Title: AMATH 875/PHYS 786 - Fall 2015 - Lecture 19

Date: Nov 09, 2015 03:00 PM

URL: http://pirsa.org/15110018

Abstract:

File Edit View Insert Actions Tools Help

GR for Cosmology, Achim Kempf, Fall 2015, Lecture 19

Note Title

Classification of solutions of GR

(and along the way we will introduce some generally useful methods of group theory)

Recall:

- The task is to solve the equations of motion of matter, jointly with the Einstein equation:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

- In practice, this problem must be simplified, i.e., the number of to-be-determined functions must be reduced.

1 / 29

File Edit View Insert Actions Tools Help

Question: How much can we weaken the symmetry assumptions of Friedmann-Lemaître and still get exact solutions?

Strategy:

- Classify cosmological models (M, g), $T_{\mu\nu}$ by the amount and type of symmetry assumed.

- For each amount and type of symmetry assumed, try to find exact solutions or at least (asymptotic) properties of exact solutions.

Remark: Among the high symmetry models, some come arbitrarily close to F.L. at finite times!

2 / 29

File Edit View Insert Actions Tools Help

Remark: Among the high symmetry models, some come arbitrarily close to F.L. at finite times!

See, e.g., text by Wainwright & Ellis.

Recall: Symmetries & Killing vector fields

- Two spacetimes (M, g), $(\tilde{M}, \tilde{g})$ are isometric (and therefore of exactly identical shape) if there is a diffeomorphism $\phi: M \to \tilde{M}$ so that the image of the metric g in $\tilde{M}$ is $\tilde{g}$: $Tg = \tilde{g}$.

- A spacetime has a sym... [cut off]

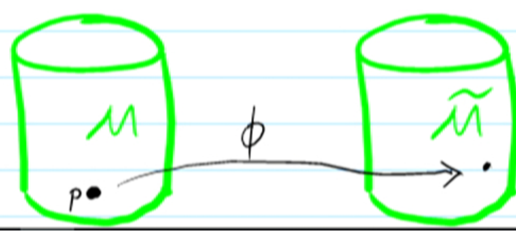
2 / 29

Recall: Symmetries & Killing vector fields

- Two spacetimes $(\mathcal{M}, g)$, $(\tilde{\mathcal{M}}, \tilde{g})$ are isometric (and therefore of exactly identical shape) if there is a diffeomorphism $\phi: \mathcal{M} \to \tilde{\mathcal{M}}$ so that the image of the metric g in $\tilde{\mathcal{M}}$ is $\tilde{g}$: $Tg = \tilde{g}$.

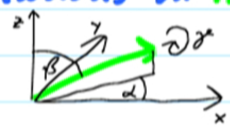
- A space-time has a symmetry, if we find such a ϕ for $\tilde{\mathcal{M}} = \mathcal{M}$.

- Example:



ϕ performs a rotation of $\mathcal{M}$ about a symmetry axis, to obtain $\tilde{\mathcal{M}} = \mathcal{M}$

Definition: A group G is called a **Lie group** if G is also a finite-dimensional smooth manifold.

Example: The sets of rotations in $\mathbb{R}^3$ forms a 3-dimensional Lie group, $SO(3)$.
The angles 

α, β, γ are coordinates for elements $g \in SO(3)$.

Remarks: ▢ The symmetries of a manifold (M, g) can be discrete, such as reflections.

▢ But often, the symmetry group of a manifold (M, g) is actually a Lie group.

such as reflections.

- But often, the symmetry group of a manifold $(\mathcal{M}, g)$ is actually a Lie group.

Note: - Each $h \in G$ yields an isometric diffeomorphism, by assumption.

$$h : \mathcal{M} \to \mathcal{M}, \text{ namely } h : p \to h(p) \qquad \forall p \in \mathcal{M}$$

- Consider the set $\mathcal{O}_p \subset \mathcal{M}$ defined by: $\mathcal{O}_p := \{ q \in \mathcal{M} \mid \exists h \in G : h(p) = q \}$

Definition: The set $\mathcal{O}_p$ is called the **Orbit** of p under the action of the group G.

