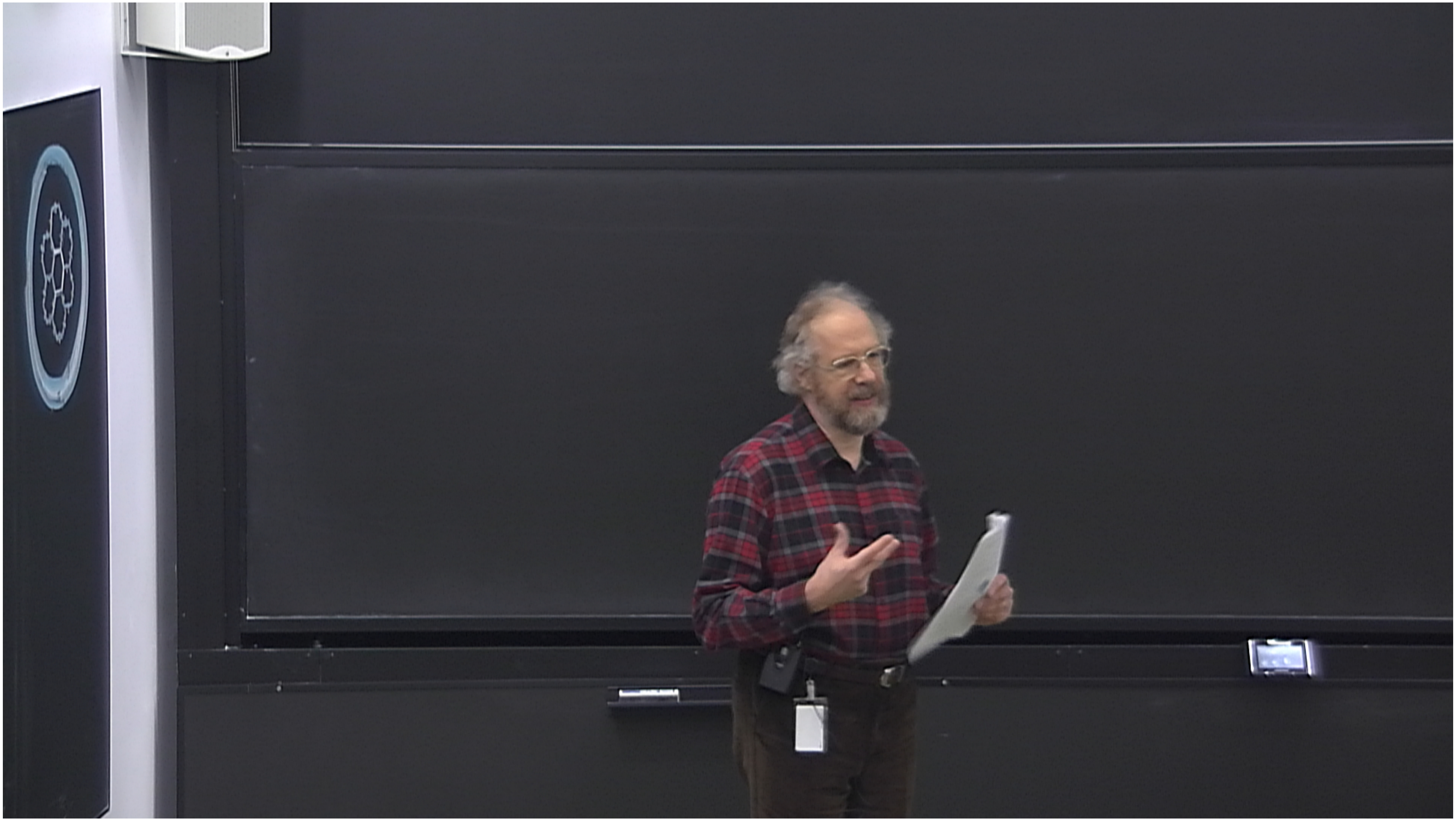
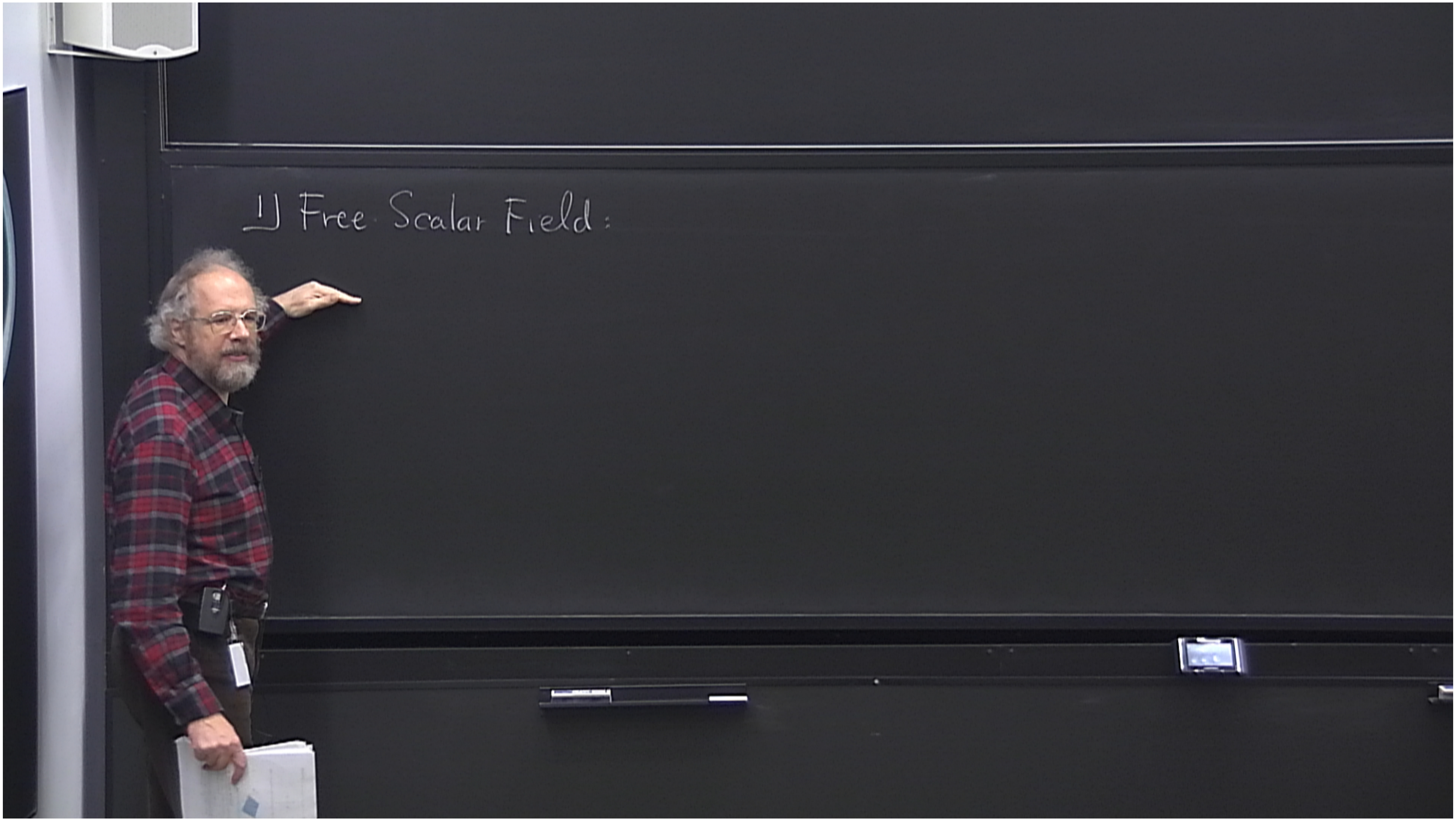
Title: PSI 2015/2016 Quantum Field Theory II - Francois David - Lecture 4

Date: Nov 12, 2015 09:00 AM

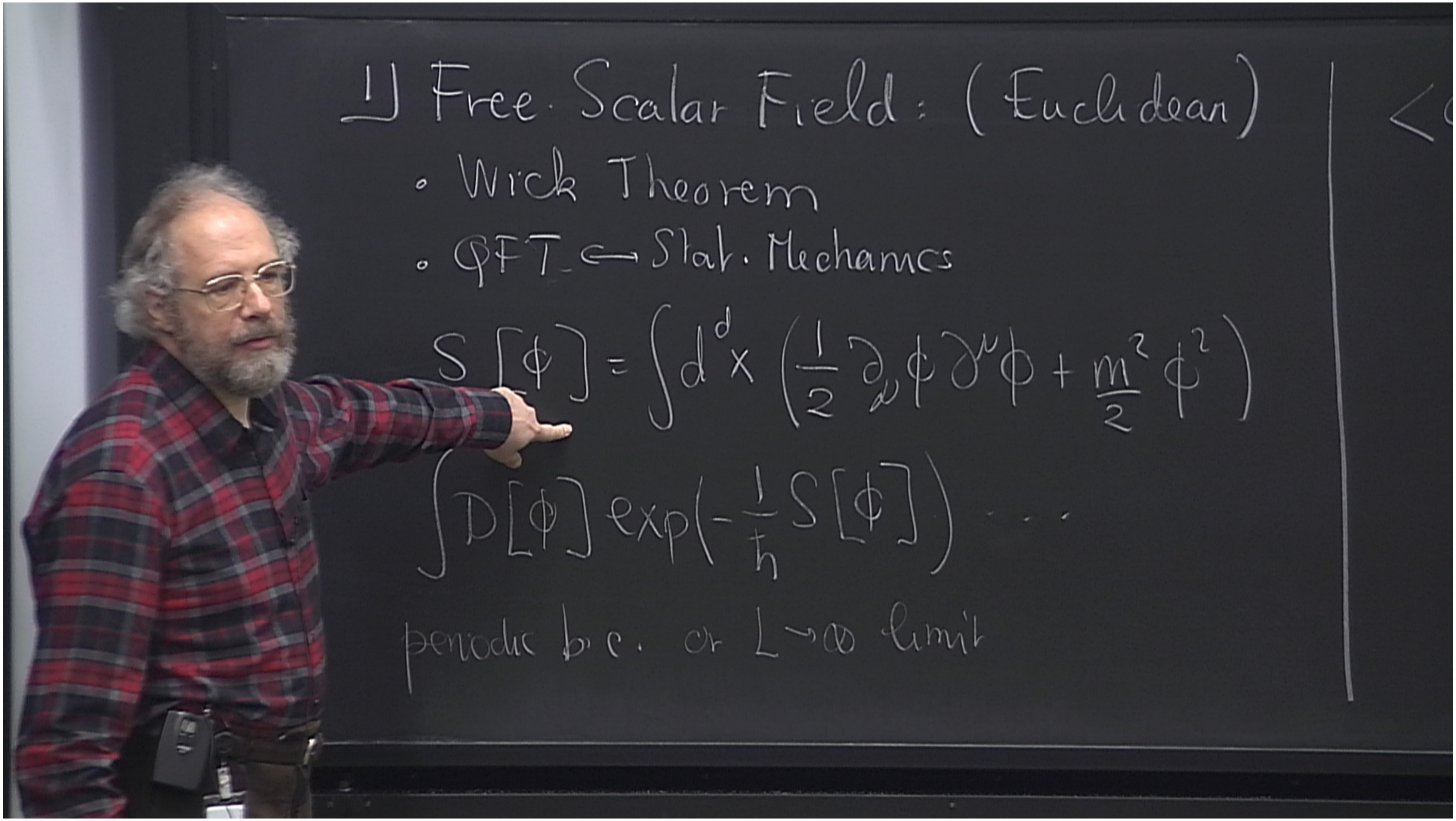
URL: http://pirsa.org/15110030

Abstract:





1) Free Scalar Field:



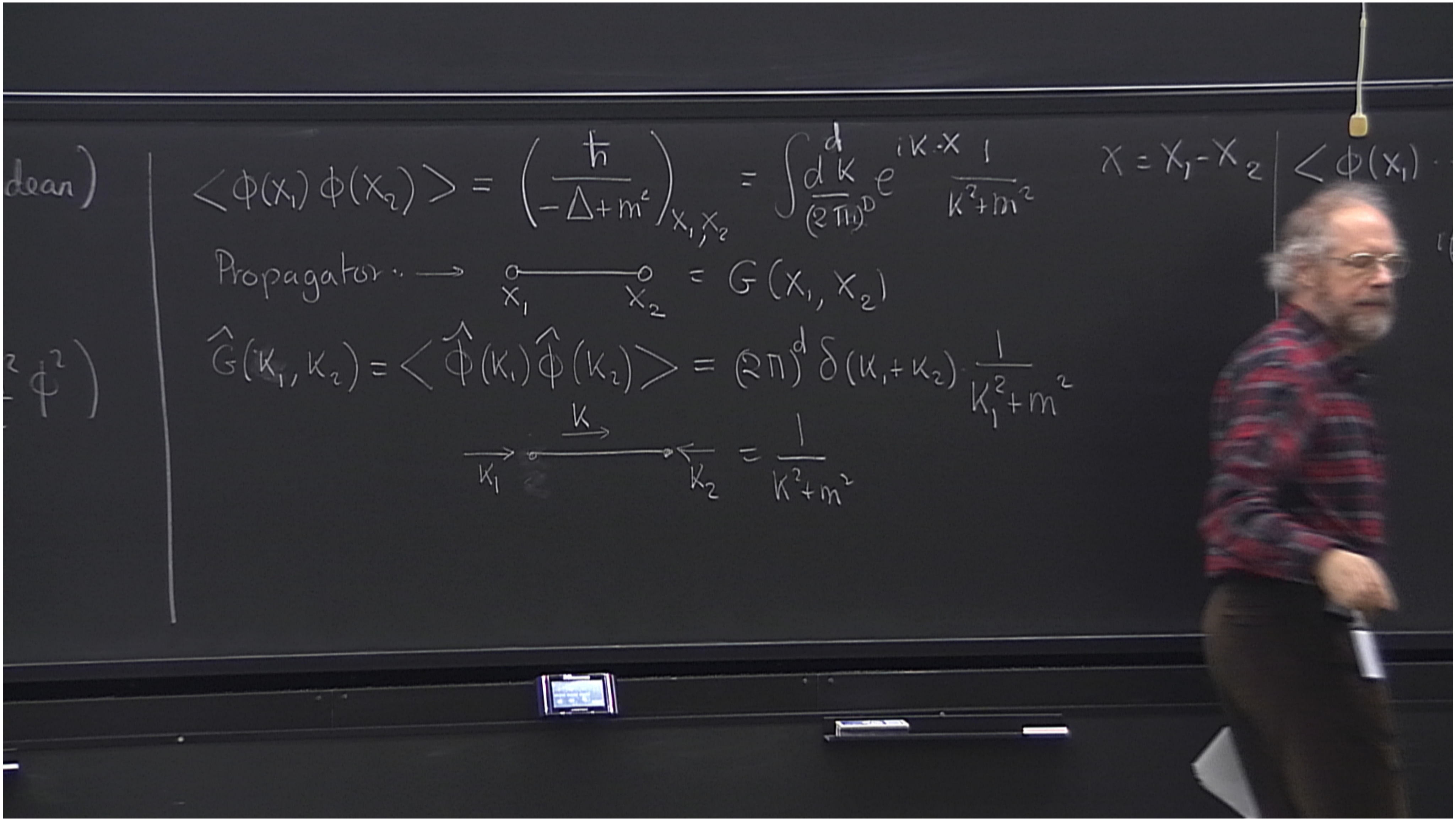
1] Free Scalar Field: (Euclidean)

- Wick Theorem
- QFT ⟷ Stat. Mechanics

$$S[\phi] = \int d^d x \left(\frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi + \frac{m^2}{2} \phi^2 \right)$$

$$\int D[\phi] \exp\left(-\frac{1}{\hbar} S[\phi]\right) \ldots$$

periodic b.c. or L→∞ limit



dean)

$^2\phi^2)$

$$\langle \phi(x_1)\,\phi(x_2)\rangle = \left(\frac{\hbar}{-\Delta+m^2}\right)_{x_1,x_2} = \int \frac{d^dk}{(2\pi)^D}\, e^{ik\cdot x}\,\frac{1}{k^2+m^2}$$

