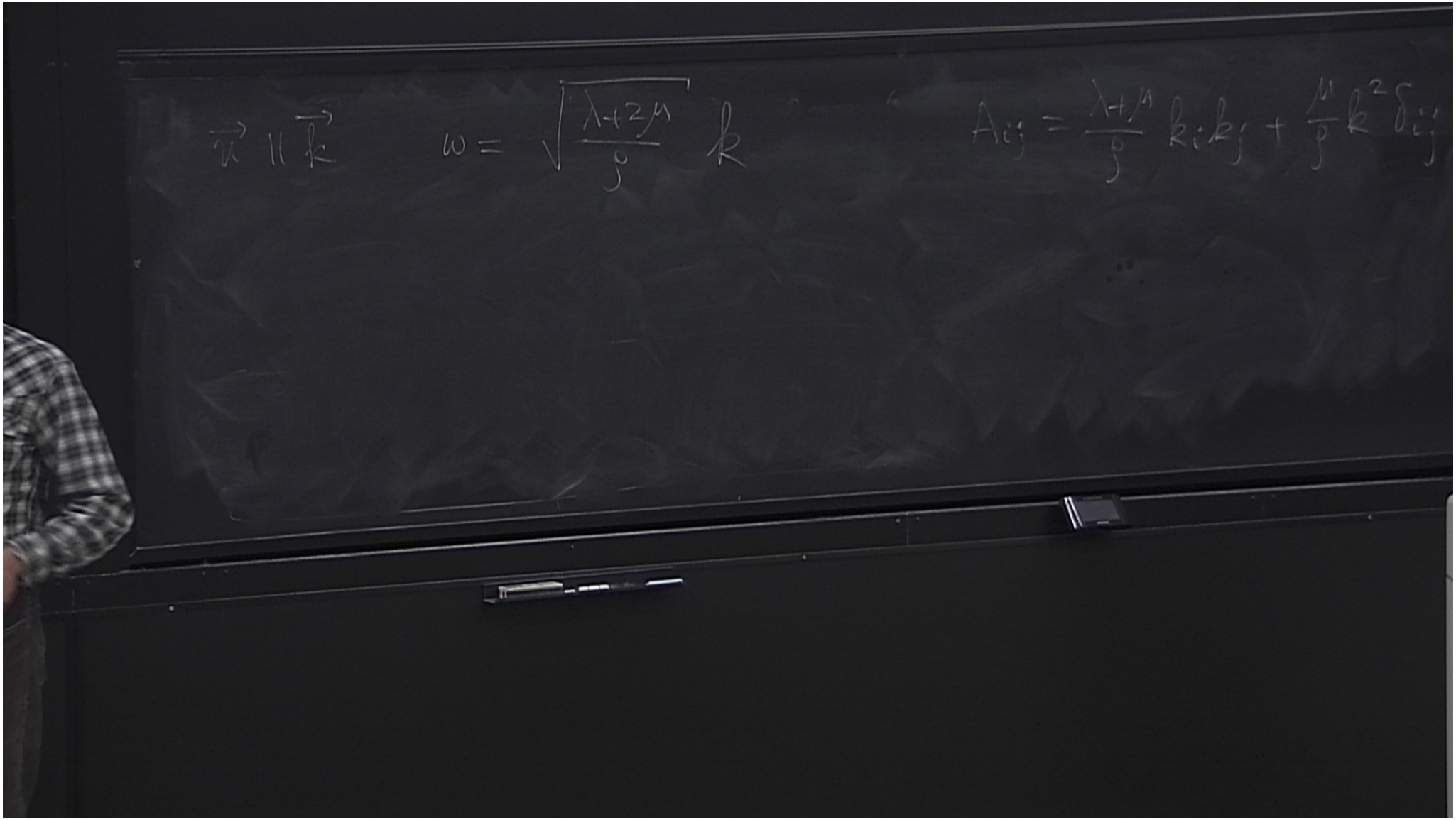

$U = \int d^d r \frac{K_{ijkl} u_{ij} u_{kl}}{2} = \int d^d r \left(\frac{1}{2} \lambda u_{ii} u_{jj} + \mu u_{ij} u_{ij} \right)$
present in fluids
distinguishing property of a solid
$\frac{\rho \dot{u}_i \dot{u}_i}{2} - \frac{1}{2} \lambda u_{ii} - \mu u_{ij} u_{ij}$
Lagrange equ: $\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{u}_i} - \frac{\partial}{\partial x_j} \frac{\partial \mathcal{L}}{\partial \partial_j u_i}$

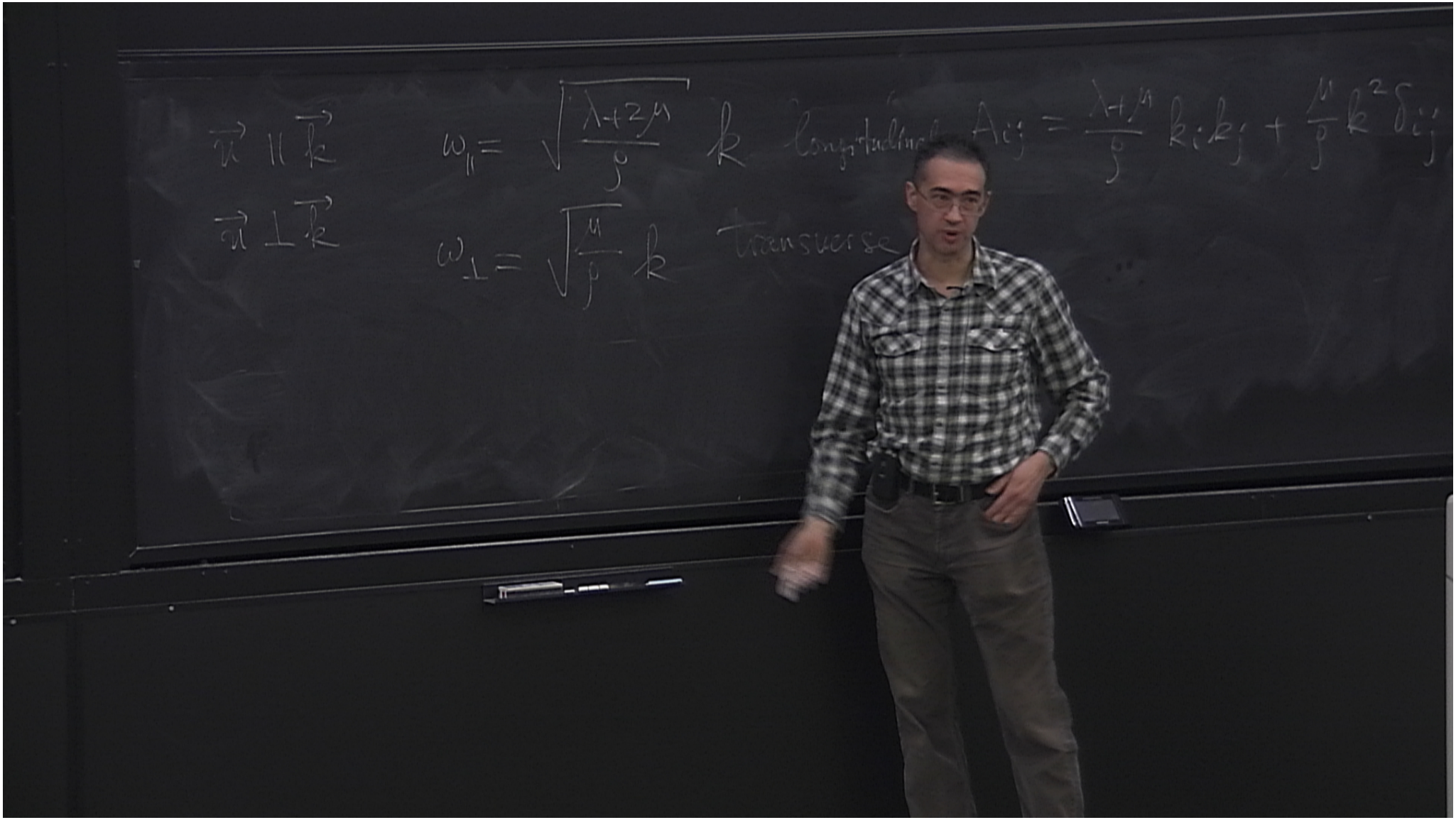

tions
$\rho \ddot{u}_i - \lambda \partial_i u_{jj} - \mu \partial_j u_{ij} = 0$
$\ddot{u}_i = \frac{\lambda}{\rho} \partial_i \partial_j u_j$

