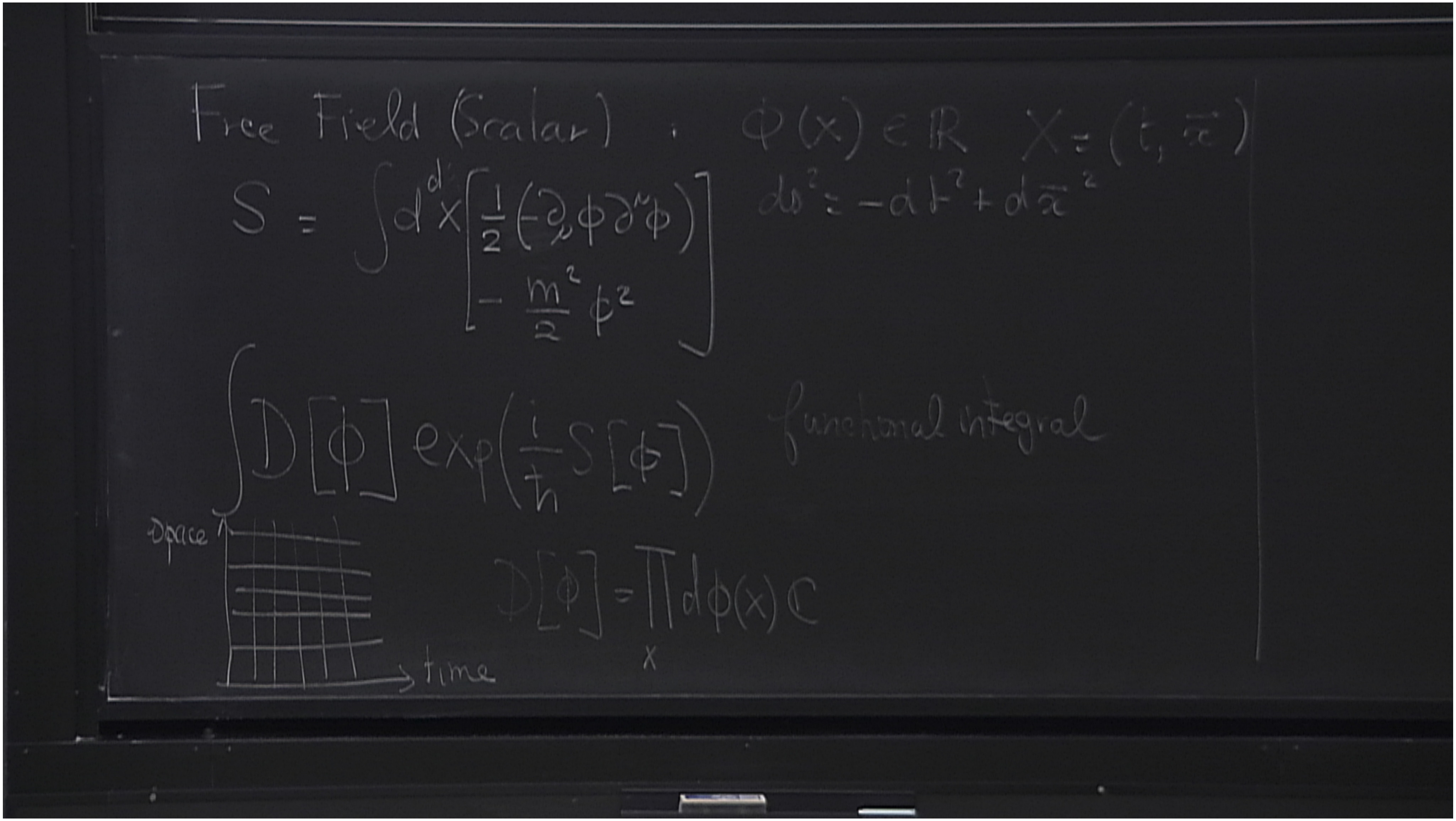
Title: PSI 2015/2016 Quantum Field Theory II - Francois David - Lecture 3

Date: Nov 11, 2015 09:00 AM

URL: http://pirsa.org/15110029

Abstract:



Free Field (Scalar) : $\phi(x) \in \mathbb{R}$ $\quad X = (t, \vec{x})$

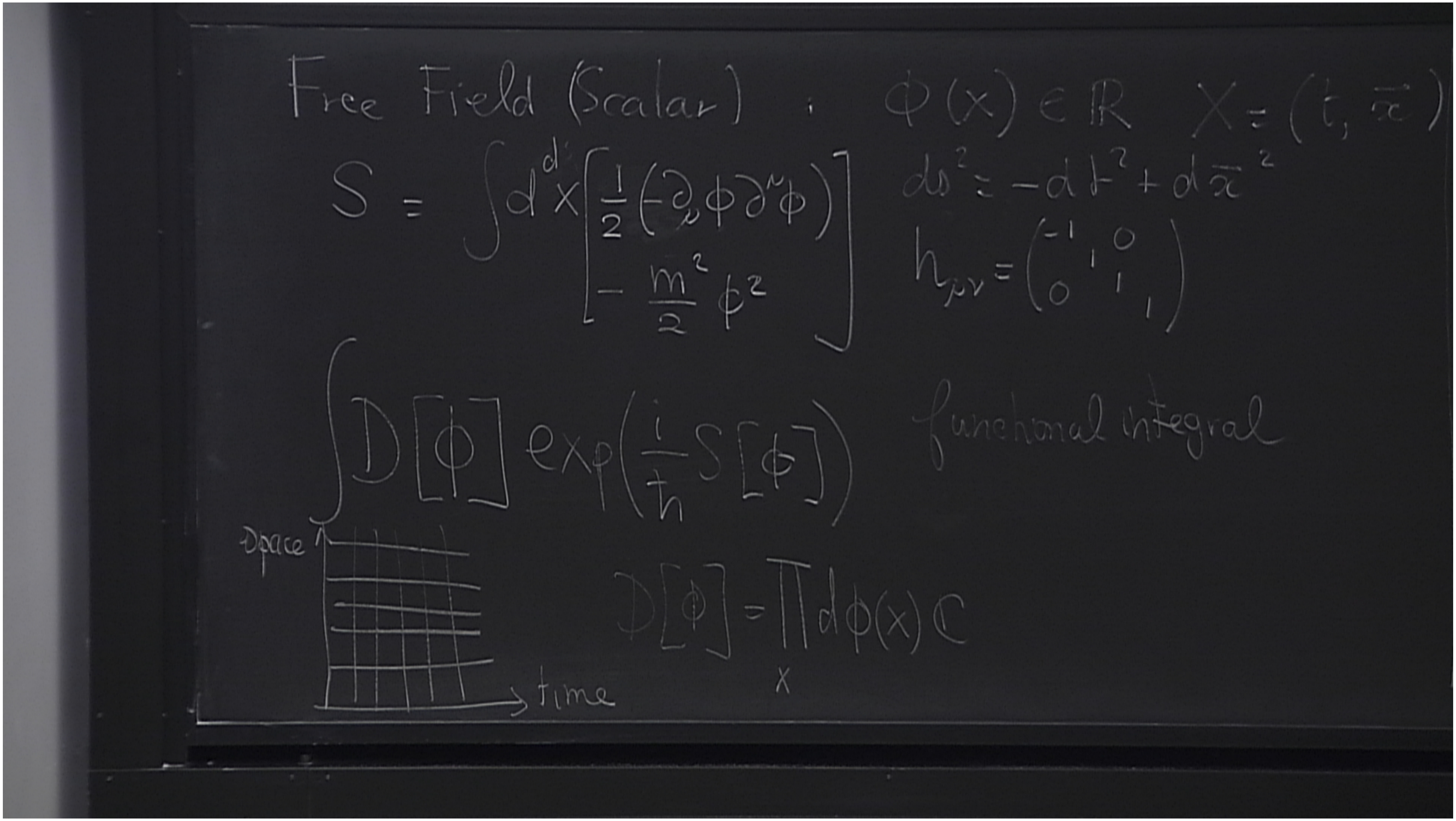
$$S = \int d^d x \left[\frac{1}{2}\left(-\partial_\mu \phi \partial^\mu \phi\right) - \frac{m^2}{2}\phi^2 \right]$$

$ds^2 = -dt^2 + d\vec{x}^{\,2}$

$$\int D[\phi] \exp\left(\frac{i}{\hbar} S[\phi]\right)$$

functional integral

$$D[\phi] = \prod_{x} d\phi(x)\, C$$



Free Field (Scalar) : $\phi(x) \in \mathbb{R}$ $\quad x = (t, \vec{x})$

$$S = \int d^dx \left[\frac{1}{2}\left(-\partial_\nu \phi \partial^\nu \phi\right) - \frac{m^2}{2}\phi^2 \right]$$

$$ds^2 = -dt^2 + d\vec{x}^2$$

$$h_{\mu\nu} = \begin{pmatrix} -1 & & 0 & \\ & 1 & & \\ 0 & & 1 & \\ & & & 1 \end{pmatrix}$$

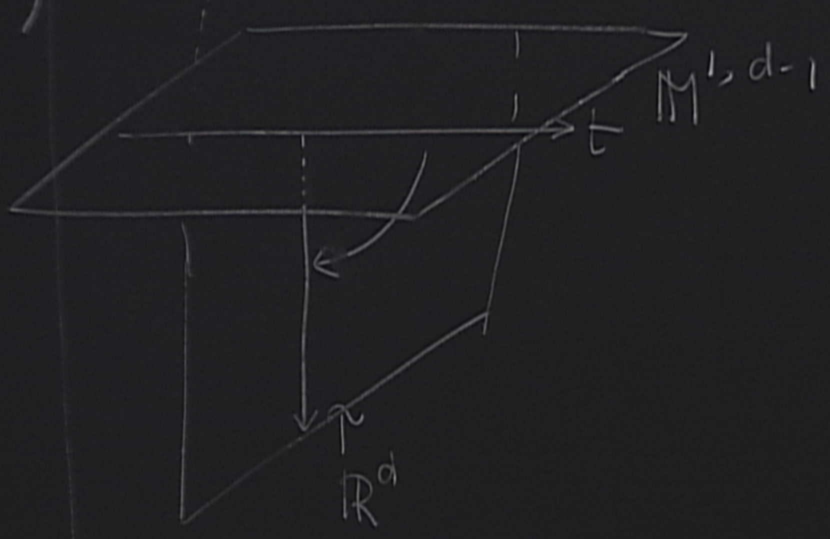
$$\int D[\phi] \exp\left(\frac{i}{\hbar} S[\phi]\right)$$

functional integral

$$D[\phi] = \prod_x d\phi(x)\, C$$

$, \vec{x})$

"Euclidean Time" $t = -i\tau$

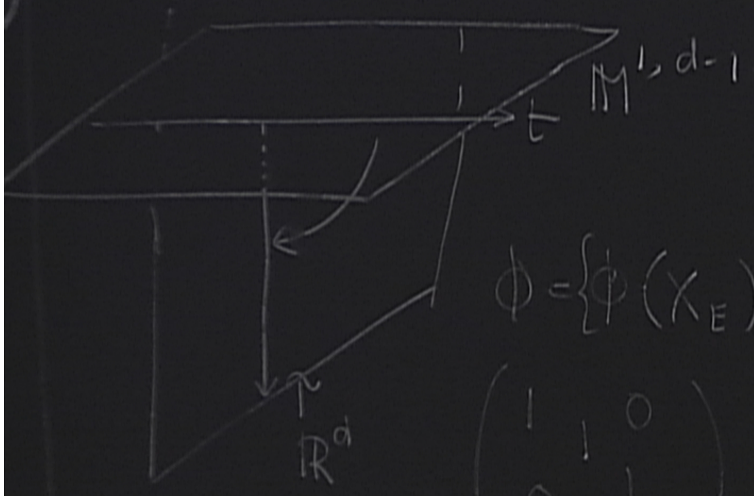


$X_E = (\tau, \vec{x})$

$ds^2 = d\tau^2 + d\vec{x}^2 = (dX_E)^2$

"Euclidean Time" $t = -i\tau$

$$S_E[\phi] = \int d^d x_E \left[\frac{1}{2}(\partial_\mu \phi \, \partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2\right]$$



$\phi = \{\phi(x_E)\}$

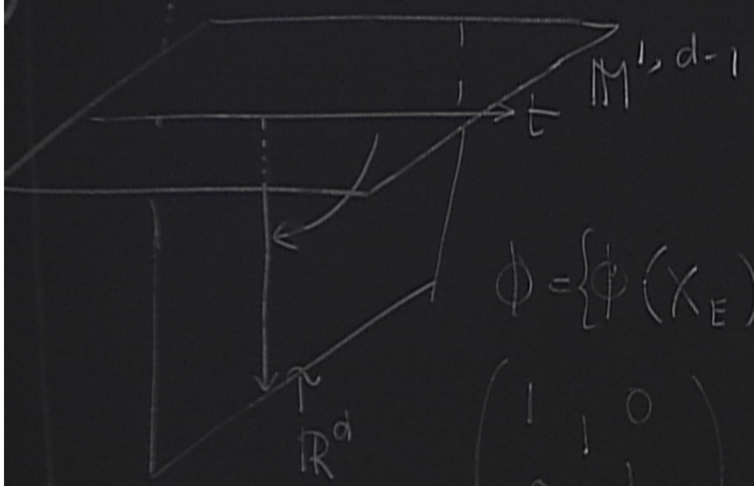
$$\begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

$x_E = (\tau, \vec{x})$

$ds^2 = d\tau^2 + d\vec{x}^2 = (dx_E)^2$

"Euclidean Time" $t = -i\tau$

$$S_E[\phi] = \int d^d x_E \left[\frac{1}{2}(\partial_\mu \phi \, \partial_\mu \phi) + \frac{m^2}{2}\phi^2\right]$$



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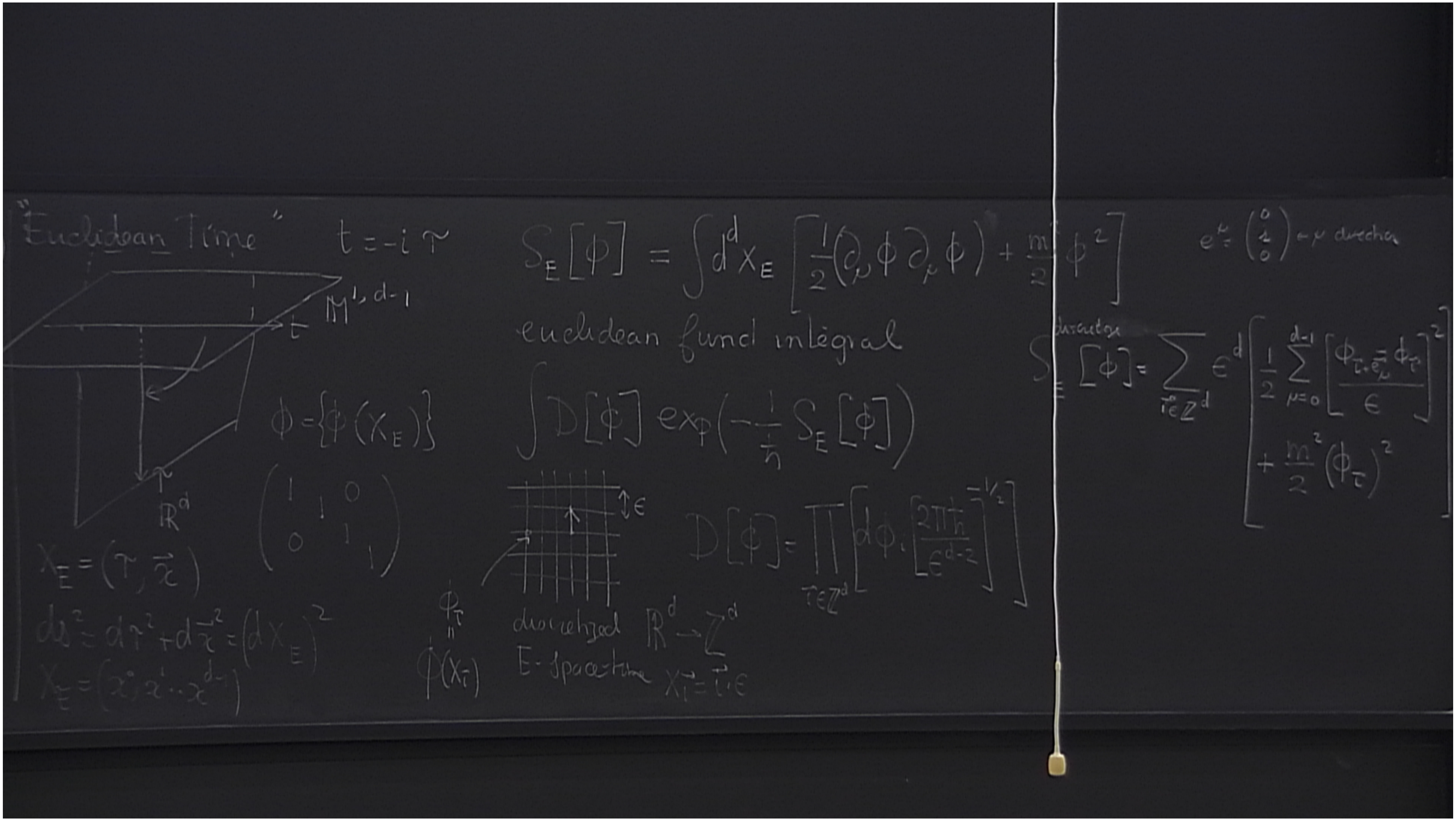


"Euclidean Time" $t = -i\tau$

$\phi = \{\phi(X_E)\}$

$$\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & 1 & \\ & & 1 \end{pmatrix}$$

