Title: PSI 2015/2016 Quantum Field Theory II - Francois David - Lecture 15

Date: Nov 27, 2015 09:00 AM

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Abstract:

Yang-Mills Theory: Perturbation Theory & Re

SU(2) Gauge Theory $\quad A = \{ A_\mu^a \, ; \, a = 1, 2, 3 \}$ Adj Repr

Dirac Field $\quad \psi = \{ \psi_\alpha^i \, , \; i = 1, 2 \leftarrow$ Fund Repr. $\; \alpha = 1, 4 \leftarrow$ Dirac $\}$

Feynman Gauge $\quad \bar{\psi}$

Ghost Fields $\quad C = \{ C^a \, ; \, a = 1, 3 \}$

$\partial_\mu A_0^\mu = \epsilon_a$ $\quad$ Grassman bul $\quad \bar{C} = \{ \bar{C}_a \, ; \, a = 1, 3 \}$

average over ϵ_a $\quad$ Spin 0

... Theory & Renormalization

..., 2, 3 } Adj Repr

1, 2 ← Fund Repr.

1, 4 ← Dirac

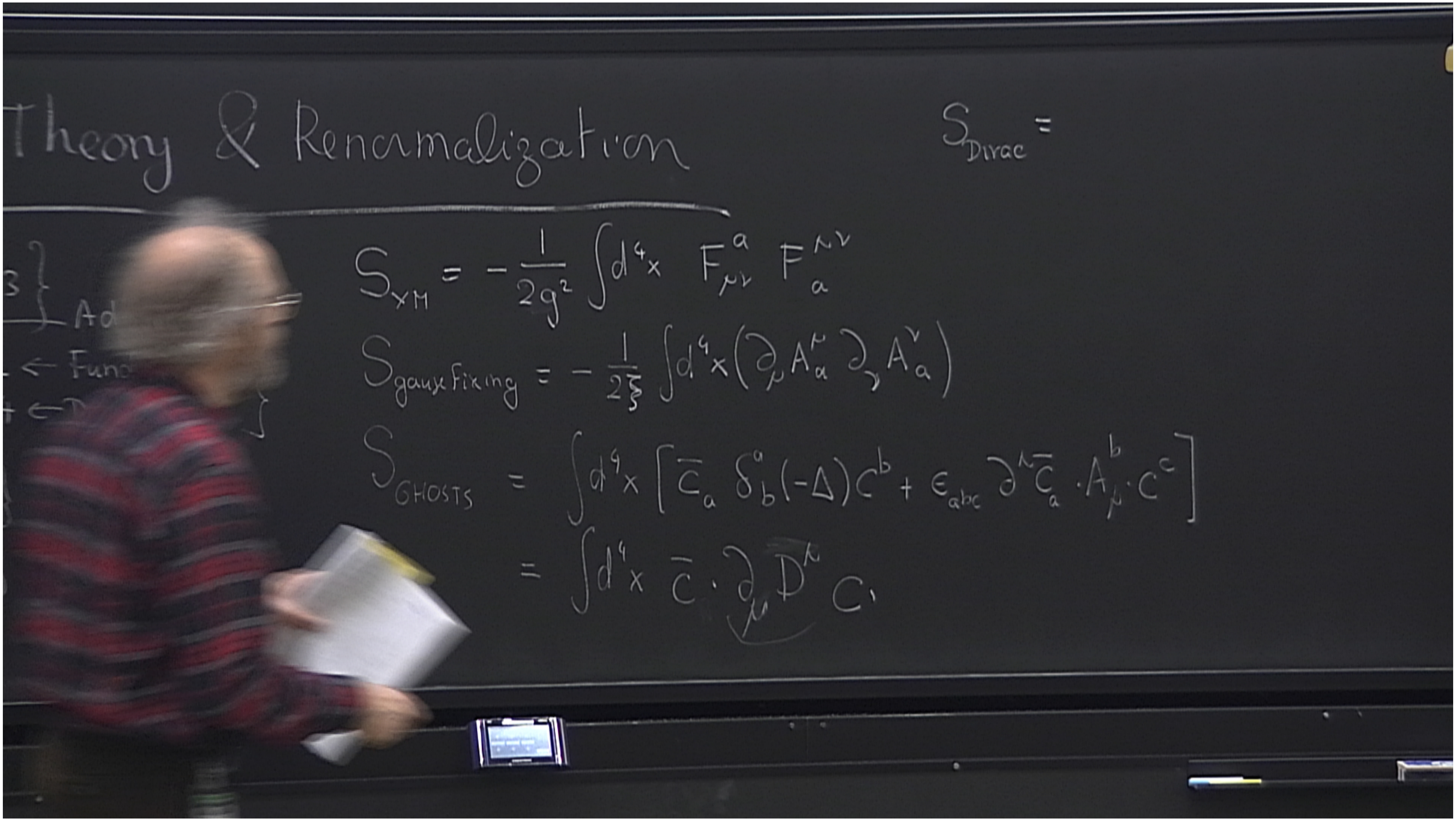
3 }

3 }

$$S_{YM} = -\frac{1}{2g^2}\int d^4x \; F^a_{\mu\nu} F^{\mu\nu}_a$$

$$S_{\text{gauge Fixing}} = -\frac{1}{2\xi}\int d^4x \left(\partial_\mu A^\mu_a \, \partial_\nu A^\nu_a\right)$$

$$S_{\text{GHOSTS}} = \int d^4x \;\; \bar{c}_a \, \delta^a_b (-\Delta) c^b + \epsilon_{abc} \, \partial^\mu \bar{c}_a \cdot A^b_\mu \cdot c^c$$


Theory & Renormalization
$S_{Dirac} =$
$S_{YM} = -\frac{1}{2g^2}\int d^4x \; F^a_{\mu\nu} F^{\mu\nu}_a$
$S_{gauge\ fixing} = -\frac{1}{2\xi}\int d^4x \left(\partial_\mu A^\mu_a \, \partial_\nu A^\nu_a\right)$
$S_{GHOSTS} = \int d^4x \left[\bar{c}_a \, \delta^a_b (-\Delta) c^b + \epsilon_{abc} \, \partial^\mu \bar{c}_a \cdot A^b_\mu \cdot c^c\right]$
$= \int d^4x \; \bar{c} \cdot \partial_\mu D^\mu c.$
Ad
Fund

σ_a Pauli Matrix

$$S_{\text{Dirac}} = \int d^4x\, \overline{\Psi}\,(i\not{D} - m)\,\Psi =$$

$$= \int d^4x\, \overline{\Psi}^{\alpha}_{i}\left[(i\gamma^{\mu})_{\alpha}^{\beta}\, \delta^{i}_{j}\, \partial_{\mu} - i\left(\frac{\sigma_a}{2}\right)^{i}_{j} \delta^{\beta}_{\alpha} A^{a}_{\mu}(x) - m\delta^{\beta}_{\alpha}\delta^{i}_{j}\right]\Psi^{\beta}_{j}$$

on

$d^4x\; F^{a}_{\mu\nu}\, F^{\mu\nu}_{a}$

$\frac{1}{2\xi}\int d^4x \left(\partial_\mu A^{\mu}_{a}\, \partial_\nu A^{\nu}_{a}\right)$

$x\left[\bar{c}_a\, \delta^{a}_{b}(-\Delta)c^{b} + \epsilon_{abc}\,\partial^{\mu}\bar{c}_a \cdot A^{b}_{\mu}\cdot c^{c}\right]$

$^4x\; \bar{c}\cdot\partial_\mu D^{\mu}\, c$

$\partial_\mu A^\mu_a = \epsilon_a$

average over ϵ_a

Ghost Fields

Grassman but

Spin 0

$C = \{ C^a ; a = 1,3 \}$

$\bar{C} = \{ \bar{C}_a ; a = 1,3 \}$

$S_{GHOSTS} = \int d^4x \left[\bar{C}_a \partial_b (-\Delta) C^b + \epsilon_{abc} \ldots \right.$

$= \int d^4x \; \bar{C} \cdot \partial_\mu D^\mu C$

$$Z = \int D[A]\, D[C,\bar{C}]\, D[\bar{\Psi},\Psi]\, \exp\left(i \left(S_{YM}[A] + S_{G.FIXING}[A] + S_{GHOSTS}[A,C,\bar{C}] + S_{Dirac}[\Psi,\bar{\Psi},A] \right) \right)$$

$$\langle \Omega | T(\text{operators}) | \Omega \rangle = \frac{\int D[\text{Fields}] \exp(i \,\text{Action}) (\text{operators})}{Z}$$