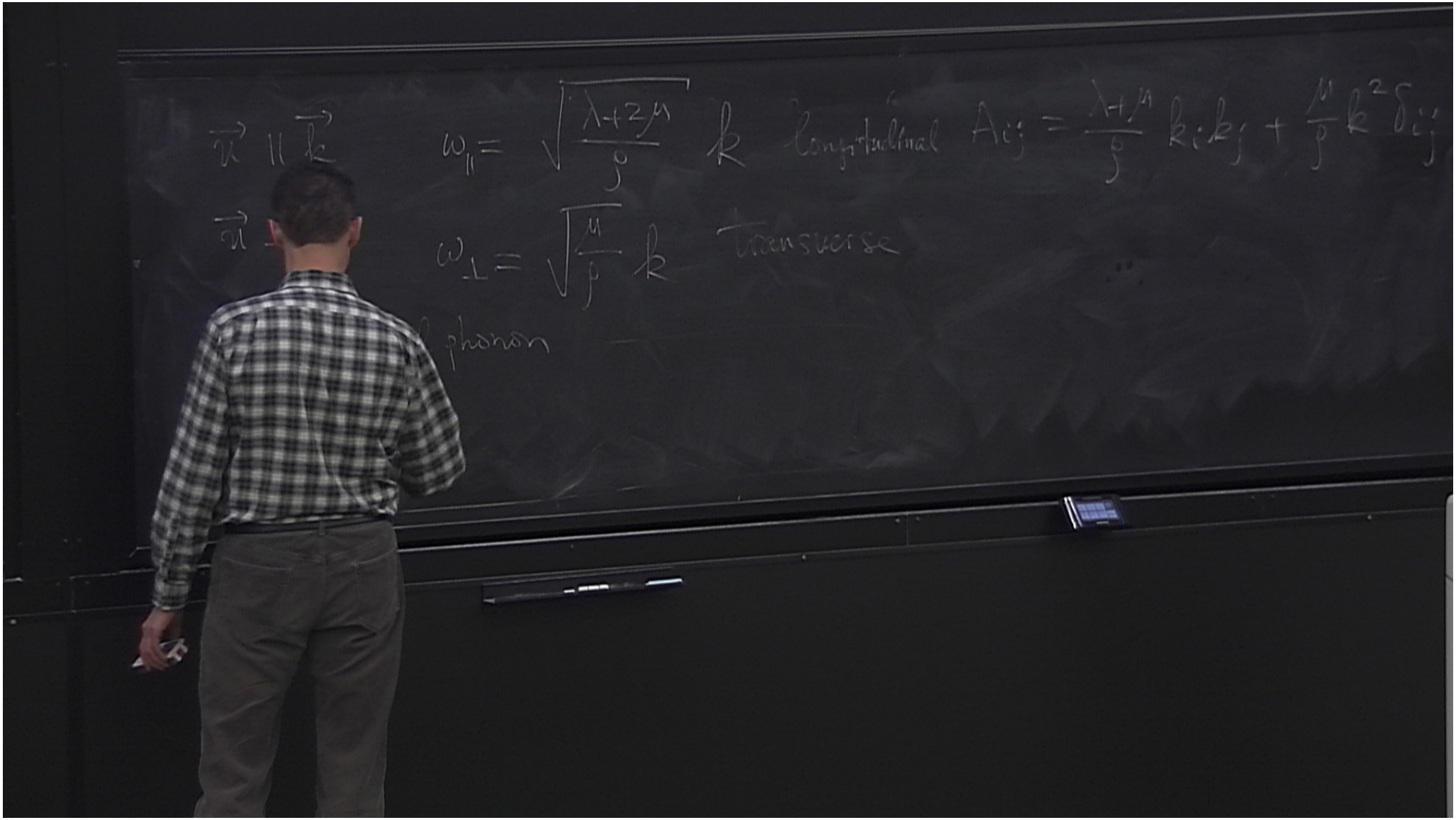

$\rho \ddot{u}_i - \lambda \partial_i u_{jj} - 2\mu \partial_j u_{ij} = 0$
$\ddot{u}_i = \frac{\lambda}{\rho} \partial_i \partial_j u_j + \frac{\mu}{\rho} \partial_j \partial_i u_j + \frac{\mu}{\rho} \partial_j \partial_j u_i$
$\ddot{u}_i = \frac{\lambda+\mu}{\rho} \partial_i \partial_j u_j + \frac{\mu}{\rho} \partial_j \partial_j u_i$

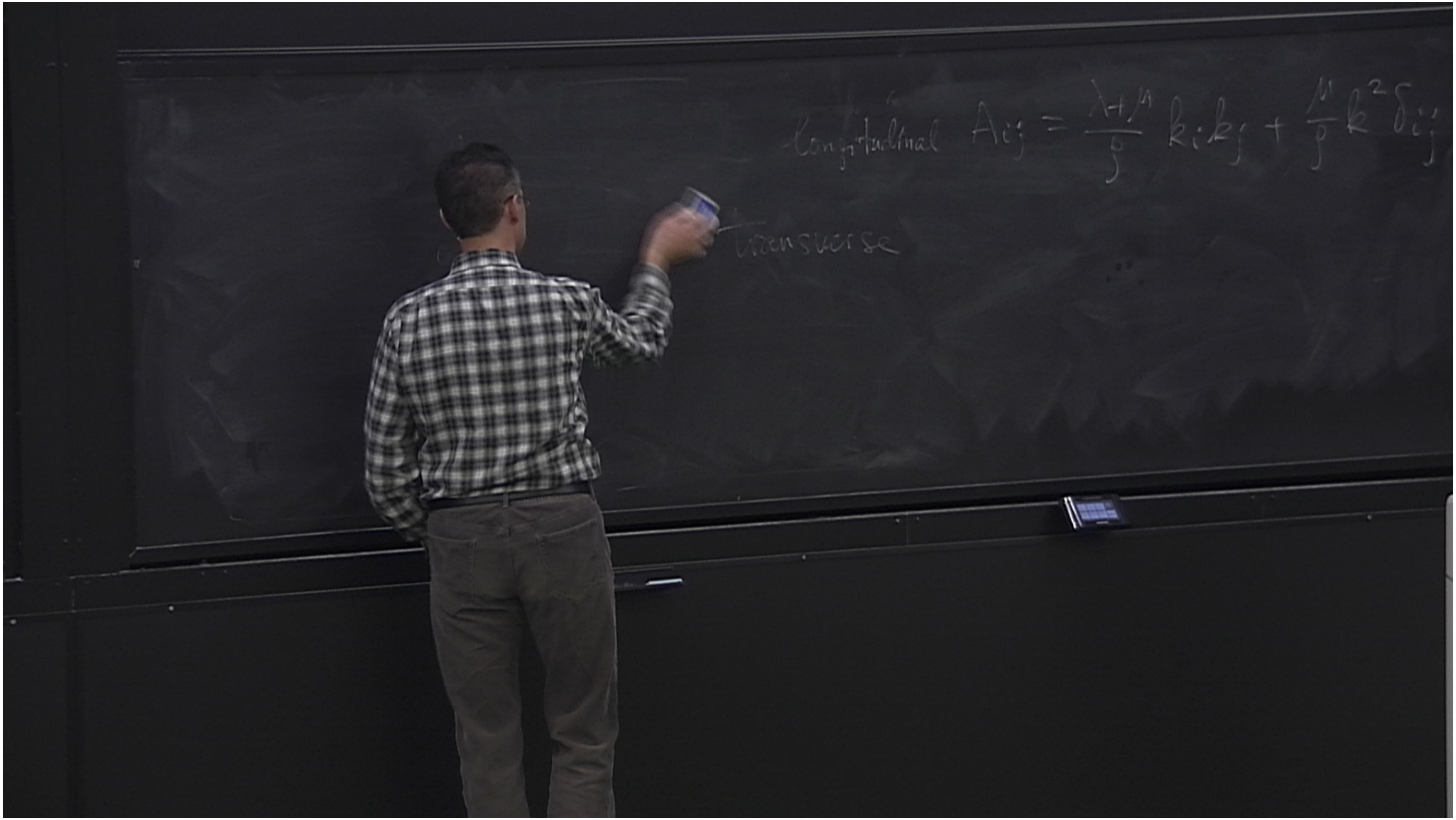

$\vec{u} = \vec{C}\, e^{i\vec{k}\cdot\vec{r} - i\omega t}$
$A_{ij} = \frac{\lambda+\mu}{\rho} k_i k_j + \frac{\mu}{\rho}$
$\frac{\lambda+\mu}{\rho} k_i k_j C_j + \frac{\mu}{\rho} k^2 C_i = \omega^2 C_i$
$A_{ij} C_j = \omega^2 C_i$