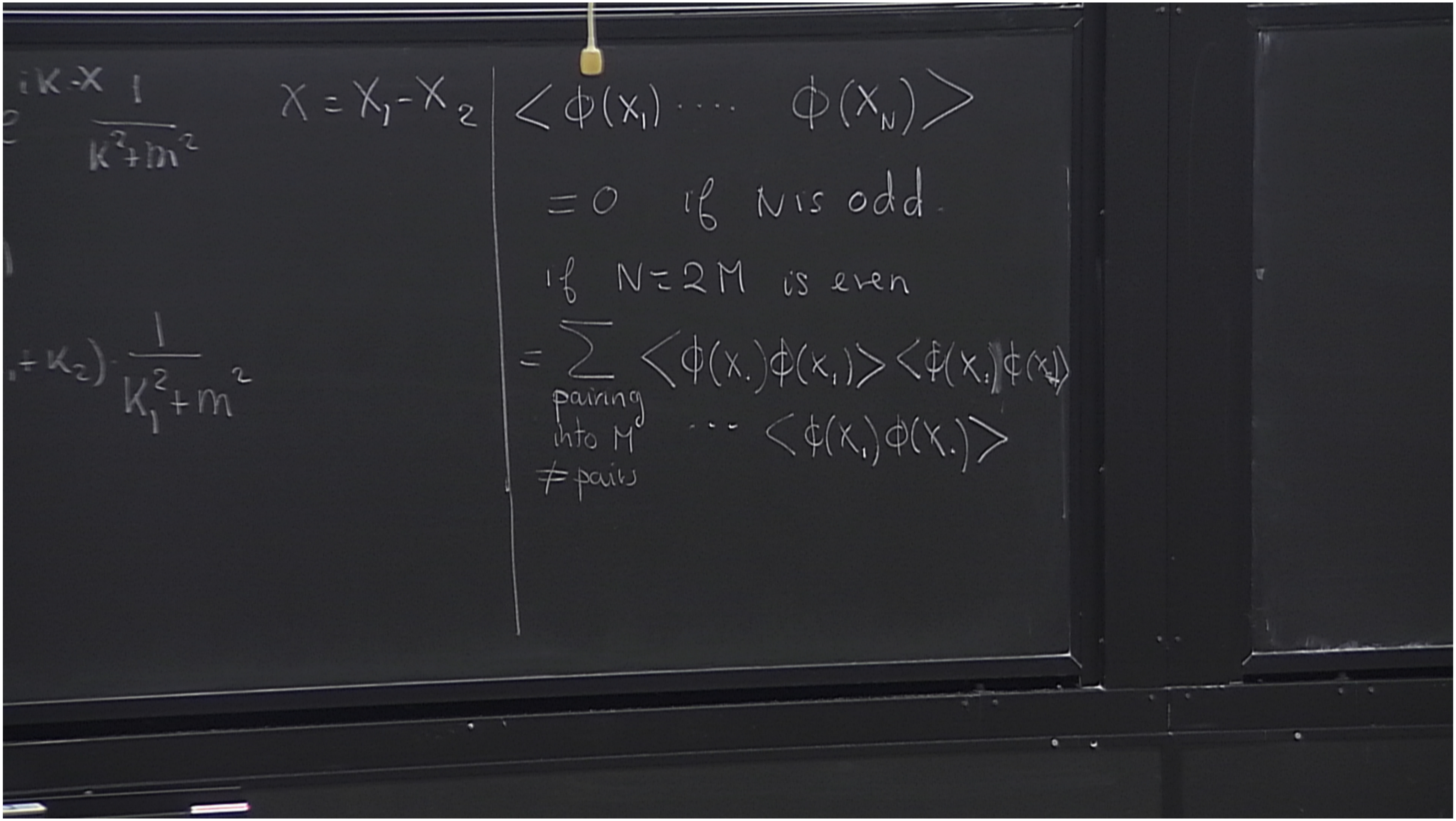
$X = X_1 - X_2$

$\langle \phi(x_1)$

Propagator: $\longrightarrow \; \underset{x_1}{\circ}\!\!-\!\!-\!\!-\!\!-\!\!\underset{x_2}{\circ} = G(x_1, x_2)$

$$\hat G(k_1, k_2) = \langle \hat\phi(k_1)\,\hat\phi(k_2)\rangle = (2\pi)^d\,\delta(k_1+k_2)\cdot\frac{1}{k_1^2+m^2}$$

$$\xrightarrow[k_1]{}\;\circ\!\overset{\xrightarrow{k}}{-\!-\!-\!-\!-}\!\circ\;\xleftarrow[k_2]{} = \frac{1}{k^2+m^2}$$



$e^{ik\cdot x}\frac{1}{k^2+m^2}$

$_1+k_2)\cdot\frac{1}{k_1^2+m^2}$

$x = x_1 - x_2$

$\langle \phi(x_1) \cdots \phi(x_N) \rangle$

$= 0$ if N is odd.

if $N = 2M$ is even

$$= \sum_{\substack{\text{pairing}\\ \text{into } M\\ \neq\text{ pairs}}} \langle \phi(x_\cdot)\phi(x_\cdot)\rangle \langle \phi(x_\cdot)\phi(x_\cdot)\rangle \cdots \langle \phi(x_\cdot)\phi(x_\cdot)\rangle$$

Generating Functional

$$j\cdot\phi = \int d^dx\, j(x)\cdot\phi(x)$$

source term

"function" classical

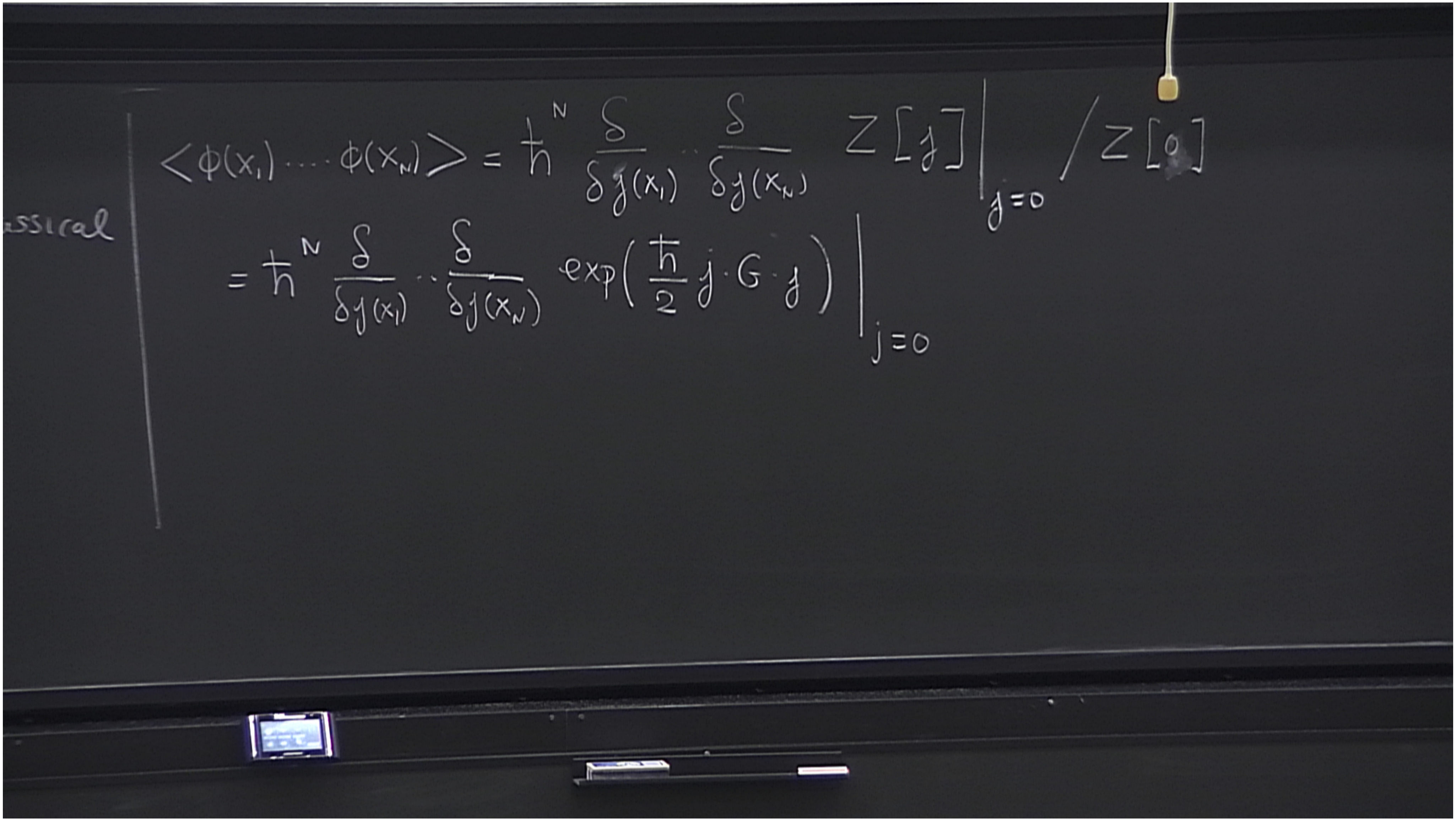
$$Z[j] = \int D[\phi]\, \exp\left[-\frac{1}{\hbar}\left(S[\phi] - j\cdot\phi\right)\right]$$

$$S[\phi] = \frac{1}{2}\left[\phi\cdot(-\Delta+m^2)\cdot\phi\right] = \int d^dx\, \frac{1}{2}\,\phi\,(-\Delta+m^2)\,\phi.$$

$$Z[j] = \text{"}\det\text{"}\left[-\Delta+m^2\right]^{-1/2}\cdot\exp\left(+\frac{\hbar}{2}\, j\cdot(-\Delta+m^2)^{-1}\cdot j\right)$$

$$\int d^dy_1\, d^dy_2\; j(y_1)\, j(y_2)\, G(y_1, y_2)$$

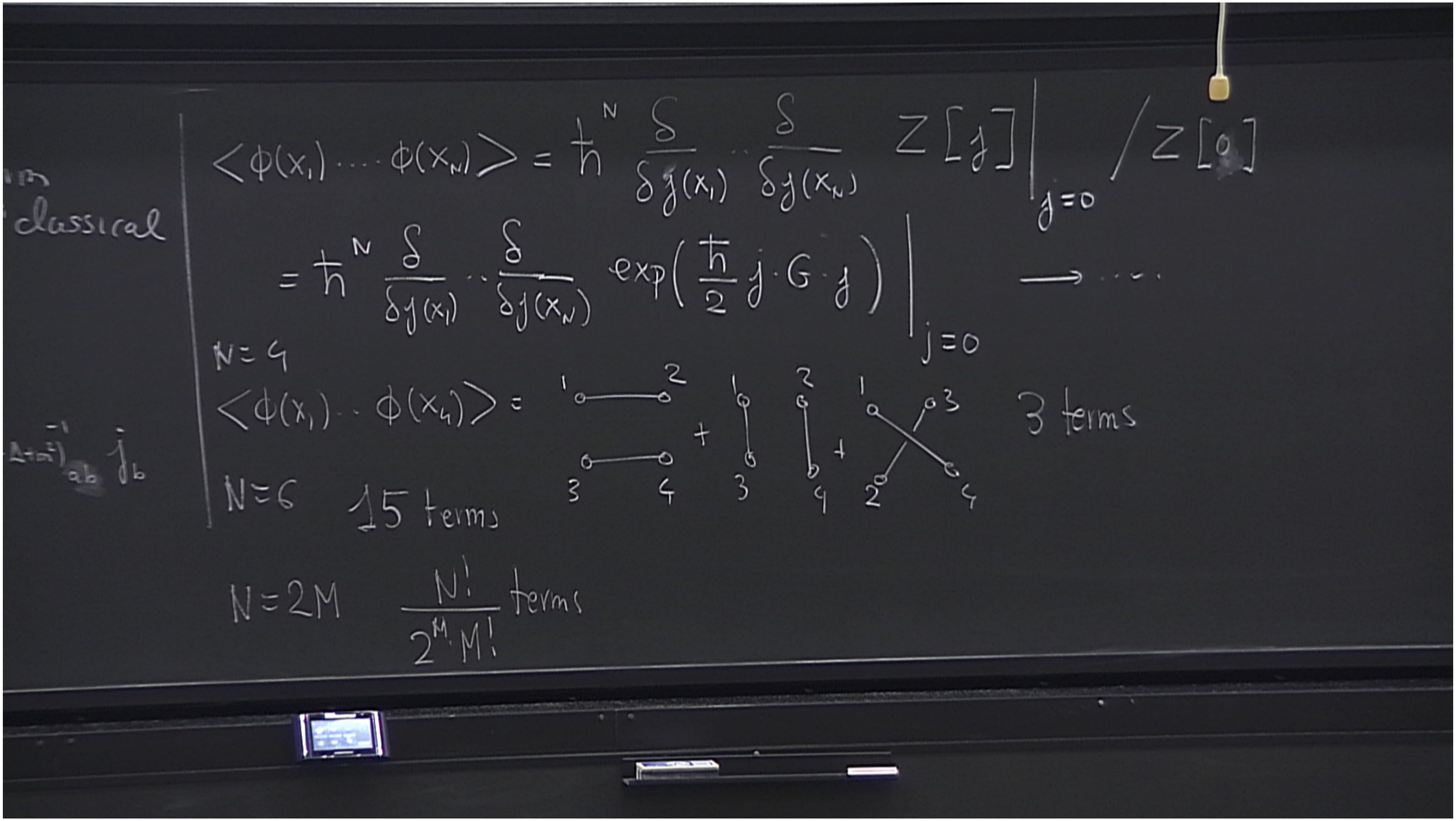
$<\phi(x,$



assical

$$\langle \phi(x_1)\cdots\phi(x_N)\rangle = \hbar^N \frac{\delta}{\delta j(x_1)}\cdots\frac{\delta}{\delta j(x_N)} Z[j]\Big|_{j=0} \Big/ Z[0]$$

$$= \hbar^N \frac{\delta}{\delta j(x_1)}\cdots\frac{\delta}{\delta j(x_N)} \exp\left(\frac{\hbar}{2} j\cdot G\cdot j\right)\Bigg|_{j=0}$$



…m

classical

$(-\Delta+m^2)^{-1}_{ab}\, j_b$

$$\langle \phi(x_1)\cdots\phi(x_N)\rangle = \hbar^N \frac{\delta}{\delta j(x_1)}\cdots\frac{\delta}{\delta j(x_N)}\, Z[j]\Big|_{j=0} \Big/ Z[0]$$

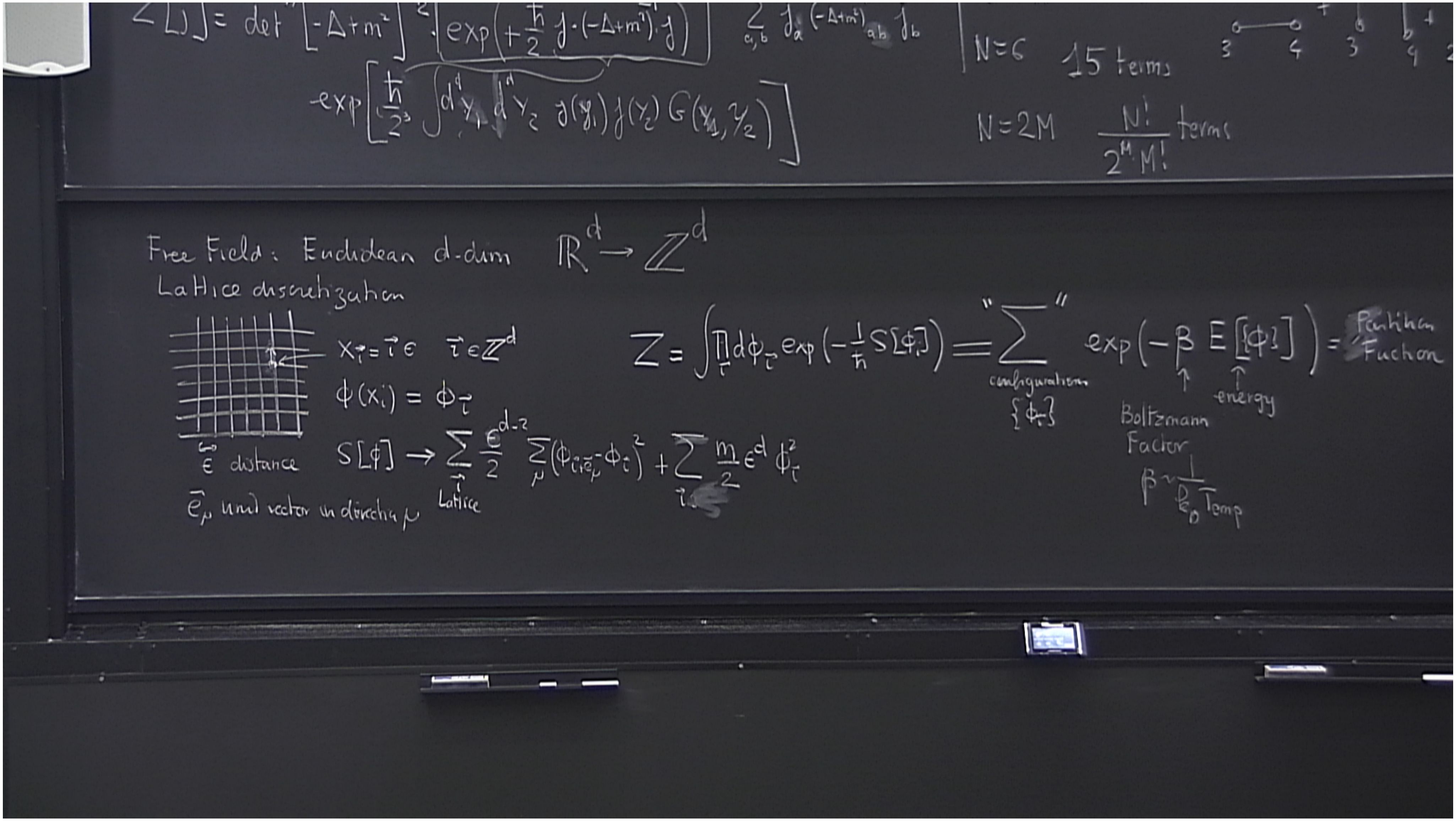
$$= \hbar^N \frac{\delta}{\delta j(x_1)}\cdots\frac{\delta}{\delta j(x_N)}\, \exp\left(\frac{\hbar}{2}\, j\cdot G\cdot j\right)\Big|_{j=0} \longrightarrow \cdots$$