Perturbation Theory = expansion in g^2 ; <-> or $\hbar$: $A^a_\mu \to g A^a_\mu$

gauge Field propagator

$$\sim\sim\sim = G^{ab}_{\mu\nu}(k) \simeq \langle A^a_\mu(x) A^b_\nu(y) \rangle_0$$

...ization

$$-\frac{1}{2g^2}\int d^4x \; F^a_{\mu\nu} \, F_a^{\mu\nu}$$

$$\ldots g = -\frac{1}{2\xi}\int d^4x \left(\partial_\mu A^\mu_a \, \partial_\nu A^\nu_a\right)$$

$$= \int d^4x \left[\bar{c}_a \, \delta^a_b(-\Delta) c^b + \epsilon_{abc}\, \partial^\mu \bar{c}_a \cdot A^b_\mu \cdot c^c\right]$$

$$= \int d^4x \; \bar{c} \cdot \partial_\mu D^\mu \, c$$

$$S_{Dirac} = \int d^4x \; \bar{\Psi} \, (i \not{D} - m)\, \Psi =$$

σ_a Pauli Matrix

$$= \int d^4x \; \bar{\Psi}^\alpha_i \left[(i\gamma^\mu)^\beta_\alpha \, \delta_{ij} \, \partial_\mu - i \left(\frac{\sigma_a}{2}\right)_{ij} \delta^\beta_\alpha \, A^a_\mu(x) - m \, \delta^\beta_\alpha \, \delta_{ij} \right] \Psi^\beta_j$$

$$F_{\mu\nu} = g\,\partial A - g\,\partial A + i g^2 [A, A]$$

$$\ldots, c, \bar{c}] + S_{Dirac}[\Psi, \bar{\Psi}, A]))$$

$\partial_\mu A^\mu_0 = \epsilon_a$ Ghost Fields $c = \{ c^a ; a = 1,3 \}$ $\mathcal{L}_{GHOSTS} = \int d^4x \, [\bar{c}_a \, \delta_b (-\Delta) c^b + \epsilon_{ab} \ldots$

average over ϵ_a Grassman but spin 0 $\bar{c} = \{ \bar{c}_a ; a = 1,3 \}$ $= \int d^4x \; \bar{c} \cdot \partial_\mu D^\mu c \cdot$

$$Z = \int D[A] \, D[c, \bar{c}] \, D[\bar{\psi}, \psi] \, \exp\left(i \left(S_{YM}[A] + S_{G\text{-}FIXING}[A] + S_{GHOSTS}[A, c, \bar{c}] + S_{Dirac}[\psi, \bar{\psi}, A] \right) \right)$$

$$\langle \Omega | T(\text{operators}) | \Omega \rangle = \frac{\int D[\text{Fields}] \exp(i \, \text{Action}) \, (\text{operators})}{Z}$$

Perturbation Theory = expansion in g^2; <-> or $\hbar$: $A^a_\mu \to g A^a_\mu$ linearized terms are of order 1

gauge field propagator

$${}^a_\mu \xrightarrow{k} {}^b_\nu = G^{ab}_{\mu\nu}(k) \simeq \langle A^a_\mu(x) A^b_\nu(y) \rangle_0 = \delta^{ab} \frac{-i}{k^2 - i\epsilon_+} \left(h_{\mu\nu} + (\xi - 1) \frac{k_\mu k_\nu}{k^2} \right) \qquad k^2 = -E^2 + \vec{k}^2$$

$(-,+,+,+)$

$\partial_\mu A_0^\mu = \epsilon_a$

average over ϵ_a

Ghost Fields
Grassman but
Spin 0

$c = \{ c^a ; a = 1, 3 \}$

$\bar{c} = \{ \bar{c}_a ; a = 1, 3 \}$

$$S_{\text{GHOSTS}} = \int d^4x \left[\bar{c}_a \, \delta^a_b (-\Delta) c^b + \epsilon_{abc} \, \partial^\mu \bar{c}_a \cdot A^b_\mu \cdot c^c \right.$$

$$= \int d^4x \; \bar{c} \cdot \partial_\mu D^\mu c \cdot$$

$$Z = \int D[A] \, D[c, \bar{c}] \, D[\bar{\psi}, \psi] \, \exp\left(i \left(S_{YM}[A] + S_{\text{G.FIXING}}[A] + S_{\text{GHOSTS}}[A, c, \bar{c}] + S_{\text{Dirac}}[\psi, \bar{\psi}, A] \right) \right)$$

$$\langle \Omega | T(\text{operators}) | \Omega \rangle = \frac{\int D[\text{Fields}] \exp(i \, \text{Action}) \, (\text{operators})}{Z}$$

g = charge of gauge bosons & fermions $= e$ in QED
and also the coupling constant
(redefinition of A_μ)

<u>Perturbation Theory</u> = expansion in g^2; <-> or $\hbar$: $A^a_\mu \to g A^a_\mu$

linearized terms
are of order 1

gauge Field propagator

metric of space-time

$$ {}^{a}_{\mu} \overset{k}{\sim\!\sim\!\sim\!\sim} {}^{b}_{\nu} = G^{ab}_{\mu\nu}(k) \simeq \langle A^a_\mu(x) A^b_\nu(y) \rangle_0 = \delta^{ab} \frac{-i}{k^2 - i\epsilon_+} \left(h_{\mu\nu} + (\xi - 1) \frac{k_\mu k_\nu}{k^2} \right)$$

$k^2 = -E^2 + \vec{k}^2$
$(-, +, +, +)$

$$h_{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

$$\exp\left(i \left(S_{YM}[A] + S_{G.FIXING}[A] + S_{GHOSTS}[A, c, \bar{c}] + S_{Dirac}[\psi, \bar{\psi}, A] \right) \right)$$

$]\exp(i\,A_{cha})$ (operators)

g = charge of gauge bosons & fermions $= e$ in QED

and also the coupling constant

(redefinition of A_μ)

$g^2 : \leftrightarrow$ or $\hbar$: $A^a_\mu \to g\, A^a_\mu$

linearized terms are of order 1

ξ is a free parameter (gauge Fixing)

metric of space-time

$$\langle A^b_\nu(y) \rangle_0 = \delta^{ab} \frac{-i}{k^2 - i\epsilon_+} \left(h_{\mu\nu} + (\xi - 1) \frac{k_\mu k_\nu}{k^2} \right)$$

$k^2 = -E^2 + \vec{k}^2$

$(-, +, +, +)$

$$h_{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

QED

...ing constant

(gauge Fixing)

space-time

Ghosts $\quad a \overset{k}{- \to - -} b = \langle c^a(x)\, \bar{c}^b(y) \rangle_0$

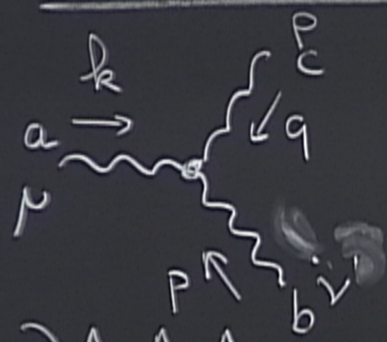
$$= \delta_{ab} \frac{-i}{k^2 - i\epsilon_+}$$

Dirac Propagator

$$\underset{\alpha}{i} \longrightarrow \underset{\beta}{j} = \langle \psi^i_\alpha(x)\, \bar{\psi}^j_\beta(y) \rangle_0$$

$= \delta^{ij} \times$ Dirac propagator as in QED

Interaction vertices



$$g \; \epsilon_{abc} \left(h^{\mu\nu}(k-p)^{\rho} + h^{\nu\rho}(p-q)^{\mu} + h^{\rho\mu}(q-k)^{\nu} \right)$$

$\uparrow$

SU(2)