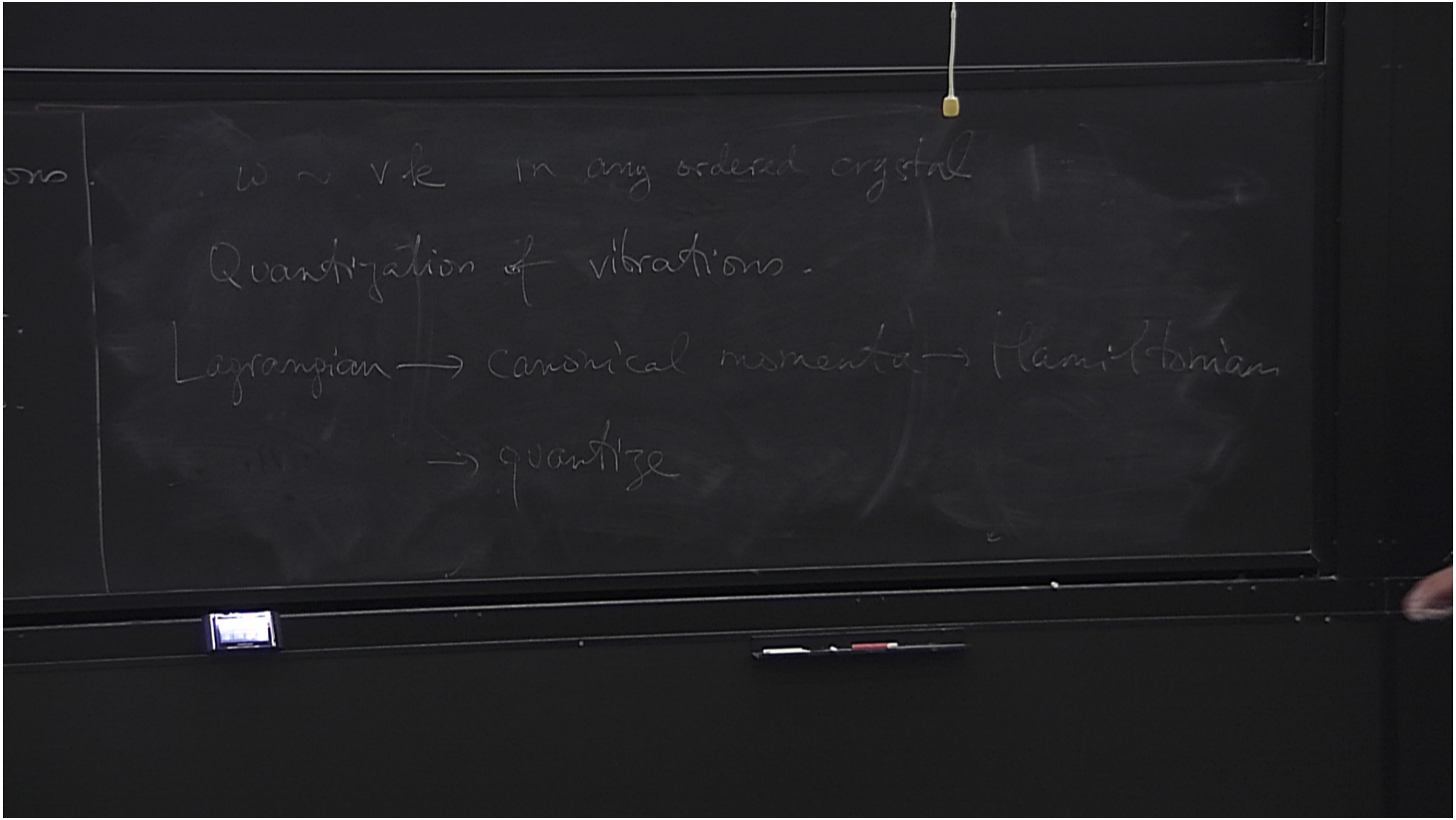

$d=2$, $\vec{k}=(k,0)$
$A=$

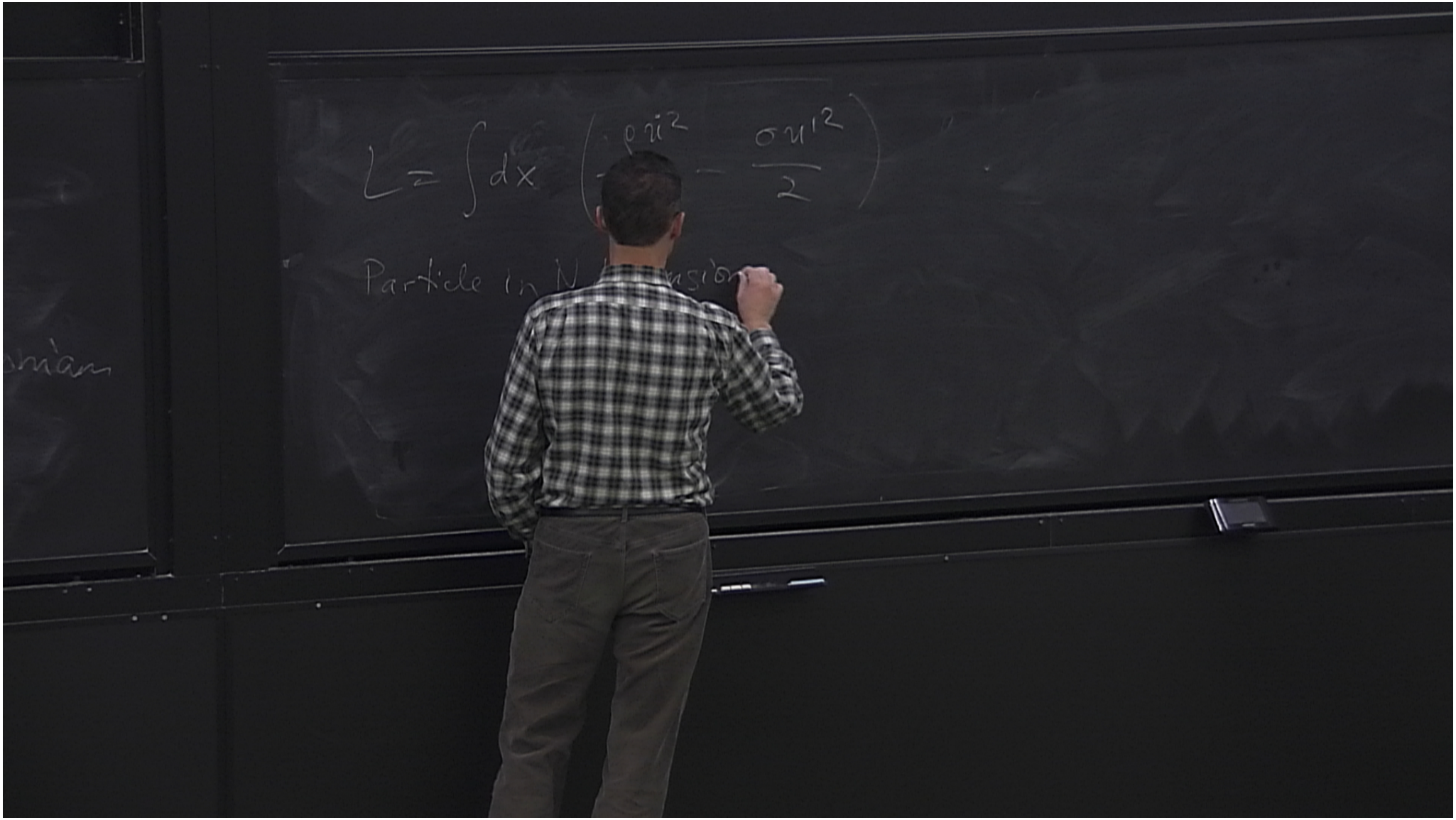

$\vec{u} \parallel \vec{k}$
$\omega = \sqrt{\frac{\lambda+2\mu}{\rho}}\, k$
$A_{ij} = \frac{\lambda+\mu}{\rho} k_i k_j + \frac{\mu}{\rho} k^2 \delta_{ij}$

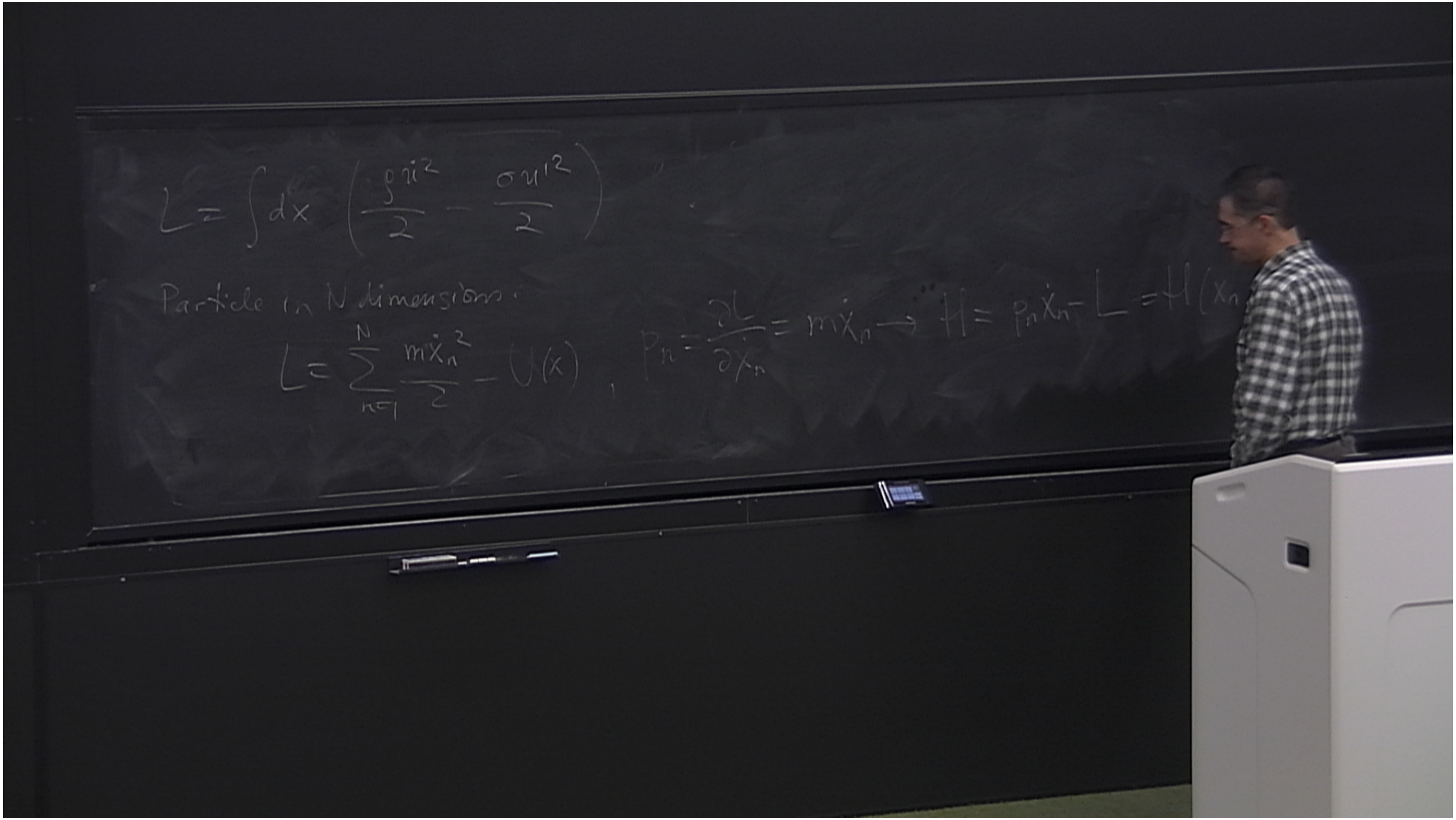

$\vec{u} \parallel \vec{k}$
$\omega_{\parallel} = \sqrt{\frac{\lambda+2\mu}{\rho}}\, k$
longitudinal
$A_{ij} = \frac{\lambda+\mu}{\rho} k_i k_j + \frac{\mu}{\rho} k^2 \delta_{ij}$
$\vec{u} \perp \vec{k}$
$\omega_{\perp} = \sqrt{\frac{\mu}{\rho}}\, k$
transverse

