Title: PSI 2015/2016 Quantum Field Theory II - Francois David - Lecture 1

Date: Nov 09, 2015 09:00 AM

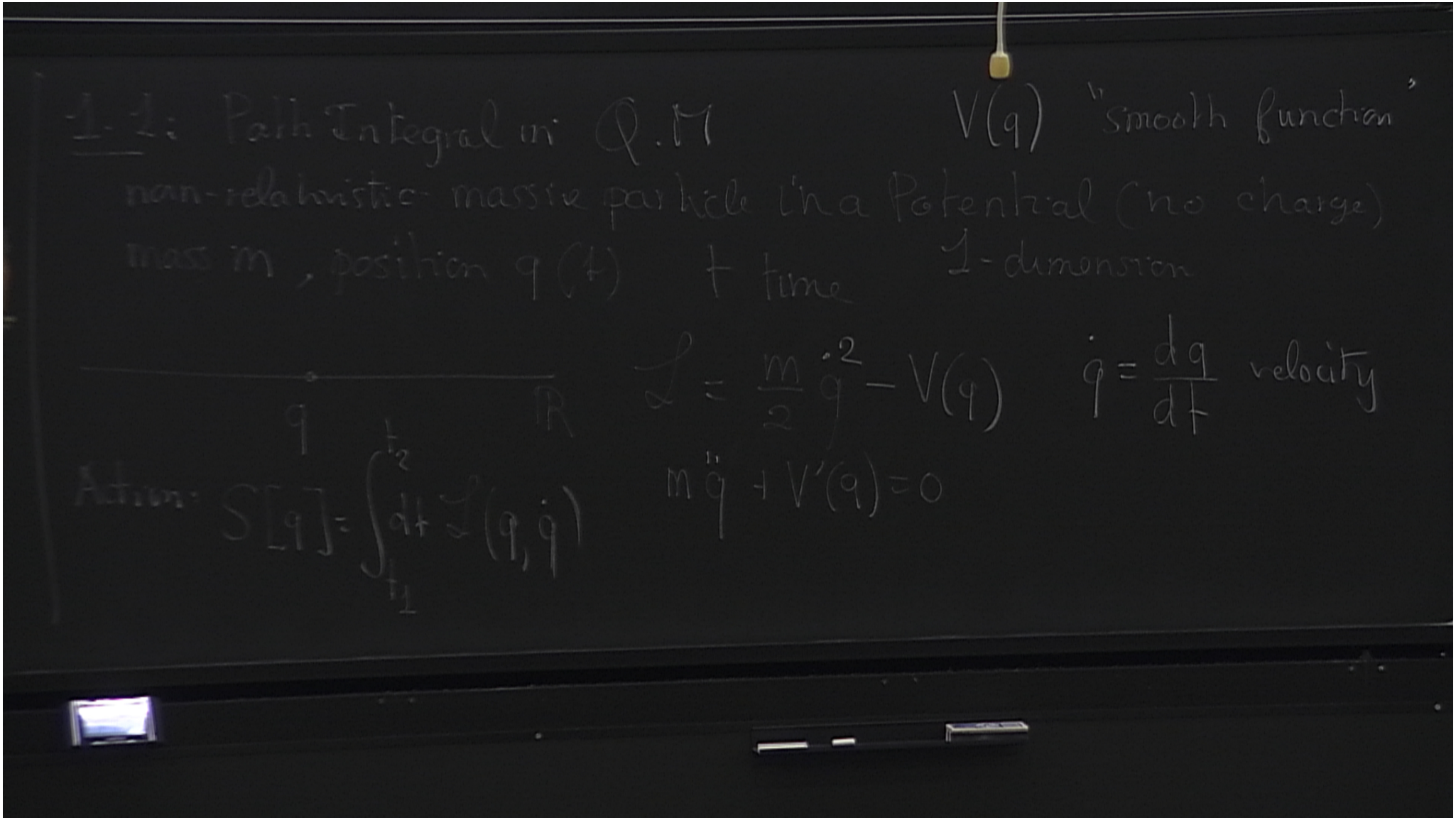
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Abstract:

1] Path & Functional Integrals in QFT

2] Perturbation Theory & Renormalization.

3] Non-abelian gauge theories


1-1: Path Integral in Q.M
V(q) "smooth function"
non-relativistic massive particle in a Potential (no charge)
mass m, position q(t) t time
1-dimension
q
R
$\mathcal{L} = \frac{m}{2}\dot{q}^2 - V(q)$
$\dot{q} = \frac{dq}{dt}$ velocity
Action: $S[q] = \int_{t_1}^{t_2} dt\, \mathcal{L}(q, \dot{q})$
$m\ddot{q} + V'(q) = 0$

Quantum Mechanics (Dirac)

states: $|\psi\rangle$ $\psi(q) = \langle q|\psi\rangle$

position-state $|q\rangle$ $Q|q\rangle = q|q\rangle$ position op

operator T P momentum operator

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H \cdot |\psi(t)\rangle$$

$$H = \frac{P^2}{2m} + V(Q)$$

$t = t_1$ initial time, $|\psi_1\rangle$ initial state $\longrightarrow$ t_2 $|\psi_2\rangle$

t, $|\psi(t)\rangle = U(t, t_1) \cdot |\psi_1\rangle$

U evolution operator unitary

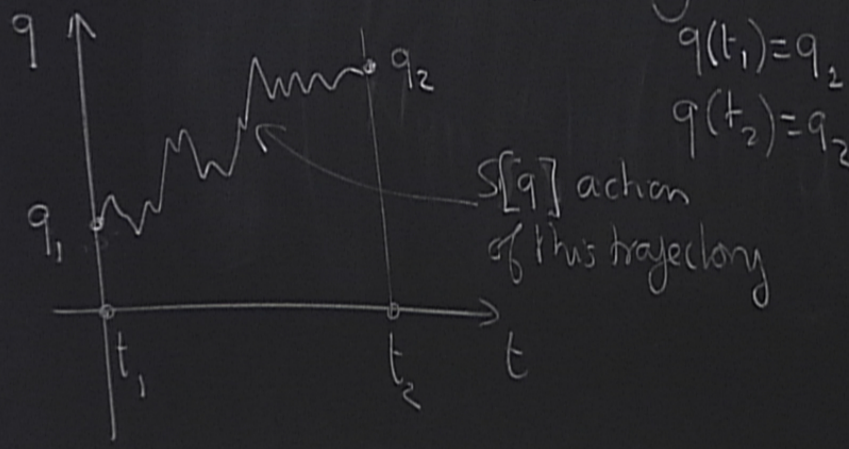
$$U(t, t_1) = U(t - t_1) = \exp\left(\frac{t - t_1}{i\hbar} H\right)$$

position state basis

$\langle q_2 | U(t_2 - t_1) | q_1 \rangle = K(q_2, t_2; q_1, t_1)$ Propagator

Path Integral

$$K(q_2, t_2; q_1, t_1) = \int_{q(t_1)=q_1,\ q(t_2)=q_2} D[q(t)]\, e^{\frac{i}{\hbar} S[q]}$$