$X_E = (\tau, \vec{x})$

$ds^2 = d\tau^2 + d\vec{x}^2 = (dX_E)^2$

$X_E = (x^0, x^1 \cdots x^{d-1})$

$\phi_{\vec{i}}$ $=$ $\phi(X_{\vec{i}})$

$$S_E[\phi] = \int d^d X_E \left[\frac{1}{2}(\partial_\mu \phi \, \partial_\mu \phi) + \frac{m^2}{2} \phi^2\right]$$

euclidean funct integral

$$\int D[\phi] \exp\left(-\frac{1}{\hbar} S_E[\phi]\right)$$

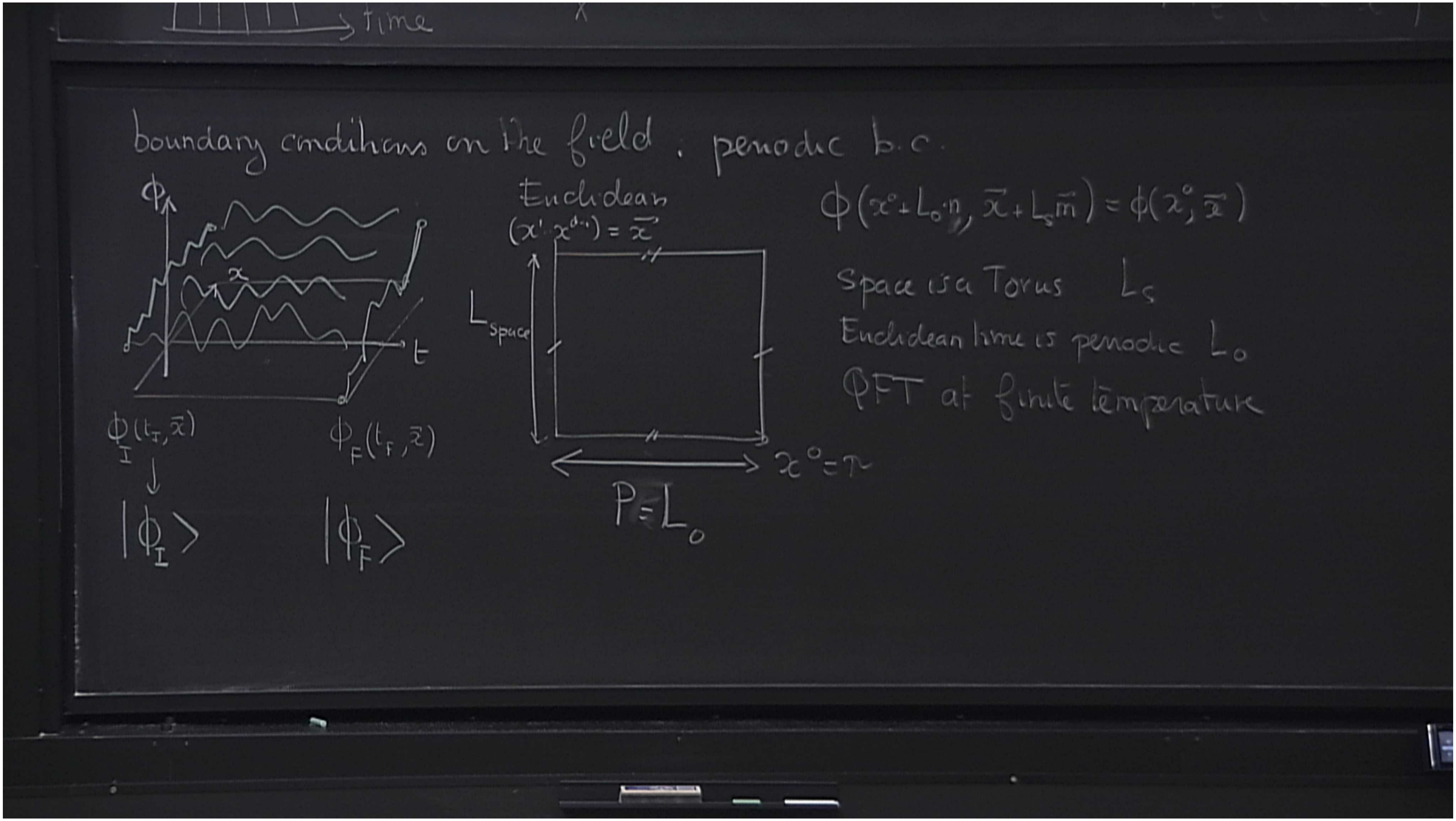
$$D[\phi] = \prod_{\vec{i} \in \mathbb{Z}^d} \left[d\phi_i \left[\frac{2\pi\hbar}{\epsilon^{d-2}}\right]^{-1/2} \right]$$

discretized $\mathbb{R}^d \to \mathbb{Z}^d$

E-spacetime $X_{\vec{i}} = \vec{i} \cdot \epsilon$

$e^\mu = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \leftarrow \mu$ direction

$$S_E^{\text{discrete}}[\phi] = \sum_{\vec{i} \in \mathbb{Z}^d} \epsilon^d \left[\frac{1}{2} \sum_{\mu=0}^{d-1} \left[\frac{\phi_{\vec{i}+\vec{e}_\mu} - \phi_{\vec{i}}}{\epsilon}\right]^2 + \frac{m^2}{2} (\phi_{\vec{i}})^2 \right]$$



boundary conditions on the field: periodic b.c.

$\phi_I(t_I, \vec{x})$ $\phi_F(t_F, \vec{x})$

$\downarrow$

$|\phi_I\rangle$ $|\phi_F\rangle$

$(x^1 \cdots x^{d-1}) = \vec{x}$

$P = L_0$

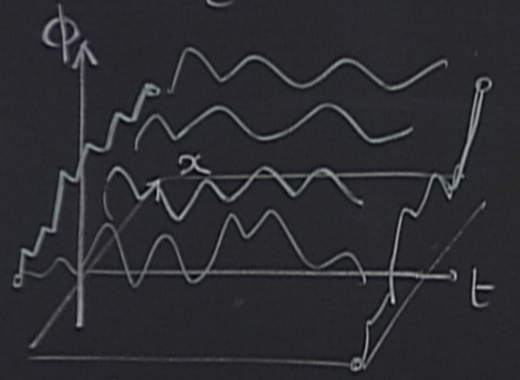
$$\phi(x^0 + L_0 \cdot n, \vec{x} + L_s \vec{m}) = \phi(x^0; \vec{x})$$

Space is a Torus L_s

Euclidean time is periodic L_0

QFT at finite temperature

boundary conditions on the field : periodic b.c. (simplest case)



$\phi_I(t_I,\vec{x})$

$\downarrow$

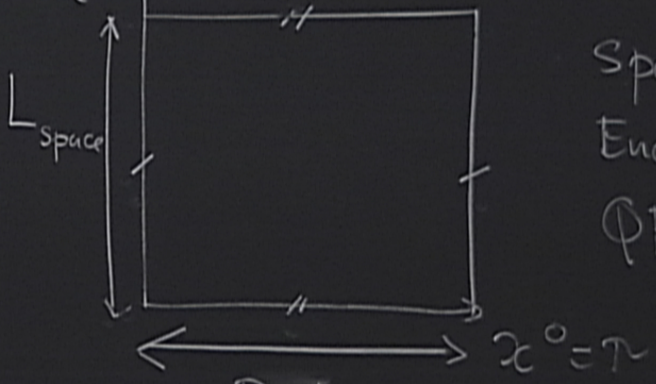
$|\phi_I\rangle$

$\phi_F(t_F,\vec{x})$

$|\phi_F\rangle$

Euclidean

$(x^1 \cdots x^{d-1}) = \vec{x}$



$P = L_0$

$$\phi(x^0 + L_0 \cdot n, \vec{x} + L_s \vec{m}) = \phi(x^0, \vec{x})$$

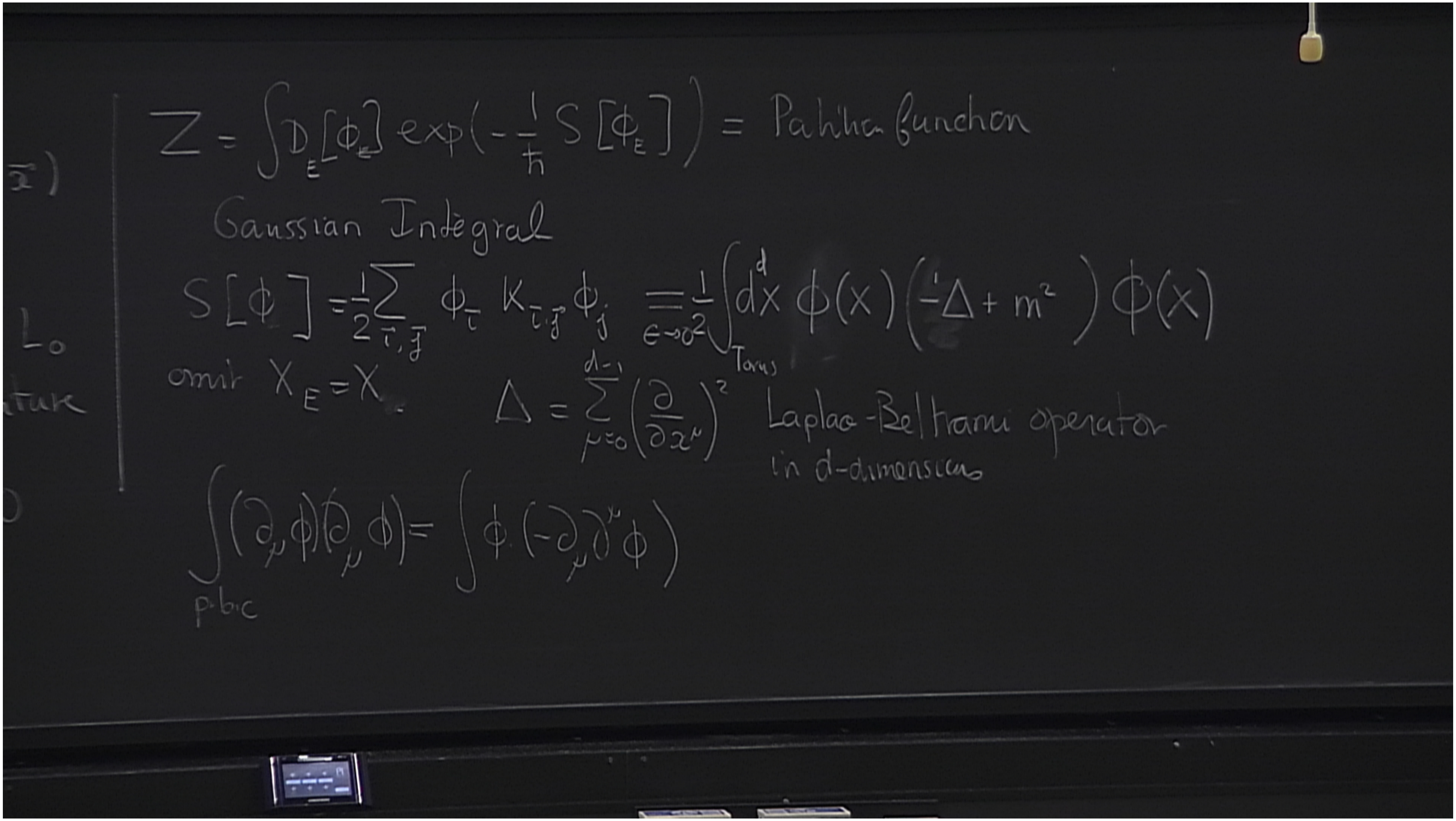
Space is a Torus L_s

Euclidean time is periodic L_0

QFT at finite temperature

$L_0 \nearrow \infty \Leftrightarrow \text{Temp} \searrow 0$

project on $|0\rangle$



$$Z = \int \mathcal{D}_E[\phi_E] \exp\left(-\frac{1}{\hbar} S[\phi_E]\right) = \text{Partition function}$$

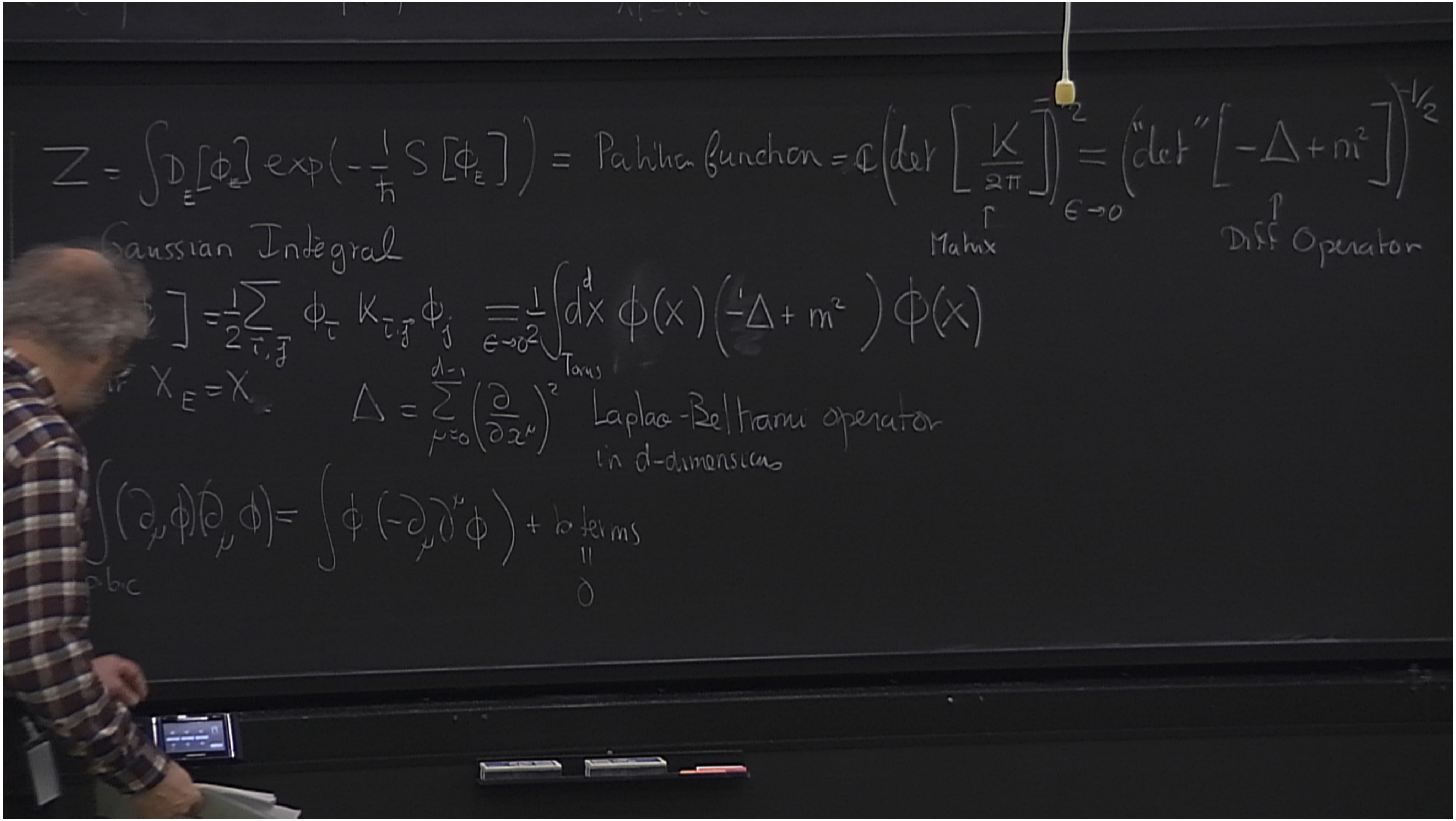
Gaussian Integral

$$S[\phi] = \frac{1}{2}\sum_{\vec{i},\vec{j}} \phi_{\vec{i}} K_{\vec{i},\vec{j}} \phi_{\vec{j}} \underset{\epsilon\to 0}{=} \frac{1}{2}\int_{\text{Torus}} d^d x\, \phi(x)\left(-\Delta + m^2\right)\phi(x)$$

omit $X_E = X$

$$\Delta = \sum_{\mu=0}^{d-1}\left(\frac{\partial}{\partial x^\mu}\right)^2$$ Laplace-Beltrami operator in d-dimensions

$$\int_{\text{p.b.c}} (\partial_\mu \phi)(\partial_\mu \phi) = \int \phi \cdot (-\partial_\mu \partial^\mu \phi)$$



$$Z = \int D_E[\phi_E] \exp\left(-\frac{1}{\hbar} S[\phi_E]\right) = \text{Partition function} = \propto \left(\det\left[\frac{K}{2\pi}\right]\right)^{-\frac{1}{2}} \underset{\epsilon\to 0}{=} \left(\text{"}\det\text{"}\left[-\Delta + m^2\right]\right)^{-1/2}$$

Matrix ↑ Diff Operator ↑

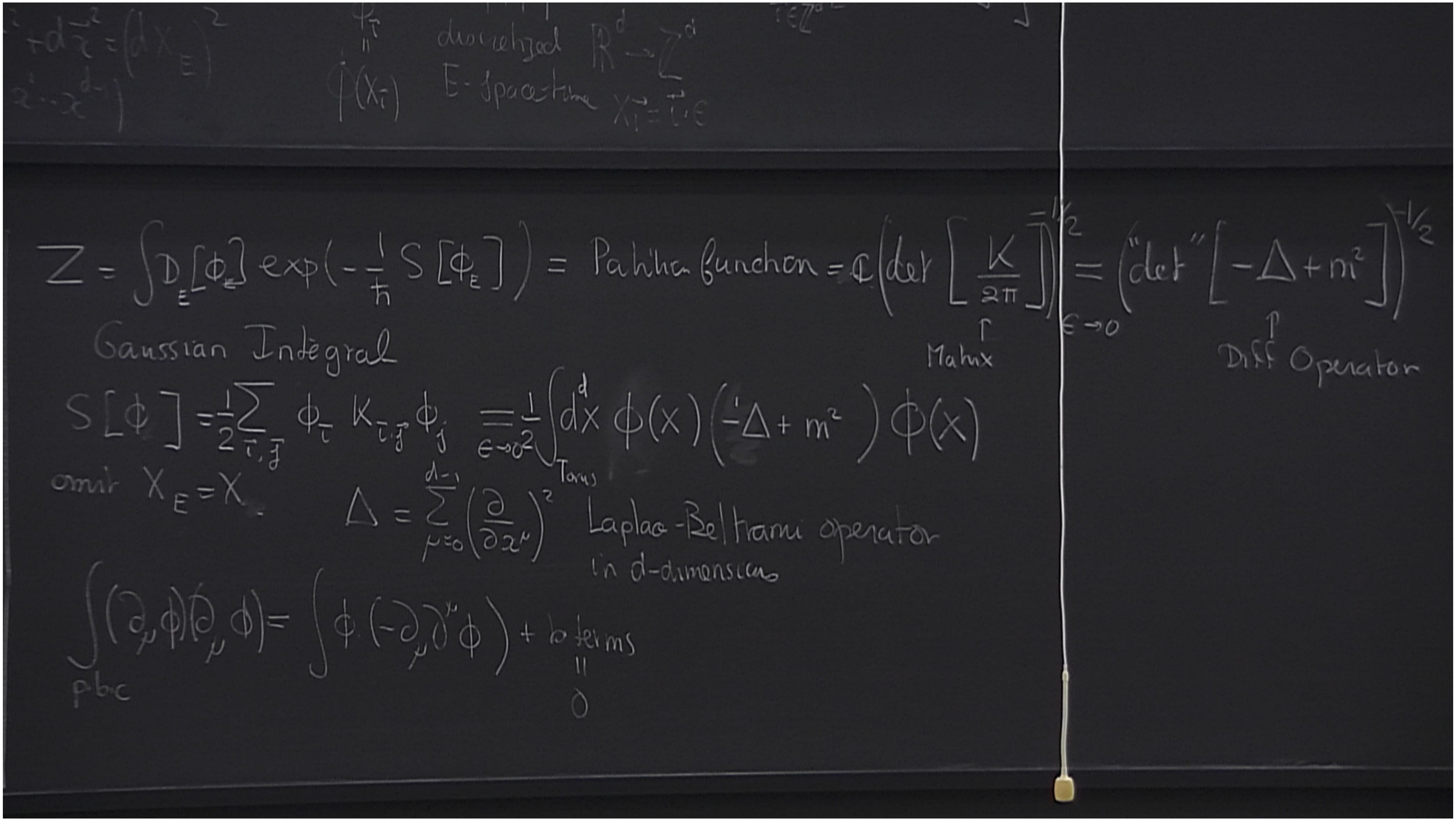
Gaussian Integral

$$] = \frac{1}{2}\sum_{i,j} \phi_i K_{i,j} \phi_j \underset{\epsilon\to 0}{=} \frac{1}{2}\int_{\text{Torus}} d^d x\, \phi(x)\left(-\Delta + m^2\right)\phi(x)$$

if $X_E = X$

$$\Delta = \sum_{\mu=0}^{d-1}\left(\frac{\partial}{\partial x^\mu}\right)^2$$ Laplace-Beltrami operator in d-dimensions

$$\int_{p.b.c} (\partial_\mu \phi)(\partial_\mu \phi) = \int \phi\left(-\partial_\mu \partial^\mu \phi\right) + \underset{\overset{\|}{0}}{\text{b terms}}$$