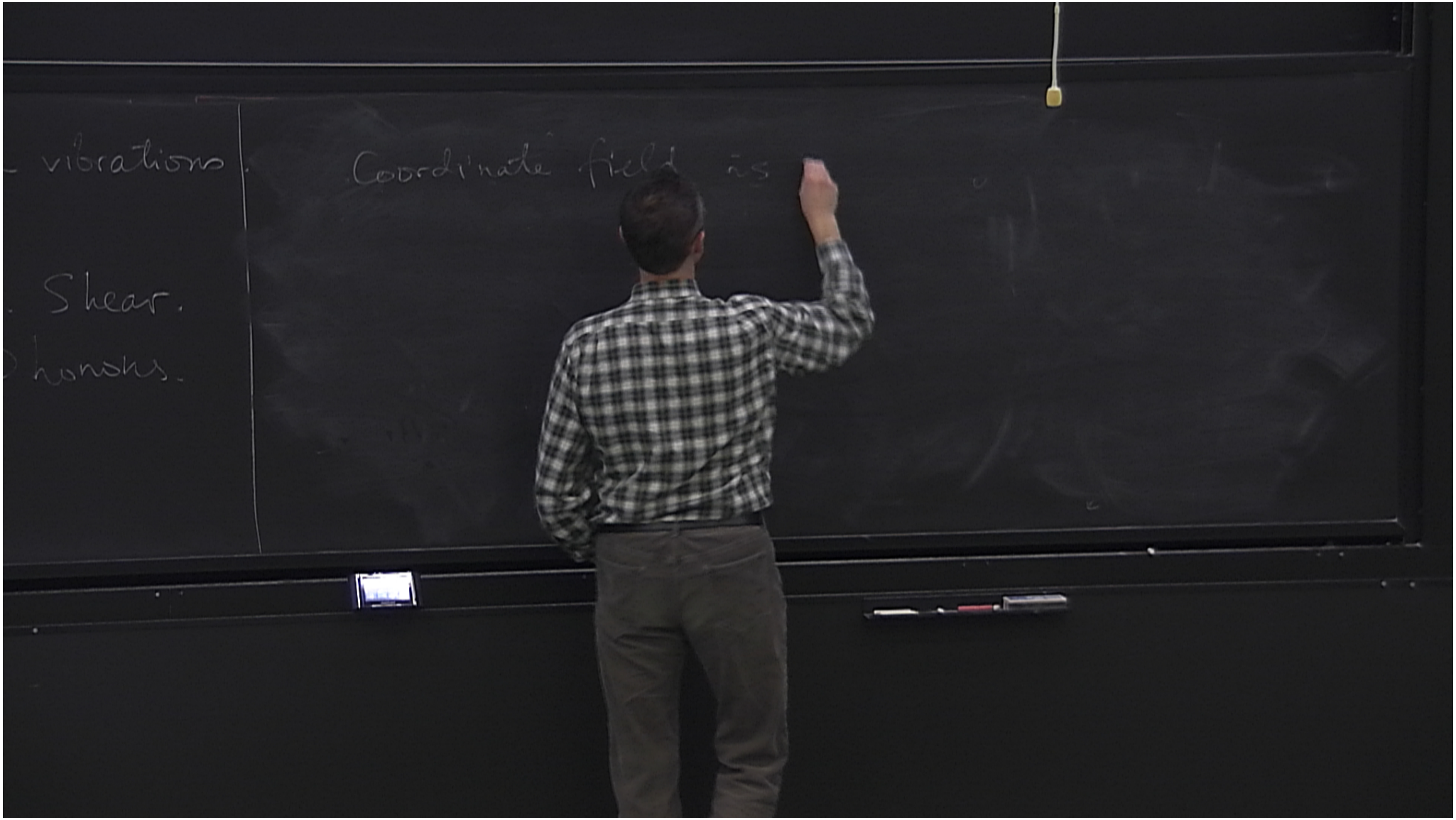

$\vec{u} \parallel \vec{k}$
$\omega_{\parallel} = \sqrt{\frac{\lambda+2\mu}{\rho}}\, k$
longitudinal
$A_{ij} = \frac{\lambda+\mu}{\rho} k_i k_j + \frac{\mu}{\rho} k^2 \delta_{ij}$
$\omega_{\perp} = \sqrt{\frac{\mu}{\rho}}\, k$
transverse
phonon

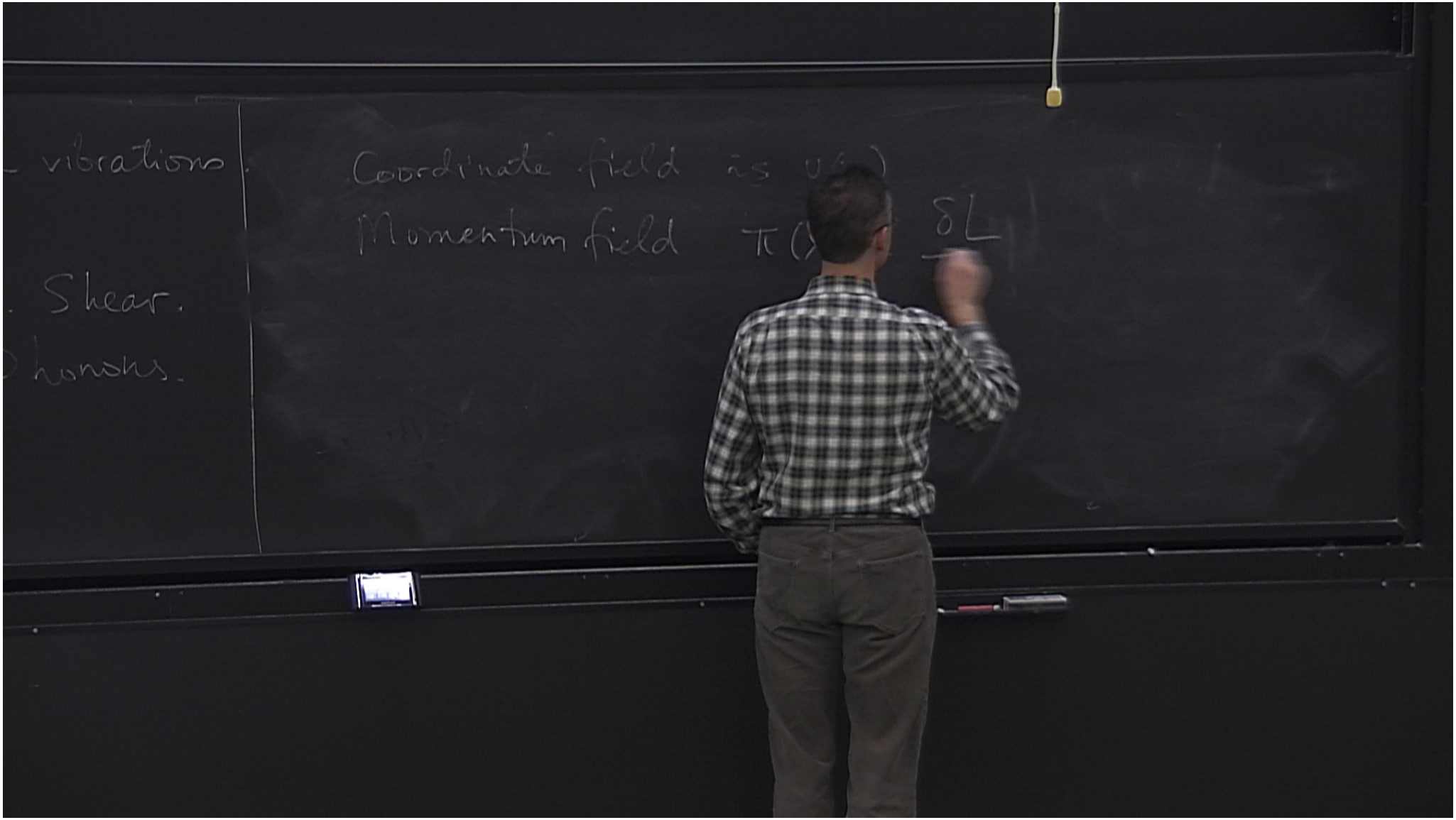

Longitudinal $A_{ij} = \frac{\lambda+\mu}{\rho} k_i k_j + \frac{\mu}{\rho} k^2 \delta_{ij}$
Transverse

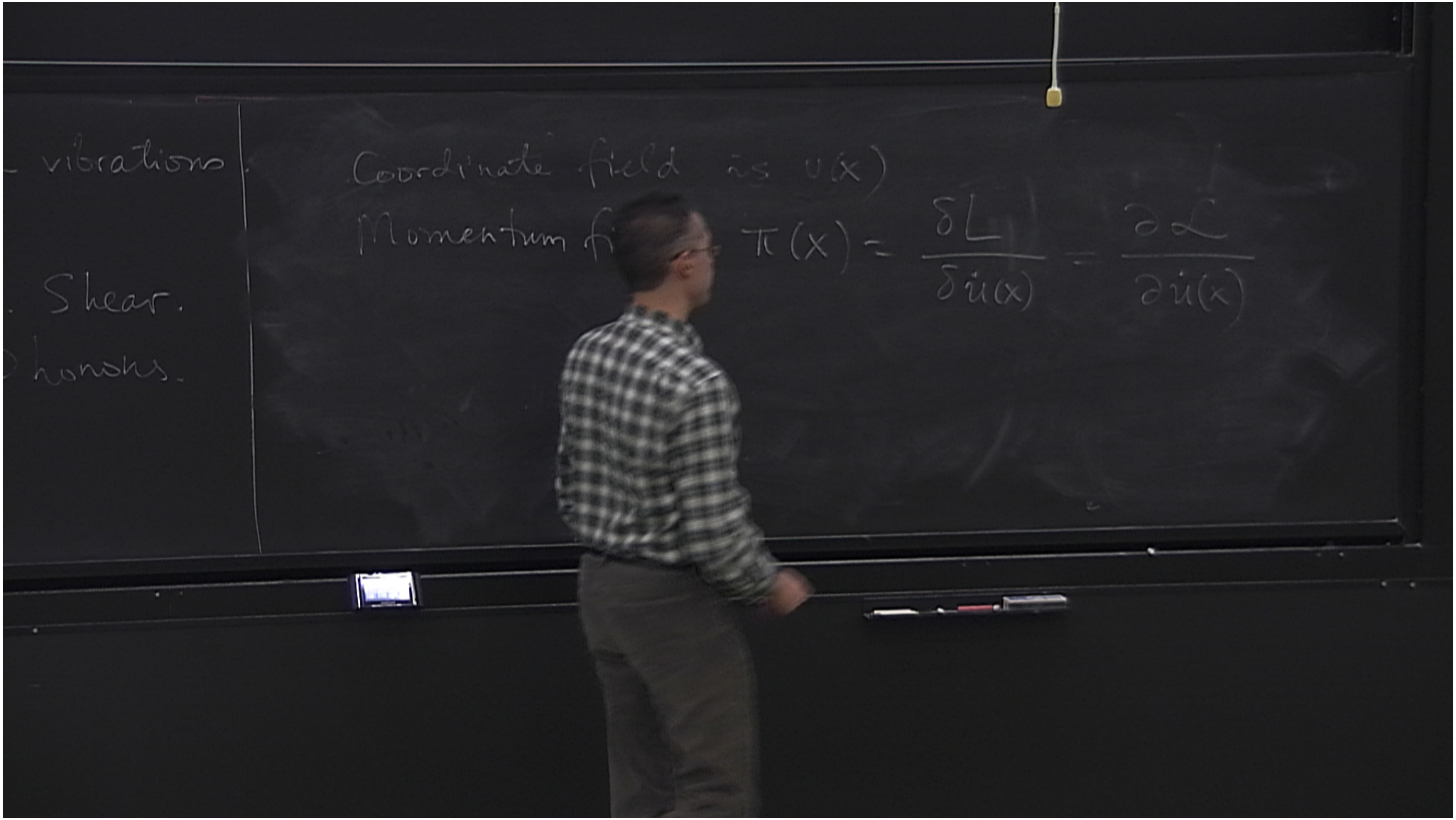

ω ~ v·k in any ordered crystal
Quantization of vibrations.
Lagrangian → canonical momenta → Hamiltonian
→ quantize