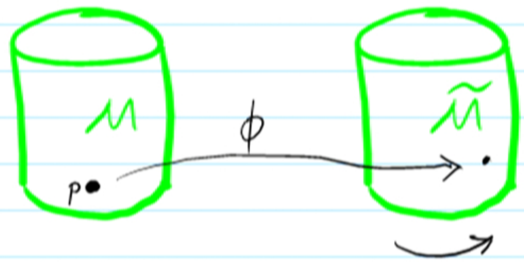
Note: If G is a Lie group then each orbit $\mathcal{O}_p$ is p or a submanifold of $(\mathcal{M}, g)$.

Question: What are the *infinitesimal* isometric diffeomorphisms?

File Edit View Insert Actions Tools Help

such a ϕ for $\tilde{M} = M$.

Example:



ϕ performs a rotation of M about a symmetry axis, to obtain $\tilde{M} = M$ with $T_g = \tilde{g}$.

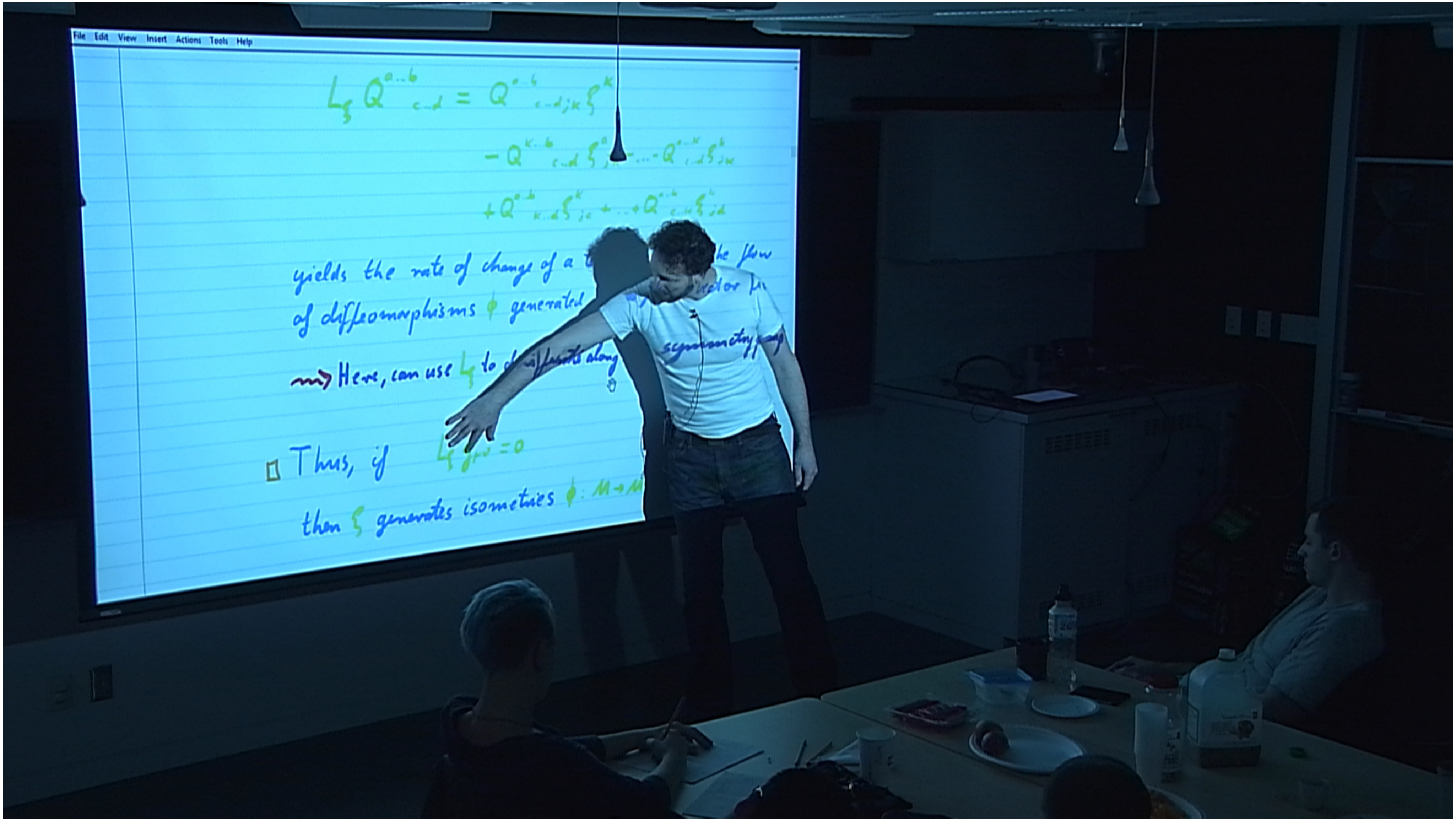
Note: The set of all symmetries of a manifold (M, g) forms a "group":