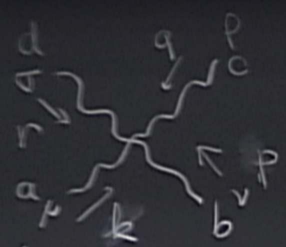
polarization – momentum

$\frac{g^3}{g^2}$ $\partial A \cdot A \cdot A$

$a, b, c = 1, 2, 3$ gauge indices

$\mu, \nu, \rho = 1, \cdot\, 4$ Lorentz indices

k, p, q 4-momenta $\quad k+p+q=0$



$$= g^2(-i)\left[\begin{array}{l}\epsilon_{abe}\epsilon_{cde}\left(h^{\mu\rho}h^{\nu\sigma}-h^{\mu\sigma}h^{\nu\sigma}\right)\\ +\epsilon_{ace}\epsilon_{bde}\left(h^{\mu\nu}h^{\rho\sigma}-h^{\mu\sigma}h^{\nu\rho}\right)\\ +\epsilon_{ade}\epsilon_{bce}\left(h^{\mu\nu}h^{\rho\sigma}-h^{\mu\sigma}h^{\nu\sigma}\right)\end{array}\right]$$

$\frac{g^4}{g^2}$ AAAA

$\epsilon_{ace}\epsilon_{bde} = \delta_{ab}\delta_{ce} - \delta_{ad}\delta_{bc}$ for $SU(2)$

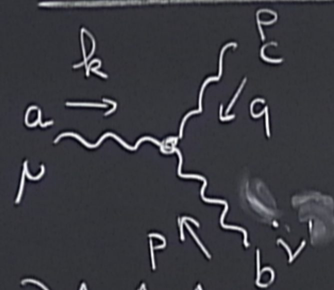
$a,b,c,d = 1,2,3$

$\mu\nu,\rho,\sigma = 1,\cdots 4$

k,p,q,r 4-momenta

Interaction vertices

3 gauge bosons interaction



$$g \, \epsilon_{abc} \left(h^{\mu\nu}(k-p)^{\rho} + h^{\nu\rho}(p-q)^{\mu} + h^{\rho\mu}(q-k)^{\nu} \right)$$

$\uparrow$

SU(2)

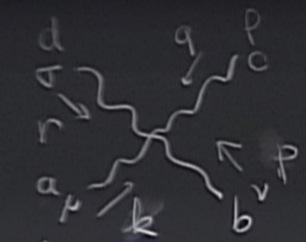
polarization – momentum

$$\frac{g^3}{g^2} \; \partial A \cdot A \cdot A$$

$a, b, c, = 1, 2, 3$ gauge indices

$\mu, \nu, \rho = 1, \cdot\, 4$ Lorentz indices

k, p, q 4-momenta $\quad k+p+q=0$



4 gauge boson interaction

$$= g^2(-i)\left[\begin{aligned}&\epsilon_{abe}\epsilon_{cde}\left(h^{\mu\rho}h^{\nu\sigma}-h^{\mu\sigma}h^{\nu\sigma}\right)\\ &+\epsilon_{ace}\epsilon_{bde}\left(h^{\mu\nu}h^{\rho\sigma}-h^{\mu\sigma}h^{\nu\rho}\right)\\ &+\epsilon_{ade}\epsilon_{bce}\left(h^{\mu\nu}h^{\rho\sigma}-h^{\mu\sigma}h^{\nu\sigma}\right)\end{aligned}\right]$$

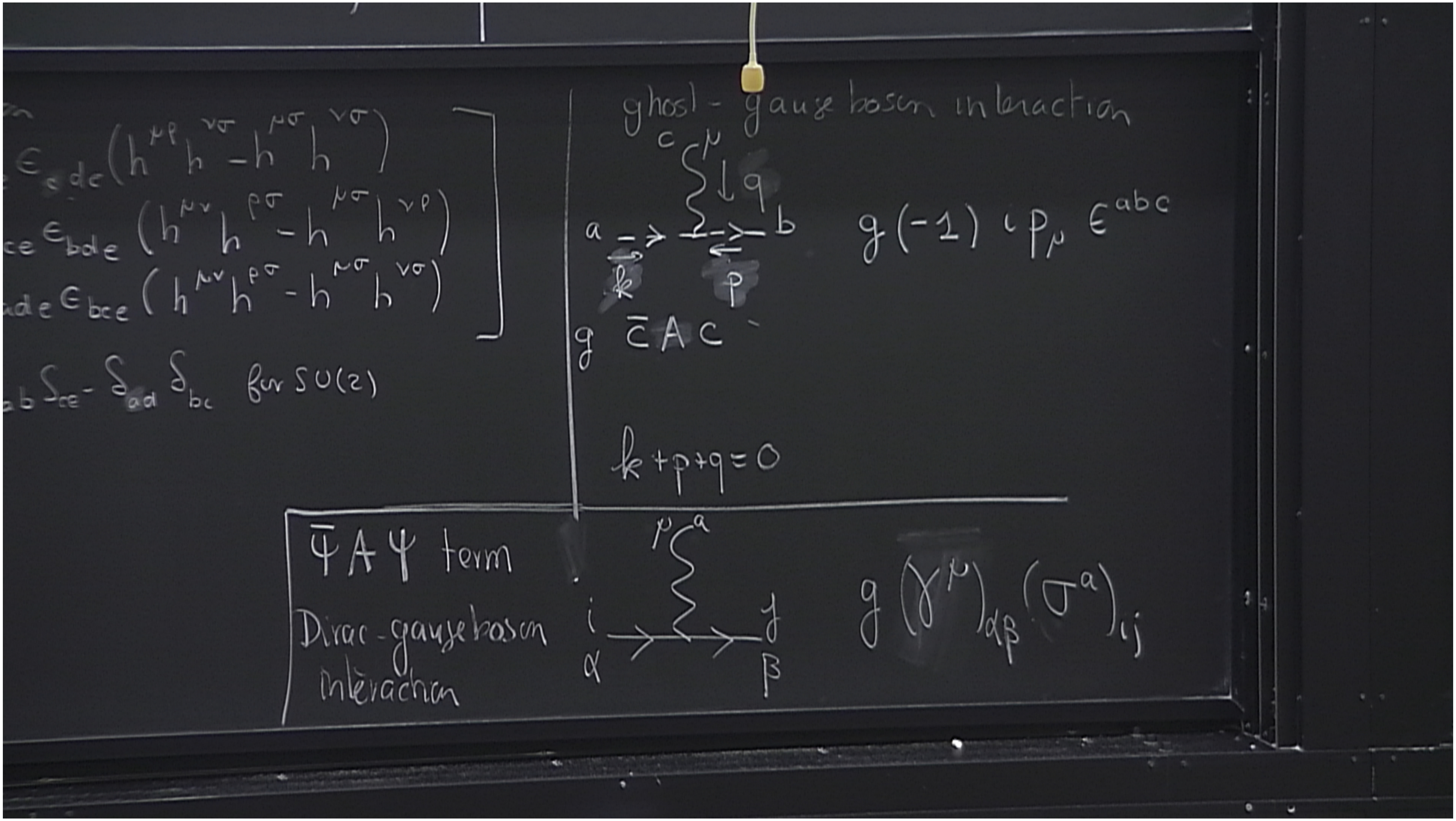
$\frac{g^4}{g^2}$ AAAA

$a,b,c,d = 1,2,3$

$\mu,\nu,\rho,\sigma = 1,\dots 4$

k, p, q, r 4-momenta

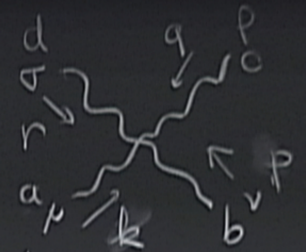
$\epsilon_{ace}\epsilon_{bde} = \delta_{ab}\delta_{ce} - \delta_{ad}\delta_{bc}$ for SU(2)


ghost - gauge boson interaction
$g(-1)\, i\, p_\mu\, \epsilon^{abc}$
$g\, \bar{c} A c$
$k+p+q=0$
$\bar{\Psi} A \Psi$ term
Dirac - gauge boson interaction
$g\, (\gamma^\mu)_{\alpha\beta} (\sigma^a)_{ij}$
for SU(2)