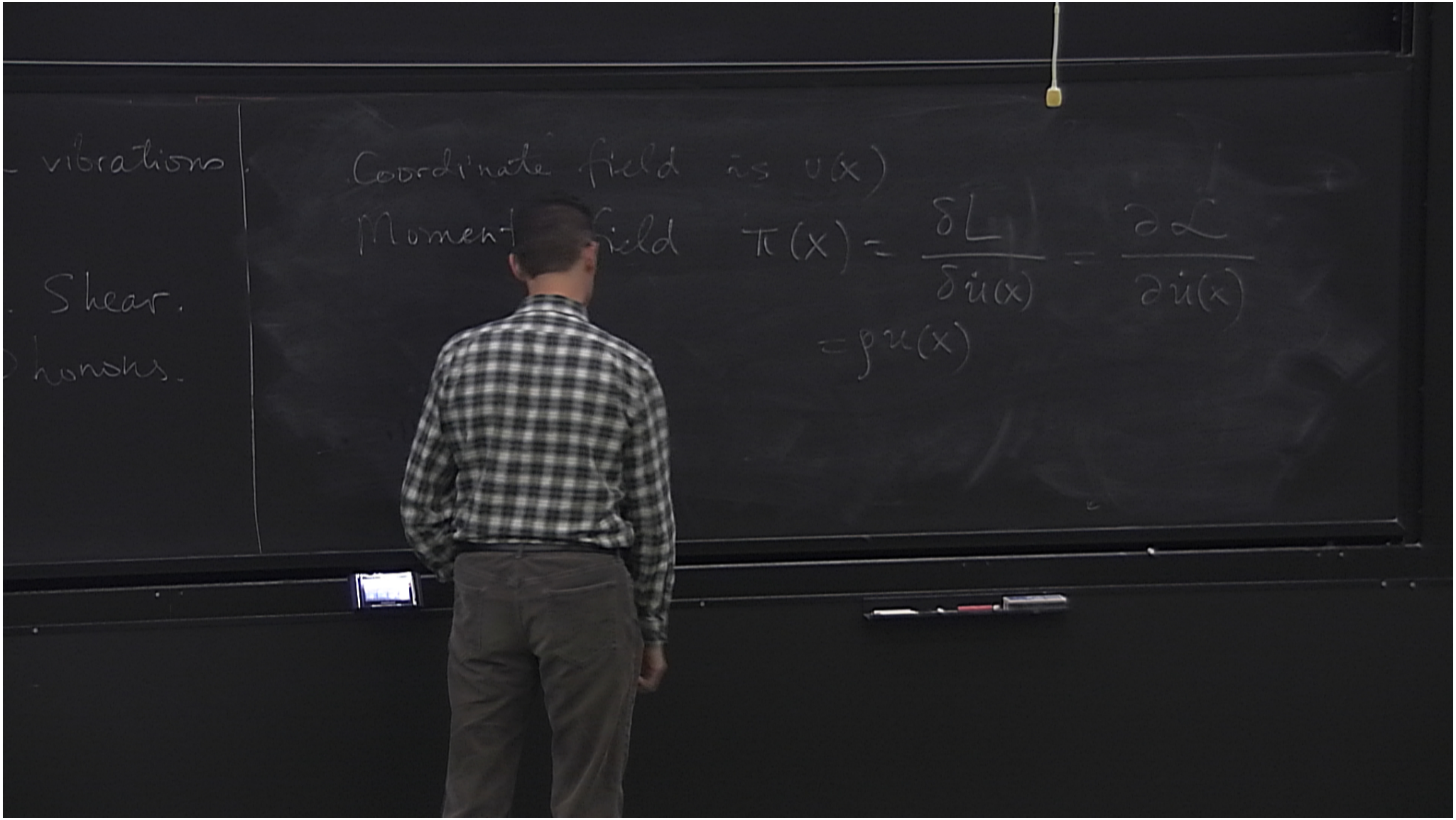

$L = \int dx \left(\frac{\rho \dot{u}^2}{2} - \frac{\sigma u'^2}{2} \right)$
Particle in N

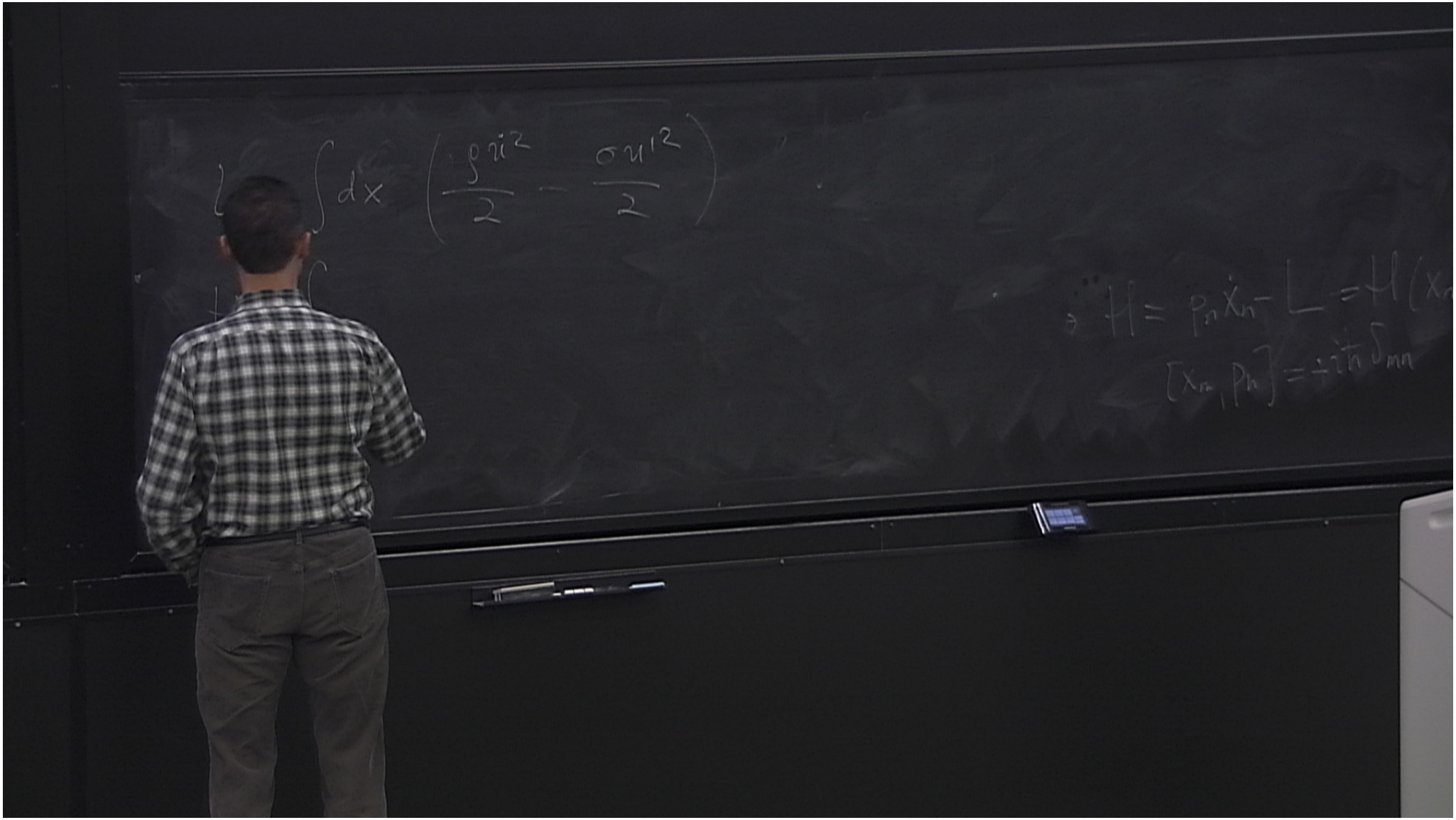

$L = \int dx \left(\frac{\rho \dot{\eta}^2}{2} - \frac{\sigma \eta'^2}{2} \right)$
Particle in N dimensions:
$L = \sum_{n=1}^{N} \frac{m \dot{x}_n^2}{2} - U(x)$, $p_n = \frac{\partial L}{\partial \dot{x}_n} = m \dot{x}_n \rightarrow H = p_n \dot{x}_n - L = H(x_n$