$+d\vec{x}^2 = (dX_E)^2$

$x^1 \dots x^{d-1})$

$\phi_{\vec{i}} \overset{\shortparallel}{} \phi(X_{\vec{i}})$

discretized $\mathbb{R}^d \to \mathbb{Z}^d$

E - space-time $X_{\vec{i}} = \vec{i} \cdot \epsilon$

$\vec{i} \in \mathbb{Z}^d$

$$Z = \int D_E[\phi_E] \exp\left(-\frac{1}{\hbar} S[\phi_E]\right) = \text{Partition function} = \mathcal{C}\left(\det \underset{\substack{\uparrow \\ \text{Matrix}}}{\left[\frac{K}{2\pi}\right]}\right)^{-1/2} \underset{\epsilon \to 0}{=} \left(\text{``det''} \underset{\substack{\uparrow \\ \text{Diff Operator}}}{\left[-\Delta + m^2\right]}\right)^{-1/2}$$

Gaussian Integral

$$S[\phi] = \frac{1}{2}\sum_{\vec{i},\vec{j}} \phi_{\vec{i}} K_{\vec{i},\vec{j}} \phi_{\vec{j}} \underset{\epsilon \to 0}{=} \frac{1}{2}\int_{\text{Torus}} d^d x\, \phi(x)\left(-\Delta + m^2\right)\phi(x)$$

omit $X_E = X$

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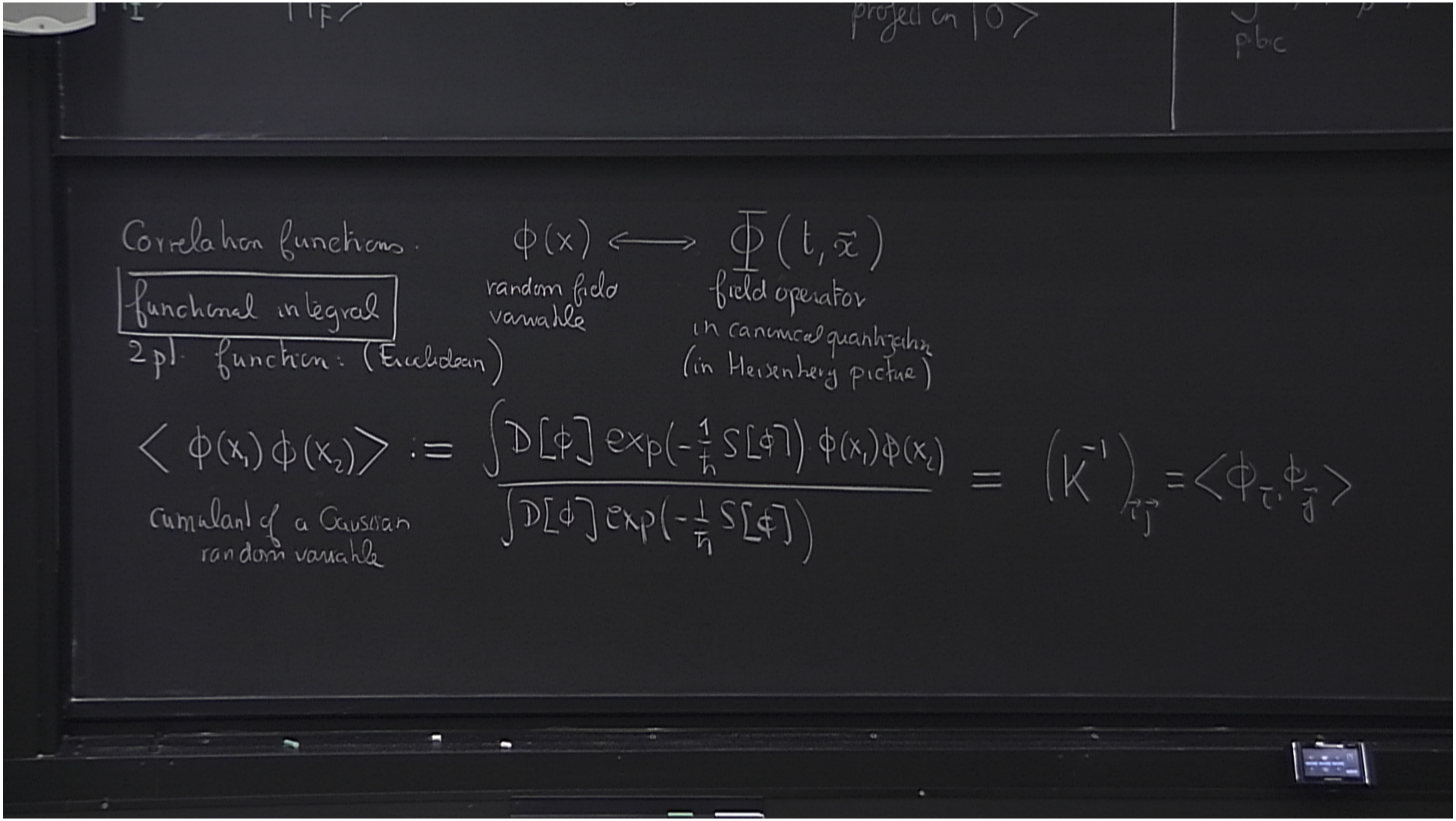


$_I\rangle$ $|_F\rangle$ project on $|0\rangle$

p.b.c

Correlation functions. $\phi(x) \longleftrightarrow \Phi(t,\vec{x})$

random field variable

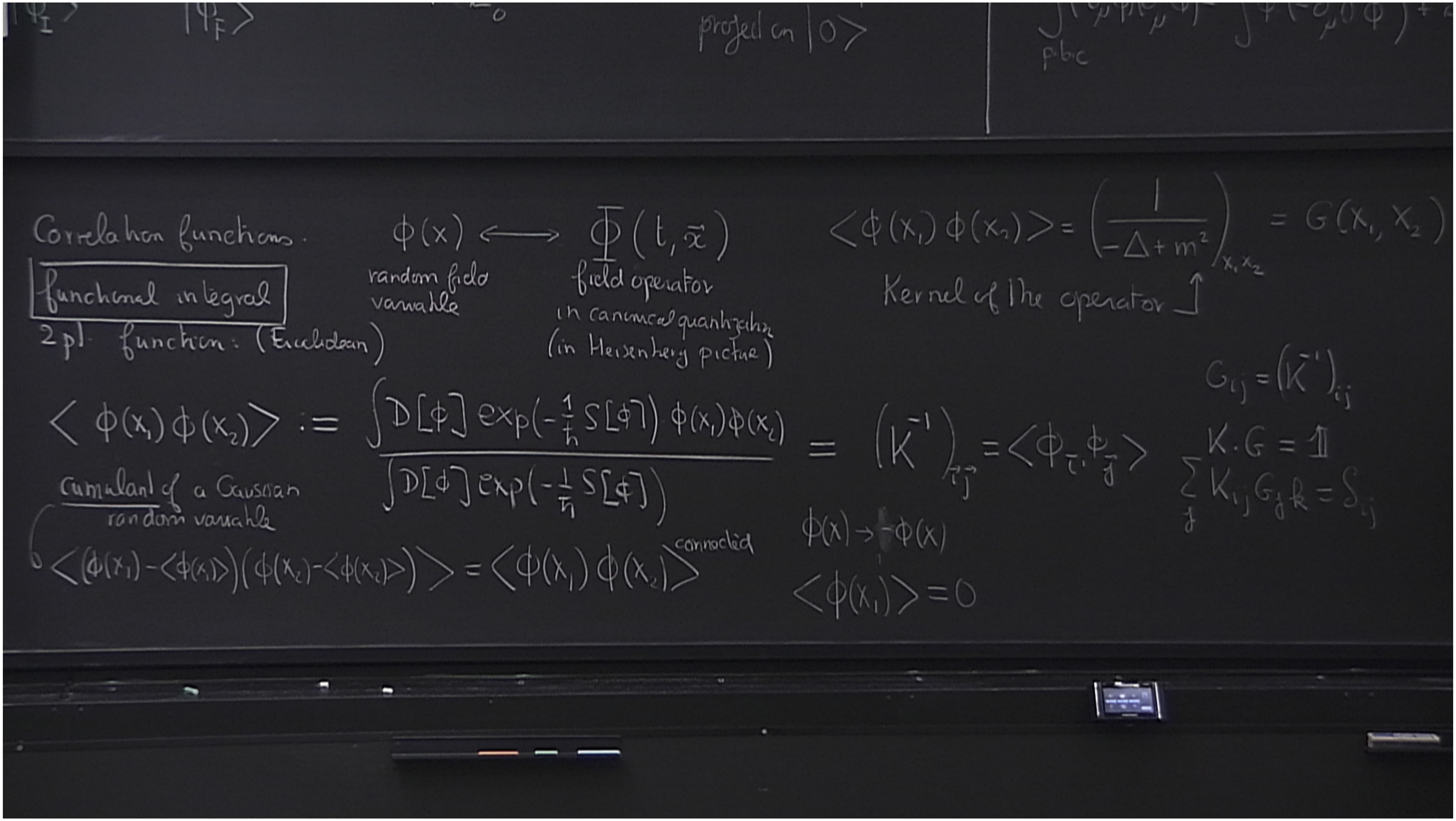
field operator in canonical quantization (in Heisenberg picture)

$\boxed{\text{functional integral}}$

2pt. function: (Euclidean)

$$\langle \phi(x_1)\phi(x_2)\rangle := \frac{\int D[\phi]\exp(-\frac{1}{\hbar}S[\phi])\,\phi(x_1)\phi(x_2)}{\int D[\phi]\exp(-\frac{1}{\hbar}S[\phi])} = \left(K^{-1}\right)_{\vec{i}\vec{j}} = \langle \phi_{\vec{i}}\,\phi_{\vec{j}}\rangle$$

cumulant of a Gaussian random variable



$|\varphi_I\rangle$ $|\varphi_F\rangle$

project on $|0\rangle$

p.b.c

Correlation functions.

$\phi(x) \longleftrightarrow \Phi(t, \vec{x})$

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field operator in canonical quantization (in Heisenberg picture)

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2 pt. function: (Euclidean)

$$\langle \phi(x_1)\phi(x_2)\rangle = \left(\frac{1}{-\Delta + m^2}\right)_{x_1 x_2} = G(x_1, x_2)$$

Kernel of the operator

$$\langle \phi(x_1)\phi(x_2)\rangle := \frac{\int D[\phi]\exp(-\frac{1}{\hbar}S[\phi])\,\phi(x_1)\phi(x_2)}{\int D[\phi]\exp(-\frac{1}{\hbar}S[\phi])} = (K^{-1})_{\vec{i}\vec{j}} = \langle \phi_{\vec{i}}\,\phi_{\vec{j}}\rangle$$

$G_{ij} = (K^{-1})_{ij}$

$K \cdot G = \mathbb{1}$

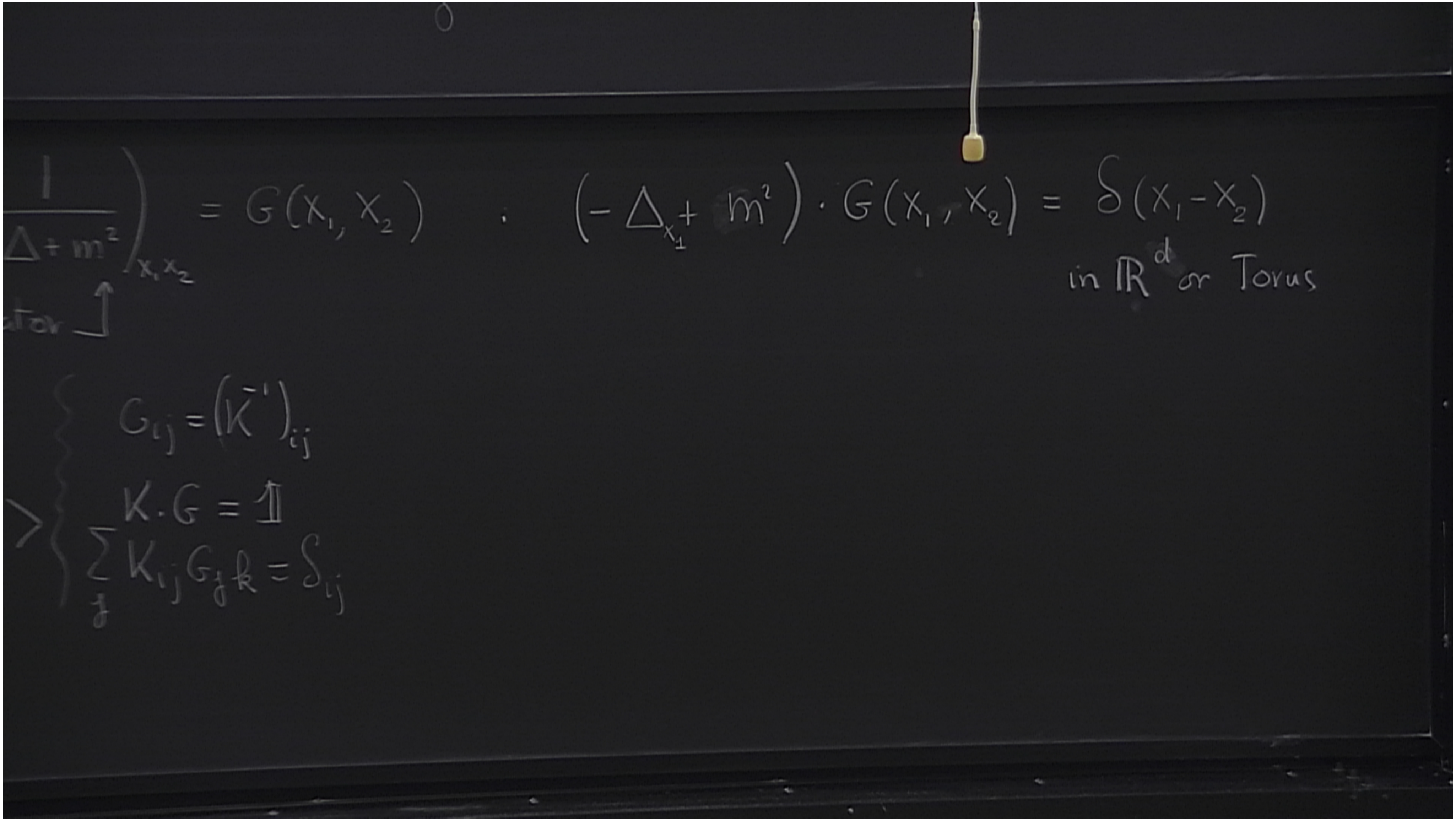
$\sum_j K_{ij} G_{jk} = \delta_{ik}$

cumulant of a Gaussian random variable

$$\langle (\phi(x_1) - \langle\phi(x_1)\rangle)(\phi(x_2) - \langle\phi(x_2)\rangle)\rangle = \langle \phi(x_1)\,\phi(x_2)\rangle^{\text{connected}}$$

$\phi(x) \to -\phi(x)$

$\langle \phi(x_1)\rangle = 0$

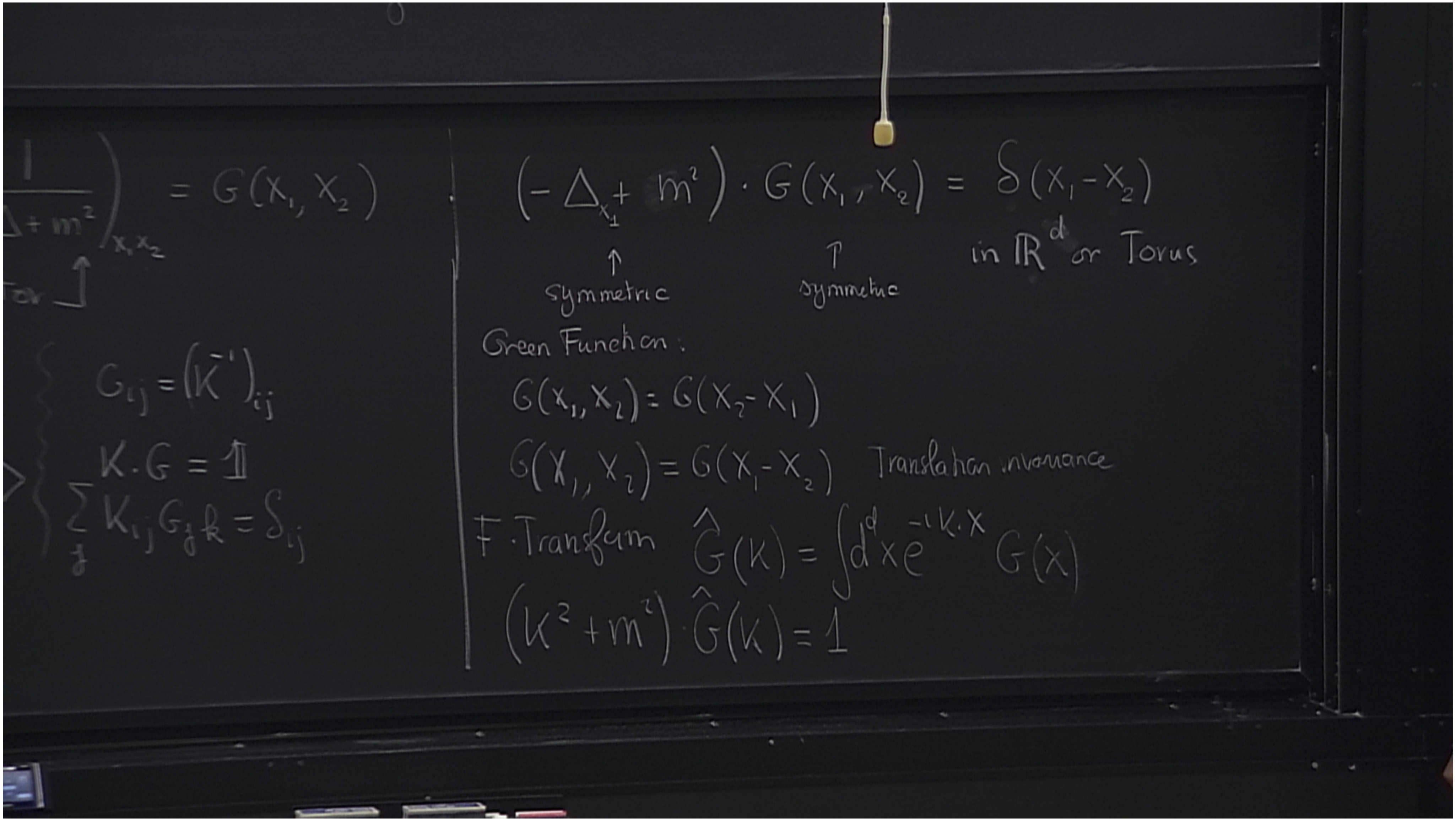


$$\left(\frac{1}{\Delta + m^2}\right)_{x_1 x_2} = G(x_1, x_2) \quad : \quad \left(-\Delta_{x_1} + m^2\right)\cdot G(x_1, x_2) = \delta(x_1 - x_2)$$

ator ↑

in $\mathbb{R}^d$ or Torus

$$> \begin{cases} G_{ij} = (K^{-1})_{ij} \\ K\cdot G = \mathbb{1} \\ \sum_j K_{ij} G_{jk} = \delta_{ij} \end{cases}$$