interaction

$-q)^\mu + h^{\rho\mu}(q-k)^\nu \Big)$

momentum



$\frac{g^4}{g^2}$ A A A A

$a, b, c, d = 1, 2, 3$

$\mu\nu, \rho, \sigma = 1, \dots 4$

k, p, q, r 4-momenta

charges: SU(2) charges

4 gauge boson interaction

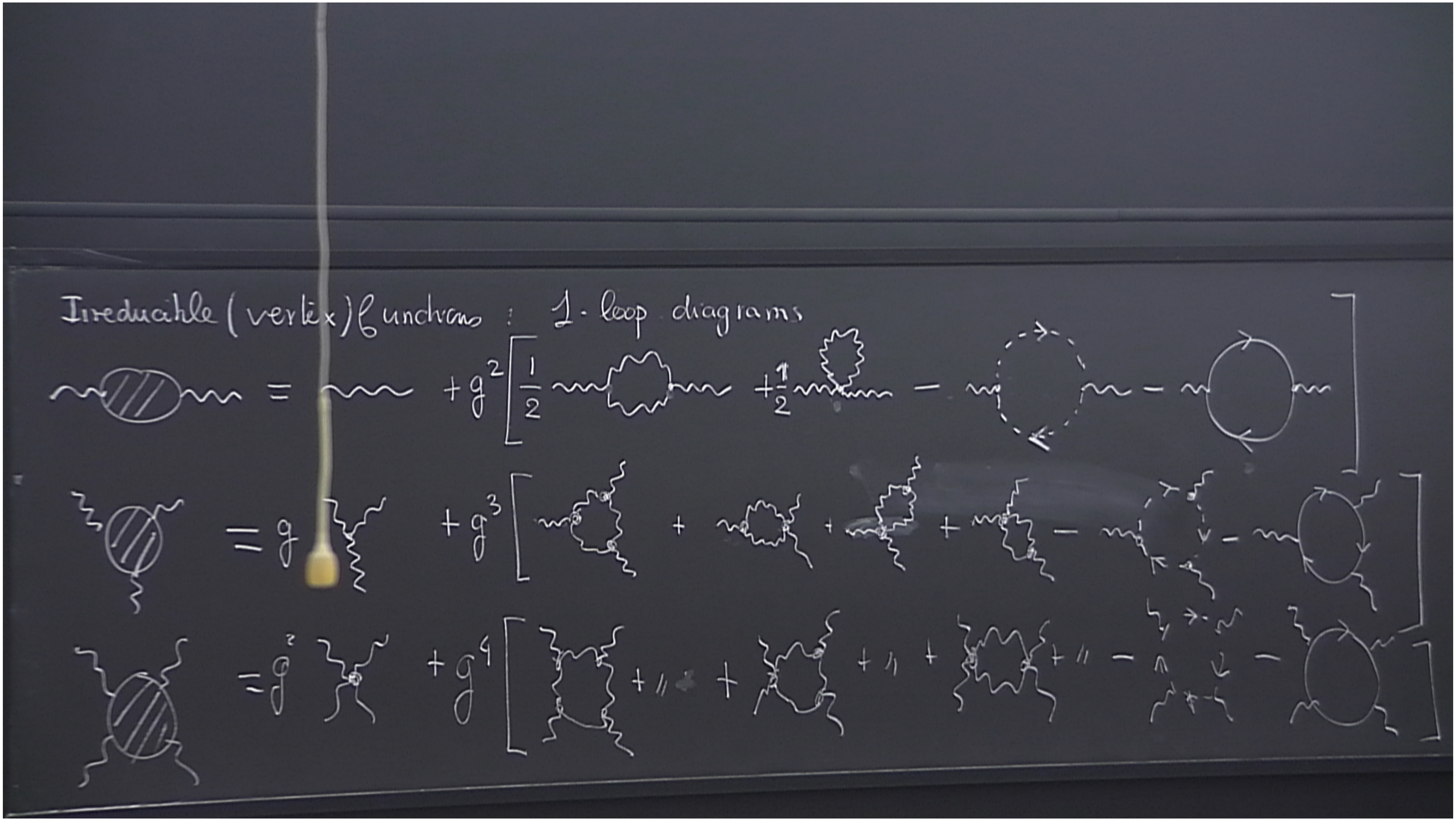
$$= g^2(-i)\Big[\epsilon_{abe}\epsilon_{cde}\left(h^{\mu\rho}h^{\nu\sigma} - h^{\mu\sigma}h^{\nu\sigma}\right) + \epsilon_{ace}\epsilon_{bde}\left(h^{\mu\nu}h^{\rho\sigma} - h^{\mu\sigma}h^{\nu\rho}\right) + \epsilon_{ade}\epsilon_{bce}\left(h^{\mu\nu}h^{\rho\sigma} - h^{\mu\sigma}h^{\nu\sigma}\right)\Big]$$

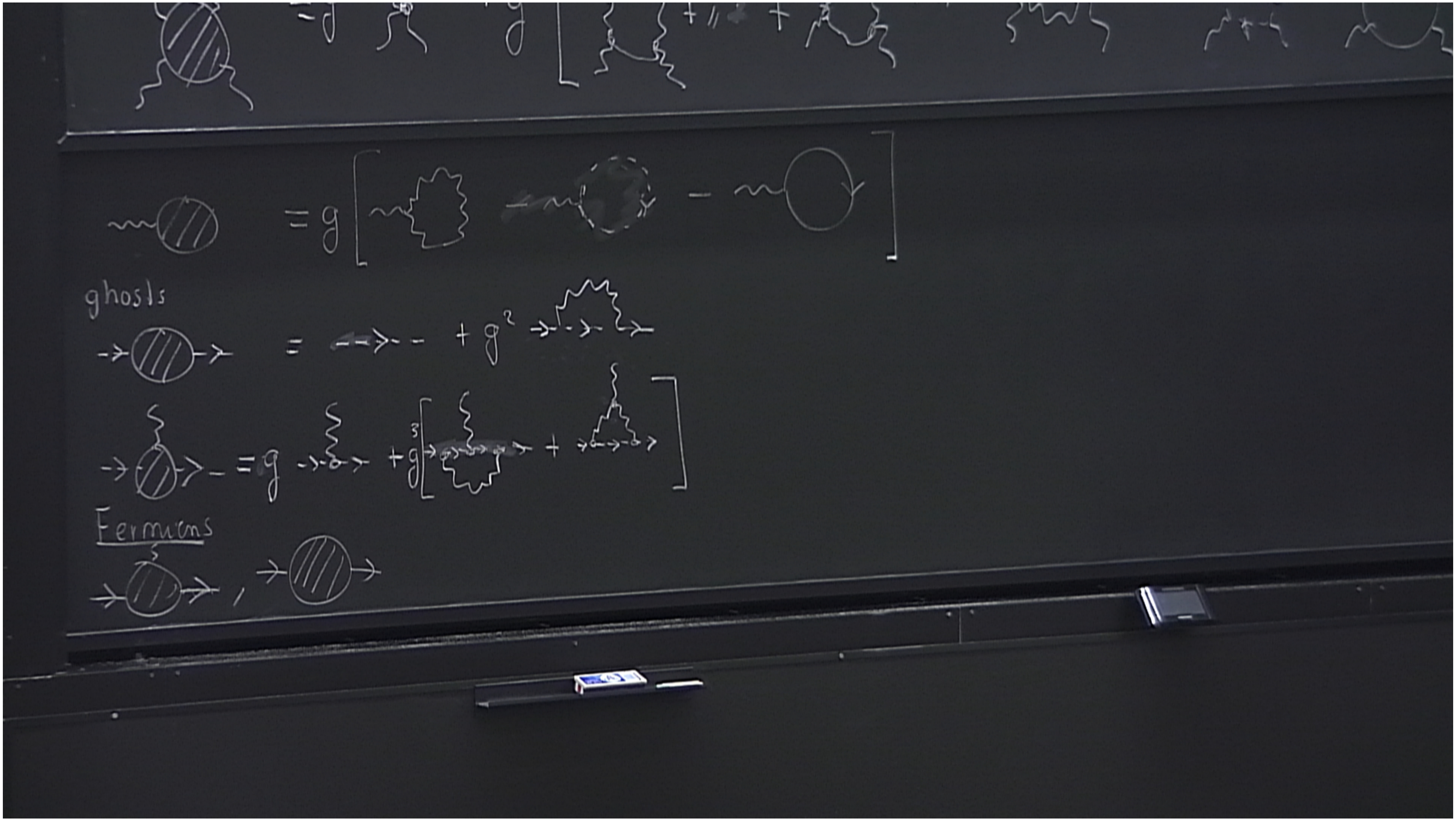
$\epsilon_{ace}\epsilon_{bde} = \delta_{ab}\delta_{ce} - \delta_{ad}\delta_{bc}$ for SU(2)