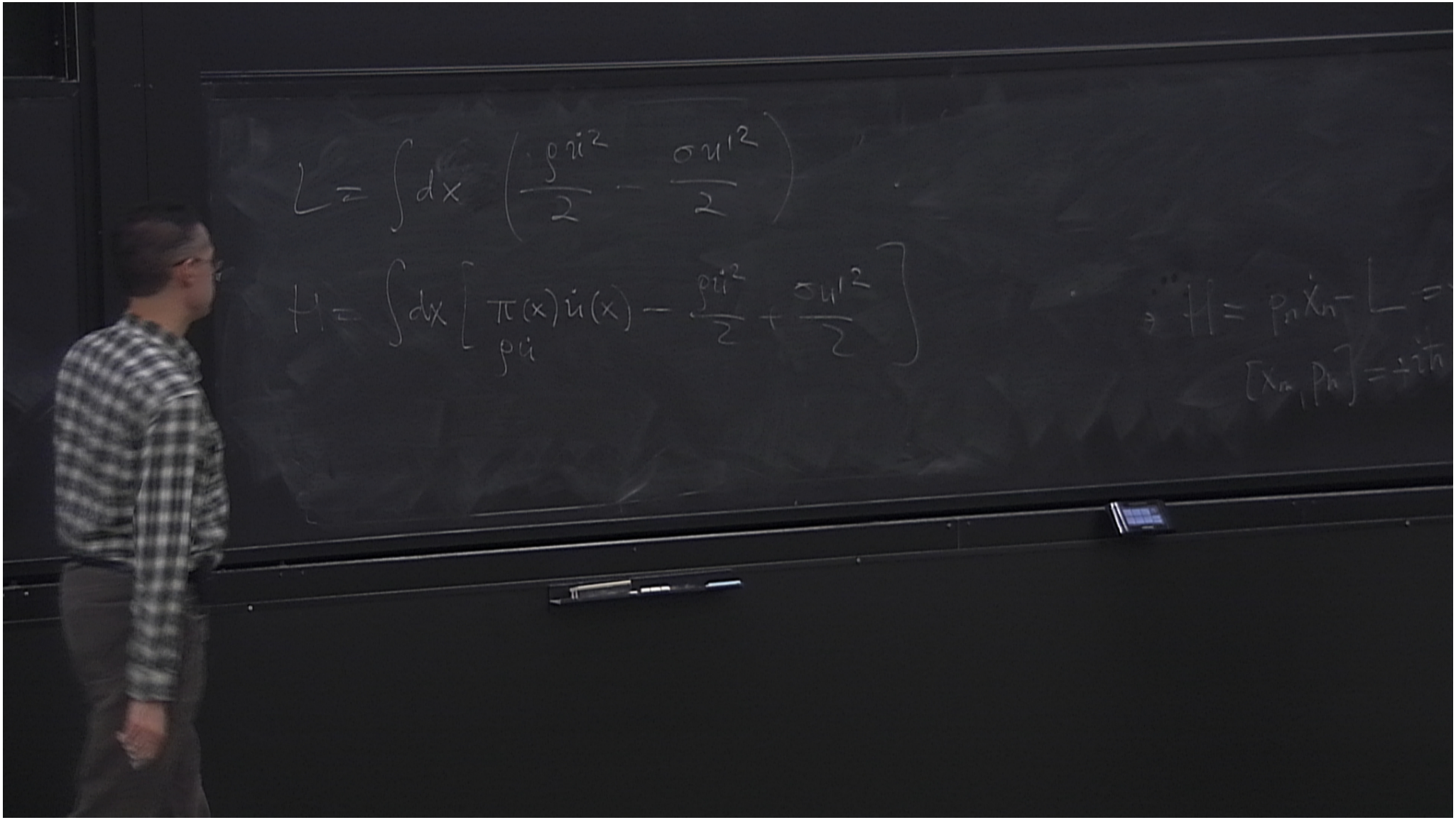

vibrations
Shear.
Coordinate field is

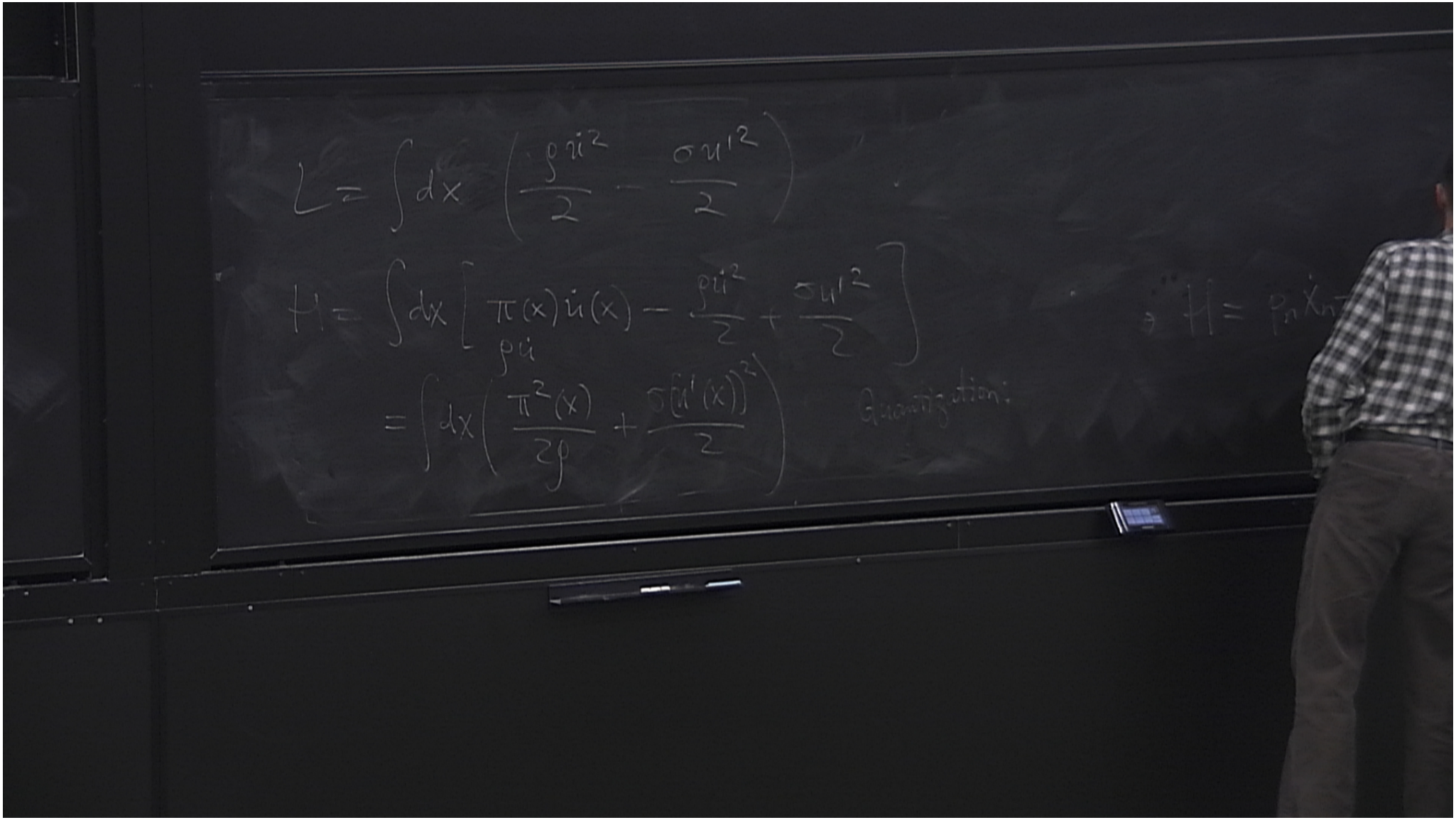

vibrations.
Shear.
honons.
Coordinate field is u()
Momentum field π()
δL

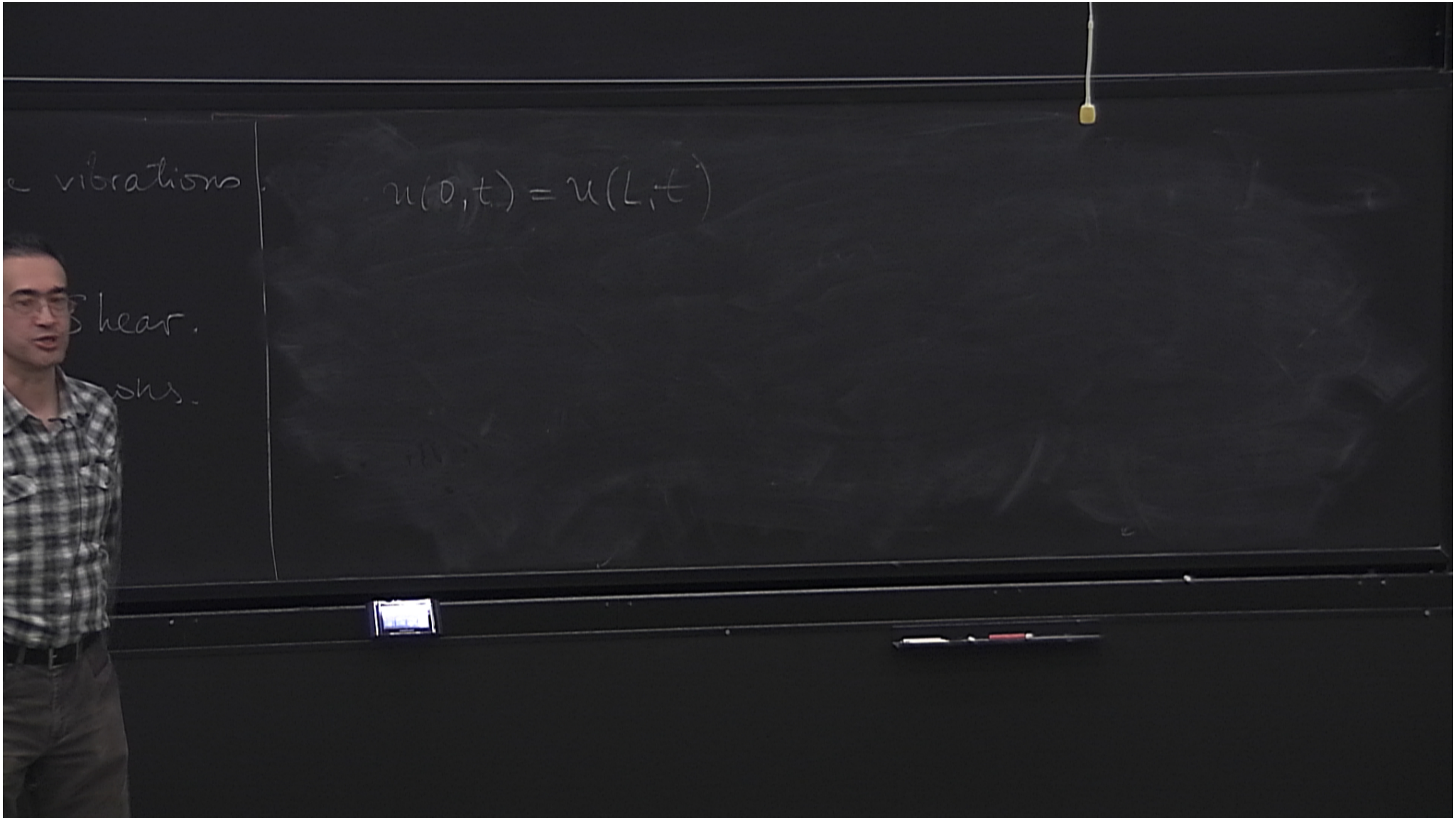

vibrations
Shear.
honons.
Coordinate field is u(x)
Momentum fi
$\pi(x) = \frac{\delta L}{\delta \dot{u}(x)} = \frac{\partial \mathcal{L}}{\partial \dot{u}(x)}$