$$\left(\frac{1}{\Delta + m^2}\right)_{x_1 x_2} = G(x_1, x_2)$$

tor

$$G_{ij} = (K^{-1})_{ij}$$

$$K \cdot G = \mathbb{1}$$

$$\sum_j K_{ij} G_{jk} = \delta_{ij}$$

$$(-\Delta_{x_1} + m^2) \cdot G(x_1, x_2) = \delta(x_1 - x_2)$$

$$\underset{\text{symmetric}}{\uparrow} \qquad \underset{\text{symmetric}}{\uparrow}$$

in $\mathbb{R}^d$ or Torus

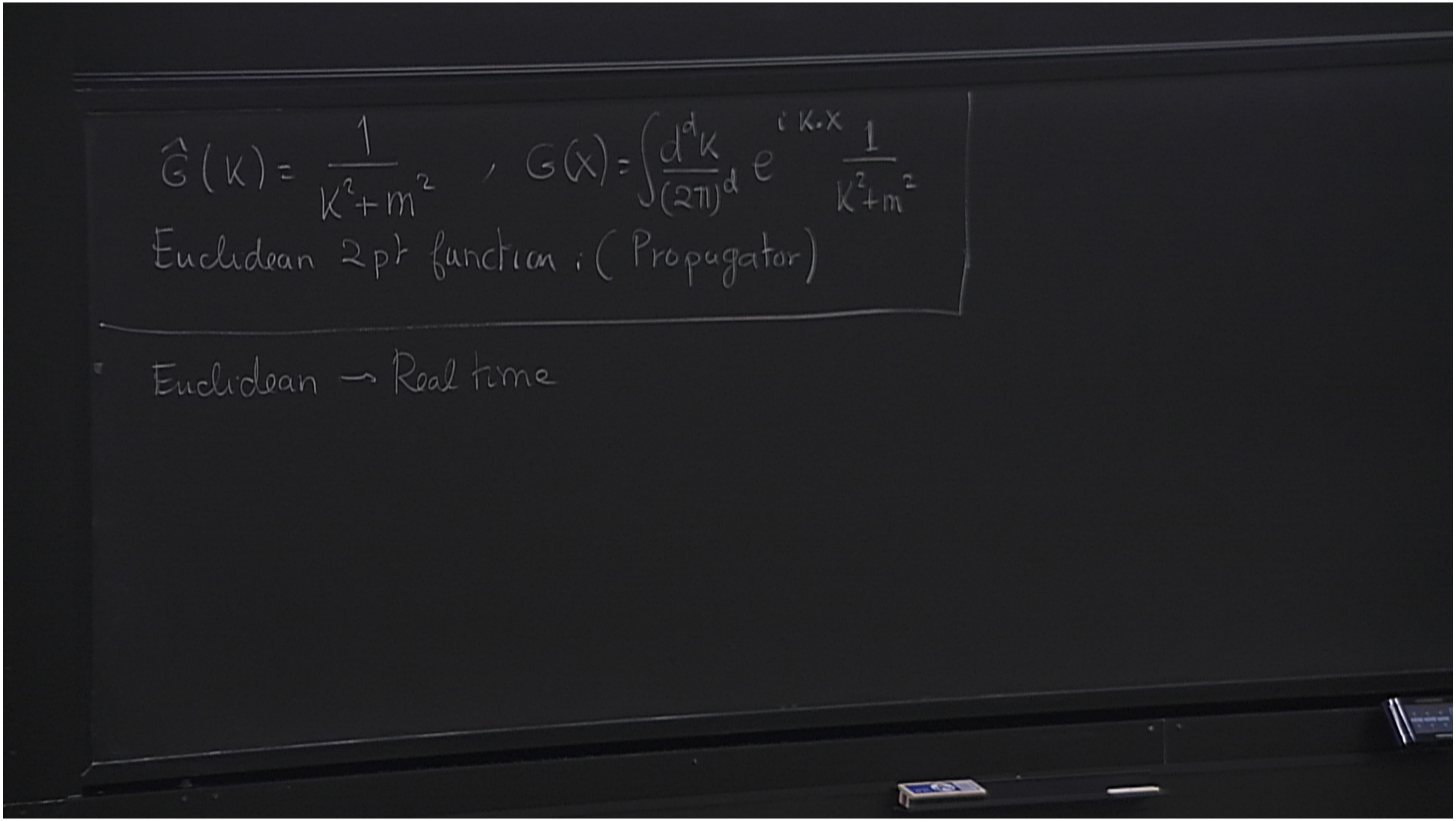
Green Function:

$$G(x_1, x_2) = G(x_2 - x_1)$$

$$G(x_1, x_2) = G(x_1 - x_2)$$ Translation invariance

F. Tranfom $\hat{G}(k) = \int d^d x\, e^{-ik \cdot x}\, G(x)$

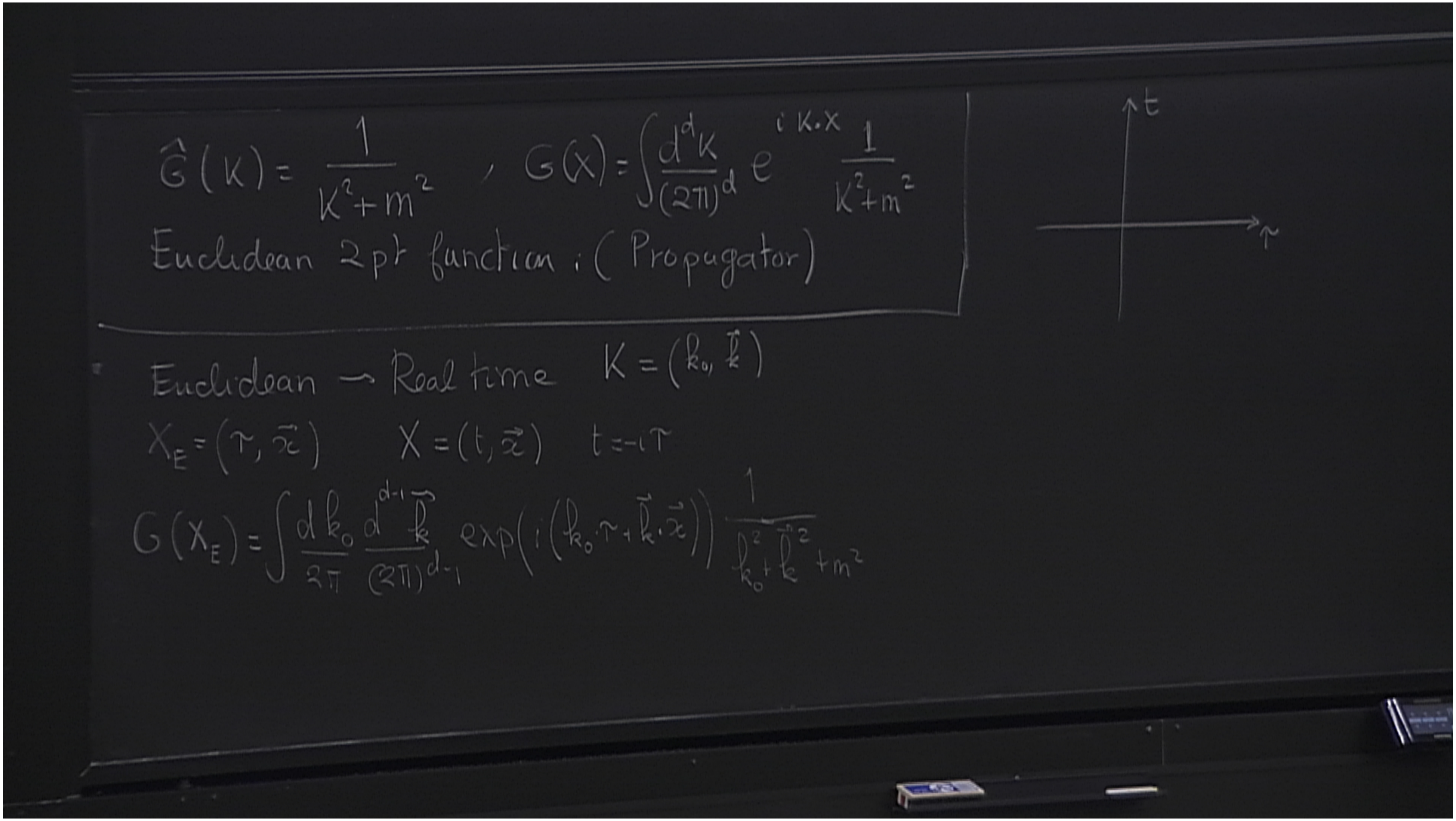
$$(k^2 + m^2) \cdot \hat{G}(k) = 1$$



$$\hat{G}(k) = \frac{1}{k^2+m^2} \quad , \quad G(x) = \int \frac{d^dk}{(2\pi)^d} e^{i k \cdot x} \frac{1}{k^2+m^2}$$

Euclidean 2pt function : (Propugator)

Euclidean → Real time



$\hat{G}(K) = \frac{1}{K^2+m^2}$, $G(x) = \int \frac{d^dK}{(2\pi)^d} e^{iK\cdot x} \frac{1}{K^2+m^2}$

Euclidean 2pt function : (Propugator)

Euclidean → Real time $K = (k_0, \vec{k})$

$X_E = (\tau, \vec{x})$ $X = (t, \vec{x})$ $t = -i\tau$

$$G(X_E) = \int \frac{dk_0}{2\pi} \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \exp(i(k_0\cdot\tau + \vec{k}\cdot\vec{x})) \frac{1}{k_0^2+\vec{k}^2+m^2}$$

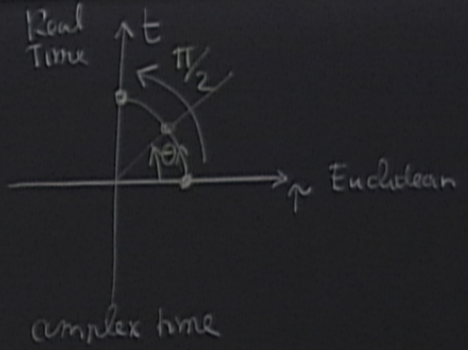
$$\boxed{\hat{G}(K) = \frac{1}{K^2+m^2} \quad , \quad G(x) = \int \frac{d^d K}{(2\pi)^d} e^{iK \cdot X} \frac{1}{K^2+m^2}}$$

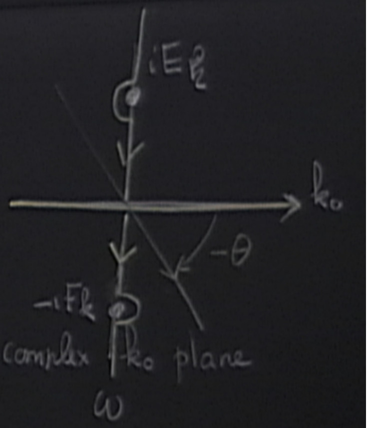
Euclidean 2pt function : (Propagator)

Euclidean → Real time $K_E = (k_0, \vec{k})$ $\boxed{k_0 = -i\omega}$

$X_E = (\tau, \vec{x})$ $X = (t, \vec{x})$ $\boxed{t = -i\tau}$

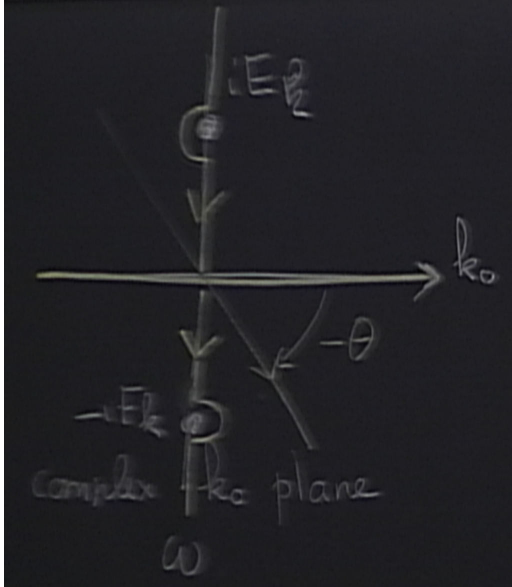
$$G(X_E) = \int_{-\infty}^{+\infty} \frac{dk_0}{2\pi} \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \exp(i(\underbrace{k_0\tau + \vec{k}\cdot\vec{x}}_{\text{a pure phase}})) \underbrace{\frac{1}{k_0^2 + \vec{k}^2 + m^2}}_{\text{analytic in } k_0} \xrightarrow[\longrightarrow]{\text{wick rotation}}$$





rotate the contour $\int dk_0$

pole at $k_0 = \pm i\sqrt{\vec{k}^2 + m^2} = \pm i E_{\vec{k}}$



$$G(X) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{+\infty} \frac{d^{d-1}\vec{k}}{(2\pi)^2} \, i \, \frac{e^{i(\omega t + \vec{k}\vec{x})}}{\left(\omega^2 - \vec{k}^2 - m^2 + i\epsilon_+\right)}$$

Functional Integral

$$= G_{\text{Feynman}}(t, \vec{x})$$

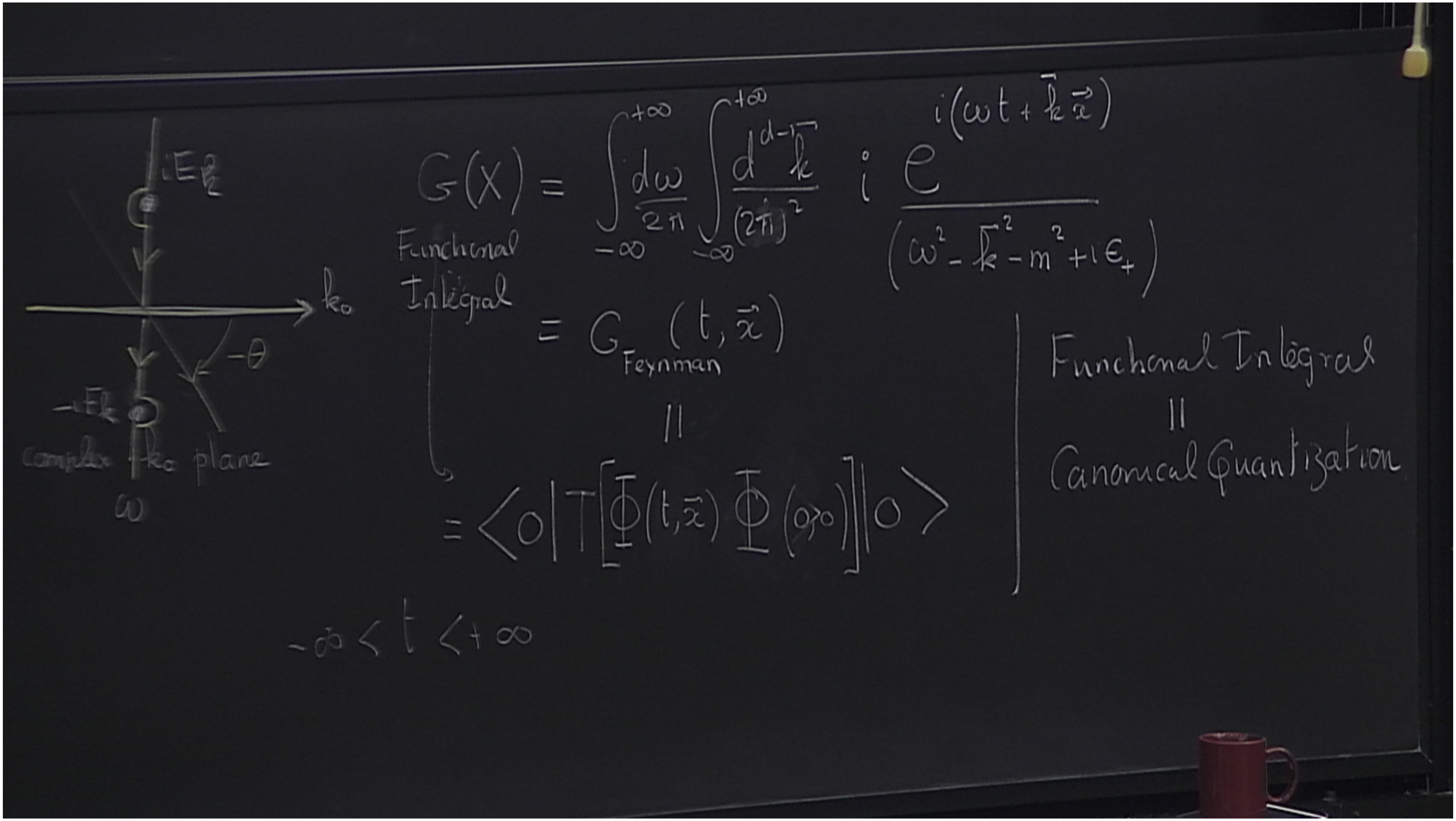
$$\|$$

$$= \langle 0 | T\left[\Phi(t,\vec{x})\,\Phi(0,0)\right] | 0 \rangle$$

Functional Integral

||

Canonical Quantization



$$G(X) = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{+\infty}\frac{d^{d-1}\vec{k}}{(2\pi)^2}\; i\;\frac{e^{i(\omega t+\vec{k}\vec{x})}}{\left(\omega^2-\vec{k}^2-m^2+i\epsilon_+\right)}$$

Functional Integral

$$= G_{\text{Feynman}}(t,\vec{x})$$

$$||$$

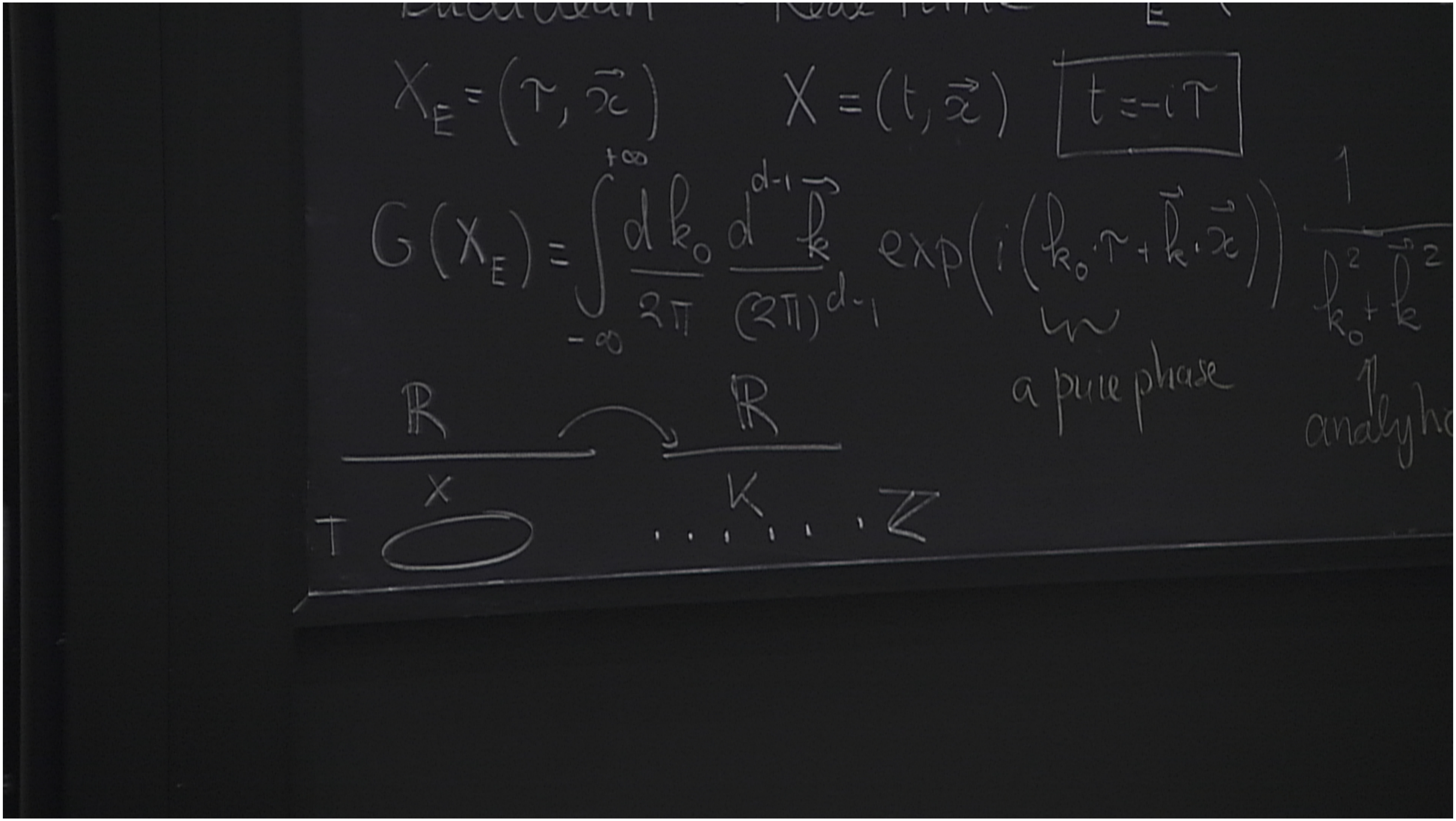
$$= \langle 0|T\left[\Phi(t,\vec{x})\,\Phi(0,0)\right]|0\rangle$$