Peskin-Schroeder Book ⟵ ⚠ Ty

$\bar{\Psi} A \Psi$ +

Dirac-gauge interaction

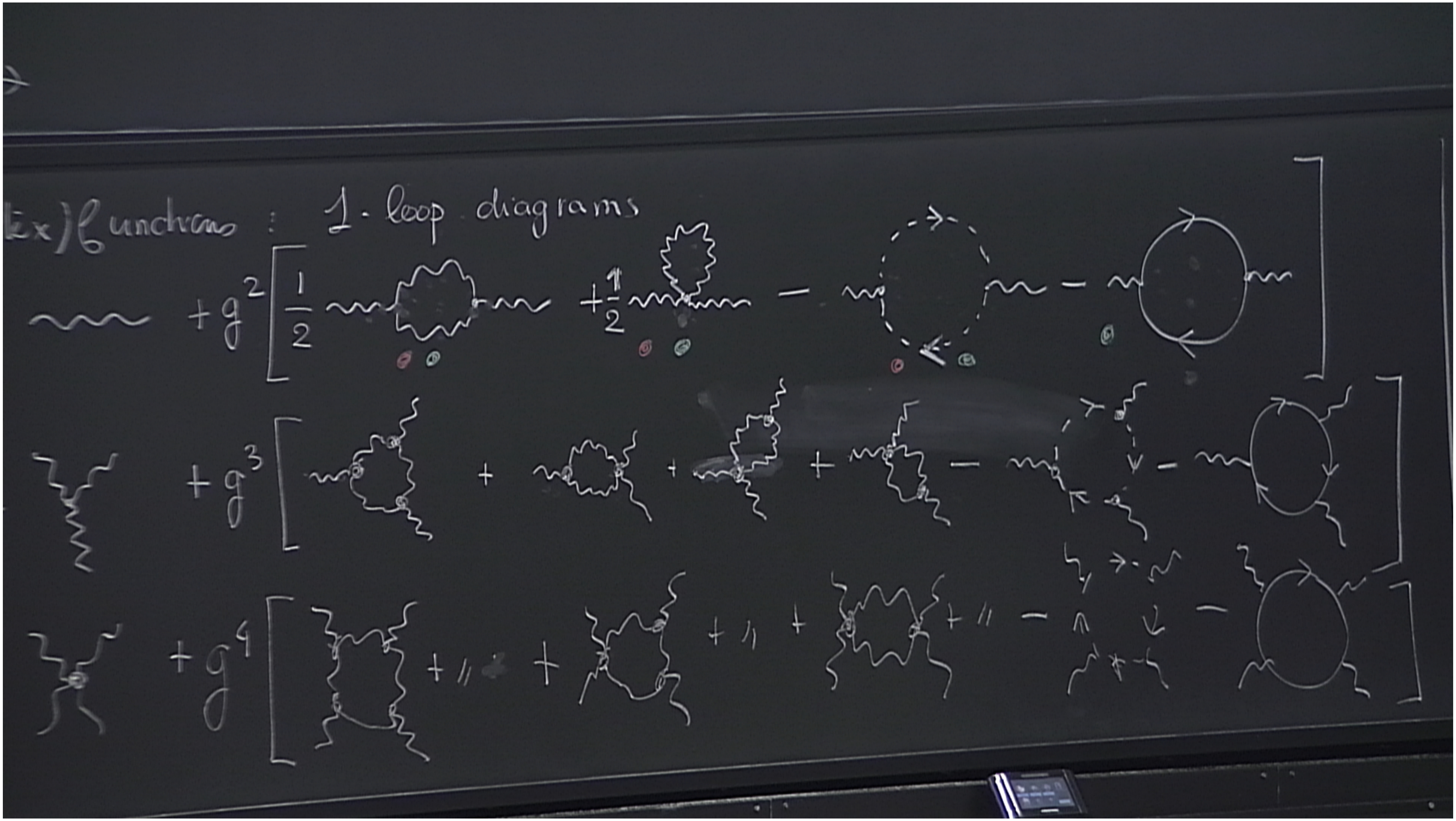

Irreducible (vertex) functions : 1-loop diagrams
= +g^2 [1/2 + 1/2 − −]
= g +g^3 [+ + + − −]
= g^2 +g^4 [+ " + + " + + " − −]