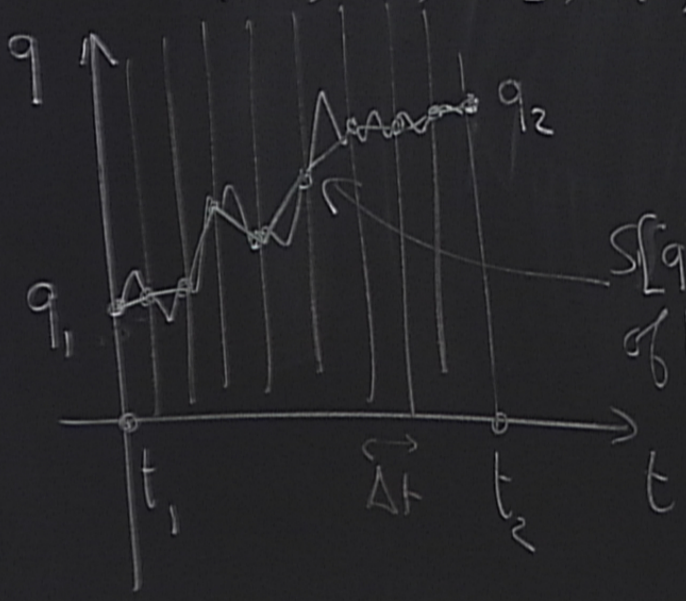


Path Integral (Feynman)

$$K(q_2, t_2; q_1, t_1) = \int D[q(t)]\, e^{\frac{i}{\hbar} S[q]}$$

$q(t_1) = q_1$

$q(t_2) = q_2$



$S[q]$

Definition: time cut off

$t \to t_i = i \cdot \Delta t \quad . \quad i = 0, \dots N = \frac{t_2 - t_1}{\Delta t}$

$q(t) \to q_i = q(t_i)$

$S[q] \Longrightarrow S\left[q \underset{\text{linear}}{\text{Piecewise}} \right]$

$$P[q] \to \prod_{i=1}^{N-1} (dq_i) \left(\frac{2\pi i \hbar \Delta t}{m} \right)^{-N/2}$$

continuum limit $\Delta t \to 0$ well defined

Quantum Mechanics (Dirac)

states: $|\psi\rangle$ $\psi(q) = \langle q|\psi\rangle$

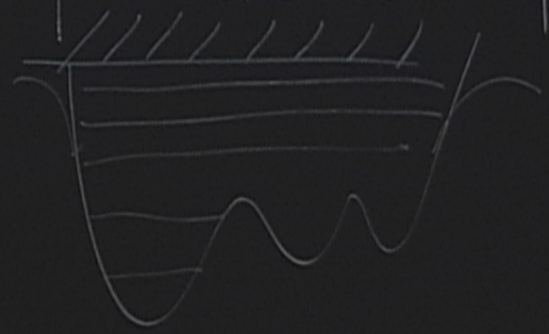
position-state $|q\rangle$ $Q|q\rangle = q|q\rangle$ position op

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H \cdot |\psi(t)\rangle$$

H operator

P momentum operator

$$H = \frac{P^2}{2m} + V(Q)$$



$t = t$

t,

U

$U(t,$

position

$\langle q_2|$

Mixed states : density matrix $\rho \geq 0 \quad \rho = \rho^\dagger$

$\rho = \sum_i p_i |i\rangle\langle i|$ $\quad \mathrm{Tr}[\rho] = 1$

$\langle i|j\rangle = \delta_{ij} \quad p_i \geq 0 \quad \sum_i p_i = 1$

p_i are "probabilities" to be in the pure state $|i\rangle$

Hamiltonian

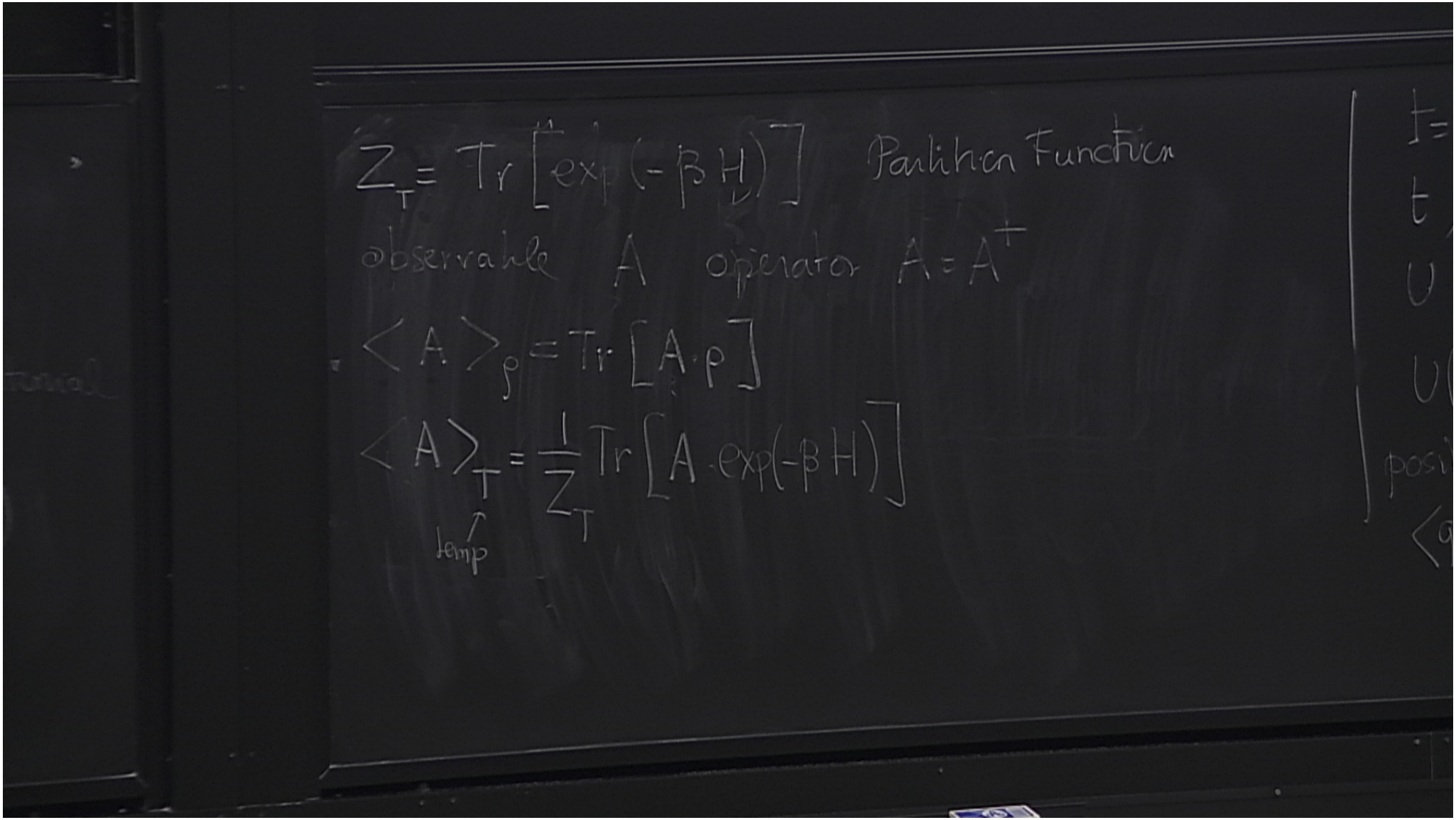
equilibrium ⟷ thermostat at temperature T