Definition: A "group" G is a set, with an operation, say "$\circ$",

$$\circ : G \times G \to G$$

and a "neutral element", say "e", $e \in G$, such that

4 / 29


File Edit View Insert Actions Tools Help
yields the rate of change of a
of diffeomorphisms ϕ generated
Here, can use
Thus, if
then ξ generates isometries

File Edit View Insert Actions Tools Help

$\Rightarrow$ A vector field ξ generates a symmetry of spacetime if it is a Killing vector field:

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0 \qquad (X)$$

Q: Maximum number, d, of Killing vector fields in n dims?

A: $\boxed{d = n(n+1)/2}$ To see this, note that there are 2 ways to obey Eq. (X):

a) $\xi_{\mu;\nu} = 0 \ \forall \nu$, i.e. $\nabla \xi = 0$
Can have maximally n such indep. vectors

b) $\nabla \xi \neq 0$, but then $K_{\mu\nu} := \xi_{\mu;\nu}$ is antisymmetric
Can have at most $n(n-1)/2$ indep. such cases.

$\Rightarrow d = n + n(n-1)/2 = n(n+1)/2$

7 / 29

File Edit View Insert Actions Tools Help

$\Rightarrow$ A vector field ξ generates a symmetry of spacetime if it is a Killing vector field:

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0 \qquad (X)$$

Q: Maximum number, d, of Killing vector fields in n dims?

A: $\boxed{d = n(n+1)/2}$ To see this, note that there are 2 ways to obey Eq. (X):

a) $\xi_{\mu;\nu} = 0 \ \forall \nu$, i.e. $\nabla\xi = 0$
(can have maximally n such indep. vectors

b) $\nabla\xi \neq 0$, but then $K_{\mu\nu} := \xi_{\mu;\nu}$ is antisymmetric
(can have at most $n(n-1)/2$ indep. such cases.)

$\Rightarrow d = n + n(n-1)/2 = n(n+1)/2$

7 / 29

File Edit View Insert Actions Tools Help

From a symmetry Lie group to a "symmetry Lie algebra":

General idea:

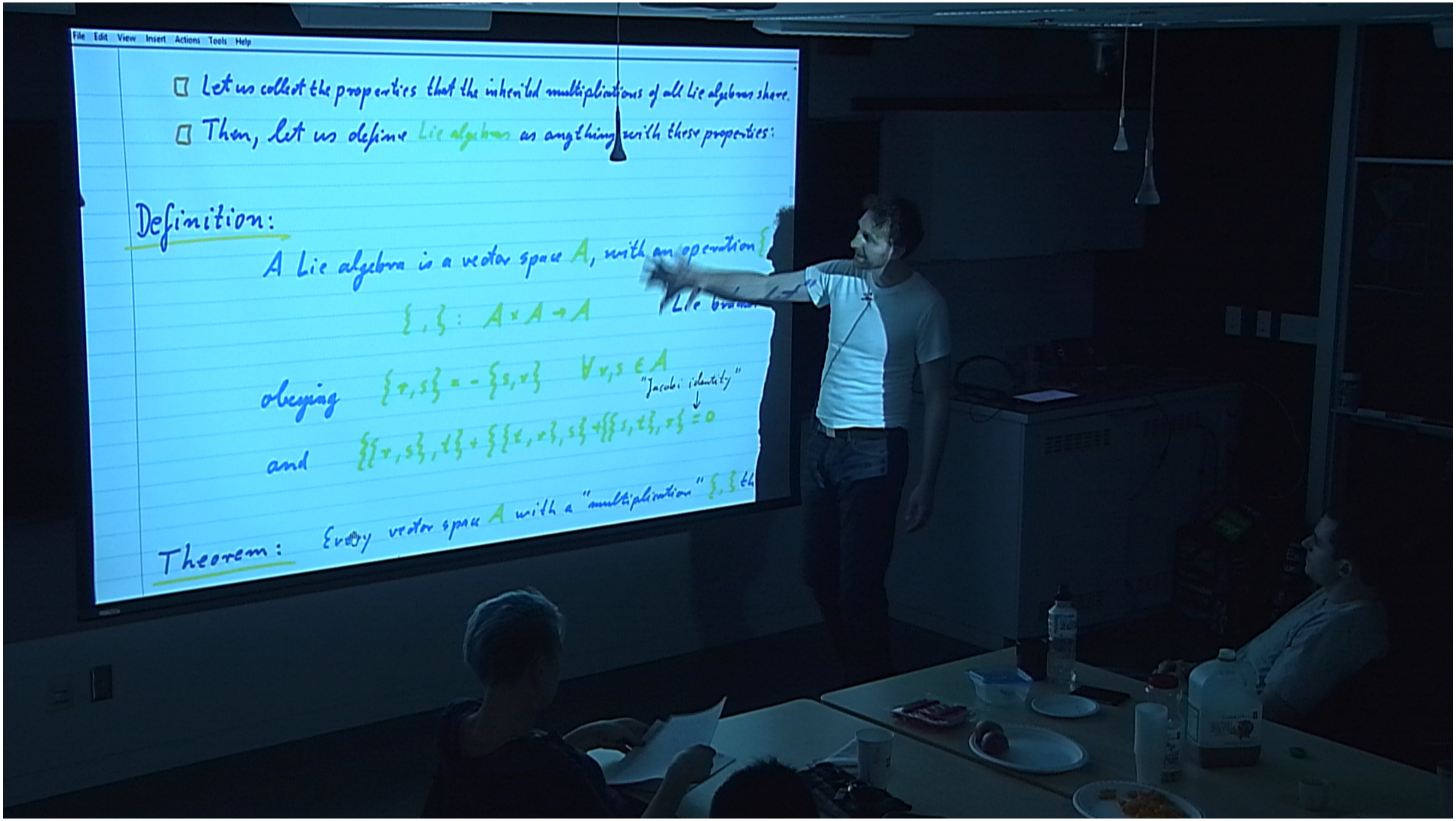
Normally the points of a manifold cannot be multiplied!