$N=4$

$\langle \phi(x_1)\cdots\phi(x_4)\rangle =$ 3 terms

$N=6$ 15 terms

$N=2M$ $\dfrac{N!}{2^M\cdot M!}$ terms



$Z[J] = \det [-\Delta + m^2]^{2} \cdot \boxed{\exp\left(+\frac{\hbar}{2} J\cdot(-\Delta+m^2)\cdot J\right)}$ $\quad \sum_{a,b} J_a (-\Delta+m^2)_{ab} J_b$

$\underbrace{\exp\left[\frac{\hbar}{2^3}\int d^d y_1\, d^d y_2\, J(y_1)\, J(y_2)\, G(y_1, y_2)\right]}$

$N=6$ 15 terms

3 4 + 3 4 + 2

$N = 2M \quad \frac{N!}{2^M \cdot M!}$ terms

Free Field: Euclidean d-dim $\mathbb{R}^d \rightarrow \mathbb{Z}^d$

Lattice discretization

$x_{\vec{i}} = \vec{i}\,\epsilon \quad \vec{i} \in \mathbb{Z}^d$

$\phi(x_i) = \phi_{\vec{i}}$

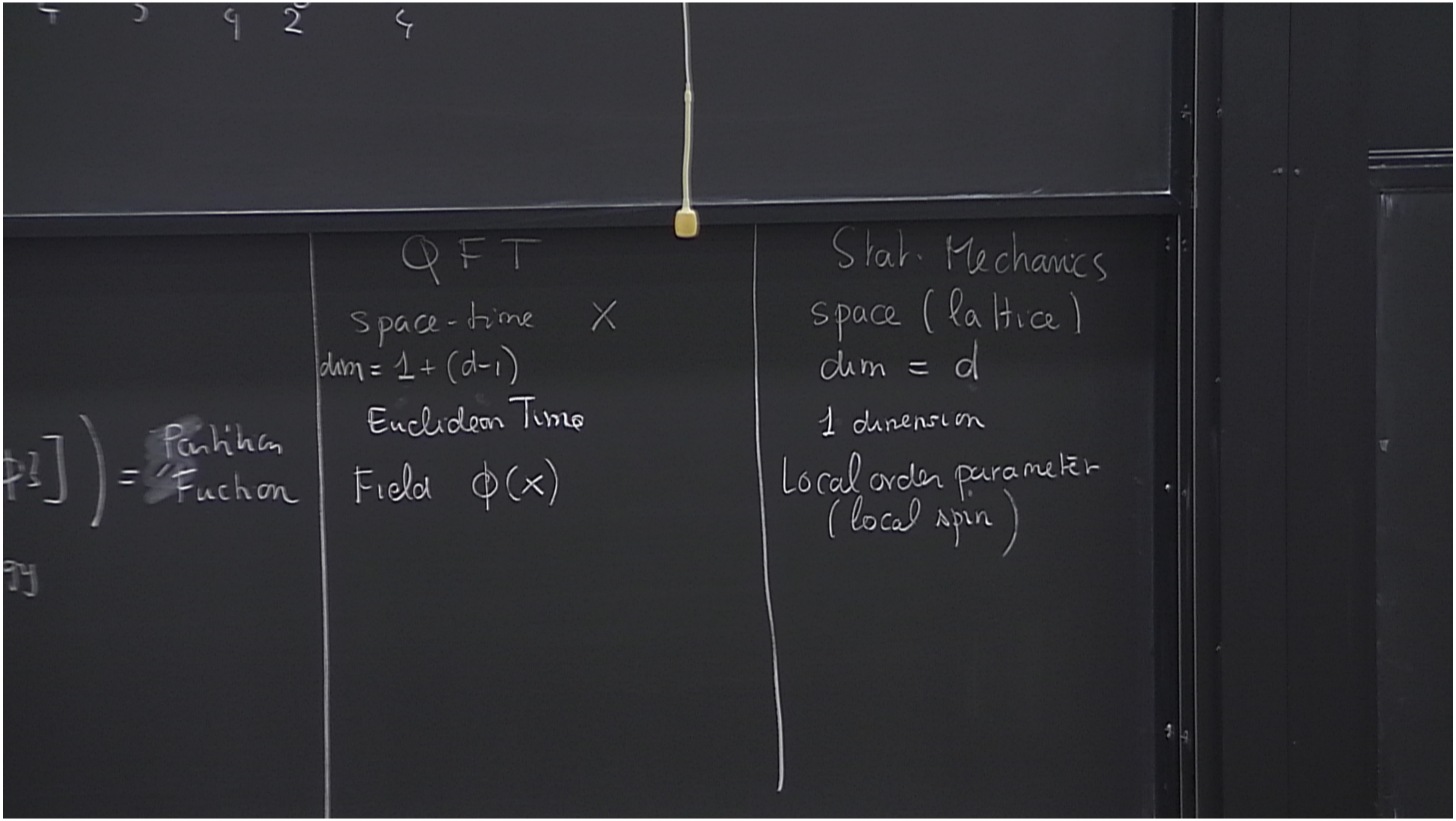
$\overset{\leftrightarrow}{\epsilon}$ distance

$$S[\phi] \rightarrow \sum_{\substack{\vec{i} \\ \text{Lattice}}} \frac{\epsilon^{d-2}}{2} \sum_{\mu} (\phi_{\vec{i}+\vec{e}_\mu} - \phi_{\vec{i}})^2 + \sum_{\vec{i}} \frac{m}{2} \epsilon^d \phi_{\vec{i}}^2$$

$\vec{e}_\mu$ unit vector in direction μ

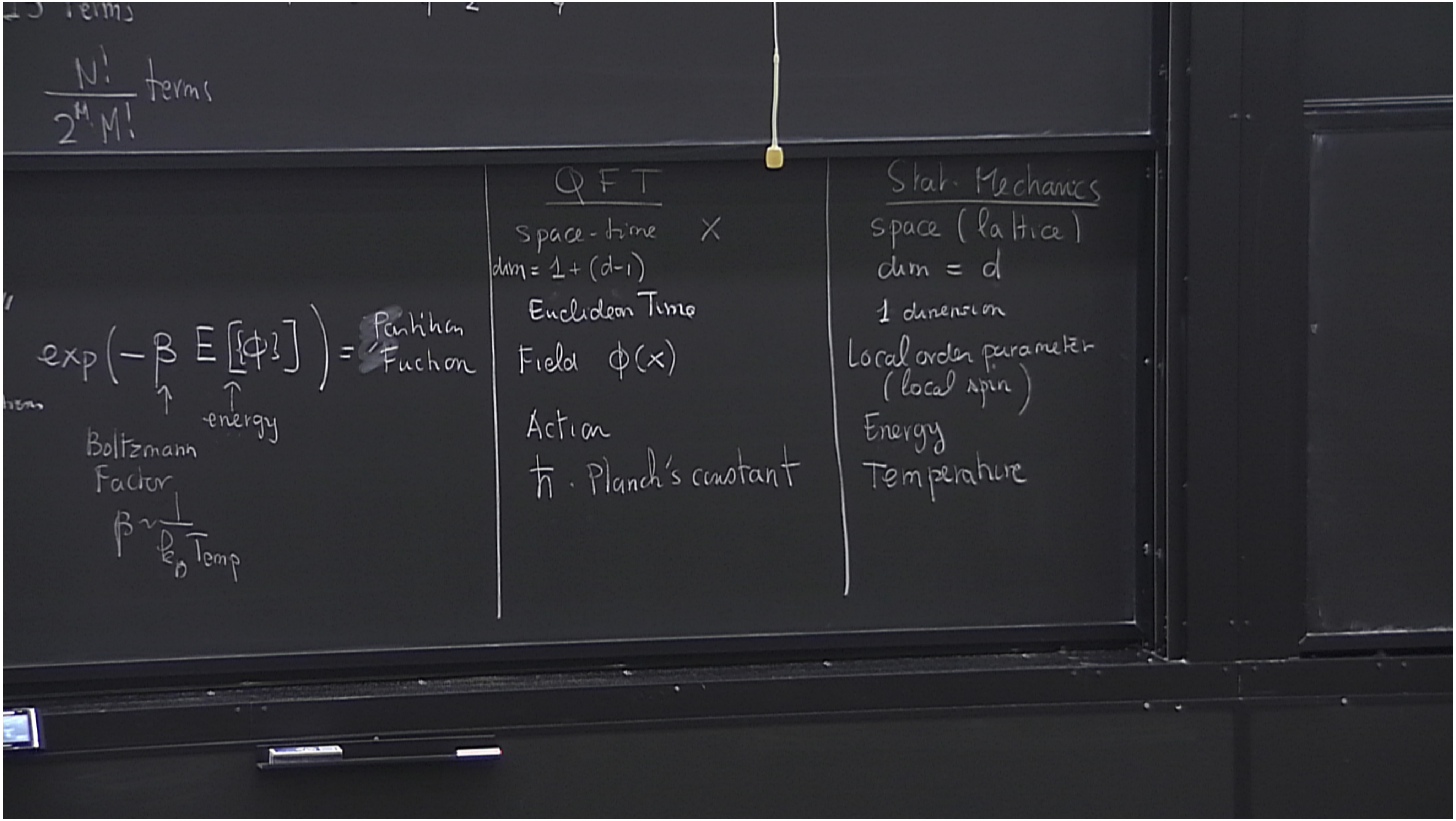
$$Z = \int \prod_{\vec{i}} d\phi_{\vec{i}} \exp\left(-\frac{1}{\hbar} S[\phi_{\vec{i}}]\right) \equiv \text{"}\sum_{\substack{\text{configurations} \\ \{\phi_{\vec{i}}\}}}\text{"} \exp\left(-\underset{\substack{\uparrow \\ \text{Boltzmann} \\ \text{Factor}}}{\beta}\, \underset{\substack{\uparrow \\ \text{energy}}}{E}[\{\phi\}]\right) = \text{Partition Function}$$

$\beta \sim \frac{1}{k_B\, \text{Temp}}$



ϕ]]) = "Partition Function"

QFT	Stat. Mechanics
space-time X	space (lattice)
dim = $1 + (d-1)$	dim = d
Euclidean Time	1 dimension
Field $\phi(x)$	Local order parameter (local spin)

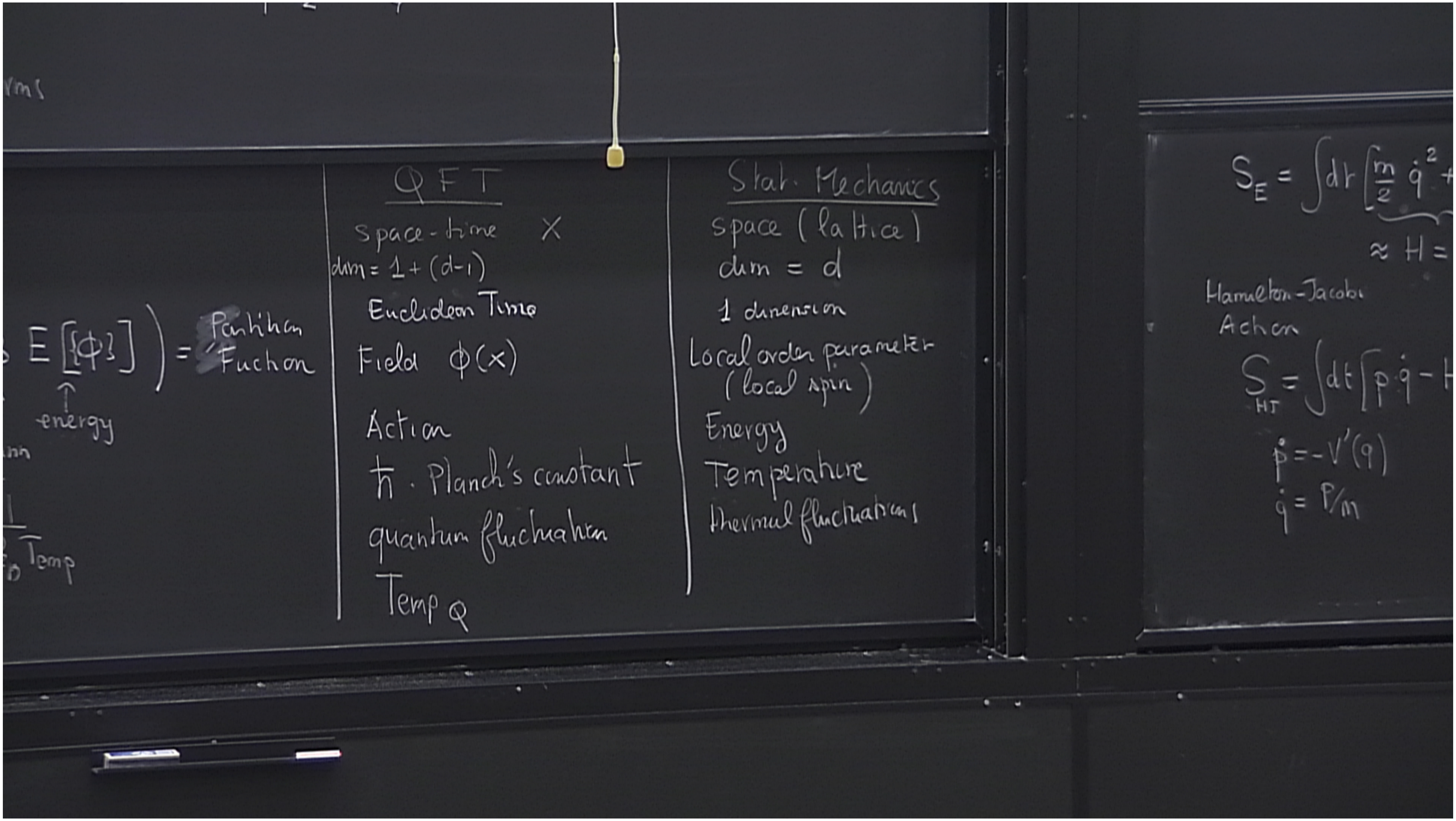


$$\frac{N!}{2^M \cdot M!} \text{ terms}$$

$$\exp\Big(-\underset{\substack{\uparrow \\ \text{Boltzmann} \\ \text{Factor}}}{\beta}\ \underset{\substack{\uparrow \\ \text{energy}}}{E}\big[\{\phi\}\big]\Big) = \text{Partition Function}$$

$$\beta \sim \frac{1}{k_B\,\text{Temp}}$$

QFT	Stat. Mechanics
space-time X	space (lattice)
dim = 1 + (d−1)	dim = d
Euclidean Time	1 dimension
Field $\phi(x)$	Local order parameter (local spin)
Action	Energy
$\hbar$. Planck's constant	Temperature



rms

$E\left[\{\phi\}\right]\Big) =$ Partition Function

↑ energy

Temp

QFT	Stat. Mechanics
space-time X	space (lattice)
$\dim = 1 + (d-1)$	$\dim = d$
Euclidean Time	1 dimension
Field $\phi(x)$	Local order parameter (local spin)
Action	Energy
$\hbar$ · Planck's constant	Temperature
quantum fluctuation	thermal fluctuations
Temp Q	