Functional Integral

||

Canonical Quantization

$$-\infty < t < +\infty$$



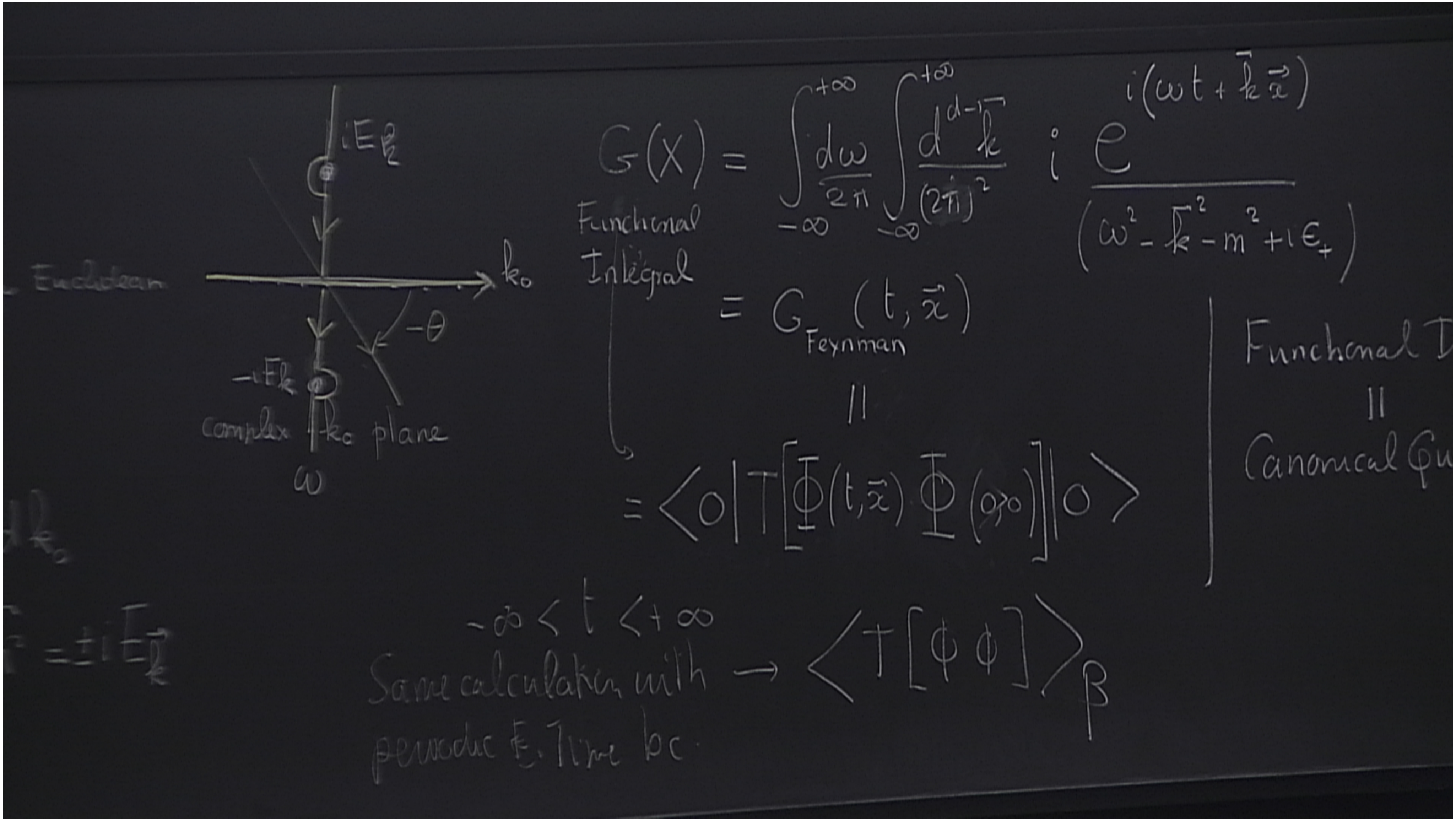
Euclidean → Real time

$X_E = (\tau, \vec{x})$ $\qquad$ $X = (t, \vec{x})$ $\qquad$ $\boxed{t = -i\tau}$

$$G(X_E) = \int_{-\infty}^{+\infty} \frac{dk_0}{2\pi} \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \exp\left(i\left(\underbrace{k_0\cdot\tau + \vec{k}\cdot\vec{x}}_{\text{a pure phase}}\right)\right) \underbrace{\frac{1}{k_0^2+\vec{k}^2}}_{\uparrow \text{ analyti}}$$

$$\frac{\mathbb{R}}{X} \curvearrowright \frac{\mathbb{R}}{K}$$

$T \quad \bigcirc \qquad \cdots\cdots \mathbb{Z}$



$$G(X) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{+\infty} \frac{d^{d-1}\vec{k}}{(2\pi)^2} \; i \; \frac{e^{i(\omega t + \vec{k}\vec{x})}}{(\omega^2 - \vec{k}^2 - m^2 + i\epsilon_+)}$$

$$= G_{\text{Feynman}}(t, \vec{x})$$

$$\|$$

$$= \langle 0 | T[\Phi(t,\vec{x})\,\Phi(0,0)] | 0 \rangle$$

Functional Integral

Functional I

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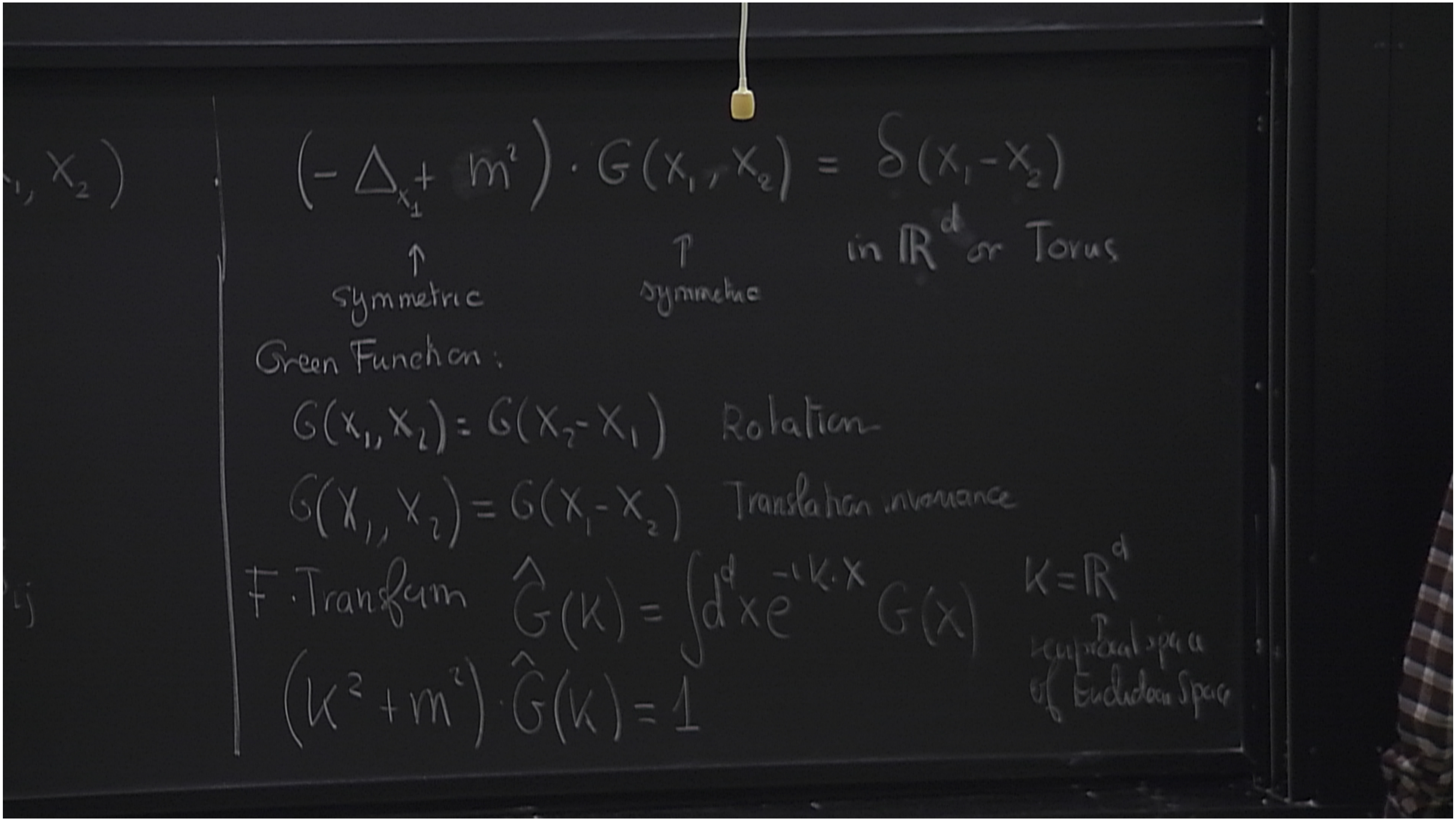
Canonical Qu

$-\infty < t < +\infty$

Same calculation with periodic E. Time bc $\longrightarrow \langle T[\phi\,\phi]\rangle_\beta$



$_1, X_2)$

$$(-\Delta_{x_1} + m^2)\cdot G(x_1, x_2) = \delta(x_1 - x_2)$$

↑ symmetric ↑ symmetric in $\mathbb{R}^d$ or Torus

Green Function:

$G(x_1, x_2) = G(X_2 - X_1)$ Rotation

$G(X_1, X_2) = G(X_1 - X_2)$ Translation invariance

$_{ij}$

F. Tranform $\hat{G}(K) = \int d^d x \, e^{-iK.X} G(X)$ $K = \mathbb{R}^d$ ↑ reciprocal space of Euclidean Space

$(K^2 + m^2)\cdot \hat{G}(K) = 1$

$$G_E(x) \qquad |x| \nearrow \infty$$

$$\parallel$$

$$G_E(|x|) \qquad m > 0$$

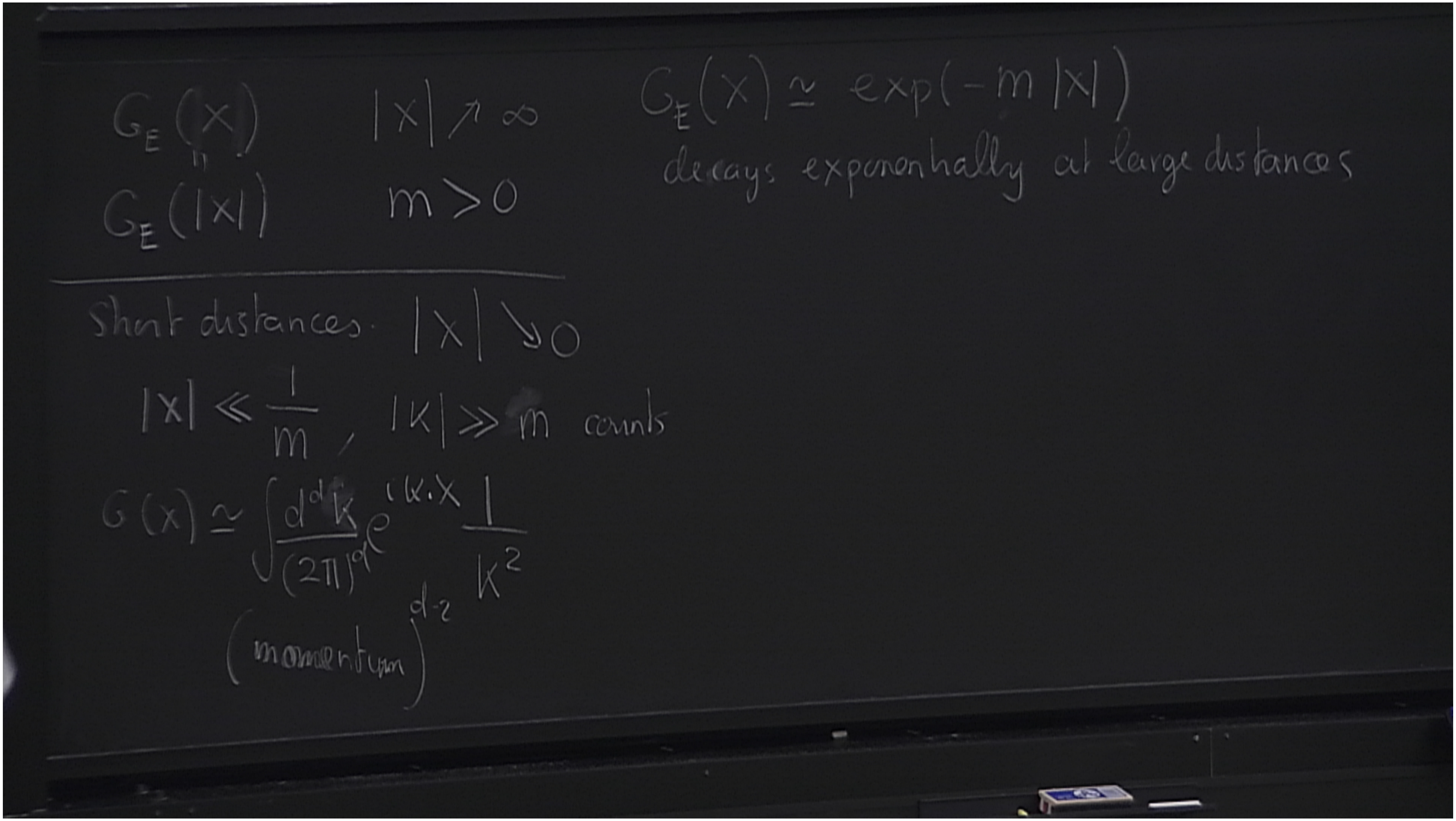
Short distances. $|x| \searrow 0$

$|x| \ll \frac{1}{m}$, $|k| \gg m$ counts

$$G(x) \simeq \int \frac{d^d k}{(2\pi)^d} e^{ik\cdot x} \frac{1}{k^2}$$

$$G_E(x) \simeq \exp(-m|x|)$$

decays exponentially at large distances



$G_E(x)$ $\quad |x| \nearrow \infty$

$\parallel$

$G_E(|x|)$ $\quad m > 0$

$G_E(x) \simeq \exp(-m|x|)$

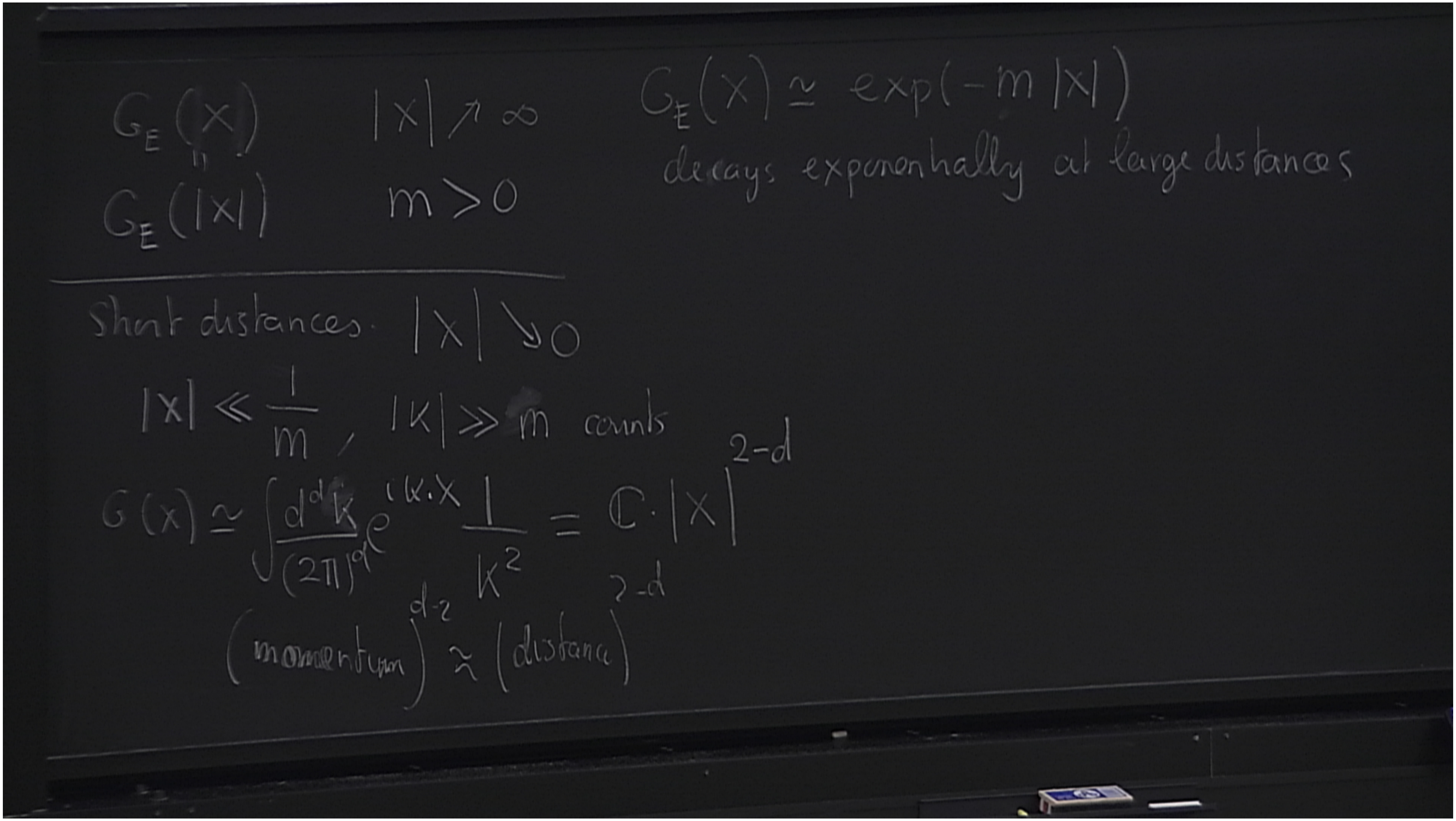
decays exponentially at large distances

Short distances. $|x| \searrow 0$

$|x| \ll \frac{1}{m}$, $|k| \gg m$ counts

$$G(x) \simeq \int \frac{d^d k}{(2\pi)^d} e^{ik.x} \frac{1}{k^2}$$

$$(\text{momentum})^{d-2}$$



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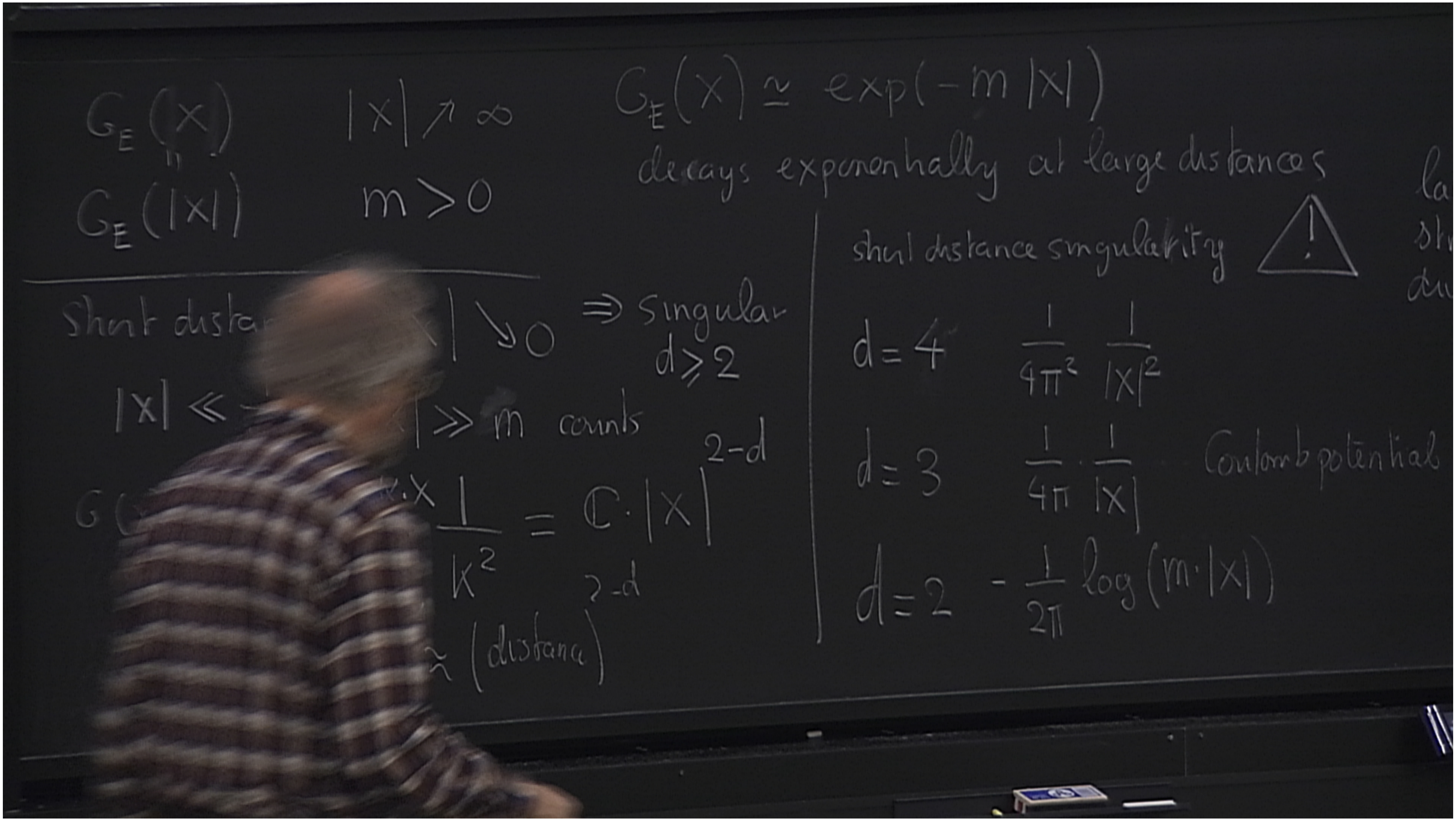
Short distances. $|x| \searrow 0$

$|x| \ll \frac{1}{m}$, $|k| \gg m$ counts

$$G(x) \simeq \int \frac{d^d k}{(2\pi)^d} e^{ik\cdot x} \frac{1}{k^2} = \mathbb{C} \cdot |x|^{2-d}$$

$$(\text{momentum})^{d-2} \simeq (\text{distance})^{2-d}$$



$G_E(x)$ $|x| \nearrow \infty$

$\shortparallel$

$G_E(|x|)$ $m > 0$

$G_E(x) \simeq \exp(-m|x|)$

decays exponentially at large distances

la

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Short dista… $|x| \searrow 0$ $\Rightarrow$ singular

$d \geq 2$

$|x| \ll$ … $|\gg m$ counts

$G($… $\ldots x \dfrac{1}{k^2} = \mathbb{C} \cdot |x|^{2-d}$

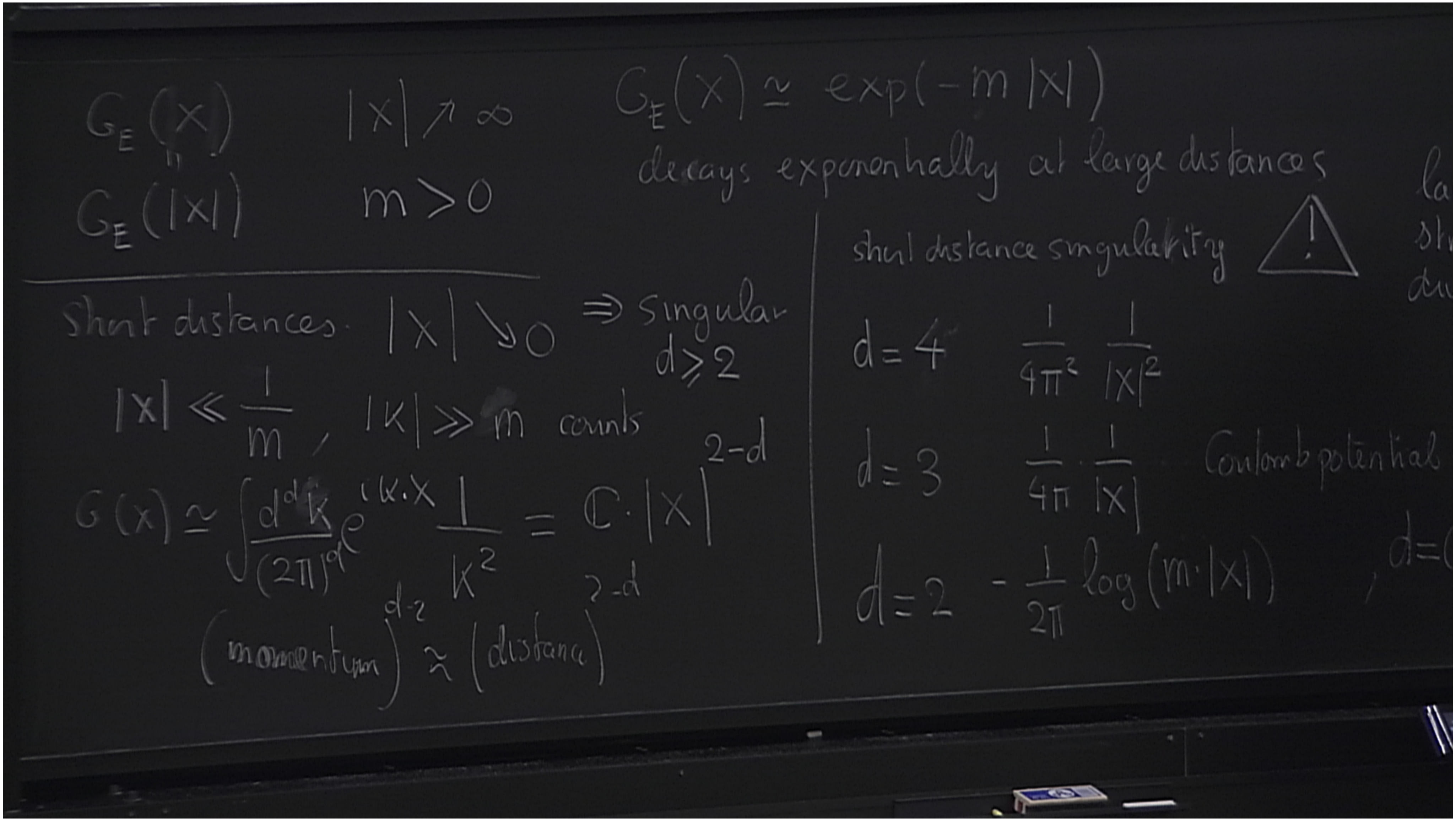
$\simeq (\text{distance})^{2-d}$

short distance singularity ⚠

$d = 4$ $\dfrac{1}{4\pi^2} \; \dfrac{1}{|x|^2}$

$d = 3$ $\dfrac{1}{4\pi} \cdot \dfrac{1}{|x|}$ Coulomb potential

$d = 2$ $-\dfrac{1}{2\pi} \log(m \cdot |x|)$



$G_E(x)$
$\parallel$
$G_E(|x|)$

$|x| \nearrow \infty$

$m > 0$

$G_E(x) \simeq \exp(-m|x|)$

decays exponentially at large distances

la
sh
di

Short distances. $|x| \searrow 0$ $\Rightarrow$ singular $d \geqslant 2$

$|x| \ll \frac{1}{m}$, $|k| \gg m$ counts

$$G(x) \simeq \int \frac{d^d k}{(2\pi)^d} e^{ik.x} \frac{1}{k^2} = \mathbb{C} \cdot |x|^{2-d}$$

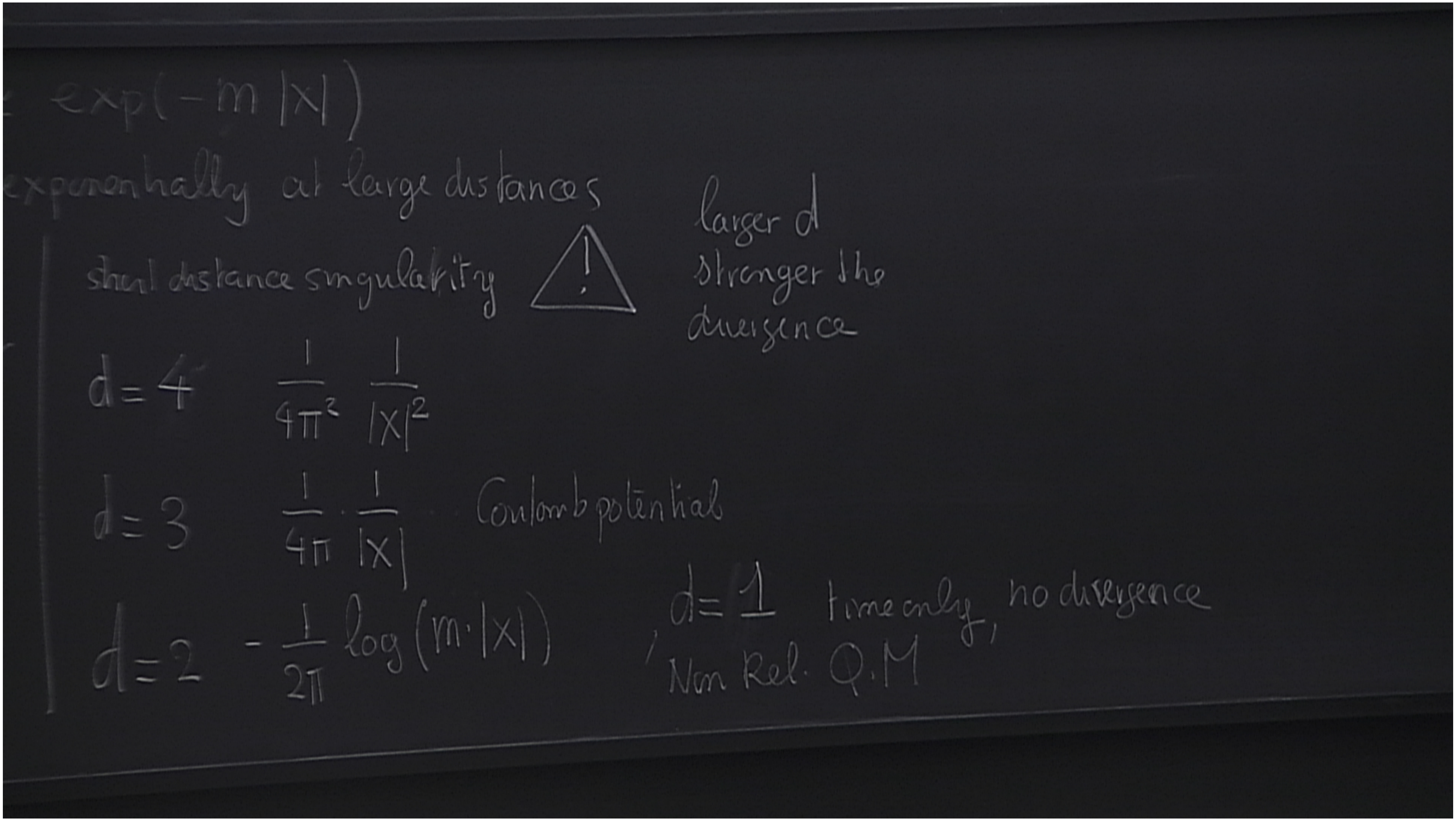
$$(\text{momentum})^{d-2} \simeq (\text{distance})^{2-d}$$

short distance singularity ⚠

$d = 4$ $\quad \frac{1}{4\pi^2} \frac{1}{|x|^2}$

$d = 3$ $\quad \frac{1}{4\pi} \cdot \frac{1}{|x|}$ Coulomb potential

$d = 2$ $\quad -\frac{1}{2\pi} \log(m \cdot |x|)$, $d=($



$\exp(-m|x|)$

exponentially at large distances

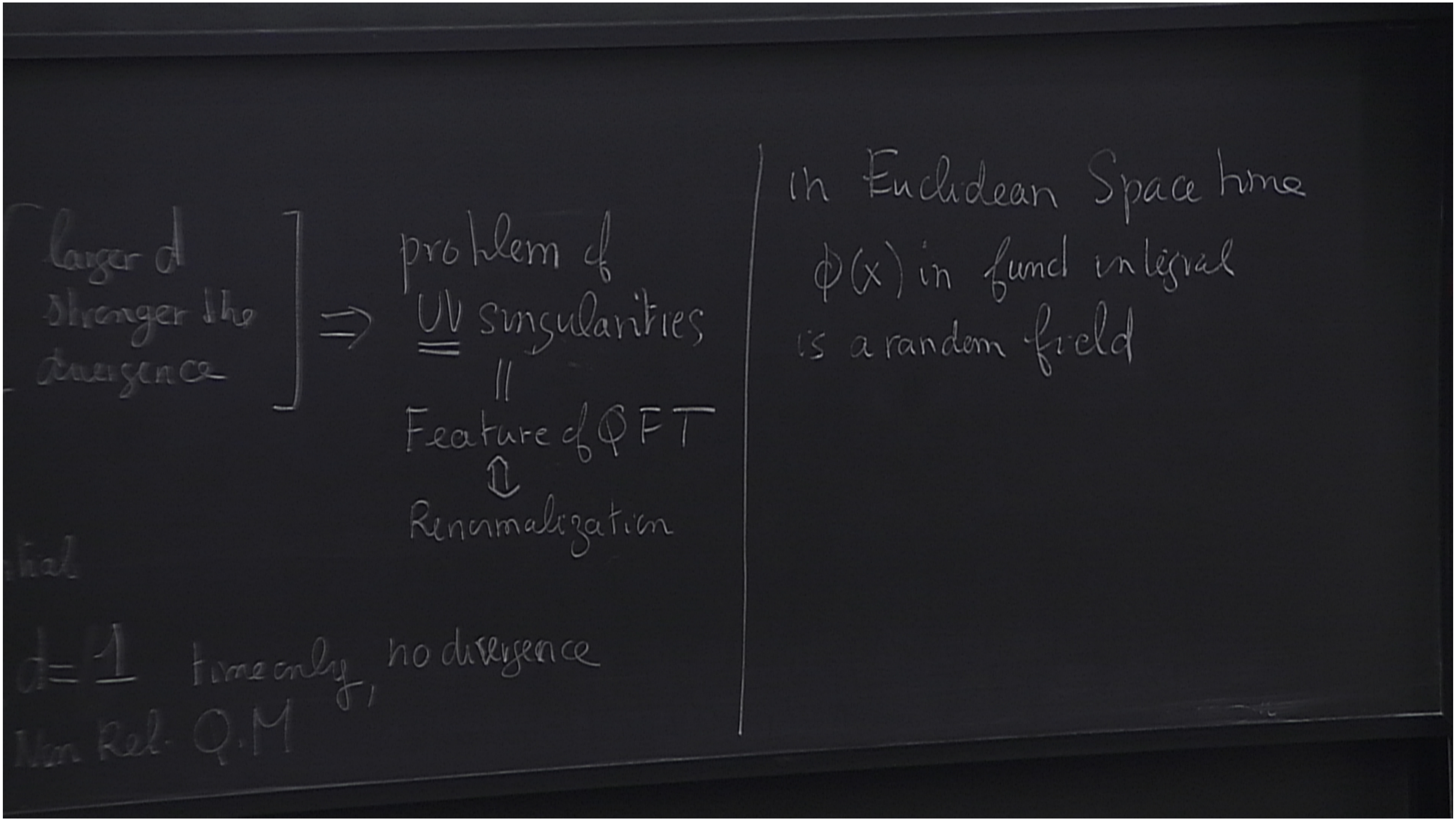
short distance singularity ⚠

larger d
stronger the
divergence

$d = 4 \quad \frac{1}{4\pi^2}\,\frac{1}{|x|^2}$

$d = 3 \quad \frac{1}{4\pi}\cdot\frac{1}{|x|}$ Coulomb potential

$d = 2 \quad -\frac{1}{2\pi}\log(m\cdot|x|)$, $d = 1$ time only, no divergence
, Non Rel. Q.M