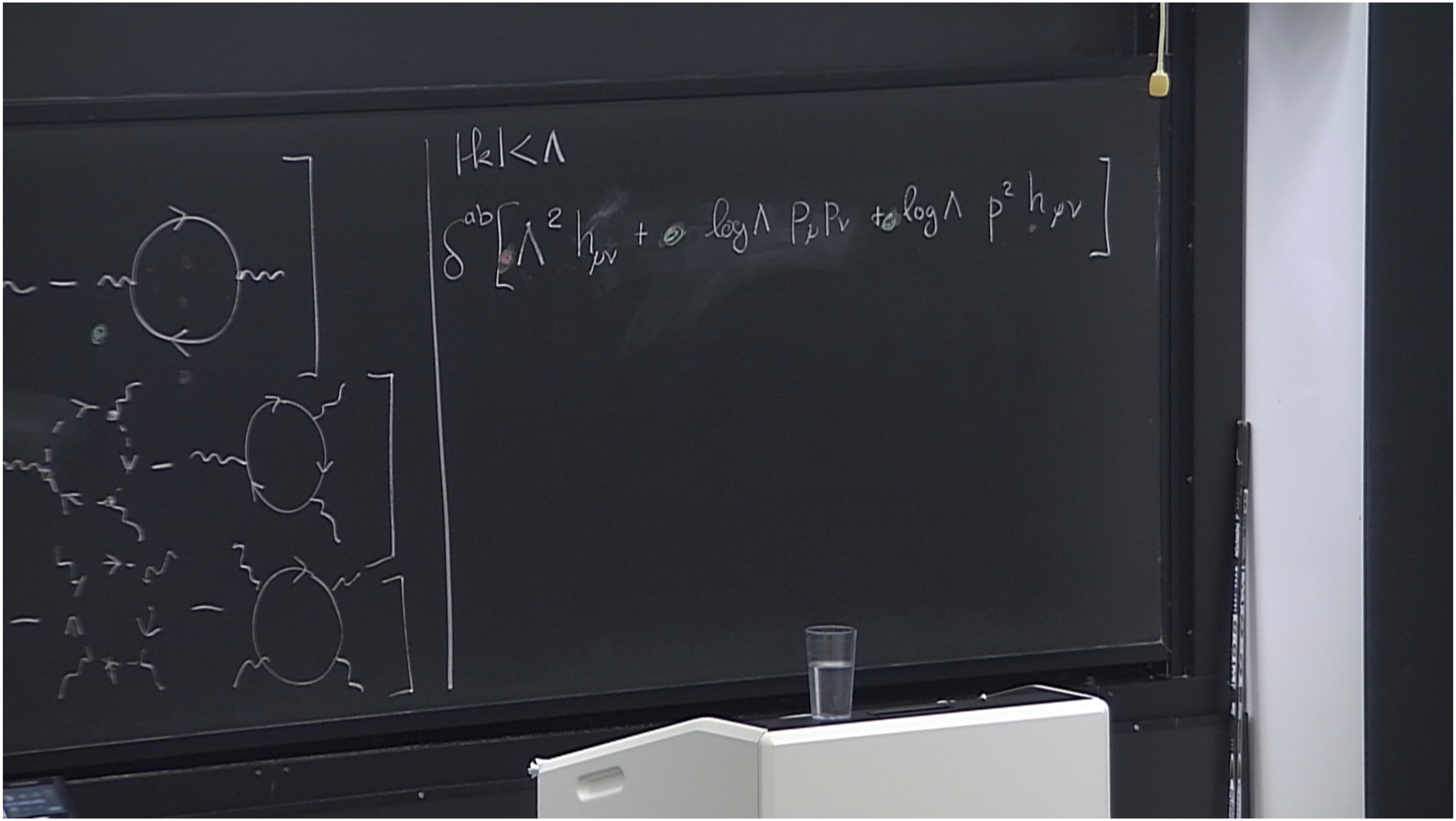

ghosts
Fermions

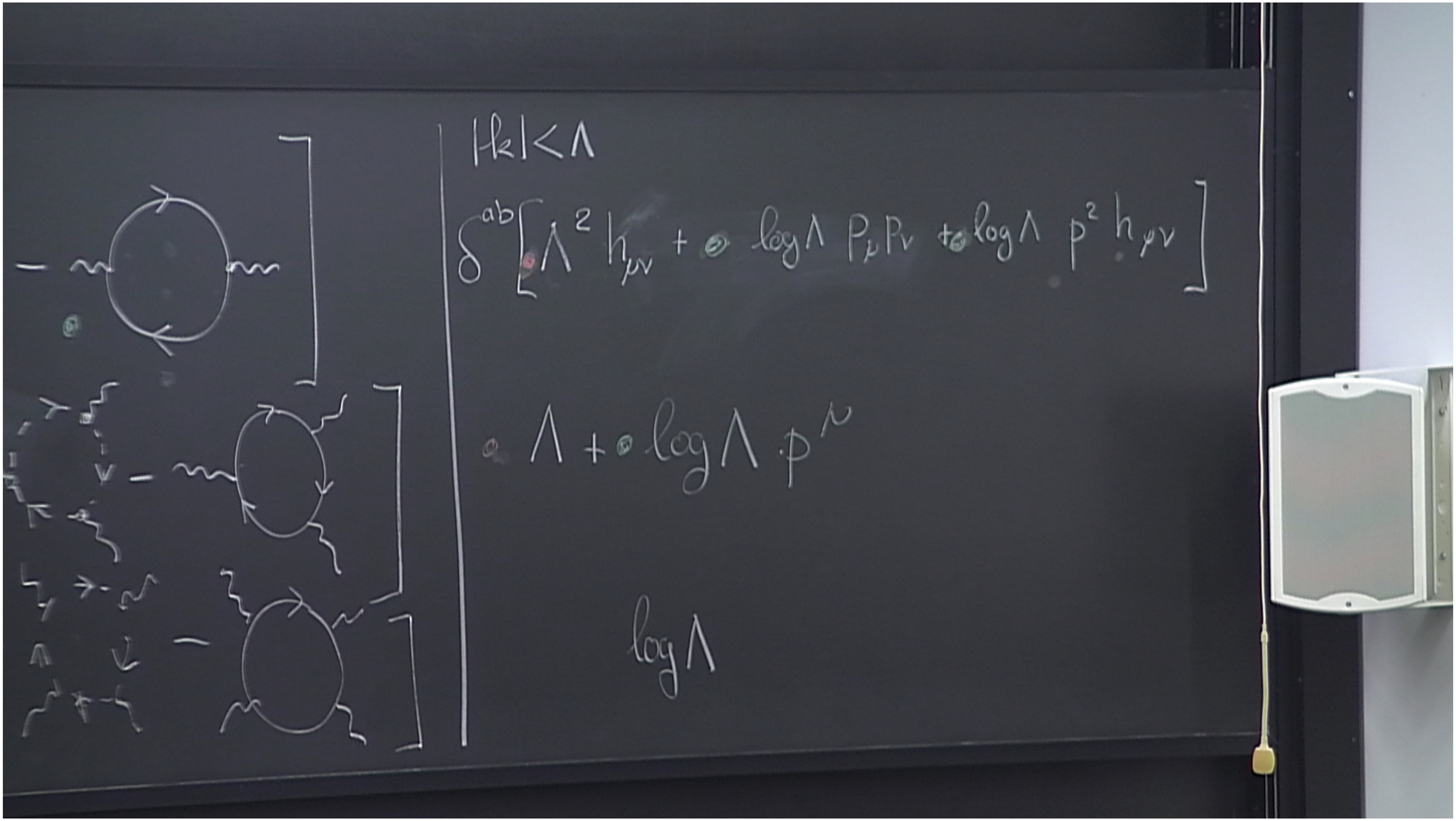
UV-divergences and renormalization.

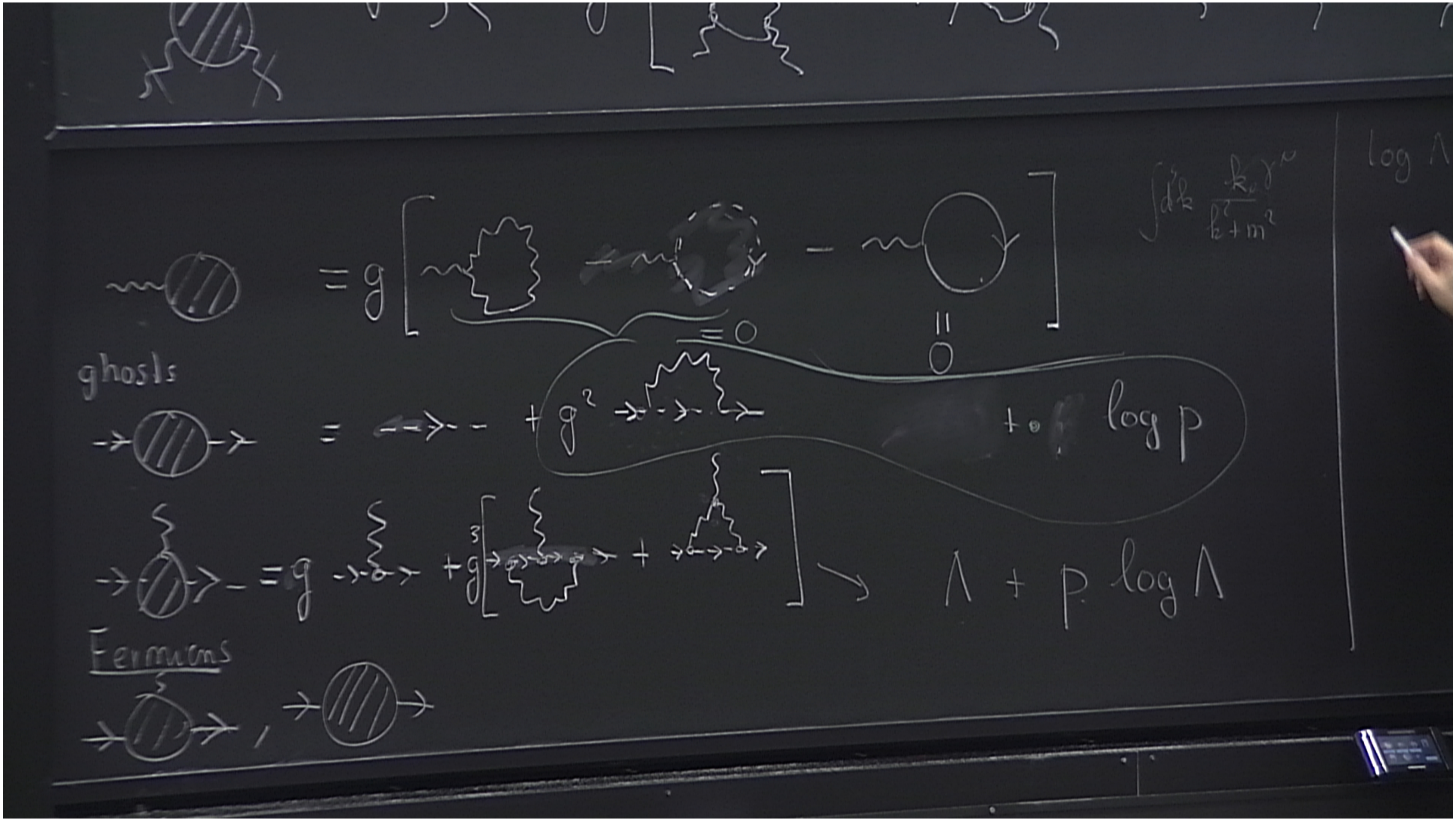
$\sim\!\sim\!\sim \,,\; - \to - \;\sim \frac{1}{k^2}$ ← of course