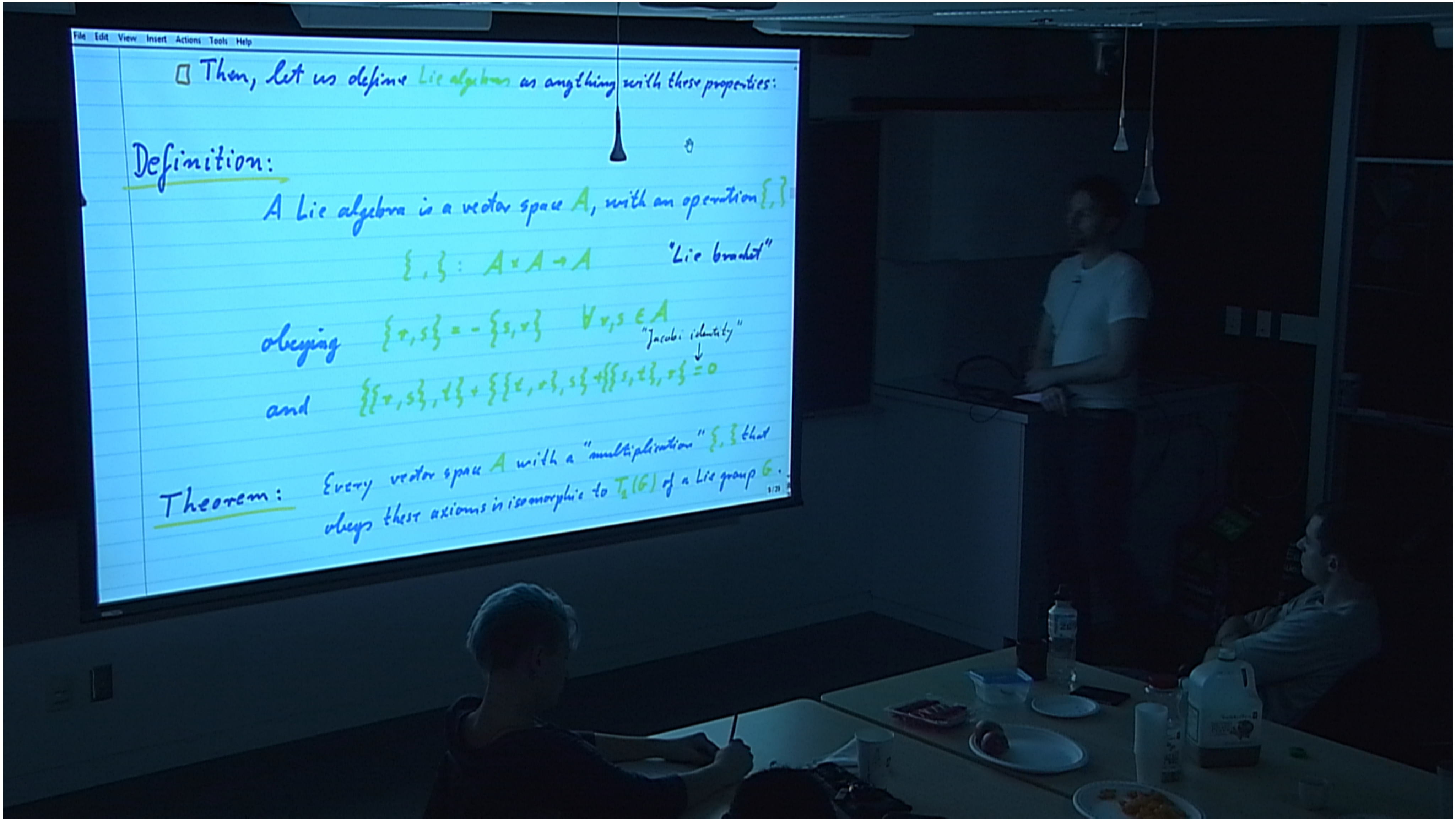
- A Lie group is a smooth manifold with extra structure: the multiplication.
- Notice: Product of group elements close to $1 \in G$ yields a group element close to 1.
- Consider the tangent space $T_1(G)$ to the point $1 \in G$ of the Lie group manifold G.
- $T_1(G)$ is a vector space and it has extra structure, inherited from the group's multiplication.
- Define the Lie algebra of a group G to be $T_1(G)$, equipped with the inherited "multiplication".