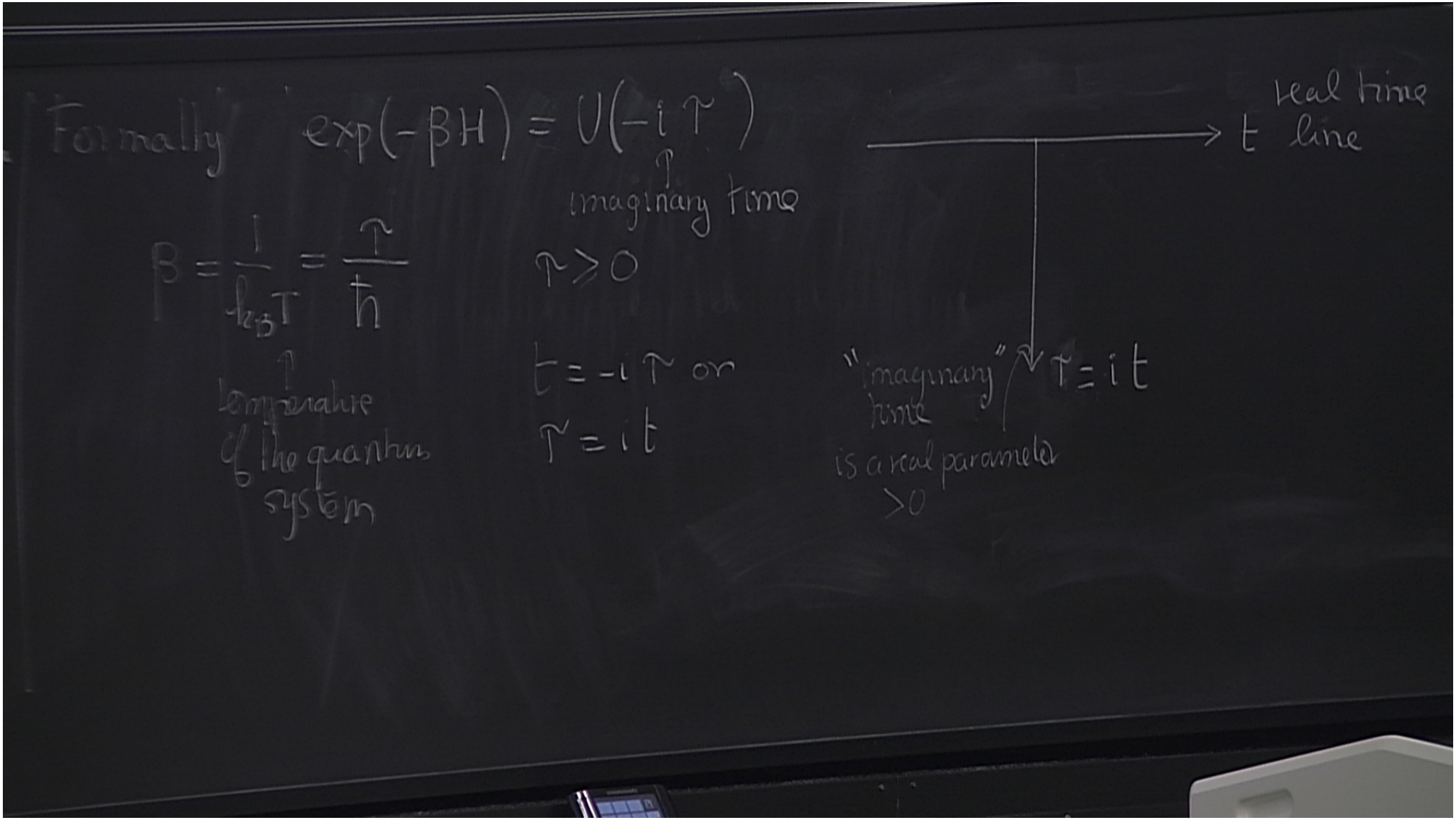
Gibbs state $\rho = \frac{1}{Z} \exp(-\beta H)$

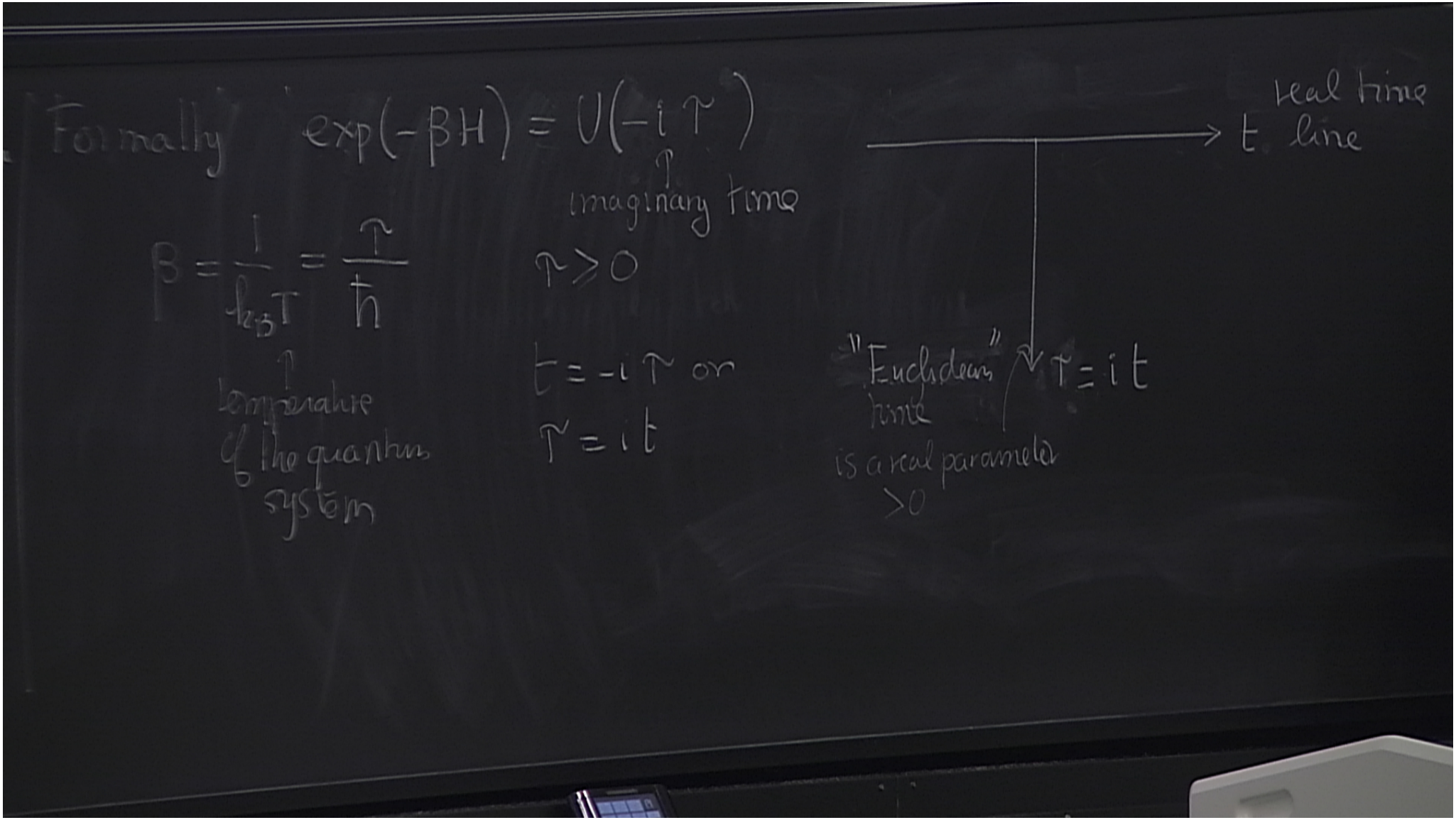
$p_i = \frac{1}{Z} \exp(-\beta E_i) \quad \beta = \frac{1}{k_B T}$