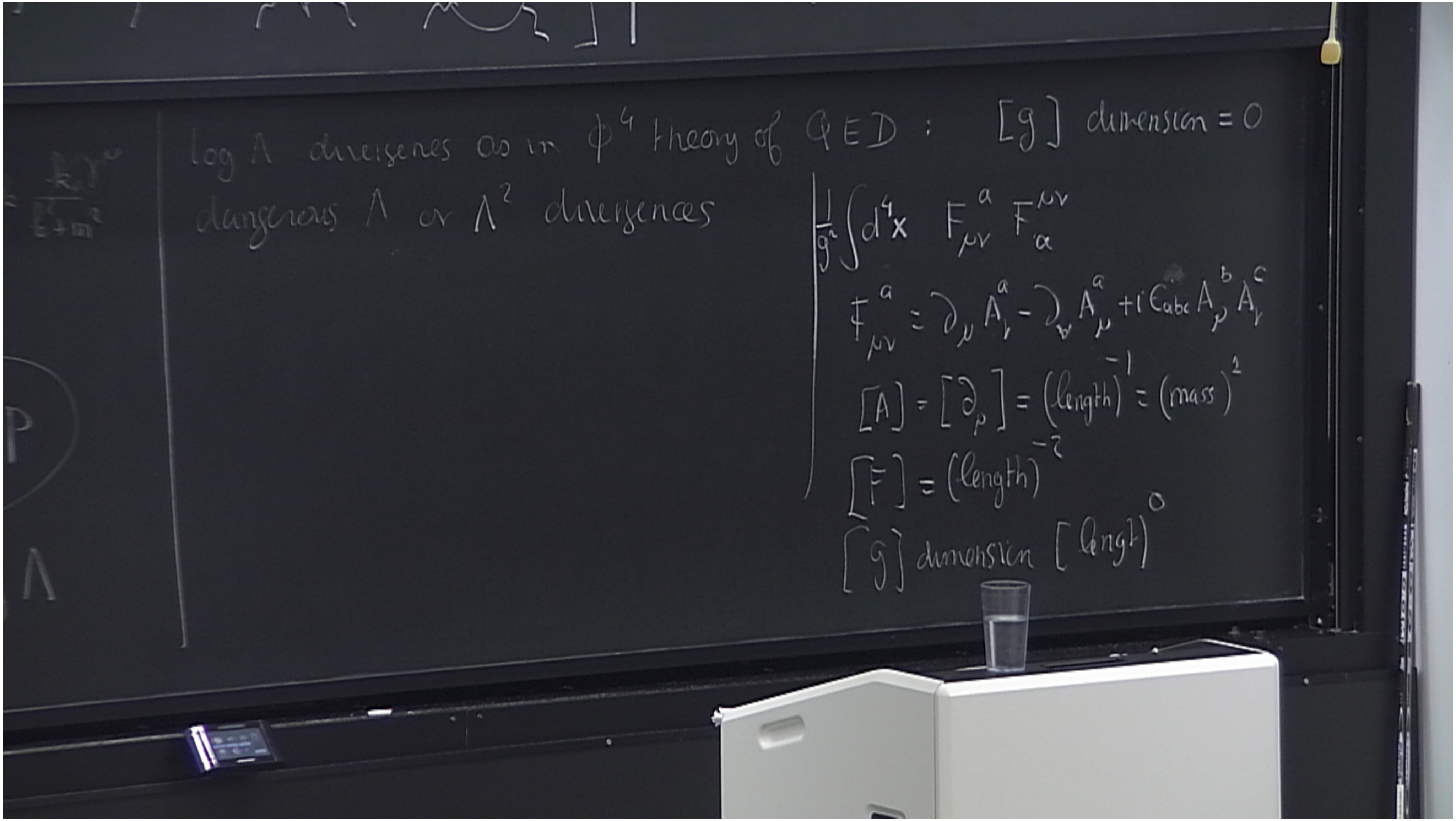
$\longrightarrow \;\sim \frac{1}{k}$


functions : 1-loop diagrams
+ g²
1/2
+ 1/2
+ g³
+ g⁴


$|k| < \Lambda$
$\delta^{ab}\left[\Lambda^2 h_{\mu\nu} + \log\Lambda\, p_\mu p_\nu + \log\Lambda\, p^2 h_{\mu\nu}\right]$


$|k| < \Lambda$
$\delta^{ab}\left[\Lambda^2 h_{\mu\nu} + \log\Lambda\, p_\mu p_\nu + \log\Lambda\, p^2 h_{\mu\nu}\right]$
$\Lambda + \log\Lambda\, p^\nu$
$\log\Lambda$


ghosts
log p
Λ + p log Λ
Fermions
log Λ


log Λ divergences as in φ⁴ theory of QED : [g] dimension = 0
dangerous Λ or Λ² divergences
1/g² ∫d⁴x F^a_μν F^μν_a
F^a_μν = ∂_μ A^a_ν − ∂_ν A^a_μ + i ε_abc A^b_μ A^c_ν
[A] = [∂_μ] = (length)^-1 = (mass)^1
[F] = (length)^-2
[g] dimension [length]^0

log Λ divergences as in φ^4 theory of QED : $[g]$ dimension $= 0$

dangerous Λ or Λ^2 divergences

$$\int_{|k|\sim\Lambda} d^4k \, \frac{1}{k^2 \, k^2} = \log\Lambda$$

(conformal invariance)

YM $d=4$ <u>Scale invariance</u> Classically

$$A^a_\mu(x) \longrightarrow \underbrace{\lambda A^a_\mu(\lambda x)}_{A_\lambda} \quad \lambda \text{ global factor}$$

$$S_{YM}[A_\lambda] = S_{YM}[A]$$

$$\frac{1}{g^2}\int d^4x \, F^a_{\mu\nu} F^{\mu\nu}_a$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + i\epsilon_{abc} A^b_\mu A^c_\nu$$

$$[A] = [\partial_\mu] = (\text{length})^{-1} = (\text{mass})^1$$

$$[F] = (\text{length})^{-2}$$

$[g]$ dimension $[\text{lengt}]^0$

UV-divergences and renormalization.

no Λ^2 and Λ divergences

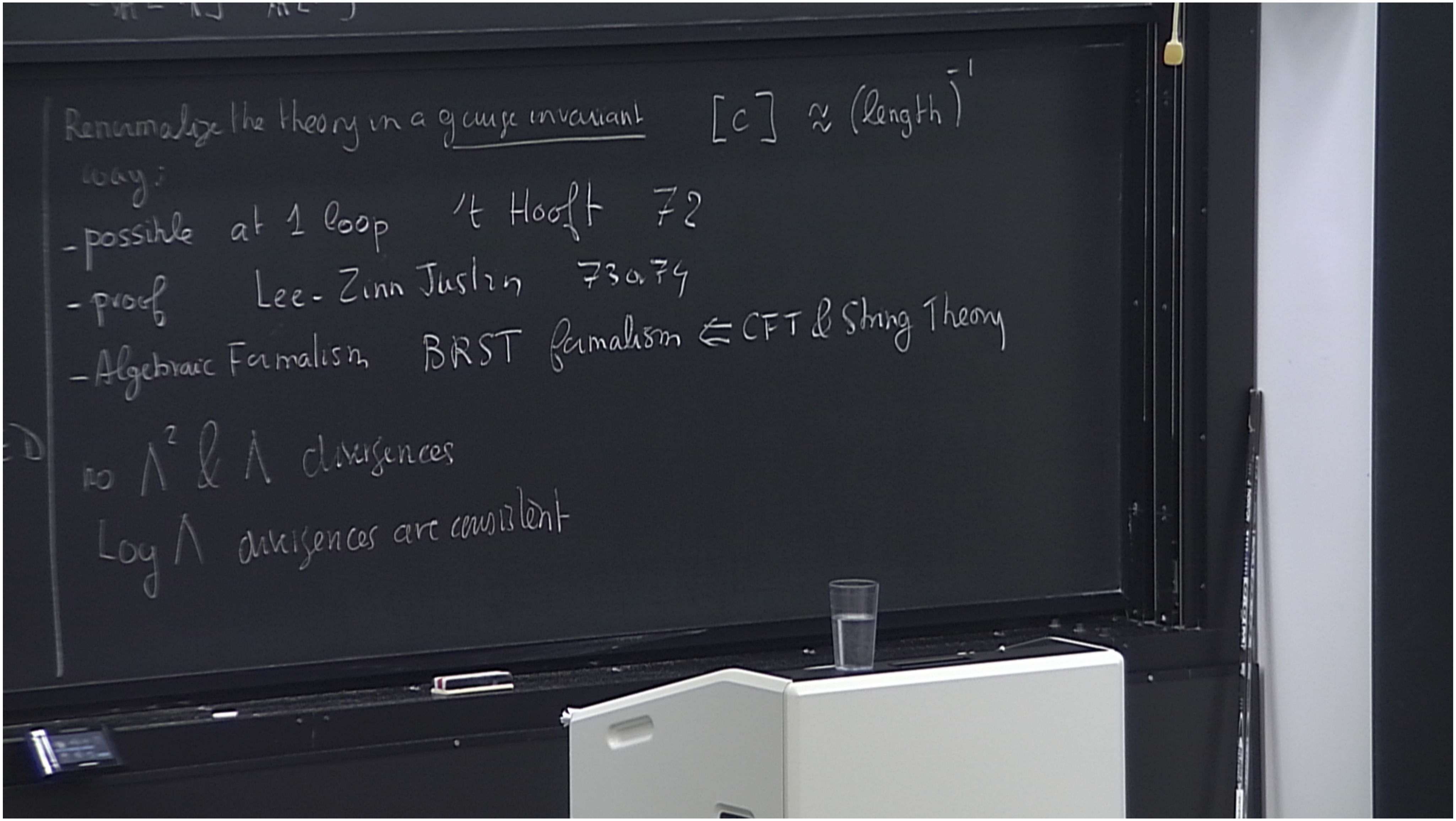
Relations between the $\log \Lambda$ divergences