$$S_E = \int d\tau \left[\underbrace{\frac{m}{2}\dot{q}^2 +}\right.$$

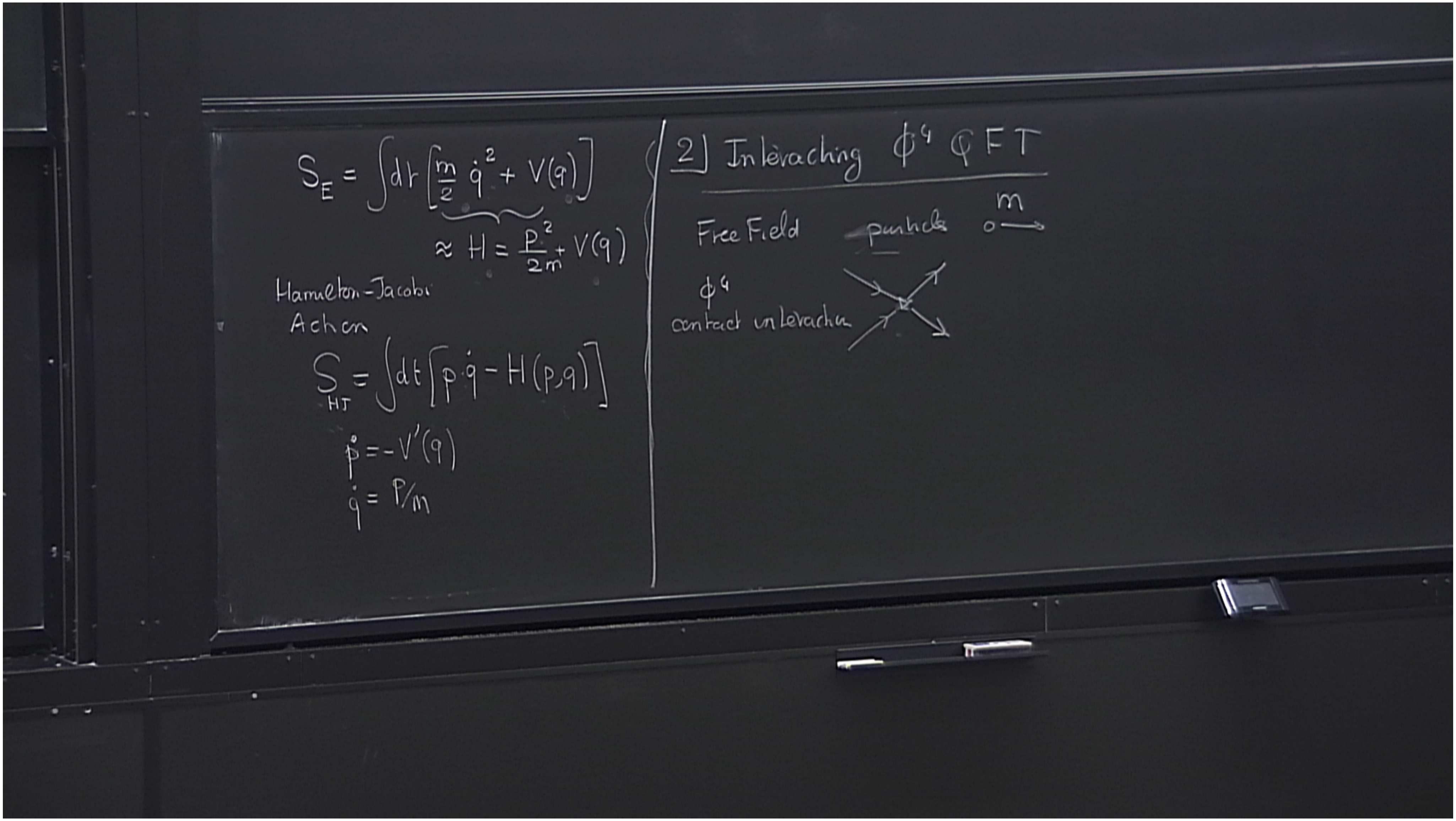
$$\approx H =$$

Hamilton-Jacobi Action

$$S_{HJ} = \int dt \left[p\dot{q} - H\right.$$

$$\dot{p} = -V'(q)$$

$$\dot{q} = p/m$$



$$S_E = \int d\tau \left[\underbrace{\frac{m}{2}\dot{q}^2}_{} + V(q)\right]$$

$$\approx H = \frac{p^2}{2m} + V(q)$$

Hamilton-Jacobi
Action

$$S_{HJ} = \int dt \left[p\,\dot{q} - H(p,q)\right]$$

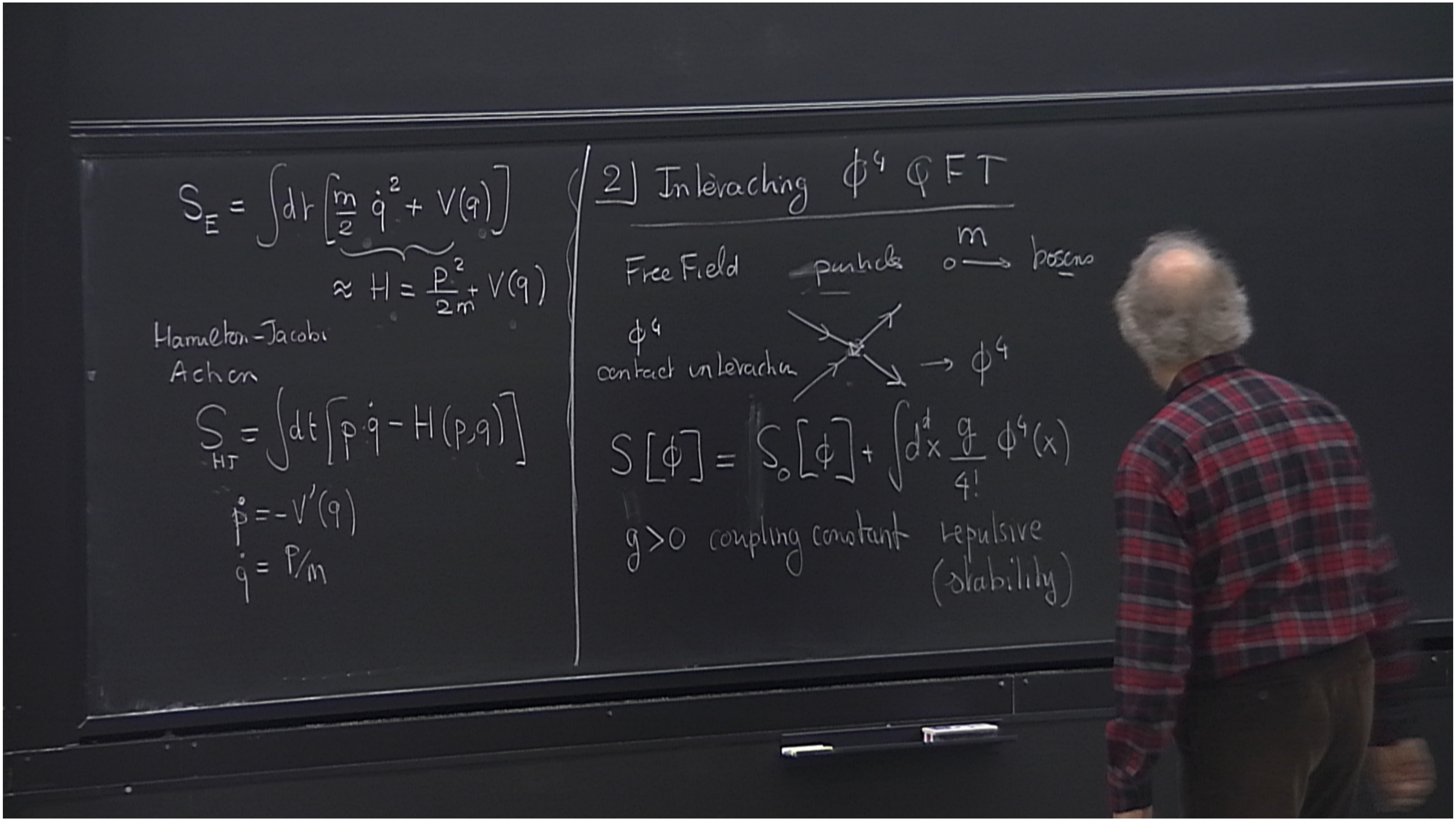
$$\dot{p} = -V'(q)$$

$$\dot{q} = p/m$$

2) Interacting ϕ^4 QFT

Free Field particles $\circ \xrightarrow{m}$

ϕ^4
contact interaction



$$S_E = \int dt \left[\underbrace{\frac{m}{2}\dot q^2 + V(q)}\right]$$

$$\approx H = \frac{p^2}{2m} + V(q)$$

Hamilton-Jacobi
Action

$$S_{HJ} = \int dt \left[p\dot q - H(p,q)\right]$$

$$\dot p = -V'(q)$$

$$\dot q = p/m$$

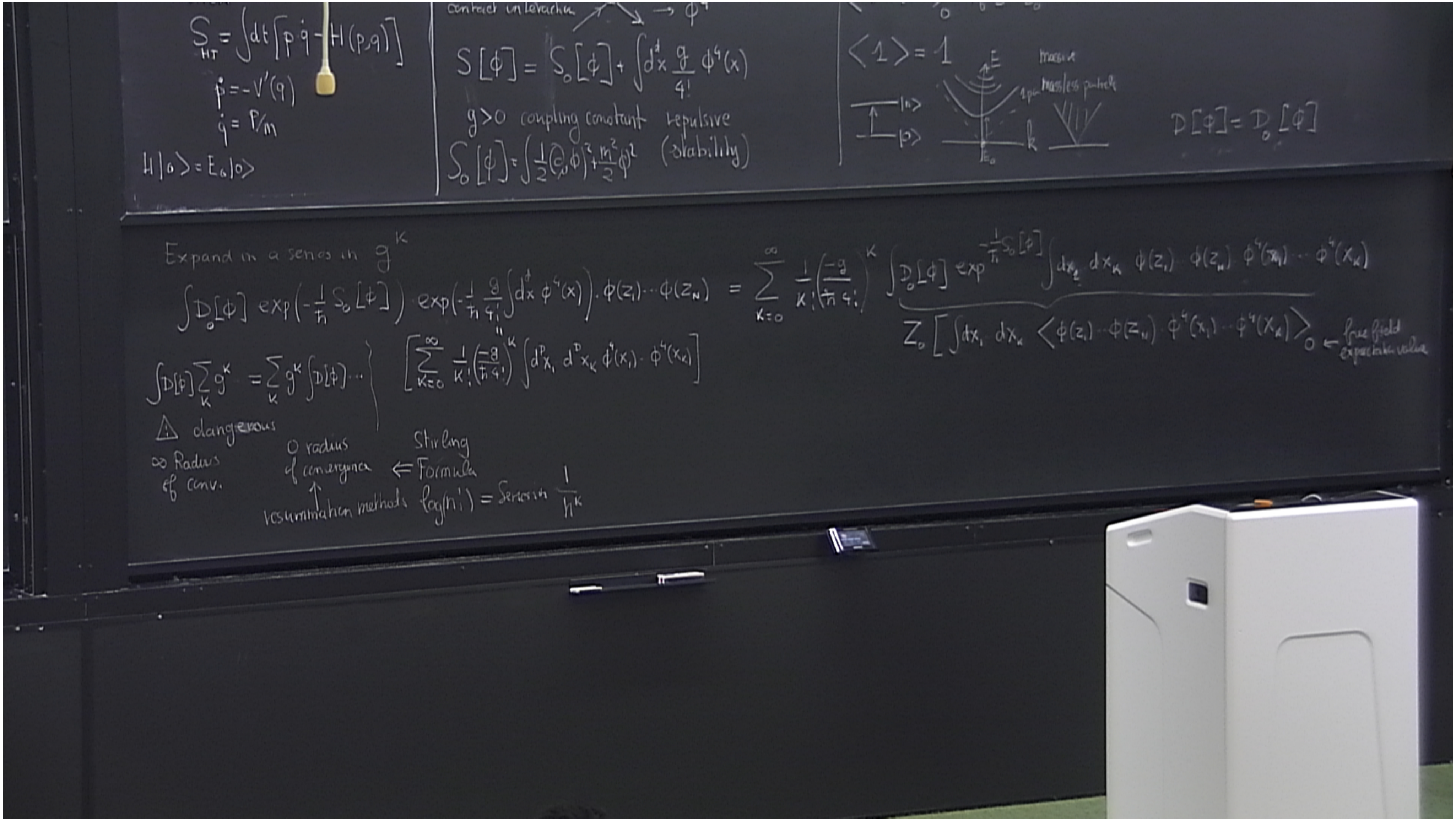
2) Interacting ϕ^4 QFT

Free Field particles $\circ \xrightarrow{m}$ bosons

ϕ^4
contact interaction $\rightarrow \phi^4$

$$S[\phi] = S_0[\phi] + \int d^dx \, \frac{g}{4!} \phi^4(x)$$

$g>0$ coupling constant repulsive
(stability)



contact interaction $\rightarrow \phi^4$

$$S_{HT} = \int dt \left[p\dot{q} - H(p,q) \right]$$

$$\dot{p} = -V'(q)$$

$$\dot{q} = p/m$$

$$H|0\rangle = E_0|0\rangle$$

$$S[\phi] = S_0[\phi] + \int d^dx \, \frac{g}{4!} \phi^4(x)$$

$g>0$ coupling constant repulsive

$$S_0[\phi] = \int \frac{1}{2}(\partial_\mu\phi)^2 + \frac{m^2}{2}\phi^2 \quad \text{(stability)}$$

$$\langle 1 \rangle = 1$$

$$D[\phi] = D_0[\phi]$$

Expand in a series in g^K

$$\int D_0[\phi] \exp\left(-\frac{1}{\hbar} S_0[\phi]\right) \exp\left(-\frac{1}{\hbar}\frac{g}{4!}\int d^dx \, \phi^4(x)\right) \cdot \phi(z_1)\cdots\phi(z_N) = \sum_{K=0}^{\infty} \frac{1}{K!}\left(\frac{-g}{\hbar\, 4!}\right)^K \underbrace{\int D_0[\phi] \exp^{-\frac{1}{\hbar}S_0[\phi]} \int dx_1 \cdots dx_K \, \phi(z_1)\cdots\phi(z_N)\,\phi^4(x_1)\cdots\phi^4(x_K)}_{Z_0\left[\int dx_1 \cdots dx_K \langle \phi(z_1)\cdots\phi(z_N)\,\phi^4(x_1)\cdots\phi^4(x_K)\rangle_0\right.}$$

$\leftarrow$ free field expectation value

$$\overset{\shortparallel}{\left[\sum_{K=0}^{\infty} \frac{1}{K!}\left(\frac{-g}{\hbar\, 4!}\right)^K \int d^Dx_1 \cdots d^Dx_K \, \phi^4(x_1)\cdots\phi^4(x_K)\right]}$$

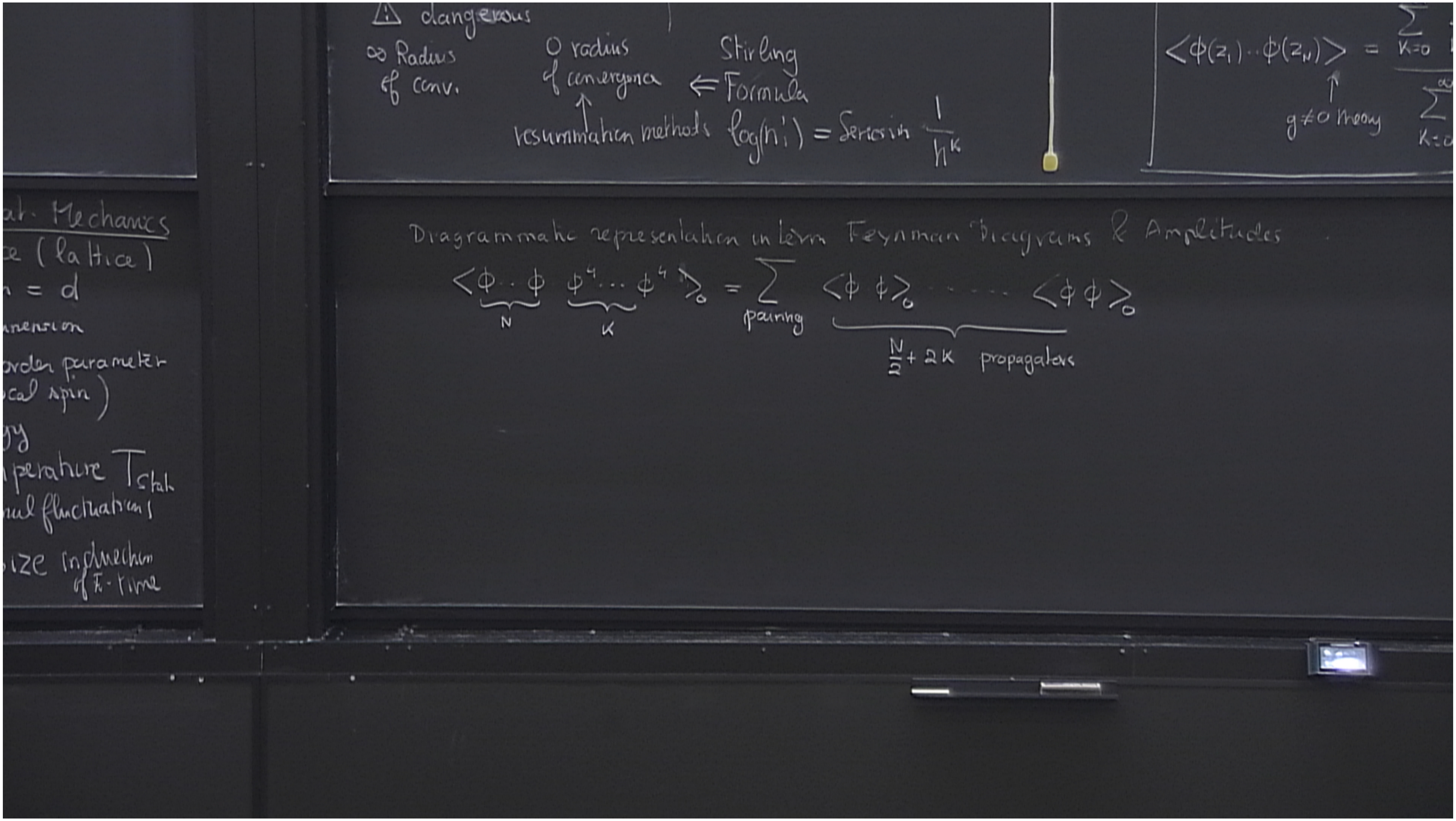
$$\int D[\phi] \sum_K g^K = \sum_K g^K \int D[\phi] \cdots$$

⚠ dangerous

∞ Radius of conv.

0 radius of convergence $\Leftarrow$ Stirling Formula

$\uparrow$ resummation methods $\log(n!) =$ Series in $\frac{1}{\hbar^K}$


⚠ dangerous

∞ Radius of conv.