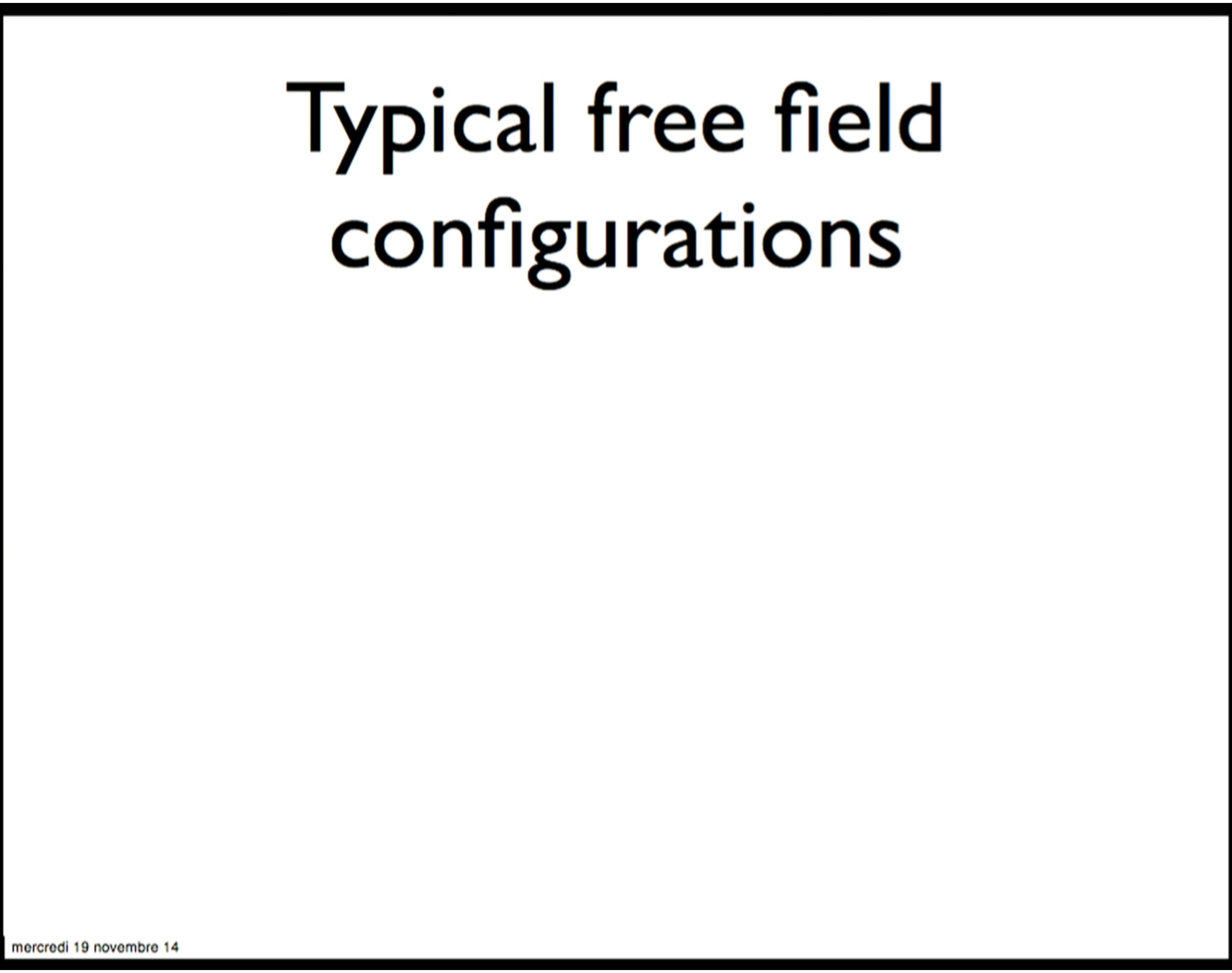


[larger d
stronger the
divergence] ⇒ problem of
$\underline{\underline{\text{UV}}}$ singularities
$\|$
Feature of QFT
⇕
Renormalization

tial

$d = \underline{1}$ time only, no divergence
Non Rel. Q.M

in Euclidean Space time
$\phi(x)$ in funct integral
is a random field



Typical free field configurations

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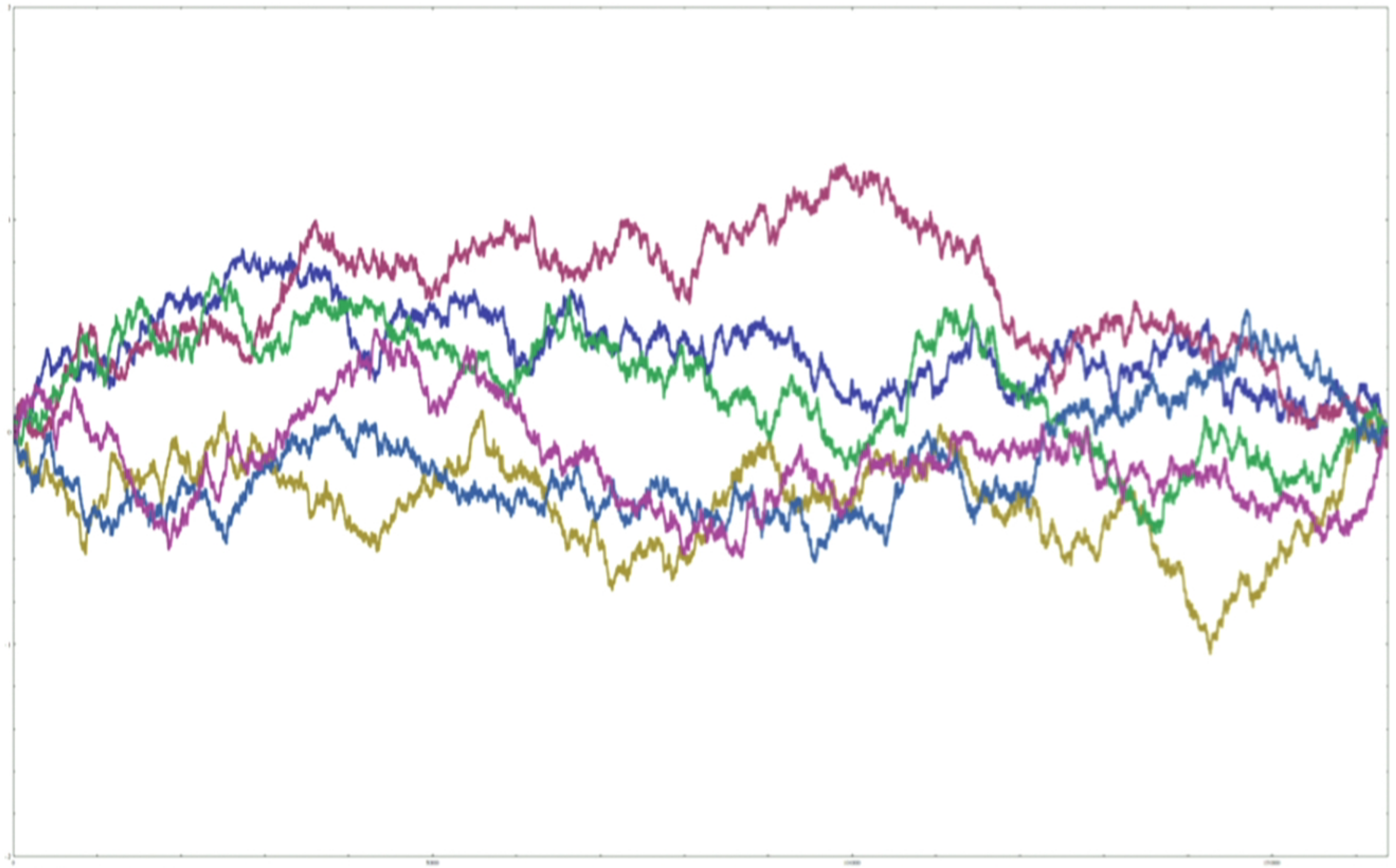
D=1 : Gaussian Free Fields = Random Walk
(i.e. Brownian or Wiener Process)



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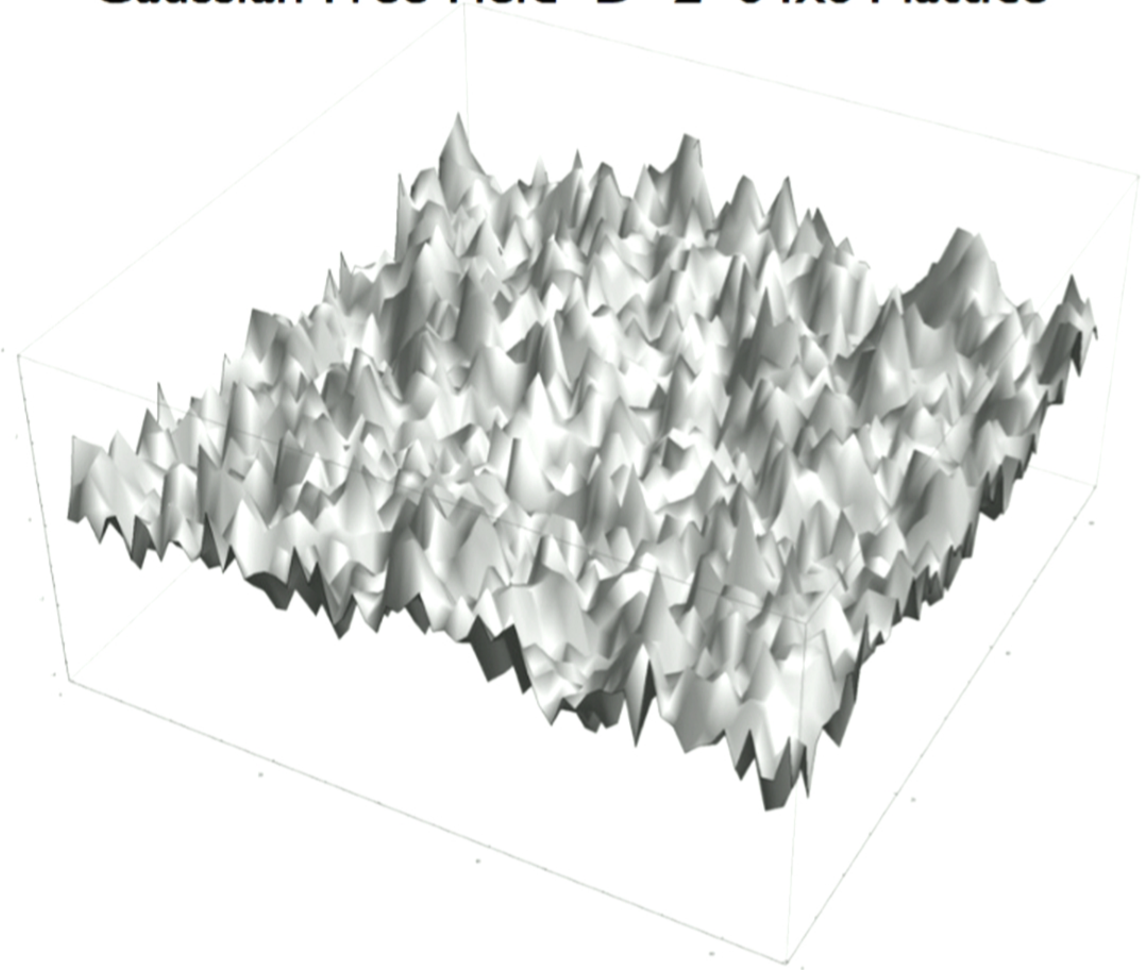
Gaussian Free Field D=2 64x64 lattice



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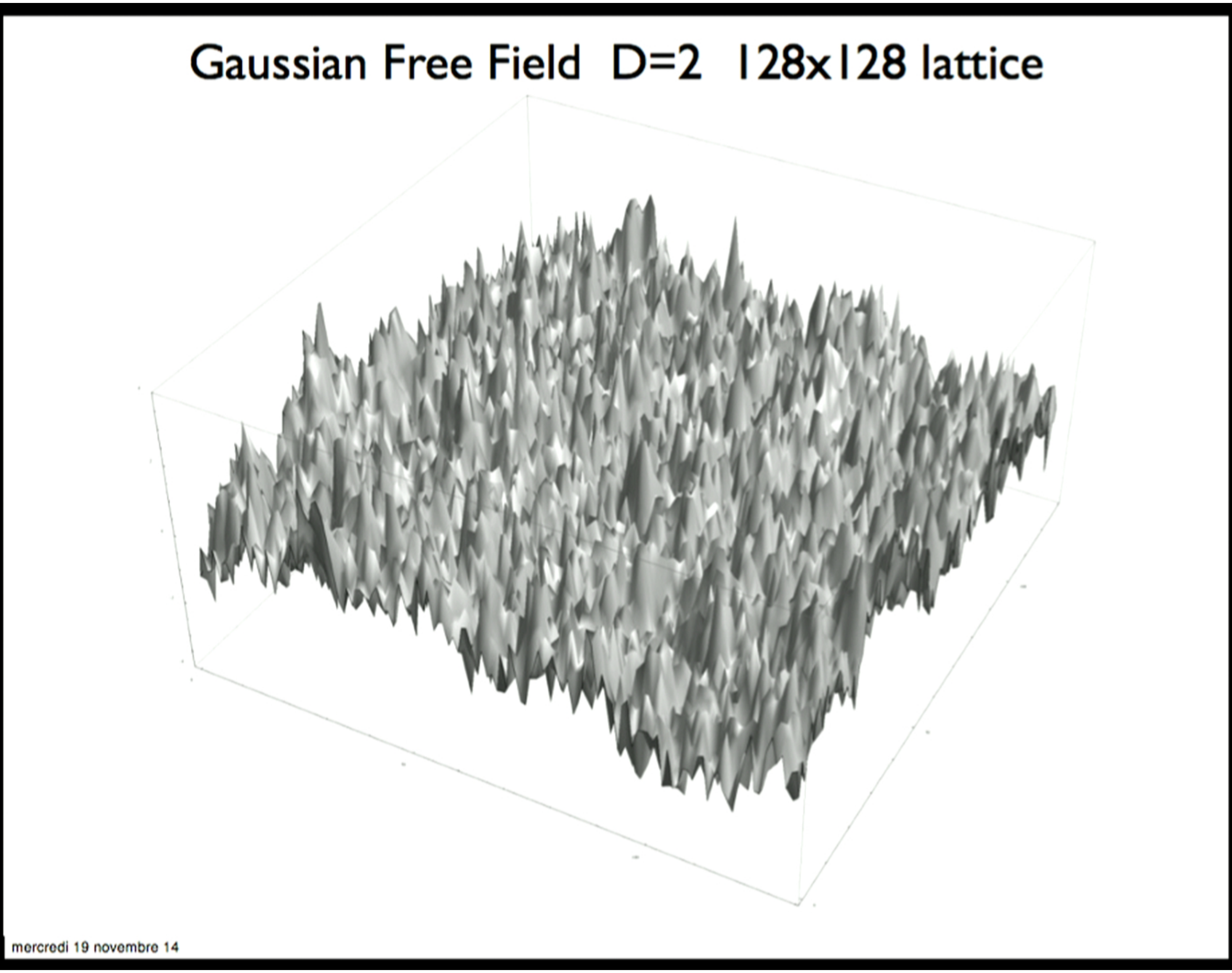

Gaussian Free Field D=2 128x128 lattice

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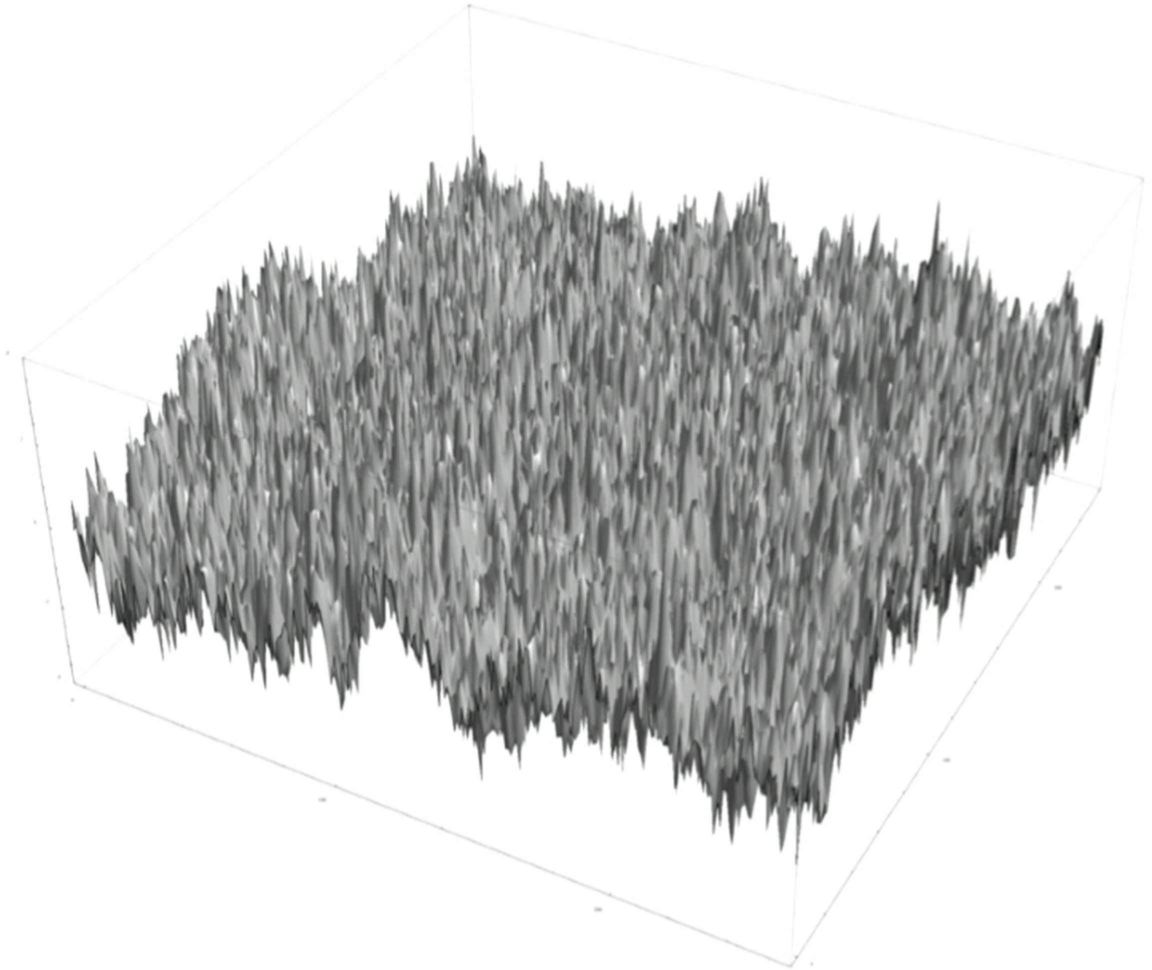
Gaussian Free Field D=2 256x256 lattice



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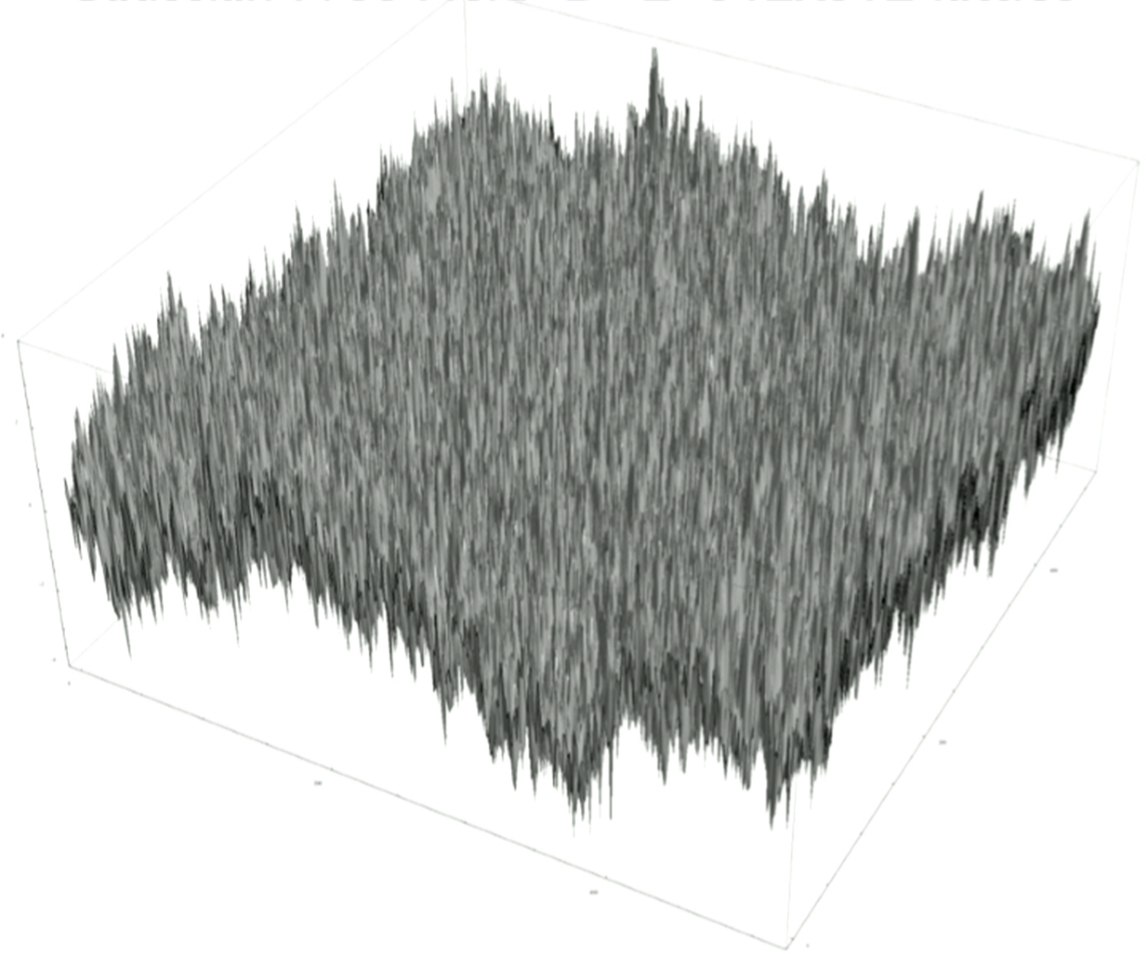
Gaussian Free Field D=2 512x512 lattice