energy

defined stationary state


$Z_T = Tr\left[\exp(-\beta H)\right]$ Partition Function
observable A operator $A = A^\dagger$
$\langle A \rangle_\rho = Tr[A \cdot \rho]$
$\langle A \rangle_T = \frac{1}{Z_T} Tr\left[A \cdot \exp(-\beta H)\right]$
temp


Formally $\exp(-\beta H) = U(-i\tau)$
imaginary time
$\beta = \frac{1}{k_B T} = \frac{\tau}{\hbar}$
$\tau \gg 0$
temperature of the quantum system
$t = -i\tau$ or $\tau = it$
"imaginary" time
$\tau = it$
is a real parameter > 0
real time line
t


Formally $\exp(-\beta H) = U(-i\tau)$
imaginary time
real time
t line
$\beta = \frac{1}{k_B T} = \frac{\tau}{\hbar}$
$\tau \geq 0$
temperature of the quantum system
$t = -i\tau$ or
$\tau = it$
"Euclidean" time
$\tau = it$
is a real parameter
> 0

real time line

$U(t)$ is defined for t complex

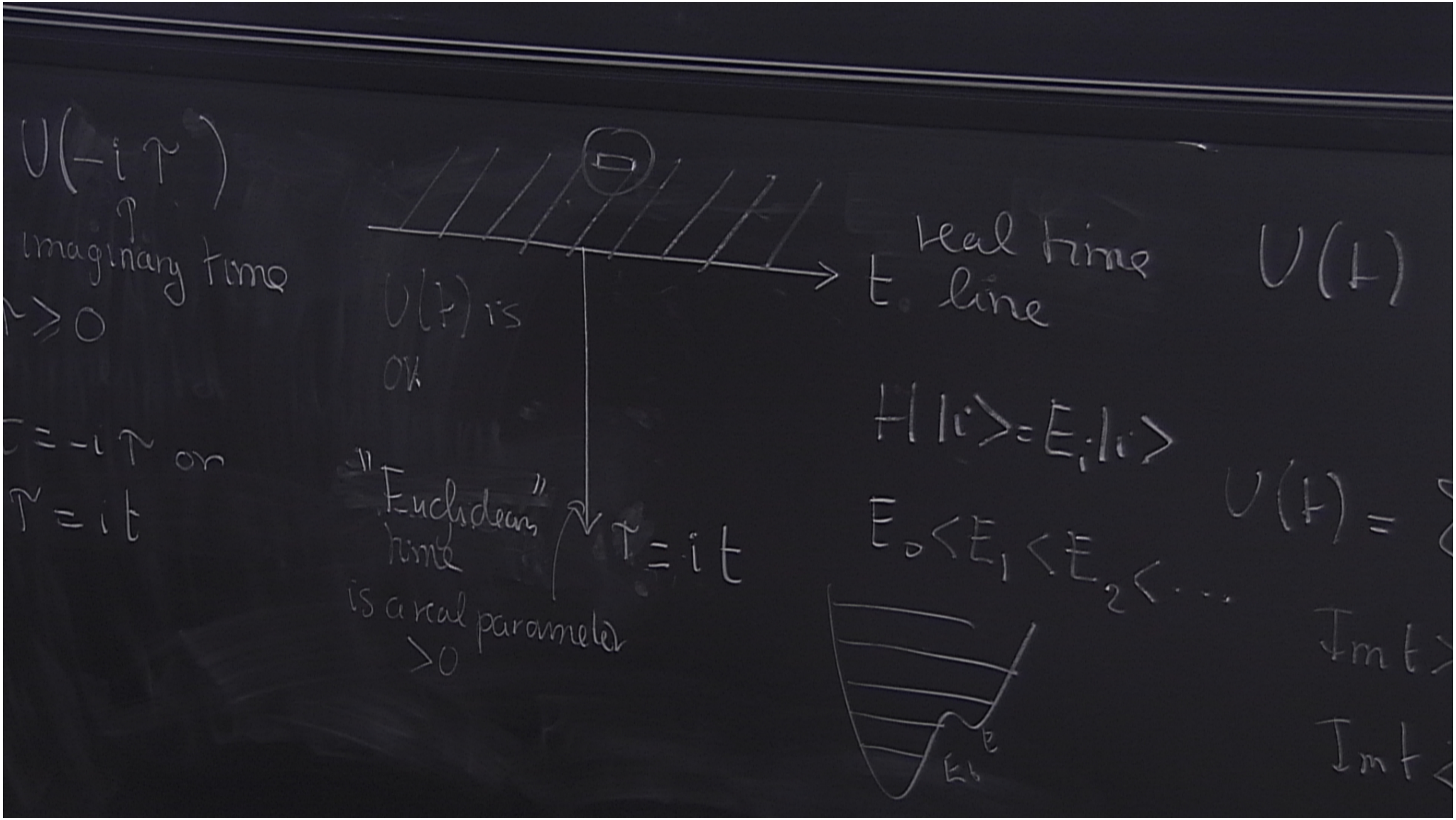
$|i\rangle = E_i |i\rangle$

$$U(t) = \sum_i e^{\frac{t}{i\hbar} E_i} |i\rangle\langle i|$$

$E_1 < E_2 < \cdots$

$\mathrm{Im}\, t > 0$ $\quad \left| e^{\frac{t}{i\hbar} E_i} \right| \nearrow \infty$ with i