0 radius of convergence $\Leftarrow$ Stirling Formula

↑

resummation methods $\log(n!) =$ Series in $\frac{1}{n^K}$

$\langle\phi(z_1)\cdots\phi(z_N)\rangle = \sum_{K=0}$

↑

$g \neq 0$ theory

$\sum_{K=0}^{\infty}$

at. Mechanics

e (lattice)

$= d$

mension

order parameter

cal spin)

gy

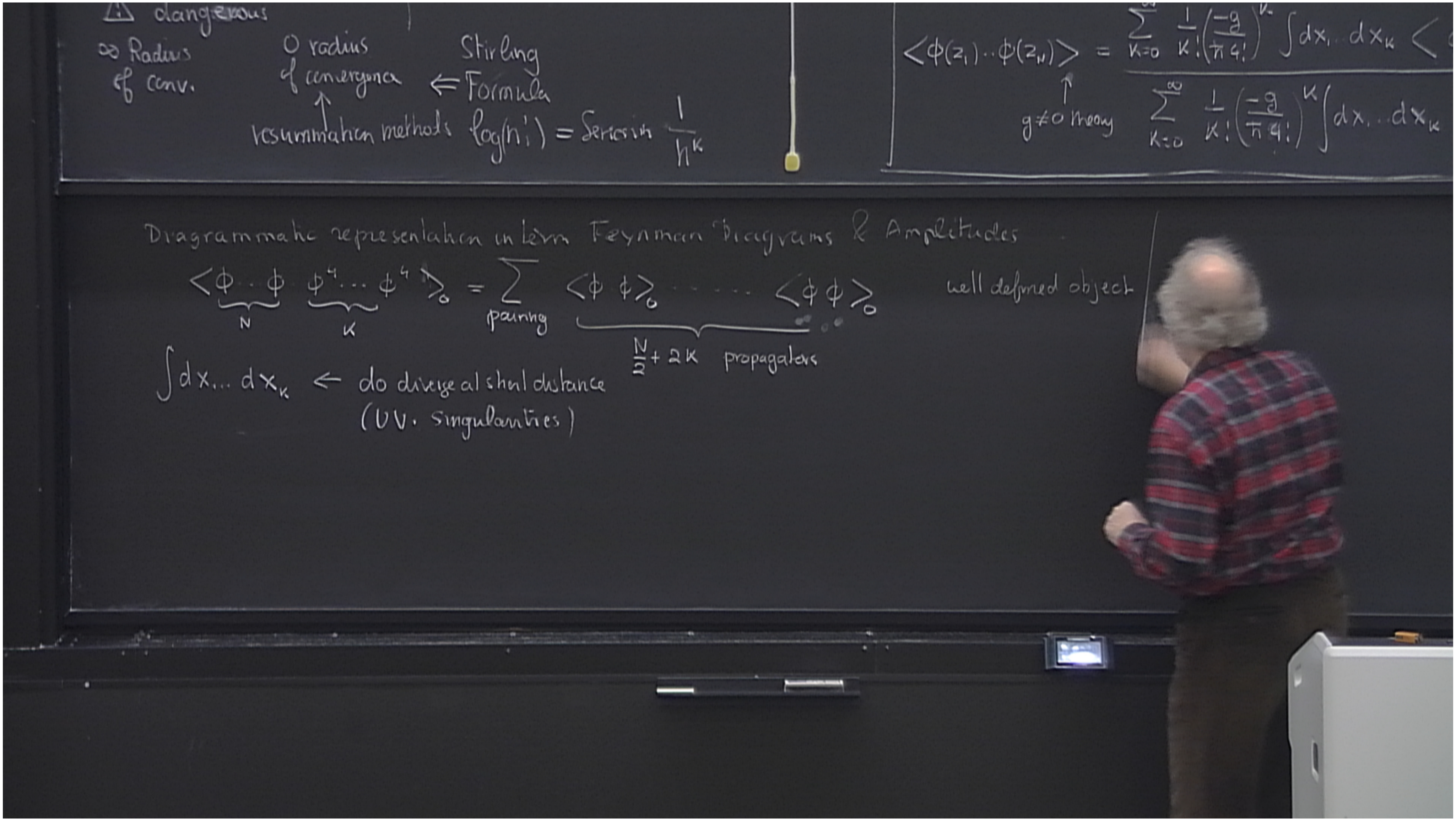
perature T_{stat}

al fluctuations

ize in direction of $\hbar$-time

Diagrammatic representation in term Feynman Diagrams & Amplitudes

$$\langle \underbrace{\phi\cdots\phi}_{N}\ \underbrace{\phi^4\cdots\phi^4}_{K} \rangle_0 = \sum_{\text{pairing}} \underbrace{\langle\phi\,\phi\rangle_0 \cdots\cdots \langle\phi\,\phi\rangle_0}_{\frac{N}{2}+2K \text{ propagators}}$$



⚠ dangerous

∞ Radius of conv.

0 radius of convergence ⇐ Stirling Formula

↑ resummation methods $\log(n!) = $ Series in $\frac{1}{n^k}$

$$\langle \phi(z_1) \cdots \phi(z_N) \rangle = \frac{\sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{-g}{\hbar\, 4!}\right)^k \int dx_1 \cdots dx_k \langle}{\sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{-g}{\hbar\, 4!}\right)^k \int dx_1 \cdots dx_k}$$

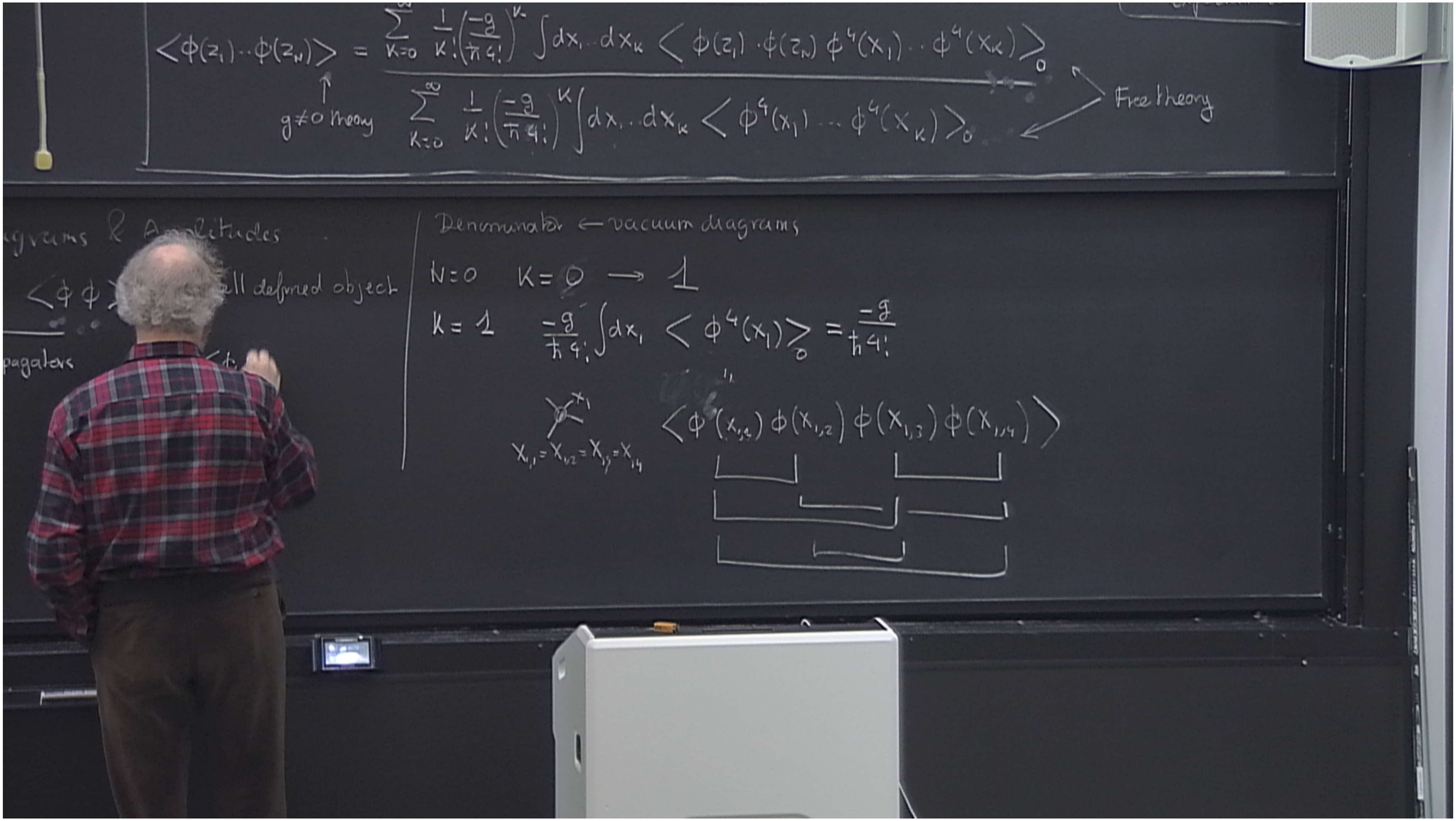
↑ $g \neq 0$ theory

Diagrammatic representation in term Feynman Diagrams & Amplitudes

$$\langle \underbrace{\phi \cdots \phi}_{N} \; \underbrace{\phi^4 \cdots}_{k} \phi^4 \rangle_0 = \sum_{\text{pairing}} \underbrace{\langle \phi\, \phi \rangle_0 \cdots\cdots \langle \phi\, \phi \rangle_0}_{\frac{N}{2} + 2k \ \text{propagators}}$$

well defined object

$\int dx_1 \ldots dx_k$ ← do diverge at short distance (UV. singularities)



$$\langle \phi(z_1)\cdots\phi(z_N)\rangle = \frac{\sum_{K=0}^{\infty}\frac{1}{K!}\left(\frac{-g}{\hbar\,4!}\right)^K \int dx_1\cdots dx_K \langle \phi(z_1)\cdots\phi(z_N)\,\phi^4(x_1)\cdots\phi^4(x_K)\rangle_0}{\sum_{K=0}^{\infty}\frac{1}{K!}\left(\frac{-g}{\hbar\,4!}\right)^K \int dx_1\cdots dx_K \langle \phi^4(x_1)\cdots\phi^4(x_K)\rangle_0}$$

$g \neq 0$ theory

Free theory

grams & Amplitudes

$\langle \phi\,\phi\rangle$ ll defined object

pagators

Denominator ← vacuum diagrams

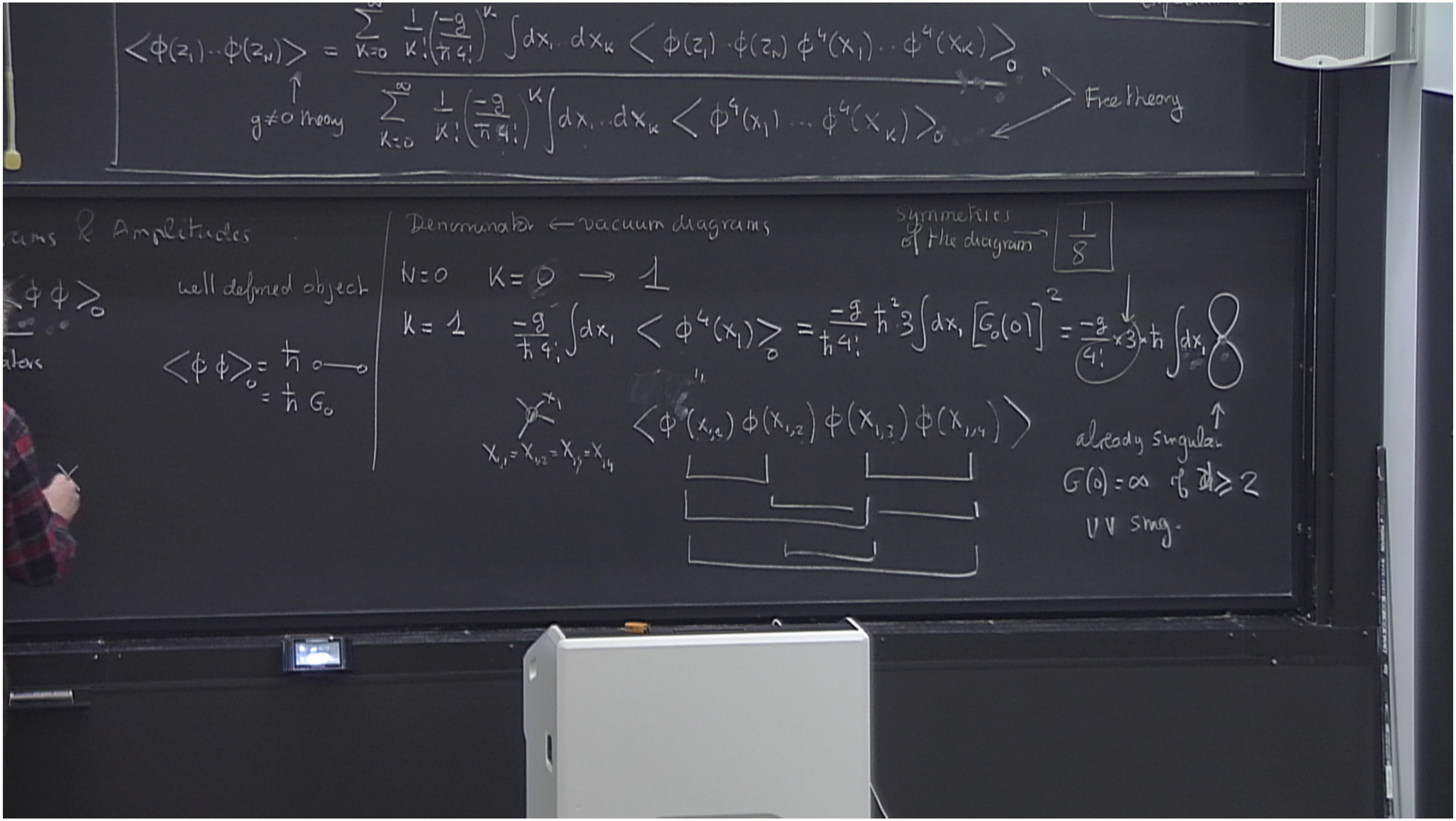
$N=0 \quad K=0 \longrightarrow 1$

$K=1 \quad \frac{-g}{\hbar\,4!}\int dx_1 \langle \phi^4(x_1)\rangle_0 = \frac{-g}{\hbar\,4!}$

x_1

$x_{1,1}=x_{1,2}=x_{1,3}=x_{1,4}$

$\langle \phi(x_{1,1})\,\phi(x_{1,2})\,\phi(x_{1,3})\,\phi(x_{1,4})\rangle$



$$\langle \phi(z_1)\cdots\phi(z_N)\rangle = \frac{\sum_{K=0}^{\infty}\frac{1}{K!}\left(\frac{-g}{\hbar 4!}\right)^K \int dx_1 \cdots dx_K \langle \phi(z_1)\cdots\phi(z_N)\,\phi^4(x_1)\cdots\phi^4(x_K)\rangle_0}{\sum_{K=0}^{\infty}\frac{1}{K!}\left(\frac{-g}{\hbar 4!}\right)^K \int dx_1 \cdots dx_K \langle \phi^4(x_1)\cdots\phi^4(x_K)\rangle_0}$$

↑ $g \neq 0$ theory

Free theory

...ams & Amplitudes

$\langle \phi\phi\rangle_0$ well defined object

...ators

$\langle \phi\phi\rangle_0 = \hbar$ o——o $= \hbar G_0$

Denominator ← vacuum diagrams

$N=0 \quad K=0 \longrightarrow 1$

$$K=1 \quad \frac{-g}{\hbar 4!}\int dx_1 \langle \phi^4(x_1)\rangle_0 = \frac{-g}{\hbar 4!}\hbar^2 3\int dx_1 \left[G_0(0)\right]^2 = \frac{-g}{4!}\times 3\times\hbar\int dx_1 \,8$$