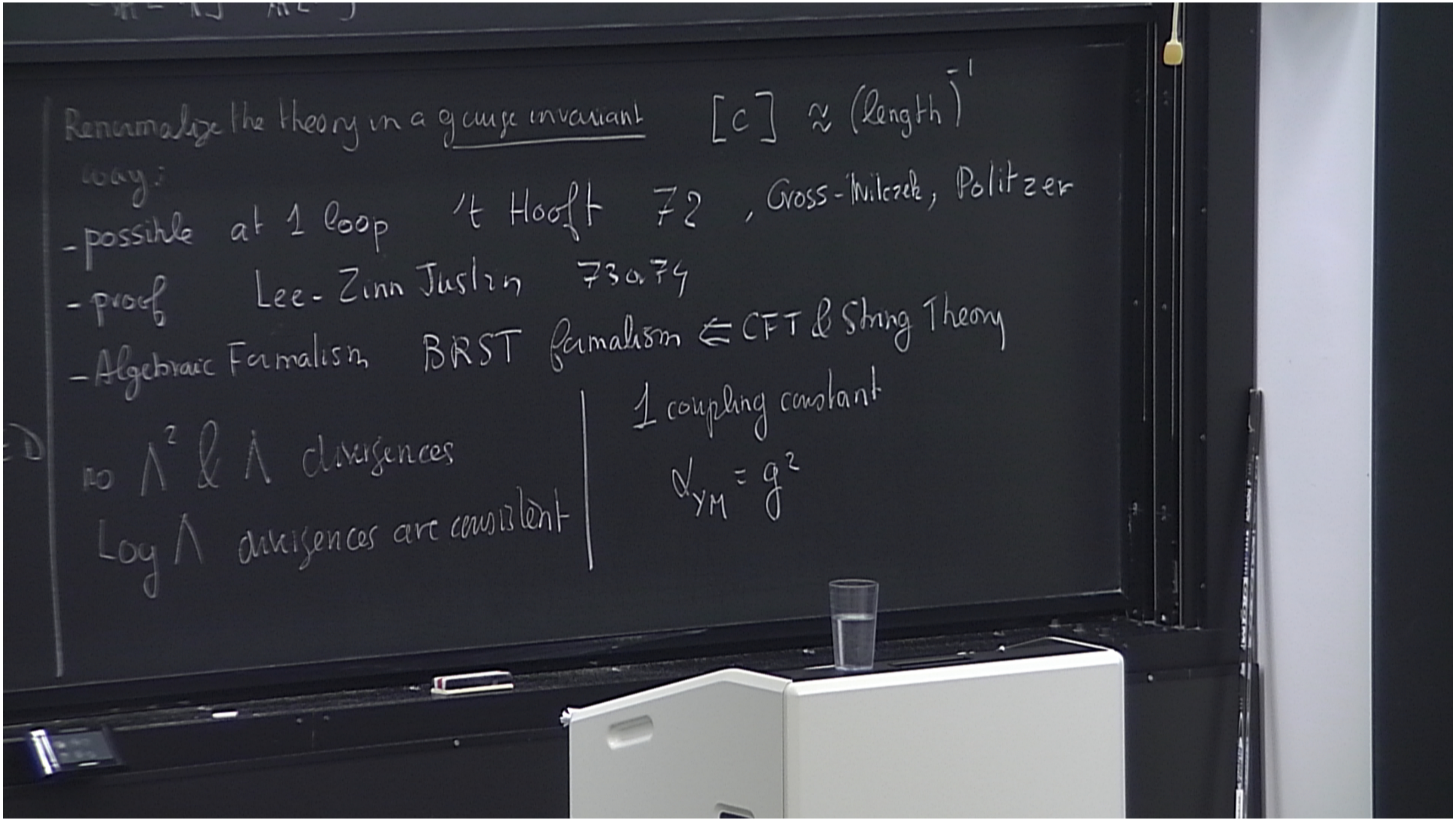
$\Rightarrow$ gauge invariance $p_\mu \left(\right)^{\mu\nu} = 0$ otherwise unphysical polarization states propagates ; as in QED

$$A_1 \log\Lambda (\partial A \partial A) + A_2 \log\Lambda (\partial A A \cdot A) + A_3 \log\Lambda (A \cdot A \cdot A \cdot A)$$

$\Rightarrow$ counterterms : not gauge invariant

Ren


Renormalize the theory in a gauge invariant way:
[c] ≈ (length)^-1
- possible at 1 loop 't Hooft 72
- proof Lee-Zinn Justin 73 a 74
- Algebraic Formalism BRST formalism ⇐ CFT & String Theory
no Λ^2 & Λ divergences
Log Λ divergences are consistent

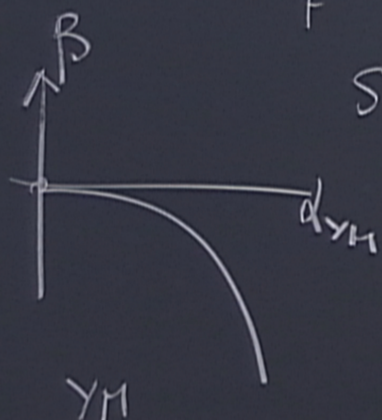

Renormalize the theory in a gauge invariant way:
$[c] \approx (\text{length})^{-1}$
- possible at 1 loop 't Hooft 72, Gross-Wilczek, Politzer
- proof Lee-Zinn Justin 73 a 74
- Algebraic Formalism BRST formalism ⇐ CFT & String Theory
no Λ^2 & Λ divergences
Log Λ divergences are consistent
1 coupling constant
$\alpha_{YM} = g^2$

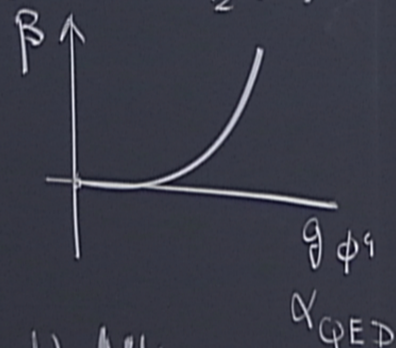
Beta Function

$$\beta(\alpha_{YM}) = \alpha_{YM}^{2} \frac{1}{(4\pi)^{2}}\left(-\frac{11}{3}C_{2}(G)\right)$$

$$f^{acd} f^{bcd} = C_{2}(G)\,\delta^{ab}$$

$$SU(N) \Rightarrow C_{2}(G) = N$$





Asymptotically Free in the UV

$g_{eff}(E) \searrow 0$ as $\frac{1}{\log(E)}$ when energy $E \nearrow \infty$

- Parton Model
- Strongly interacting theory at low energies.
- confinement.