$\mathrm{Im}\, t < 0$ $\quad \searrow 0$ with i


U(-iτ)
imaginary time
τ ≥ 0
t = -iτ or
τ = it
U(t) is
ok
"Euclidean"
time
is a real parameter
>0
τ = it
t
real time
line
U(t)
H|i> = E_i|i>
E_0 < E_1 < E_2 < ...
U(t) =
Im t >
Im t <

A: $\frac{\langle\psi|AA^{+}|\psi\rangle}{\langle\psi|\psi\rangle} \leq \|A\|^{2}$

$t \longrightarrow -i\tau$

$\mathrm{Im}\, t \leq 0$

$ds^2 = d\tau^2 + d\vec{x}^2$

Euclidean metric on $\mathbb{R}^{1+3}$

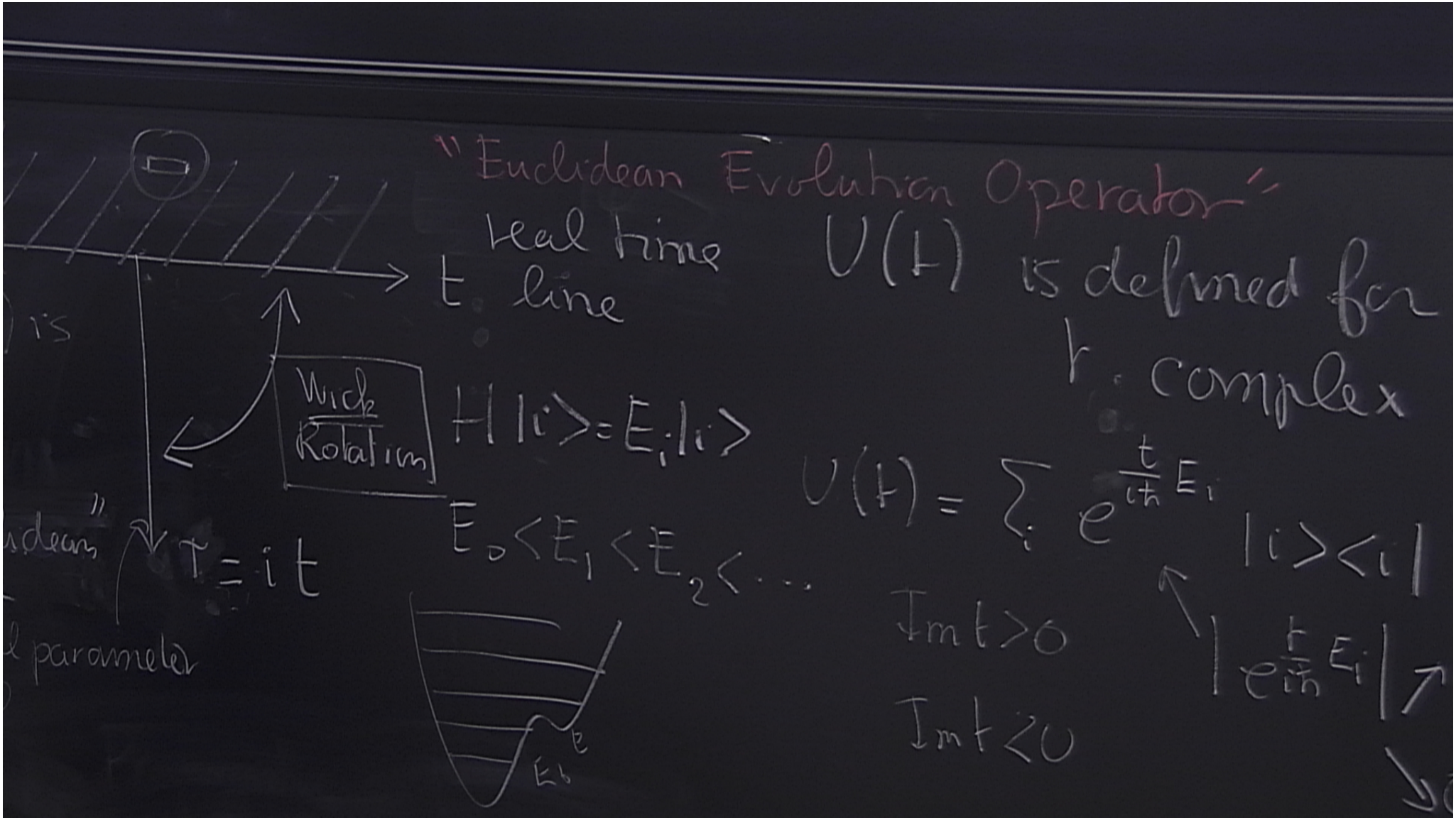
$M^{1,3}$ Minkowski

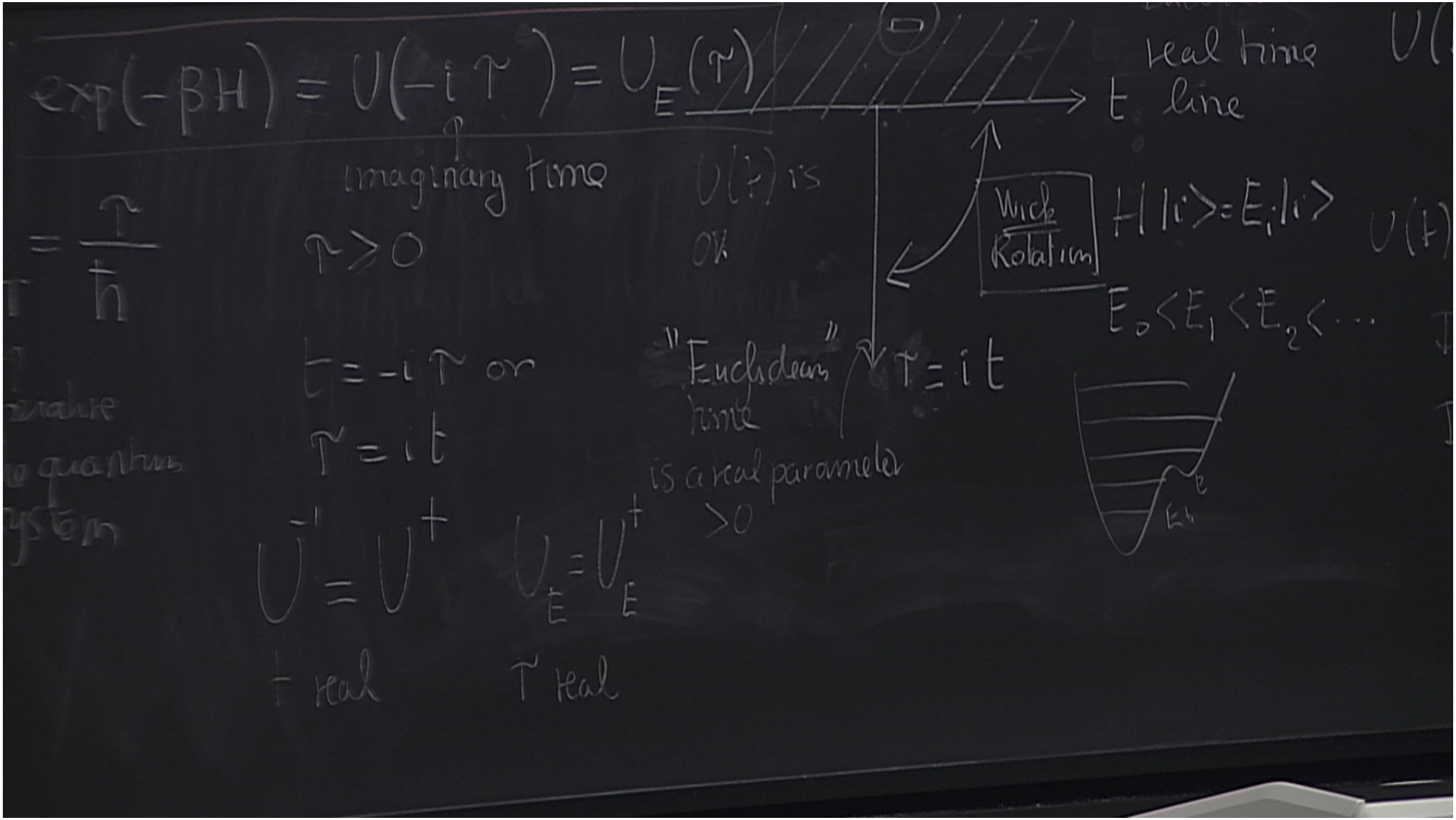
$X = (\tau, x_1, \dots x_d)$ τ = another space coordinate

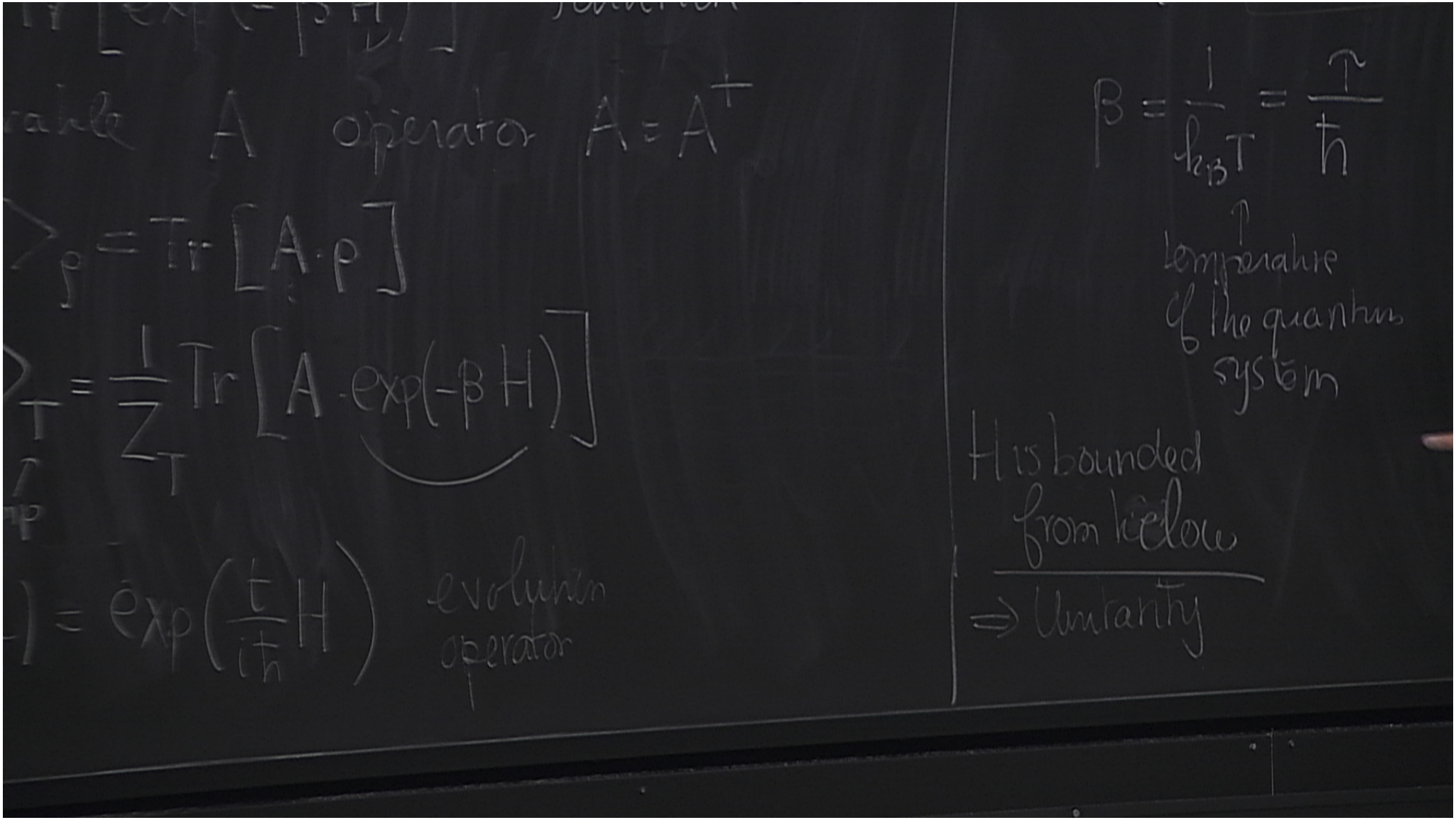
Coast Signature

$X = (t, x_1, \dots x_d)$

Poincaré Group $\longrightarrow$ Euclidean Group


"Euclidean Evolution Operator"
real time
t line
U(t) is defined for
t: complex
Wick Rotation
H|i> = E_i|i>
E_0 < E_1 < E_2 < ...
T = it
U(t) = Σ_i e^{t/iħ E_i} |i><i|
Im t > 0
Im t < 0
|e^{t/iħ E_i}|
E_0
E
parameter


$\exp(-\beta H) = U(-i\tau) = U_E(\tau)$
imaginary time
$\tau \geq 0$
$t = -i\tau$ or
$\tau = it$
$U^{-1} = U^{+}$
$U_E^{-1} = U_E^{+}$
t real
τ real
$U(\tau)$ is
ok
"Euclidean" time
is a real parameter
>0
real time
t line
Wick Rotation
$\tau = it$
$H|i\rangle = E_i|i\rangle$
$E_0 < E_1 < E_2 < \cdots$
$U(t)$
$= \frac{\tau}{\hbar}$
erature
quantum
ystem