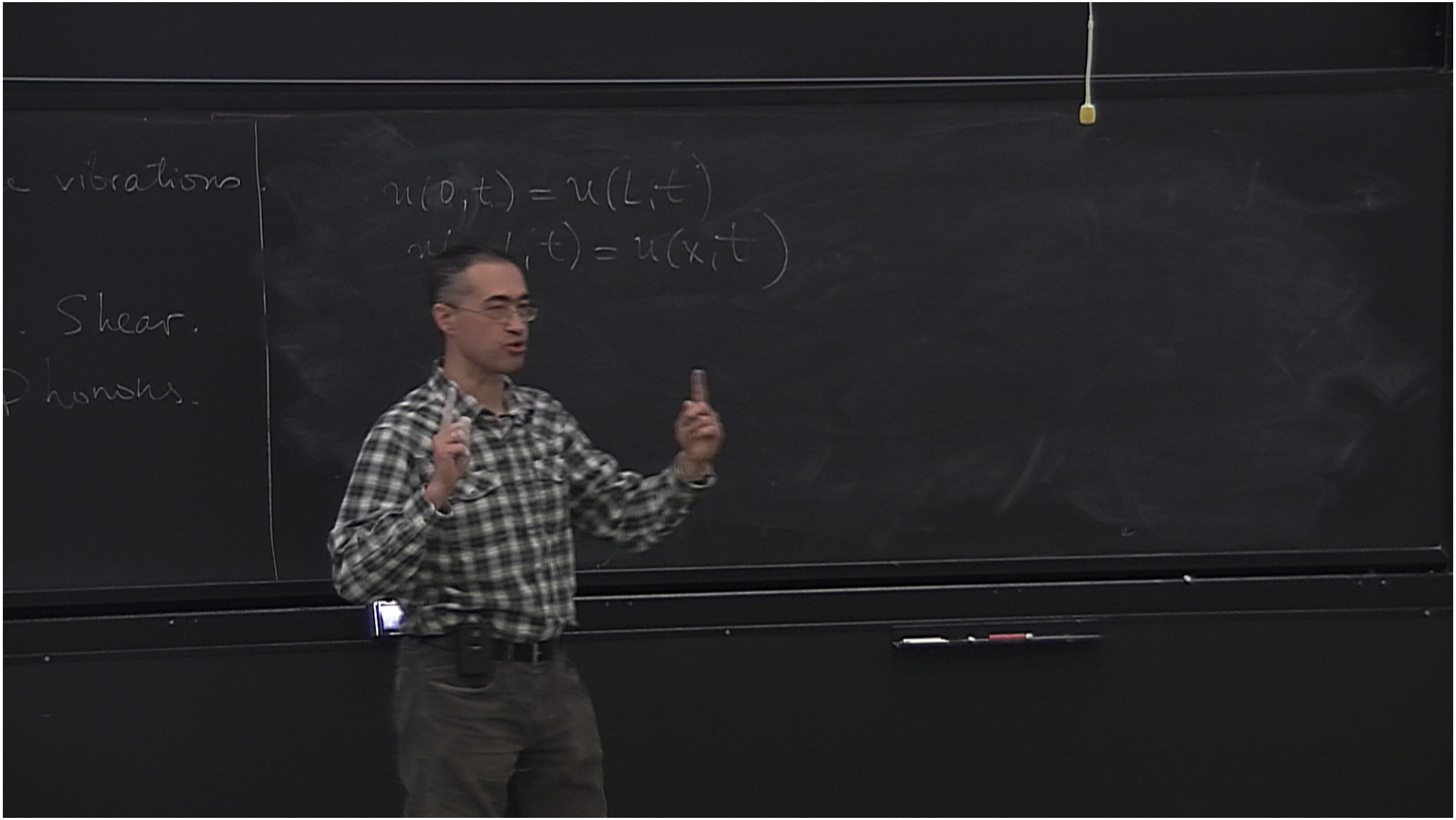

vibrations
Shear.
Coordinate field is u(x)
Moment field $\pi(x) = \frac{\delta L}{\delta \dot{u}(x)} = \frac{\partial \mathcal{L}}{\partial \dot{u}(x)}$
$= \rho \dot{u}(x)$

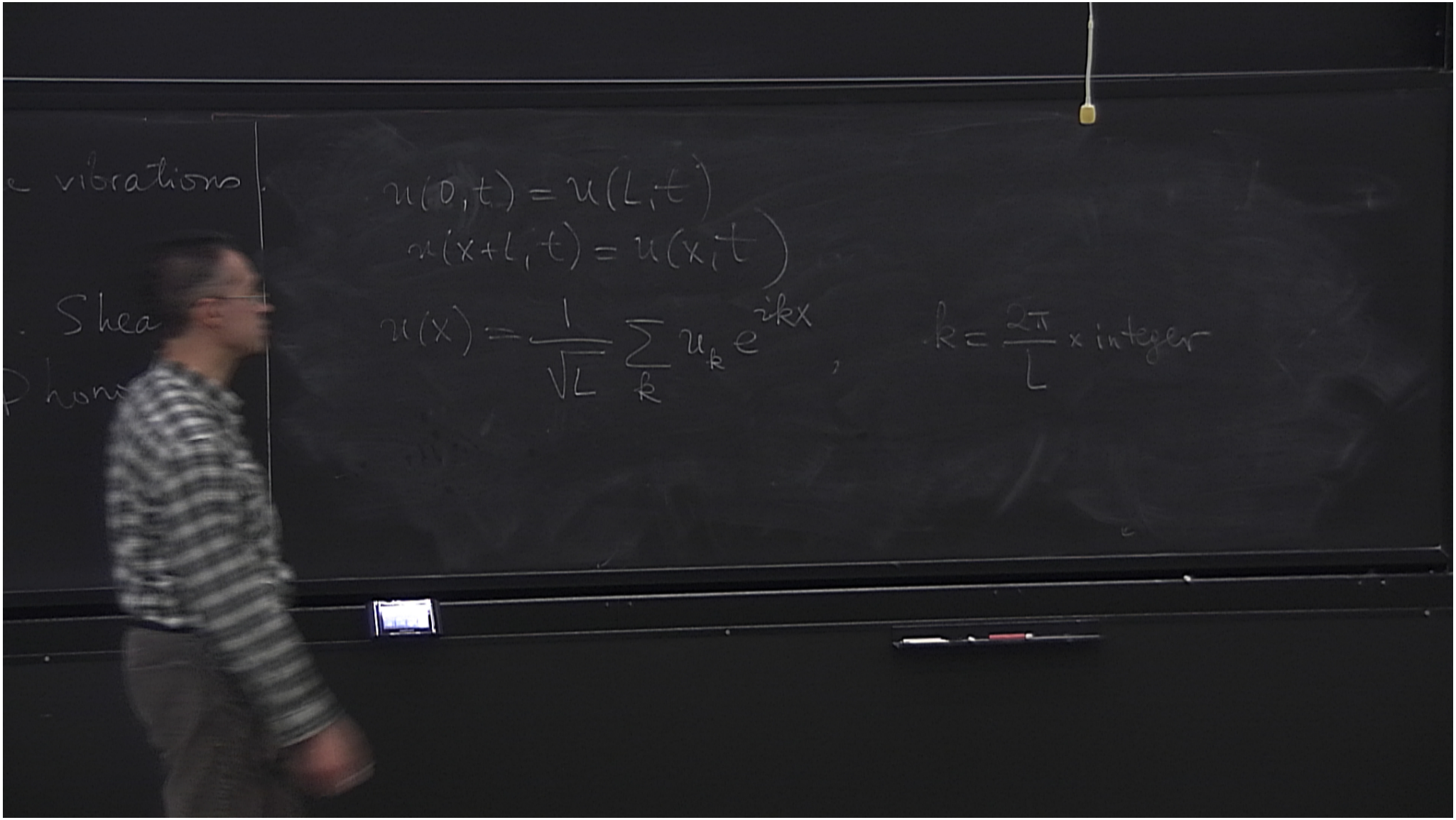

$\int dx \left(\frac{\rho \dot{u}^2}{2} - \frac{\sigma u'^2}{2} \right)$
$\Rightarrow H = p_n \dot{x}_n - L = H(x_n$
$[x_m, p_n] = +i\hbar \delta_{mn}$


$L = \int dx \left(\frac{\rho \dot{u}^2}{2} - \frac{\sigma u'^2}{2} \right)$
$H = \int dx \left[\pi(x) \dot{u}(x) - \frac{\rho \dot{u}^2}{2} + \frac{\sigma u'^2}{2} \right]$
$\pi = \rho \dot{u}$
$\Rightarrow H = p_n \dot{x}_n - L =$
$[x_n, p_n] = i\hbar$


$L = \int dx \left(\frac{\rho \dot{u}^2}{2} - \frac{\sigma u'^2}{2} \right)$
$H = \int dx \left[\pi(x)\dot{u}(x) - \frac{\rho \dot{u}^2}{2} + \frac{\sigma u'^2}{2} \right]$
$\rho \dot{u}$
$= \int dx \left(\frac{\pi^2(x)}{2\rho} + \frac{\sigma (u'(x))^2}{2} \right)$
Quantization:


vibrations
hear.
u(0,t) = u(L,t)


vibrations
Shear.
Phonons.
u(0,t) = u(L,t)
u(x,t) = u(x,t)


e vibrations
Shea
Phon
$u(0,t) = u(L,t)$
$u(x+L,t) = u(x,t)$
$u(x) = \frac{1}{\sqrt{L}} \sum_k u_k e^{ikx}$,
$k = \frac{2\pi}{L} \times \text{integer}$

e vibrations

. Shear.

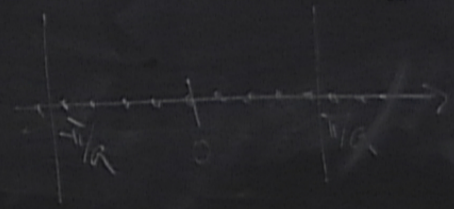
Phonons.