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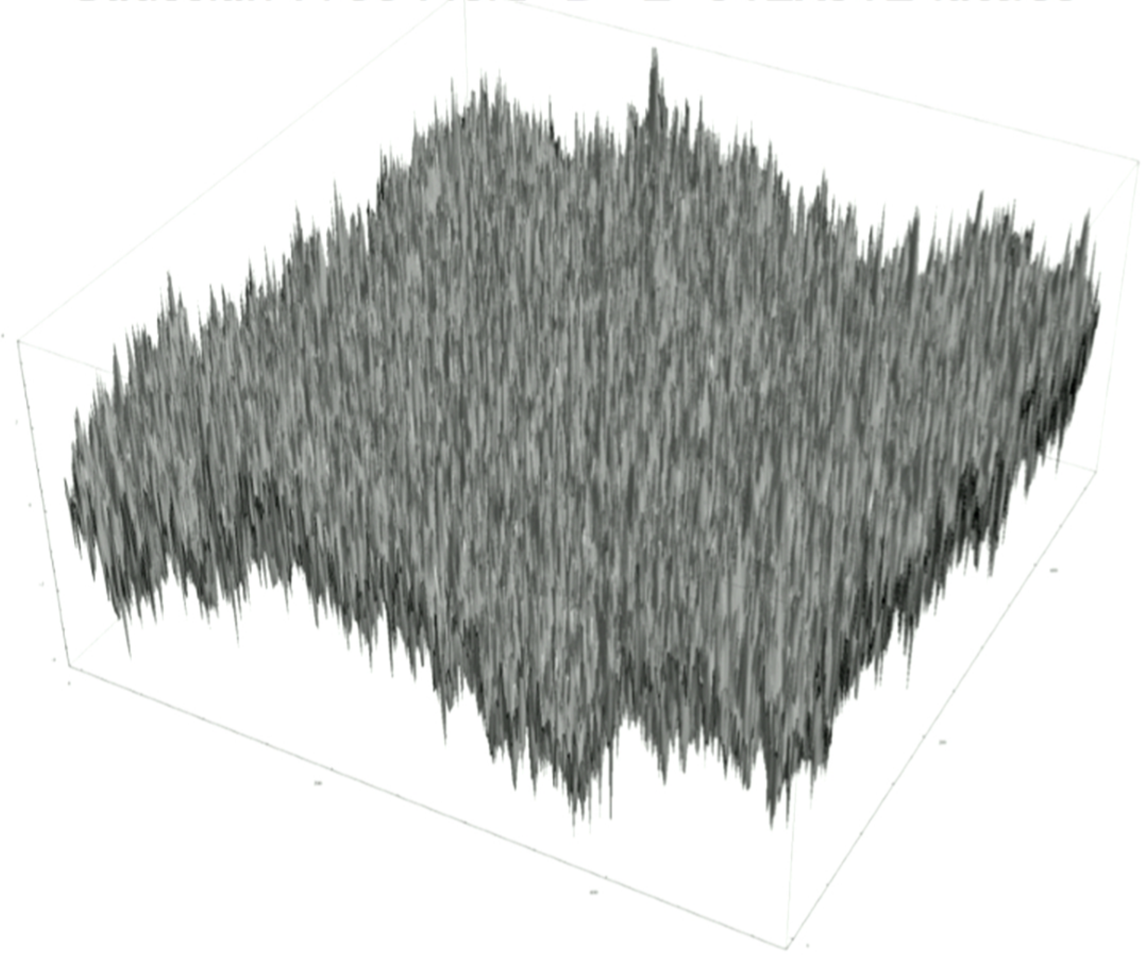
Gaussian Free Field D=2 512x512 lattice



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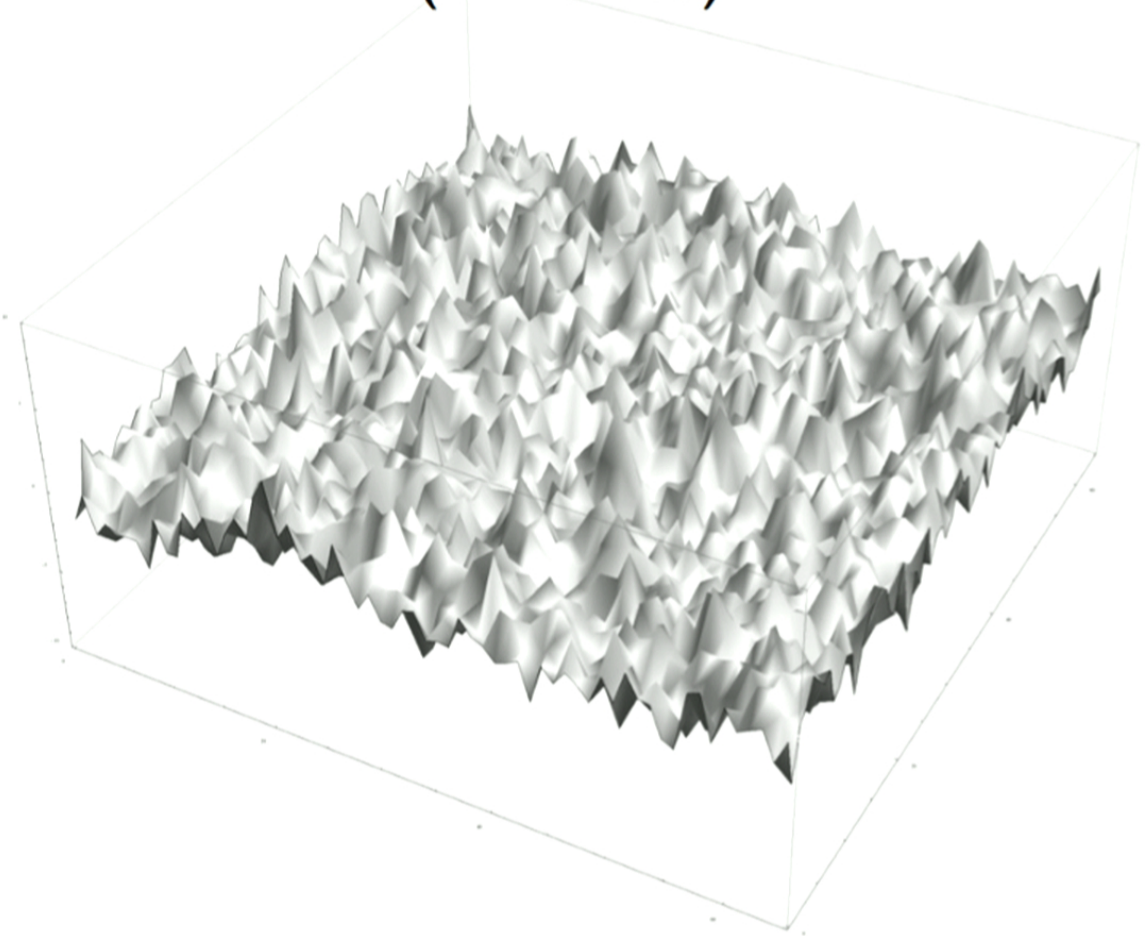
Gaussian Free Field D=3 64x64x64 lattice
(2D section)



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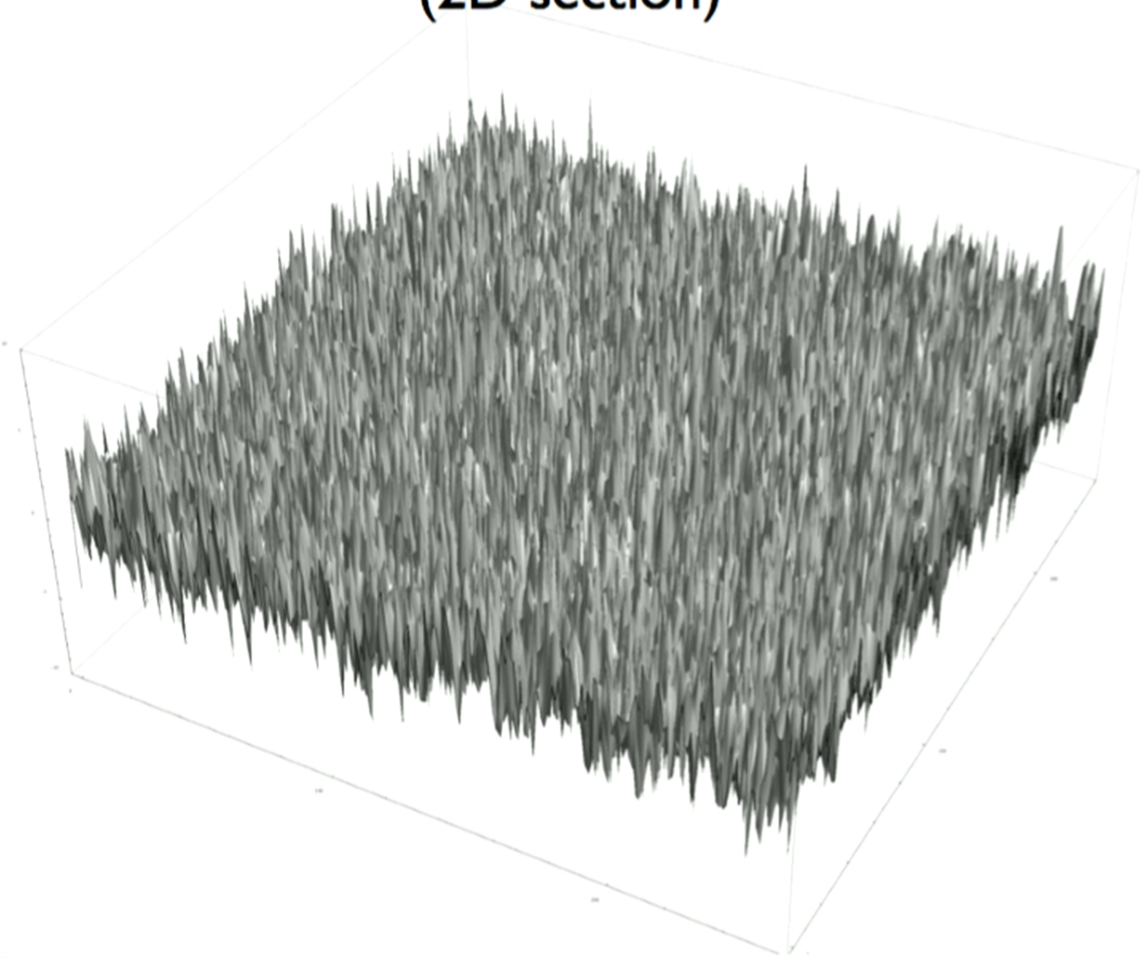
Gaussian Free Field D=3 256x256x256 lattice (2D section)



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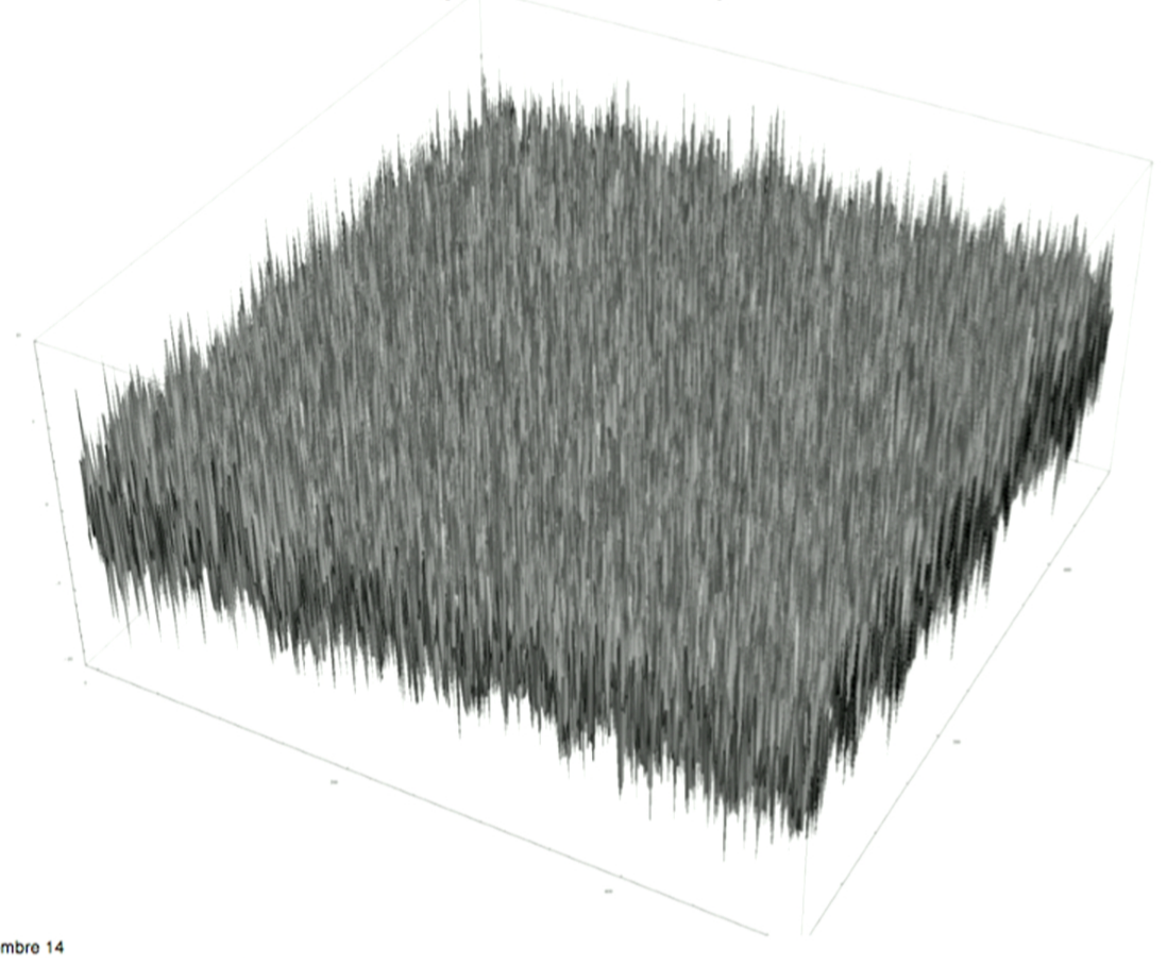
Gaussian Free Field D=3 512x512x512 lattice
(2D section)



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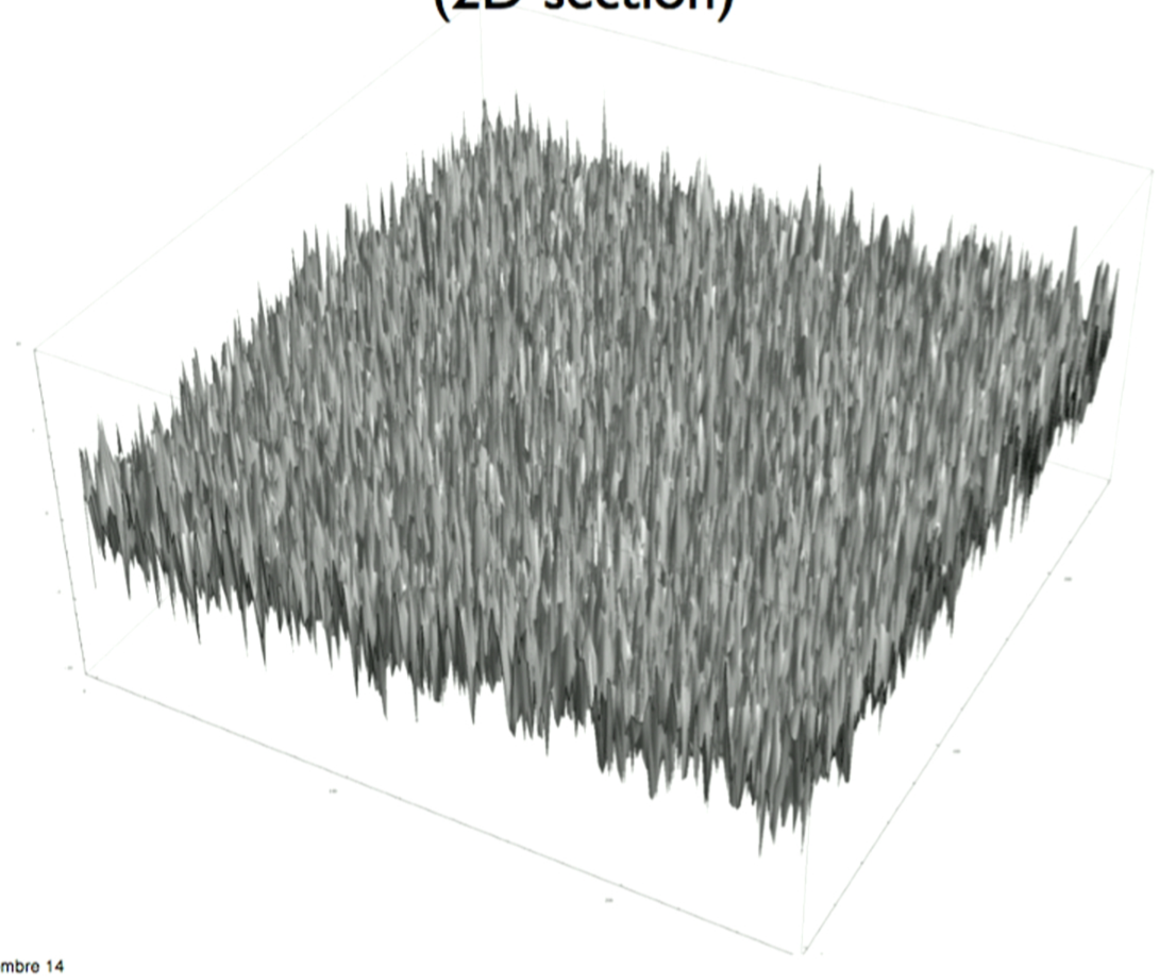
Gaussian Free Field D=3 256x256x256 lattice (2D section)



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