Symmetries of the diagram → $\boxed{\frac{1}{8}}$

x_1

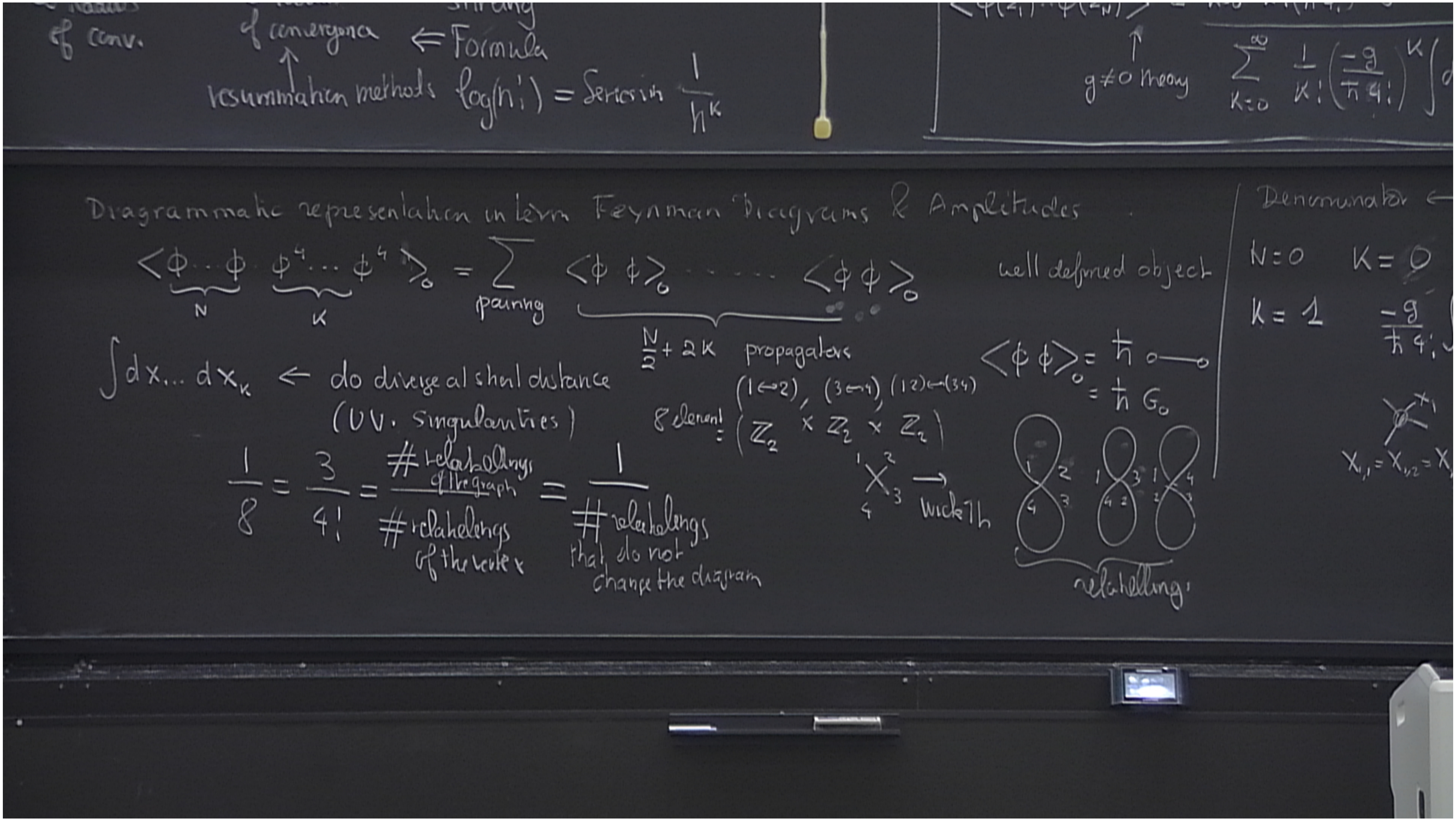
$X_{1,1} = X_{1,2} = X_{1,3} = X_{1,4}$

$\langle \phi(x_{1,1})\phi(x_{1,2})\phi(x_{1,3})\phi(x_{1,4})\rangle$

already singular ↑

$G(0) = \infty$ if $d \geq 2$

UV sing.



of conv.

of convergence ⇐ Formula

resummation methods $\log(n_i) =$ Series in $\frac{1}{\hbar^k}$

$g \neq 0$ meaning $\sum_{K=0}^{\infty} \frac{1}{K!}\left(\frac{-g}{\hbar\, 4!}\right)^K \int$

Diagrammatic representation in term Feynman Diagrams & Amplitudes

$$\langle \underbrace{\phi \cdots \phi}_{N} \; \underbrace{\phi^4 \cdots \phi^4}_{K} \rangle_0 = \sum_{\text{pairing}} \underbrace{\langle \phi\, \phi \rangle_0 \cdots\cdots \langle \phi\, \phi \rangle_0}_{\frac{N}{2} + 2K \text{ propagators}}$$

well defined object

$\int dx_1 \ldots dx_K \leftarrow$ do diverge at short distance (UV. singularities)

$\langle \phi\, \phi \rangle_0 = \hbar \; \circ\!\!-\!\!\circ = \hbar\, G_0$

$(1 \leftrightarrow 2)$, $(3 \leftrightarrow 4)$, $(12) \leftrightarrow (34)$

8 element $= (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

$$\frac{1}{8} = \frac{3}{4!} = \frac{\#\text{ relabellings of the graph}}{\#\text{ relabelings of the vertex}} = \frac{1}{\#\text{ relabelings that do not change the diagram}}$$

Wick Th

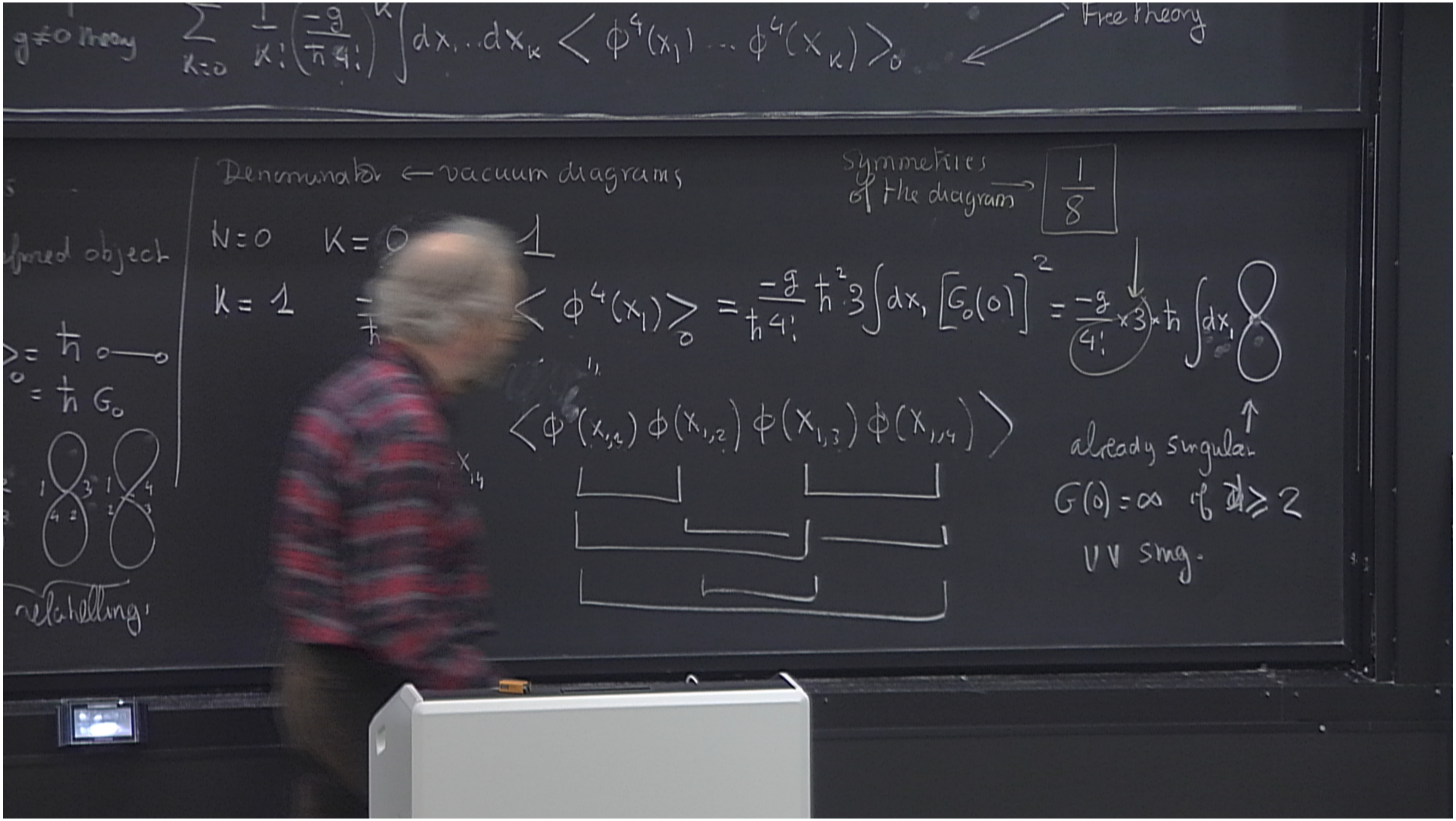
relabelling

Denominator ←

$N = 0 \quad K = 0$

$K = 1 \quad \frac{-g}{\hbar\, 4!}$

$x_{1,1} = x_{1,2} = x$



$g\neq 0$ theory $\sum_{k=0} \frac{1}{k!}\left(\frac{-g}{\hbar\, 4!}\right)^k \int dx_1\dots dx_k \left\langle \phi^4(x_1)\dots\phi^4(x_k)\right\rangle_0$ ← Free theory

Denominator ← vacuum diagrams

Symmetries of the diagram → $\boxed{\frac{1}{8}}$

$N=0 \quad K=0 \quad 1$

$K=1 \quad \frac{-g}{\hbar 4!}\left\langle \phi^4(x_1)\right\rangle_0 = \frac{-g}{\hbar\, 4!}\hbar^2\, 3\int dx_1 \left[G_0(0)\right]^2 = \frac{-g}{4!}\times 3\times\hbar\int dx_1$

$\left\langle \phi(x_{1,1})\,\phi(x_{1,2})\,\phi(x_{1,3})\,\phi(x_{1,4})\right\rangle$

already singular

$G(0)=\infty$ if $d\geq 2$

UV sing.

fined object

$\rangle_0 = \hbar$

$= \hbar\, G_0$

relabelling:

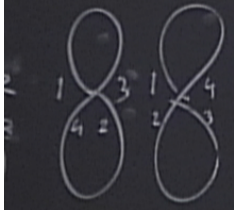
$g\neq 0$ theory $\displaystyle\sum_{k=0} \frac{1}{k!}\left(\frac{-g}{\hbar\, 4!}\right)^{k}\int dx_1\dots dx_k \left\langle \phi^4(x_1)\dots\phi^4(x_k)\right\rangle_0$ ← Free theory

s

fined object

$\rangle_0 = \hbar$ o—o

$_0 = \hbar\, G_0$



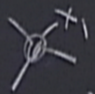
relabelling:

Denominator ← vacuum diagrams

Symmetries of the diagram → $\boxed{\frac{1}{8}}$

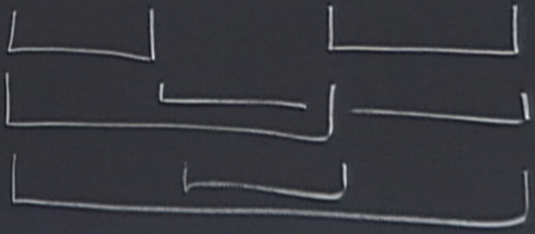
$N=0 \quad K=0 \longrightarrow 1$

$K=1 \quad \dfrac{-g}{\hbar\, 4!}\displaystyle\int dx_1 \left\langle \phi^4(x_1)\right\rangle_0 = \frac{-g}{\hbar\, 4!}\hbar^2\, 3\int dx_1 \left[G_0(0)\right]^2 = \frac{-g}{4!}\times 3\times\hbar\int dx_1$ 



$x_{1,1}=x_{1,2}=x_{1,3}=x_{1,4}$

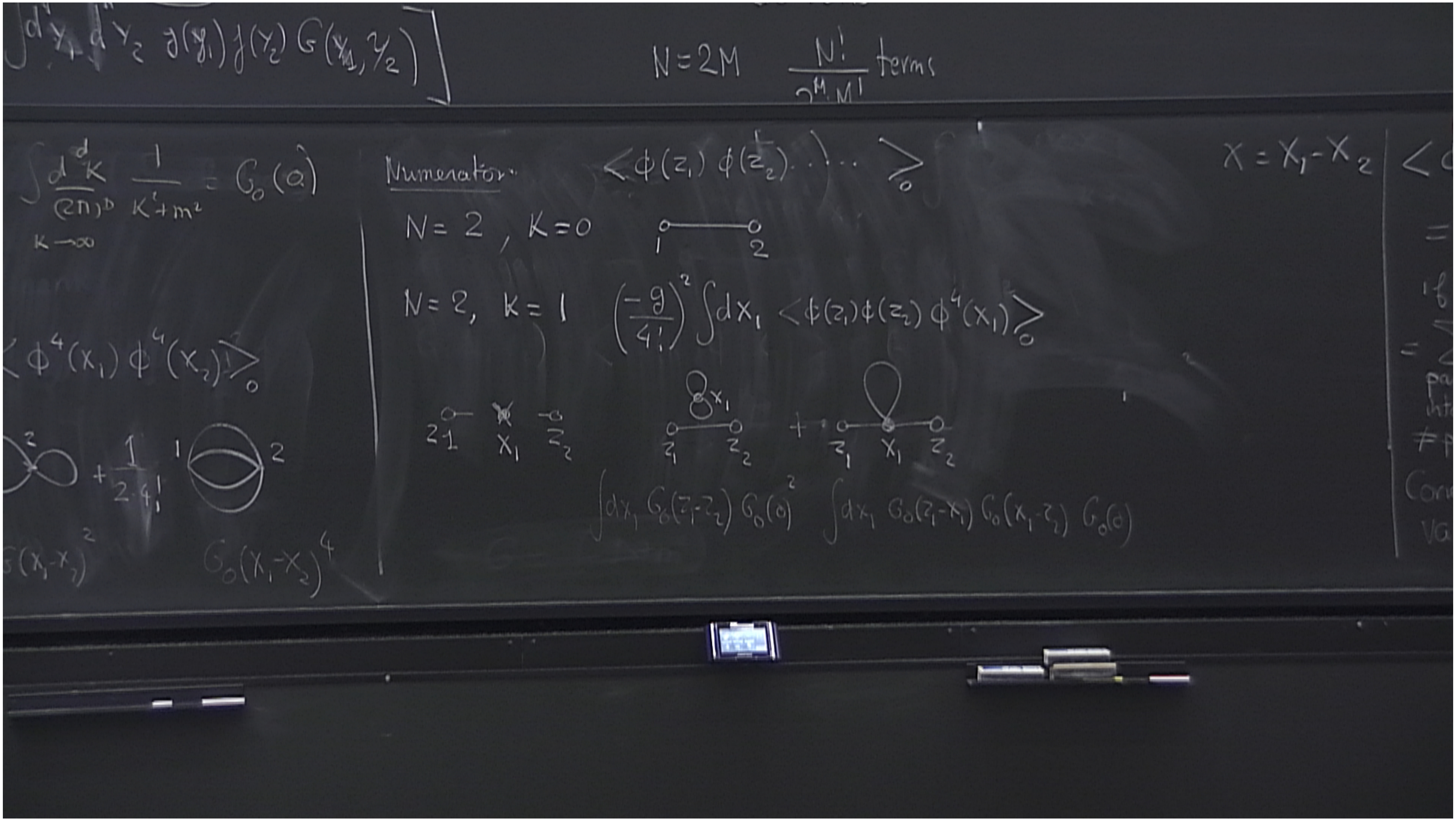
$\left\langle \phi(x_{1,1})\,\phi(x_{1,2})\,\phi(x_{1,3})\,\phi(x_{1,4})\right\rangle$

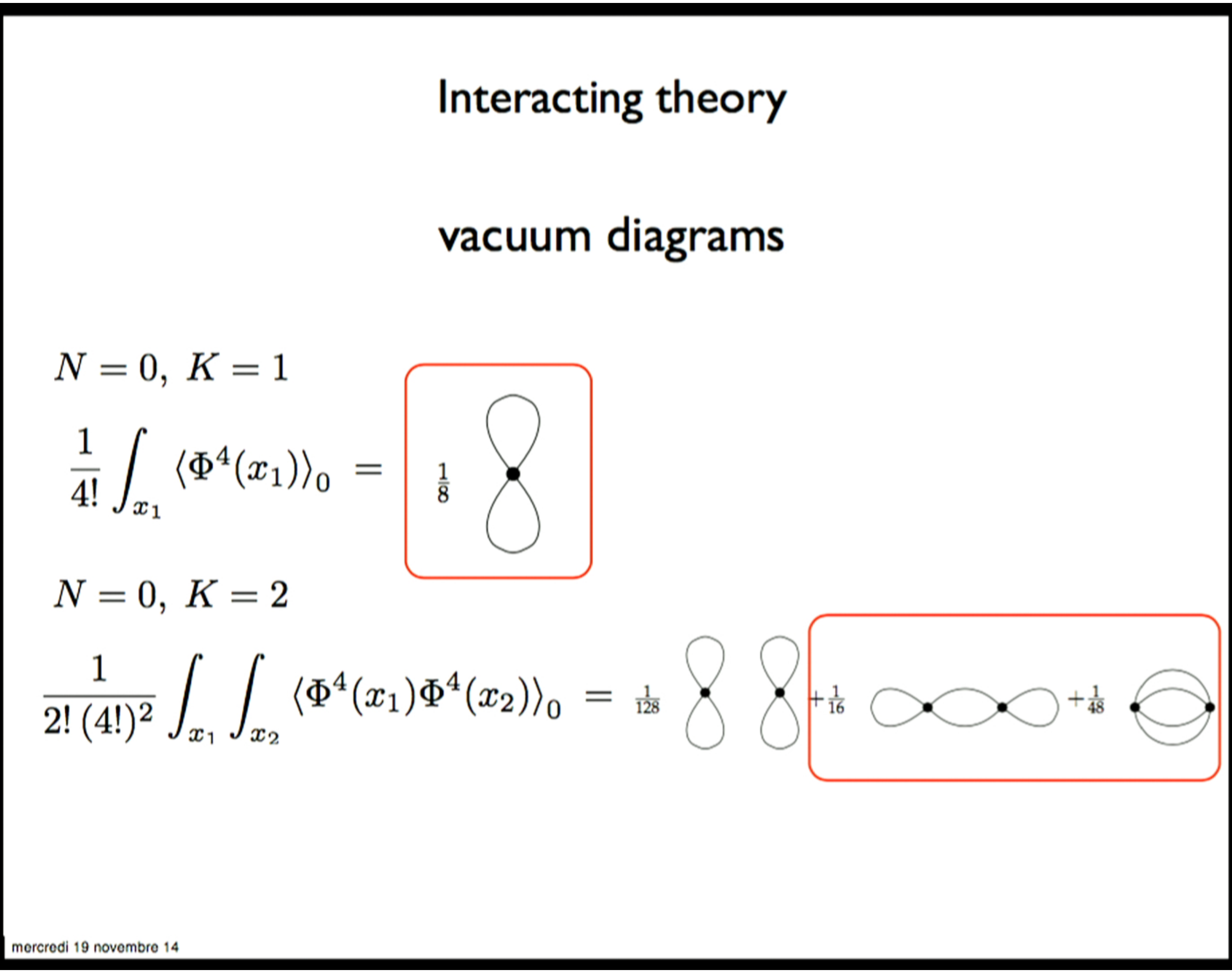


already singular

$G(0)=\infty$ if $d\geq 2$

UV sing.