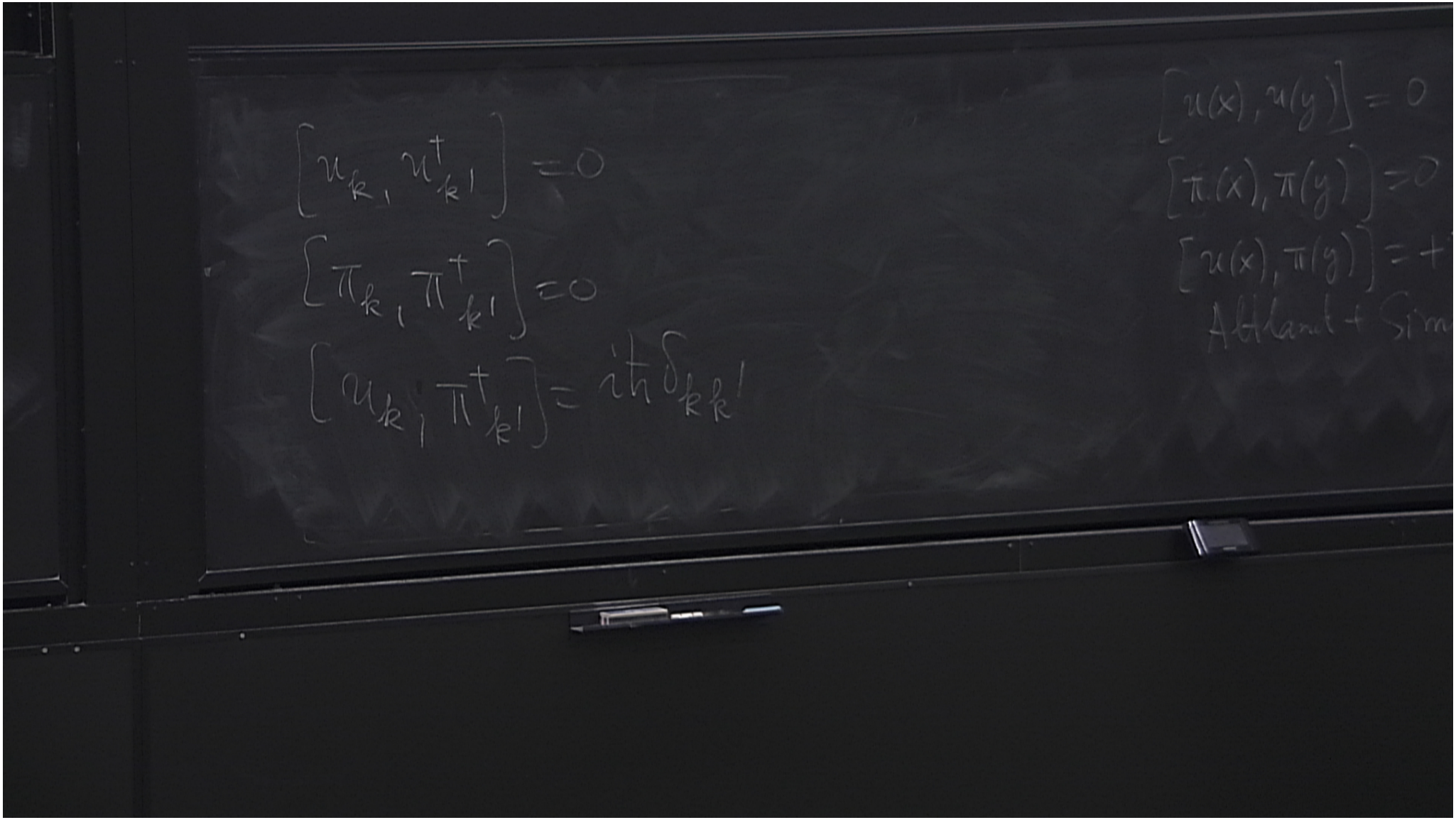
$u(0,t) = u(L,t)$

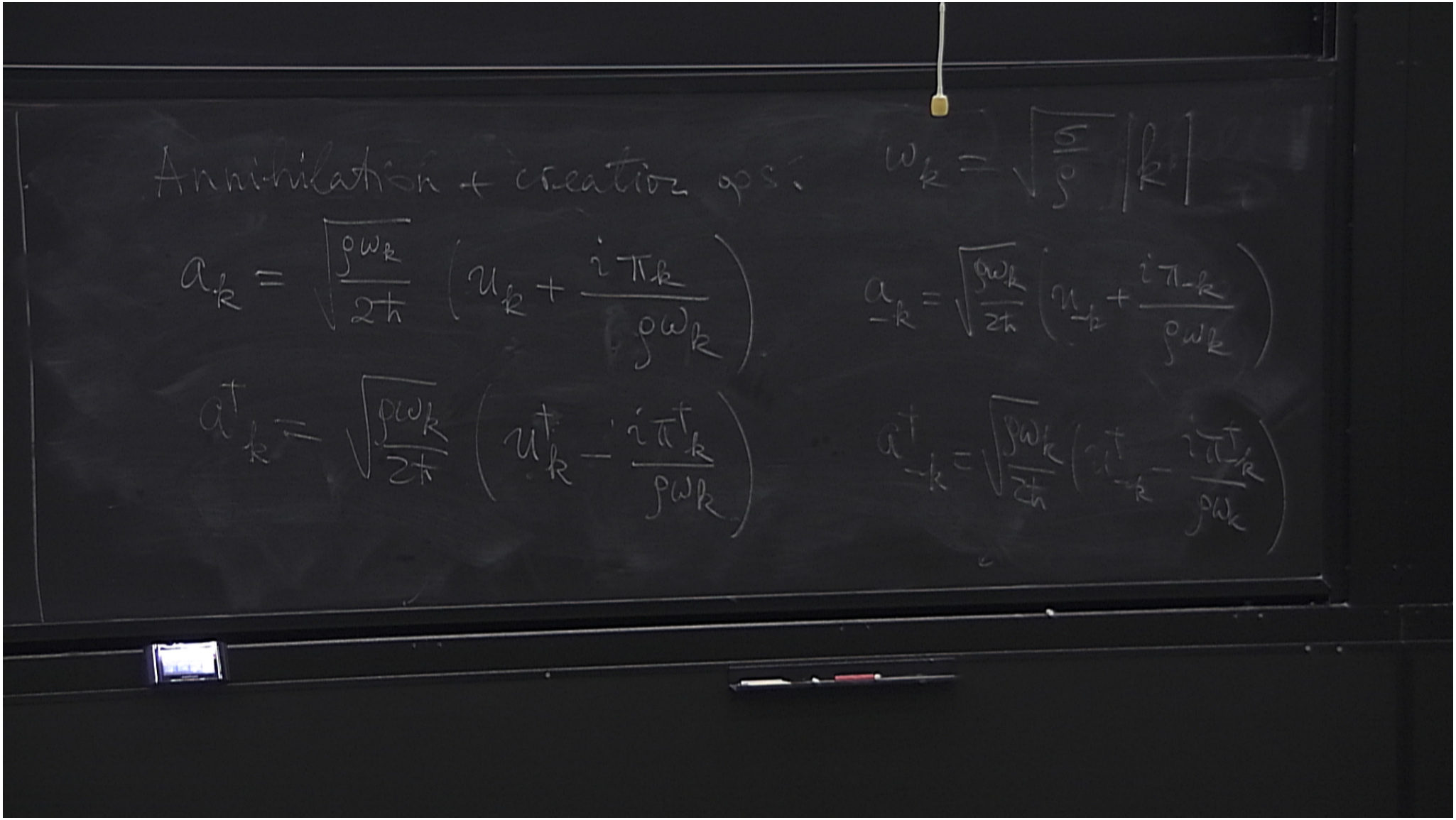
$u(x+L,t) = u(x,t)$

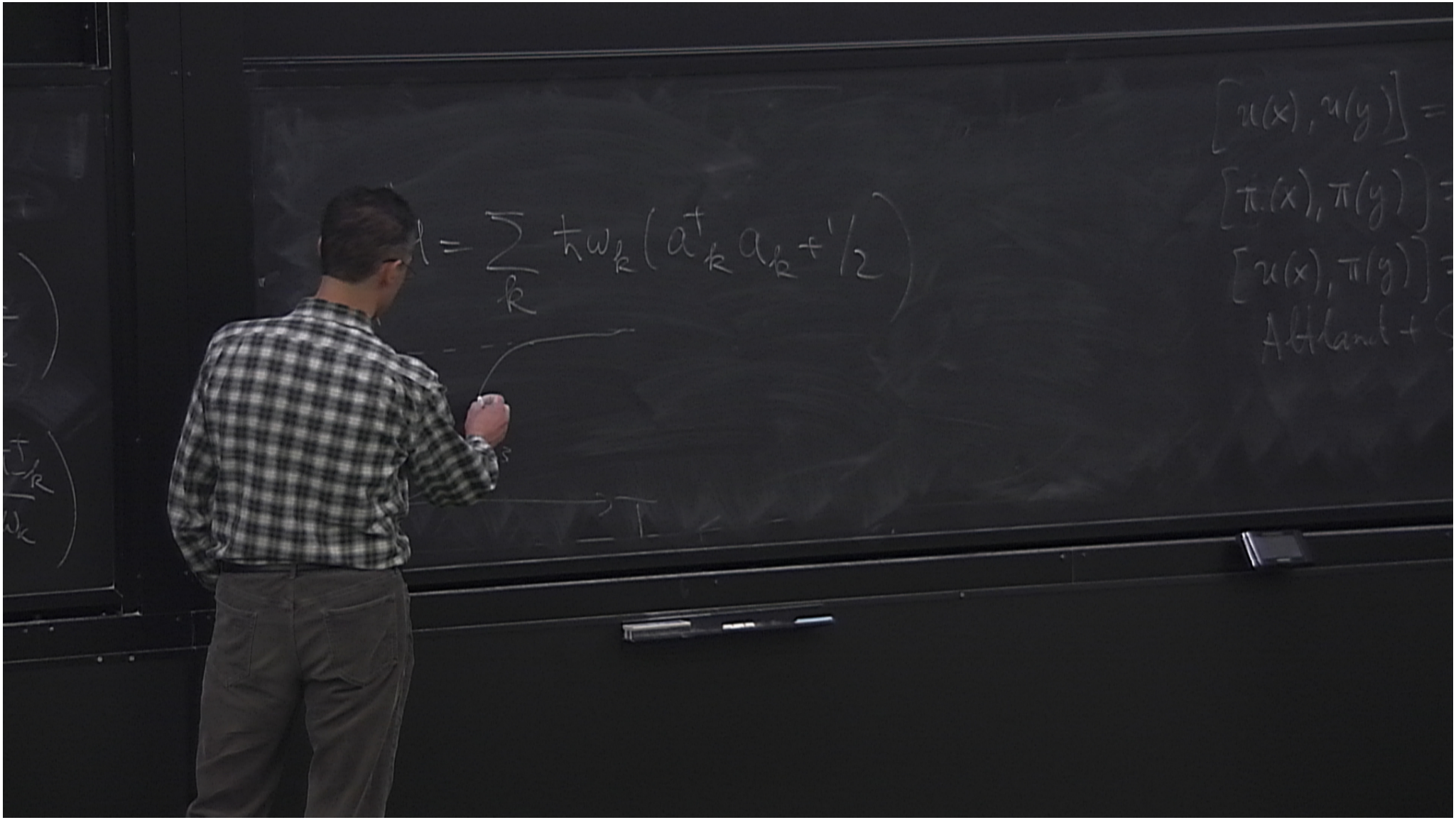
$$u(x) = \frac{1}{\sqrt{L}} \sum_k u_k e^{ikx}, \qquad k = \frac{2\pi}{L} \times \text{integer}$$

$$-\frac{\pi}{a} < k \leq \frac{\pi}{a}$$




$[u_k, u^\dagger_{k'}] = 0$
$[\pi_k, \pi^\dagger_{k'}] = 0$
$[u_k, \pi^\dagger_{k'}] = i\hbar\delta_{kk'}$
$[u(x), u(y)] = 0$
$[\pi(x), \pi(y)] = 0$
$[u(x), \pi(y)] = +$
Abhay + Sim


Annihilation + creation ops:
$\omega_k = \sqrt{\frac{s}{\rho}}\,|k|$
$a_k = \sqrt{\frac{\rho\omega_k}{2\hbar}}\left(u_k + \frac{i\pi_k}{\rho\omega_k}\right)$
$a_{-k} = \sqrt{\frac{\rho\omega_k}{2\hbar}}\left(u_{-k} + \frac{i\pi_{-k}}{\rho\omega_k}\right)$
$a_k^\dagger = \sqrt{\frac{\rho\omega_k}{2\hbar}}\left(u_k^\dagger - \frac{i\pi_k^\dagger}{\rho\omega_k}\right)$
$a_{-k}^\dagger = \sqrt{\frac{\rho\omega_k}{2\hbar}}\left(u_{-k}^\dagger - \frac{i\pi_{-k}^\dagger}{\rho\omega_k}\right)$


$H = \sum_k \hbar\omega_k \left(a^\dagger_k a_k + 1/2\right)$
$[u(x), u(y)] =$
$[\pi(x), \pi(y)] =$
$[u(x), \pi(y)] =$
Atlant