A operator $A = A^\dagger$
$\beta = \frac{1}{k_B T} = \frac{\hbar}{\hbar}$
temperature of the quantum system
$\rho = Tr[A \cdot \rho]$
$\rho_T = \frac{1}{Z_T} Tr[A \cdot \exp(-\beta H)]$
H is bounded from below
$\Rightarrow$ Unitarity
$= \exp\left(\frac{t}{i\hbar} H\right)$ evolution operator

$$\frac{\langle\psi|AA^{+}|\psi\rangle}{\langle\psi|\psi\rangle} \leq \|A\|^{2} \simeq \sup\left|\mathrm{evd}(AA^{+})\right| \simeq \left|\mathrm{diam}(\mathrm{Spectrum}(A))\right|^{2}$$

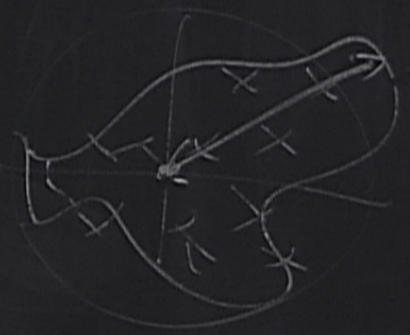
$t \longrightarrow -i\tau$

$ds^{2} = d\tau^{2} + d\vec{x}^{2}$

Euclidean metric on $\mathbb{R}^{1+3}$

$X = (\tau, x_1, \ldots x_d)$ τ = another space coordinate

Poincare Group $\longrightarrow$ Euclidean Group



Is there a path integral representation ?

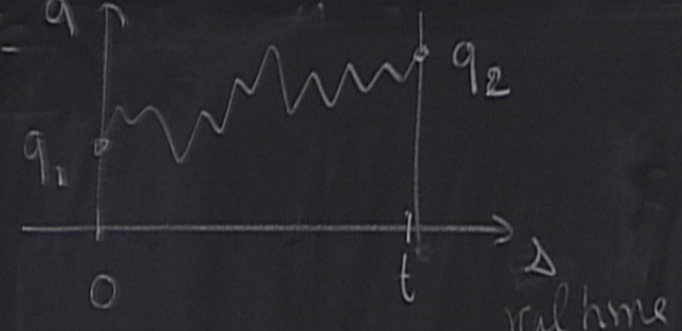
for $U_E(\tau) = \exp\left(-\frac{\tau}{\hbar} H\right)$

Yes! Formal derivation

$$K(q_2, t; q_1, 0) = \langle q_2 | U(t) | q_1 \rangle = \int_{\substack{q(0)=q_1 \\ q(t)=q_2}} \mathcal{D}[q] \exp\left(\frac{i}{\hbar} S[q]\right)$$

real time

$$S[q] = \int_0^t ds \left[\frac{m}{2}\left(\frac{dq(s)}{ds}\right)^2 - V(q(s))\right]$$



Wick rotation

$t = -i\tau \quad 0 < s < t$

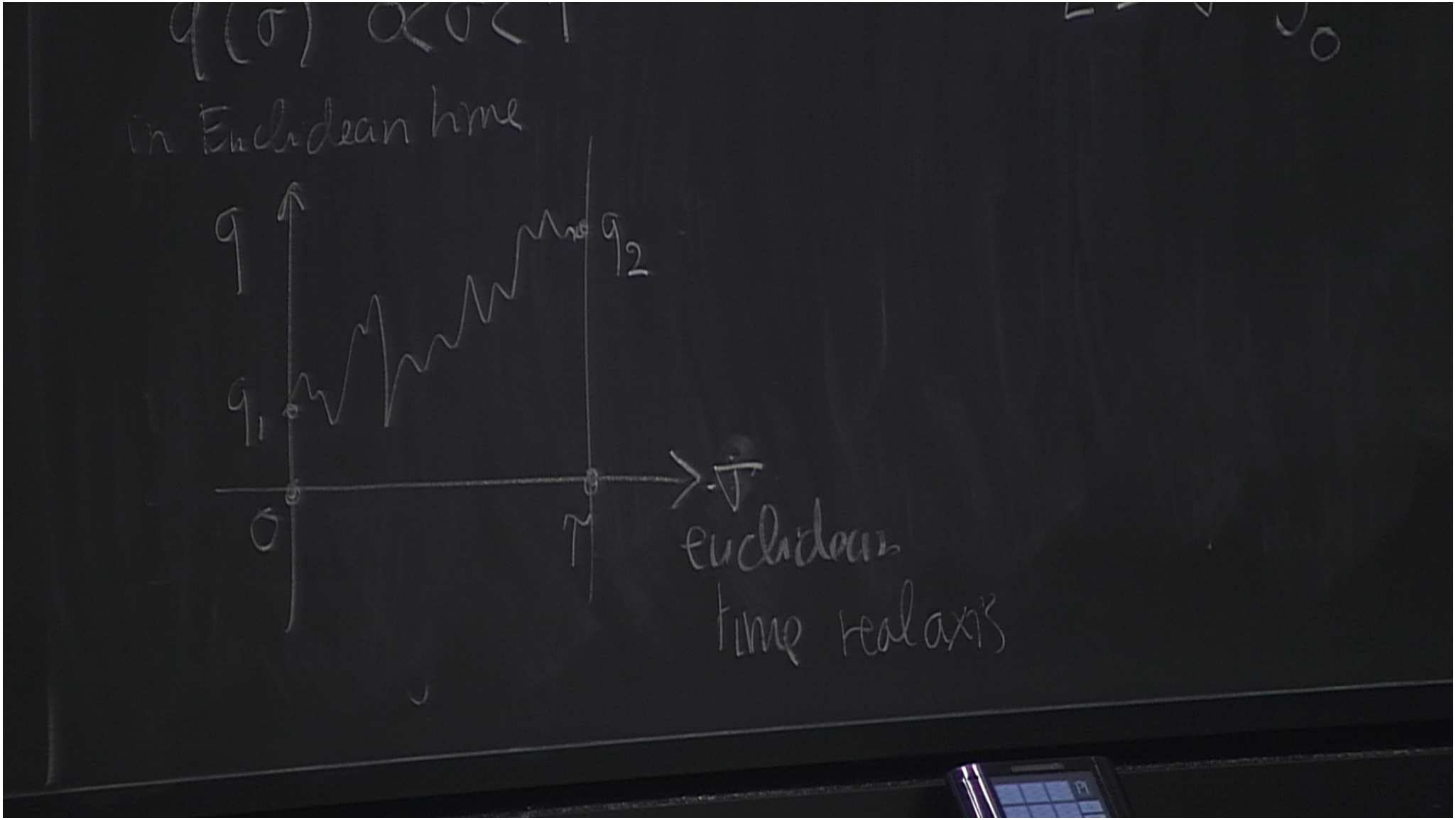
$s = -i\sigma \quad 0 < \sigma < \tau$

now $q(\sigma)$

$$iS[q] = i\int_0^t ds \left[\frac{m}{2}\left(\frac{dq}{ds}\right)^2 - V(q)\right]$$

$$= i(-i)\int_0^\tau d\sigma \left[\frac{m}{2}\left(i\frac{dq}{d\sigma}\right)^2 - V(q)\right]$$

$$= -\int_0^\tau d\sigma \left[\frac{m}{2}\left(\frac{dq}{d\sigma}\right)^2 + V(q)\right]$$