Interacting theory

vacuum diagrams

$N = 0,\ K = 1$

$$\frac{1}{4!}\int_{x_1} \langle \Phi^4(x_1) \rangle_0 \ = \ \frac{1}{8}$$

$N = 0,\ K = 2$

$$\frac{1}{2!\,(4!)^2}\int_{x_1}\int_{x_2} \langle \Phi^4(x_1)\Phi^4(x_2) \rangle_0 \ = \ \frac{1}{128} \ + \frac{1}{16} \ + \frac{1}{48}$$

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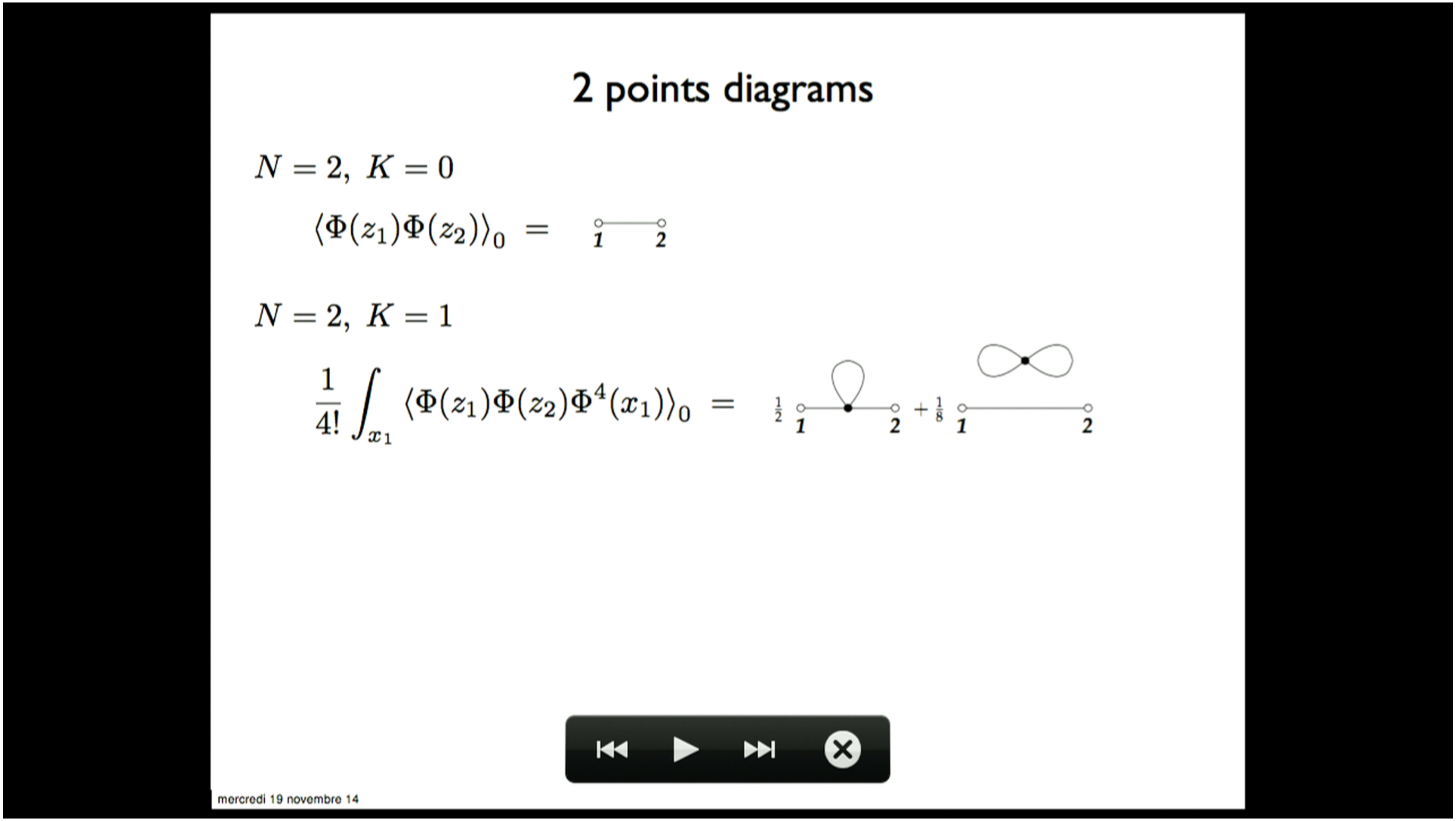

2 points diagrams

$N=2,\ K=0$

$$\langle \Phi(z_1)\Phi(z_2)\rangle_0 \ =$$

$N=2,\ K=1$

$$\frac{1}{4!}\int_{x_1}\langle \Phi(z_1)\Phi(z_2)\Phi^4(x_1)\rangle_0 \ = \ \tfrac{1}{2} \ \ + \tfrac{1}{8}$$

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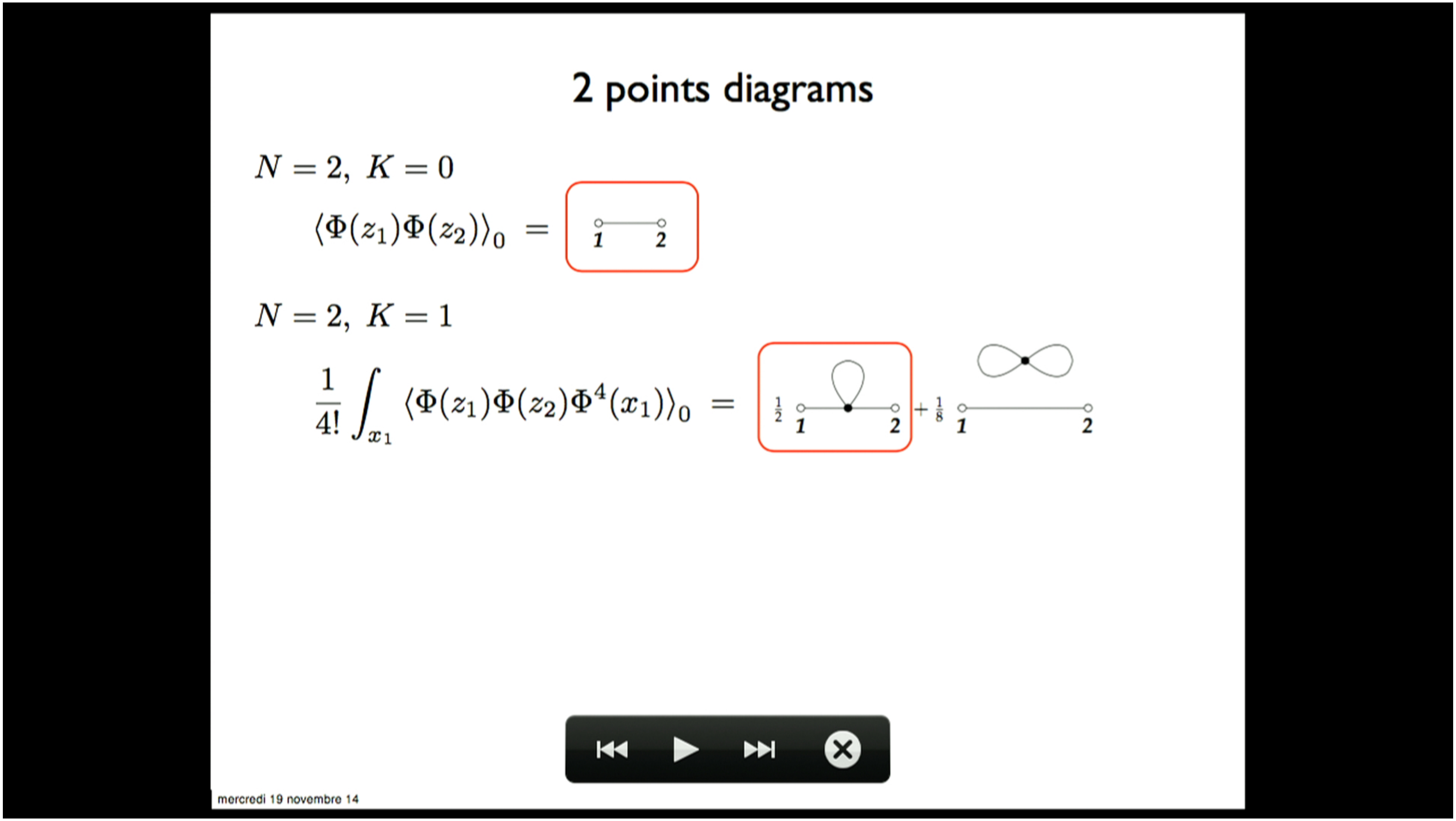

2 points diagrams

$N = 2,\ K = 0$

$$\langle \Phi(z_1)\Phi(z_2)\rangle_0 =$$

$N = 2,\ K = 1$

$$\frac{1}{4!}\int_{x_1}\langle \Phi(z_1)\Phi(z_2)\Phi^4(x_1)\rangle_0 = \frac{1}{2}\qquad + \frac{1}{8}$$

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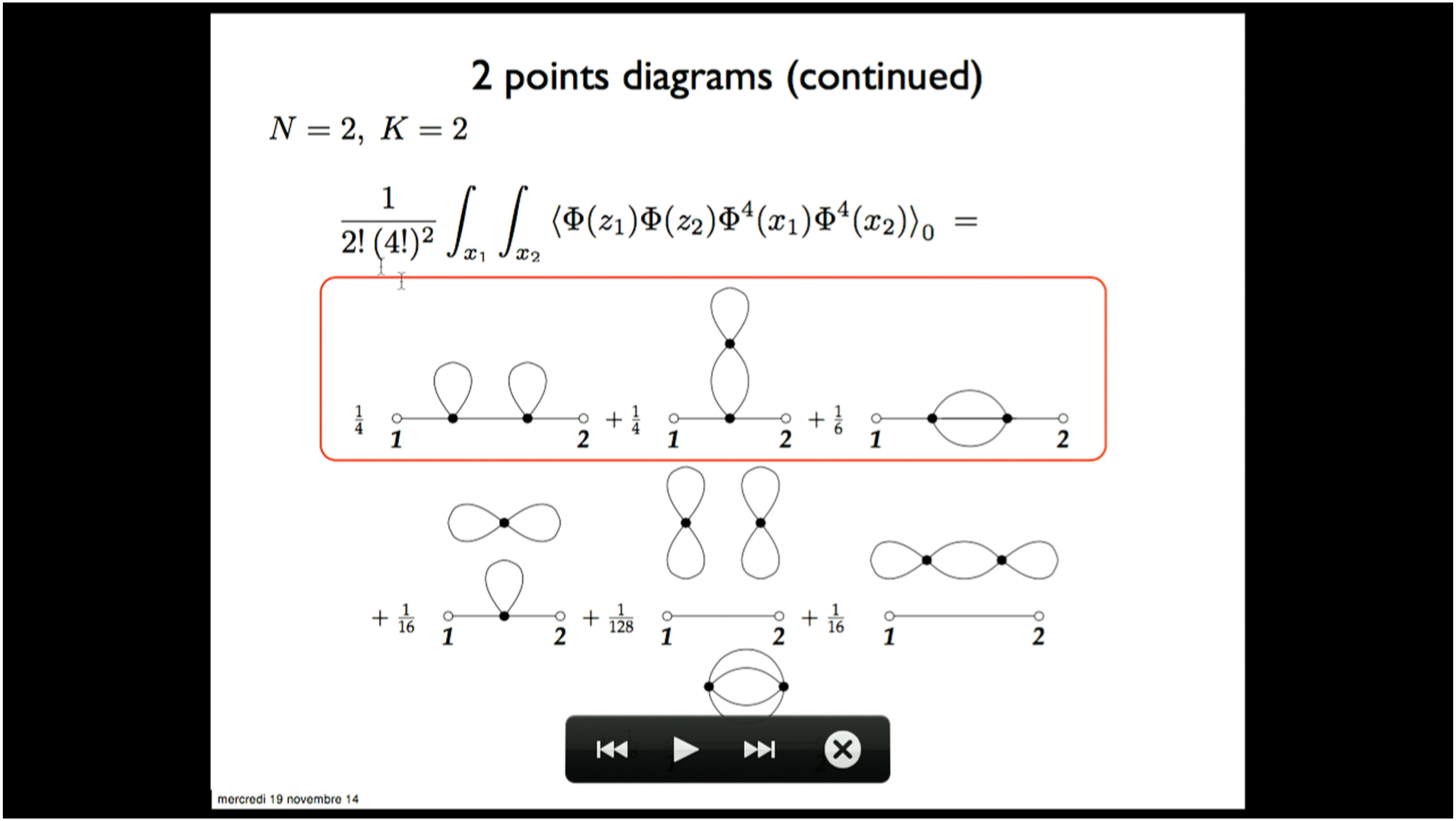

2 points diagrams (continued)

$N = 2,\ K = 2$

$$\frac{1}{2!\,(4!)^2}\int_{x_1}\int_{x_2}\langle\Phi(z_1)\Phi(z_2)\Phi^4(x_1)\Phi^4(x_2)\rangle_0 \ =$$

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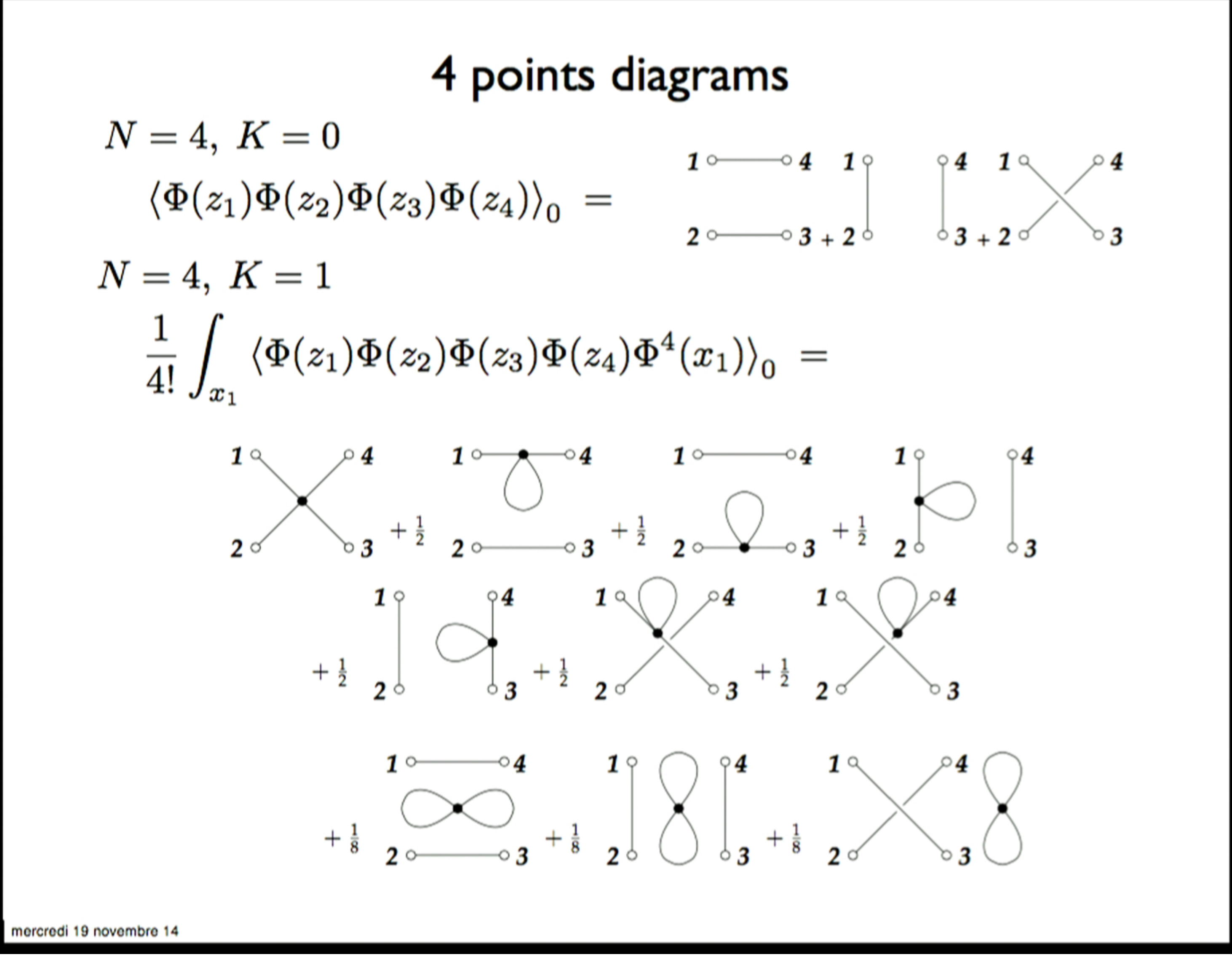