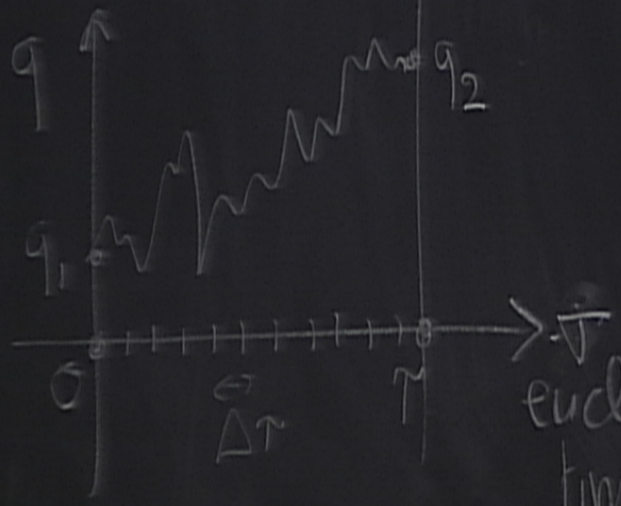

in Euclidean time
q
q_2
q_1
0
T
euclidean
time real axis

over trajectories

$q(\sigma)$ $0 < \sigma < \tau$

in Euclidean time

$q(\tau) = q_2$



Euclidean time real axis

$$S_E[q] := \int_0^\tau d\sigma \left[\frac{m}{2}(\dot{q})^2 + V(q)\right]$$

$$D_E[q] = \prod_i dq_i \left(\frac{2\pi\hbar\,\Delta\tau}{m}\right)^{-1/2}$$

discretize Euclidean time

by $\sigma_i = \Delta\tau \cdot i \quad i = 0, \ldots N = \frac{\tau}{\Delta\tau}$

$\lim \Delta\tau \to 0$

$$\langle q_2 | q_1 \rangle = \int \mathcal{D}_E[q] \exp\left(-\frac{1}{\hbar} S_E[q]\right)$$

positive measure probability on path (histories)

Probability

$q(0) = q_1$

$q(T) = q_2$

Euclidean Action

$$S_E[q] := \int_0^T d\tau \left[\frac{m}{2}(\dot{q})^2 + V(q)\right]$$

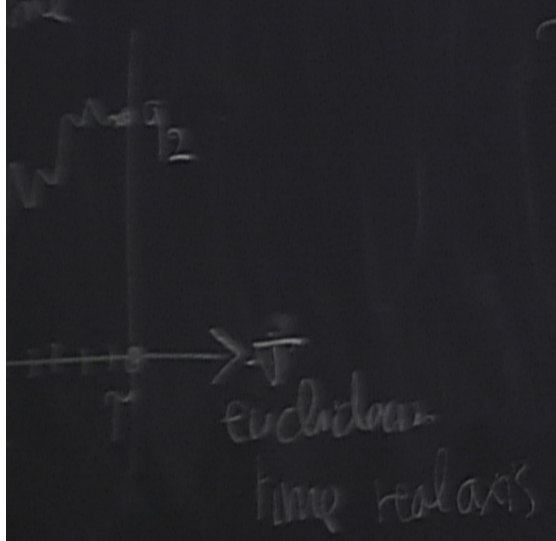
$$\mathcal{D}_E[q] = \prod_i dq_i \left(\frac{2\pi\hbar\,\Delta\tau}{m}\right)^{-1/2}$$

real measure

discretize Euclidean time

by $\tau_i = \Delta\tau \cdot i \quad i = 0, \ldots N = \frac{T}{\Delta\tau}$

$\lim \Delta\tau \to 0$



Probability space

path with largest probability?

q such that $S_E[q]$ minimal

Saddle point of this integral

$$-m\frac{d^2q}{d\tau^2} + V'(q) = 0 \Leftarrow m\ddot{q} + V'(q) = 0$$

Gradient

Euclidean Trajectory $\overset{\text{Wick}}{\Leftarrow}$ classical trajectory

at real time

$$\rho_T = \frac{1}{Z} U_E(\tau)$$

$$\boxed{\frac{\hat{T}}{\tau} = \frac{1}{k_B T}}$$

$$Z = \mathrm{Tr}\left[U_E(\tau)\right] \quad \text{Partition Function}$$

$$= \int_{\mathbb{R}} dq \, \langle q | U_E(\tau) | q \rangle = \int \mathcal{D}_E[q] \exp\left[-\frac{1}{\hbar} S_E[q]\right]$$

because $\langle q | q' \rangle = \delta(q - q')$ $\quad q(\sigma) : 0 < \sigma < \tau$

$\mathbb{I} = \int dq \, |q\rangle\langle q|$ $\quad q(0) = q(\tau)$

$U_E(\tau)$

$\frac{\hbar}{\tau} = k_B T$

Period $= \tau = \frac{\hbar}{k_B T}$

$Tr[U_E(\tau)]$ Partition Function

$$\int_R dq \langle q | U_E(\tau) | q \rangle = \int \mathcal{D}_E[q] \exp\left[-\frac{1}{\hbar} S_E[q]\right]$$

$|q'\rangle = \delta(q - q')$

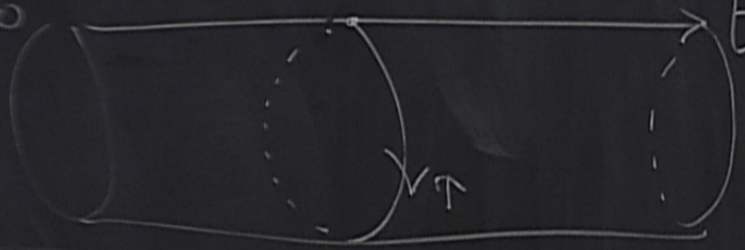
$\int |q\rangle\langle q|$

$q(\sigma) : 0 < \sigma < \tau$

$q(0) = q(\tau)$

Path integral over periodic trajectories in Euclidean time with period τ

at finite temperature
complex time is a cylinder with period $-i\tau$



KMS states $\Leftarrow$ density Gibbs for mahx

QFT in a black-hole classical background $\rightarrow$ Euclidean Spacetime