4 points diagrams

$N=4,\ K=0$

$$\langle\Phi(z_1)\Phi(z_2)\Phi(z_3)\Phi(z_4)\rangle_0 \ = \ + \ +$$

$N=4,\ K=1$

$$\frac{1}{4!}\int_{x_1}\langle\Phi(z_1)\Phi(z_2)\Phi(z_3)\Phi(z_4)\Phi^4(x_1)\rangle_0 \ =$$

$$+\tfrac{1}{2} \quad +\tfrac{1}{2} \quad +\tfrac{1}{2}$$

$$+\tfrac{1}{2} \quad +\tfrac{1}{2} \quad +\tfrac{1}{2}$$

$$+\tfrac{1}{8} \quad +\tfrac{1}{8} \quad +\tfrac{1}{8}$$

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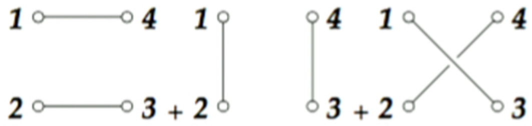
4 points diagrams

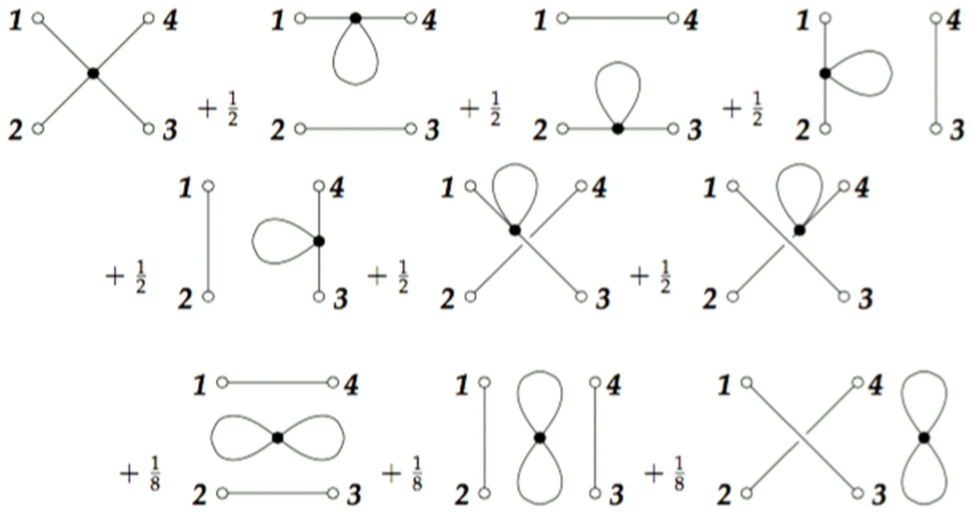
$N = 4,\ K = 0$

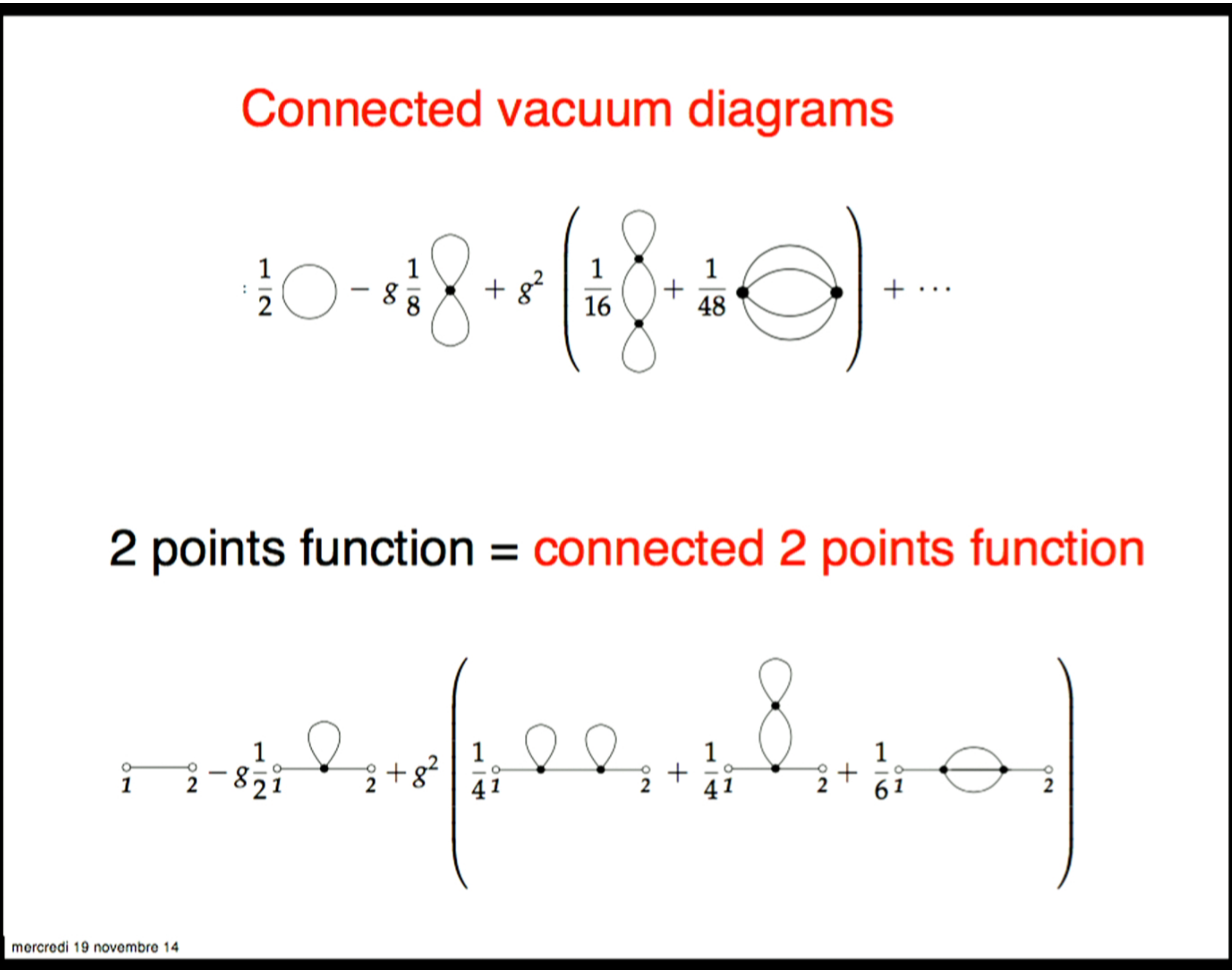
$$\langle \Phi(z_1)\Phi(z_2)\Phi(z_3)\Phi(z_4)\rangle_0 \ =$$



$N = 4,\ K = 1$

$$\frac{1}{4!}\int_{x_1} \langle \Phi(z_1)\Phi(z_2)\Phi(z_3)\Phi(z_4)\Phi^4(x_1)\rangle_0 \ =$$





Connected vacuum diagrams

2 points function = connected 2 points function

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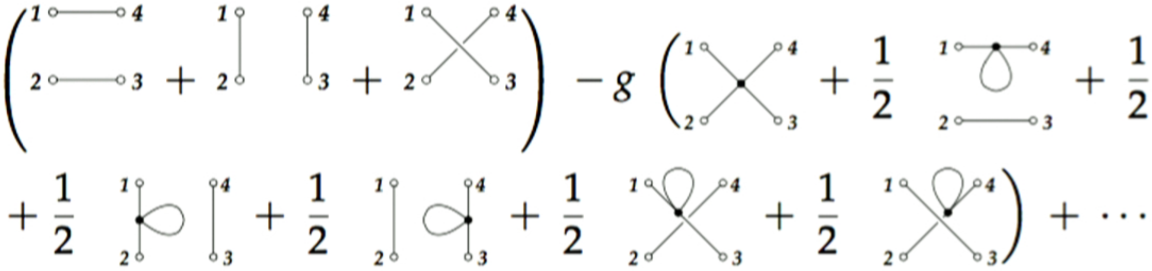
4 points function (up to order 1)



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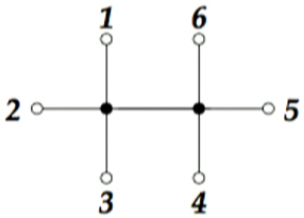
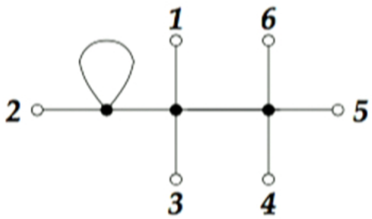
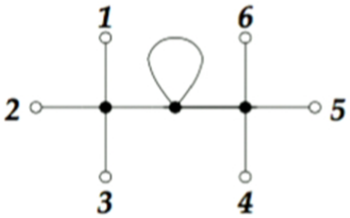
Connected 6 points function (up to order 3)

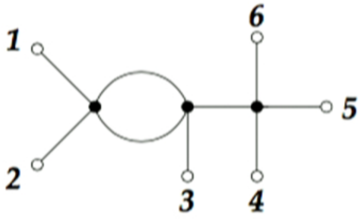
g^2


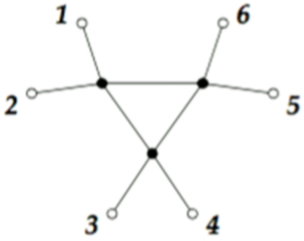
$+\cdots$

$-g^3\frac{1}{2}$


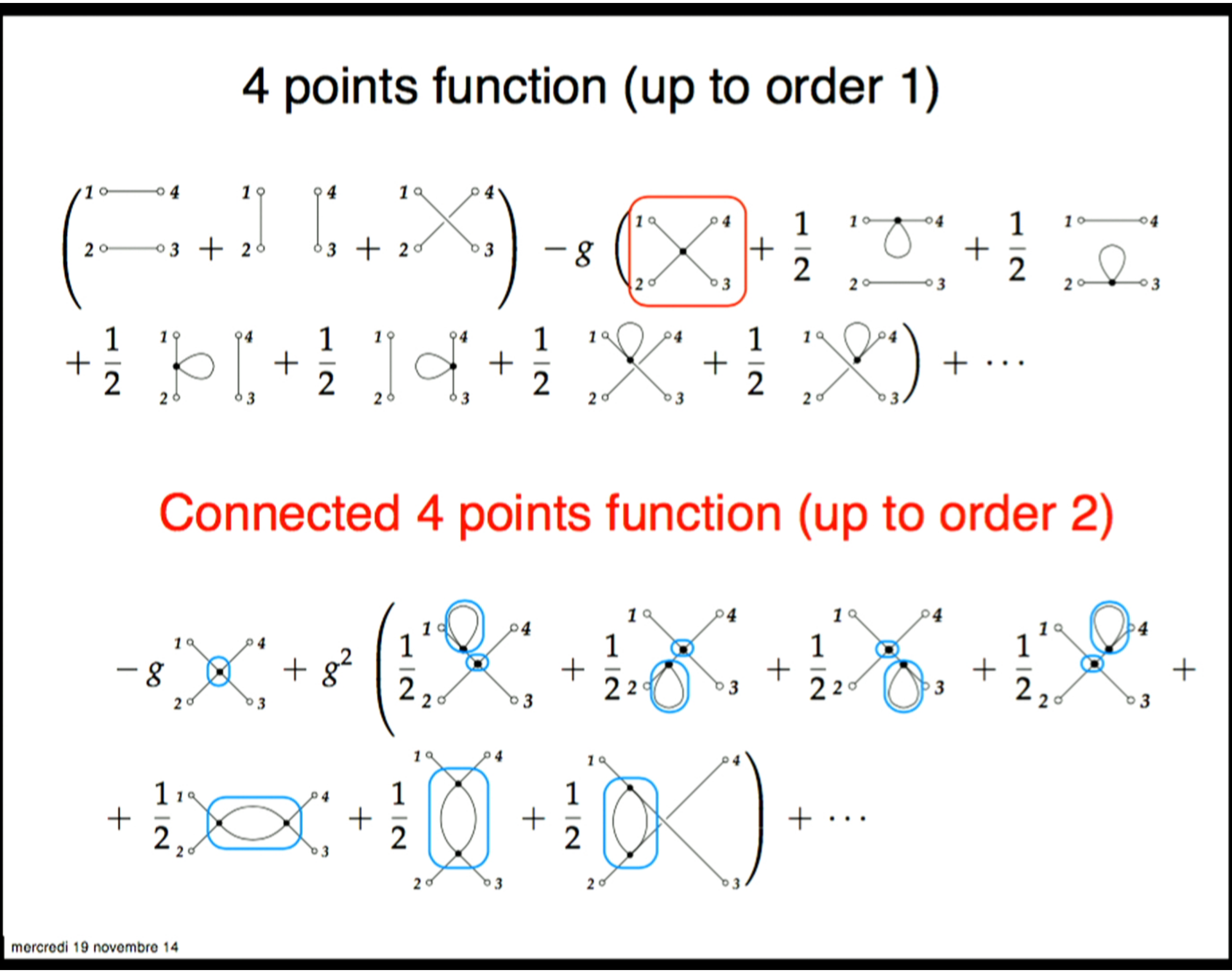
$+\cdots$ $-g^3\frac{1}{2}$


$+\cdots$

$-g^3\frac{1}{2}$


$+\cdots$ $-g^3$


$+\cdots$



4 points function (up to order 1)

Connected 4 points function (up to order 2)

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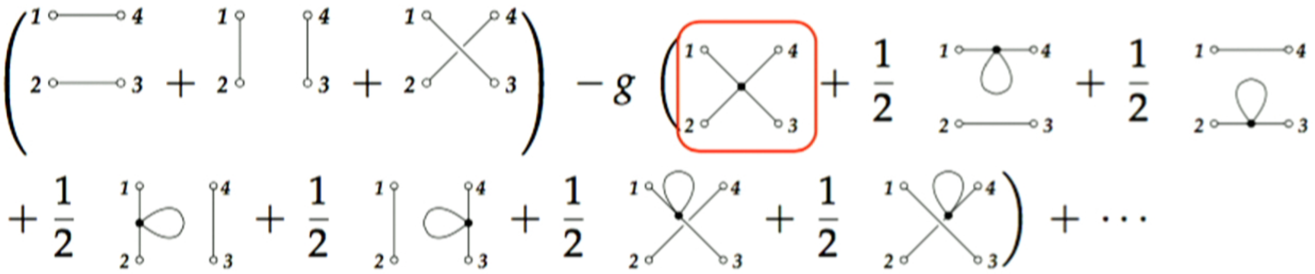
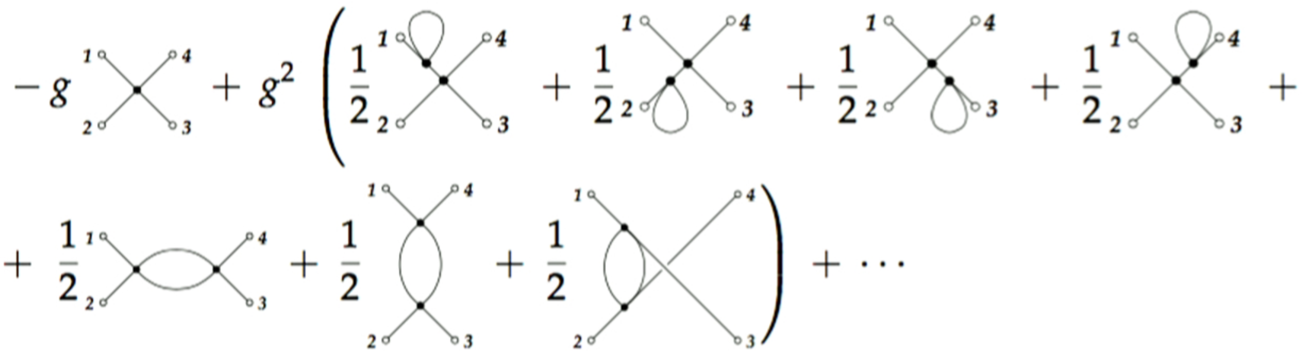
4 points function (up to order 1)



Connected 4 points function (up to order 2)



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