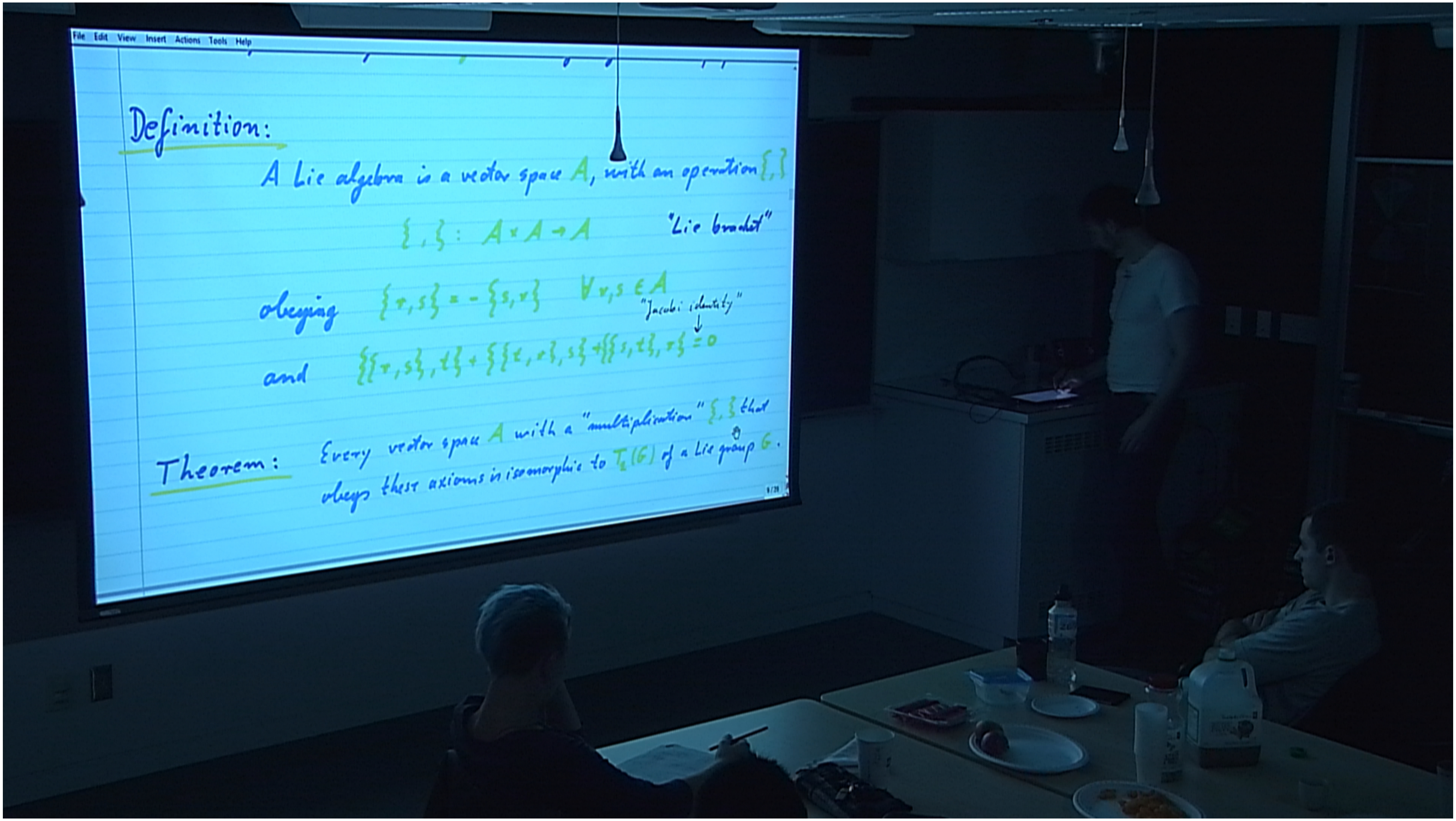
The identity element of the group, $p = 1$ is also a point of the group's manifold. $T_1(G)$ is the tangent space to this point.

Crucial fact: From knowledge of only the Lie algebra, i.e., only $T_1(G)$ and

8 / 29


File Edit View Insert Actions Tools Help
Let us collect the properties that the inherited multiplications of all Lie algebras share.
Then, let us define Lie algebras as anything with these properties:
Definition:
A Lie algebra is a vector space A, with an operation
{ , } : A × A → A
obeying {r,s} = - {s,r} ∀ r,s ∈ A
"Jacobi identity"
and {{r,s},t} + {{t,r},s} + {{s,t},r} = 0
Theorem: Every vector space A with a "multiplication" {,}


File Edit View Insert Actions Tools Help
Then, let us define Lie algebras as anything with these properties:
Definition:
A Lie algebra is a vector space A, with an operation {,}
{,} : A × A → A
"Lie bracket"
obeying {r,s} = -{s,r} ∀ r,s ∈ A
"Jacobi identity"
and {{r,s},t} + {{t,r},s} + {{s,t},r} = 0
Theorem:
Every vector space A with a "multiplication" {,} that obeys these axioms is isomorphic to T1(G) of a Lie group G.


File Edit View Insert Actions Tools Help
Definition:
A Lie algebra is a vector space A, with an operation {,}
{ , } : A × A → A "Lie bracket"
obeying {r,s} = -{s,r} ∀ r,s ∈ A
"Jacobi identity"
and {{r,s},t} + {{t,r},s} + {{s,t},r} = 0
Theorem: Every vector space A with a "multiplication" {,} that obeys these axioms is isomorphic to T_e(G) of a Lie group G.

File Edit View Insert Actions Tools Help

Exercise: Prove this, i.e., show the following:

Assume $\xi^{(1)}, \xi^{(2)}$ are Killing vector fields of (M, g) and $\alpha, \beta \in \mathbb{R}$.

Then: $\alpha \xi^{(1)} + \beta \xi^{(2)}$ (i.e., they form a vector space)

and $\{\xi^{(1)}, \xi^{(2)}\} := \xi^{(1)}\xi^{(2)} - \xi^{(2)}\xi^{(1)}$

are also Killing vector fields,

and the $\xi^{(i)}$ obey the Jacobi identity.

10 / 29

File Edit View Insert Actions Tools Help

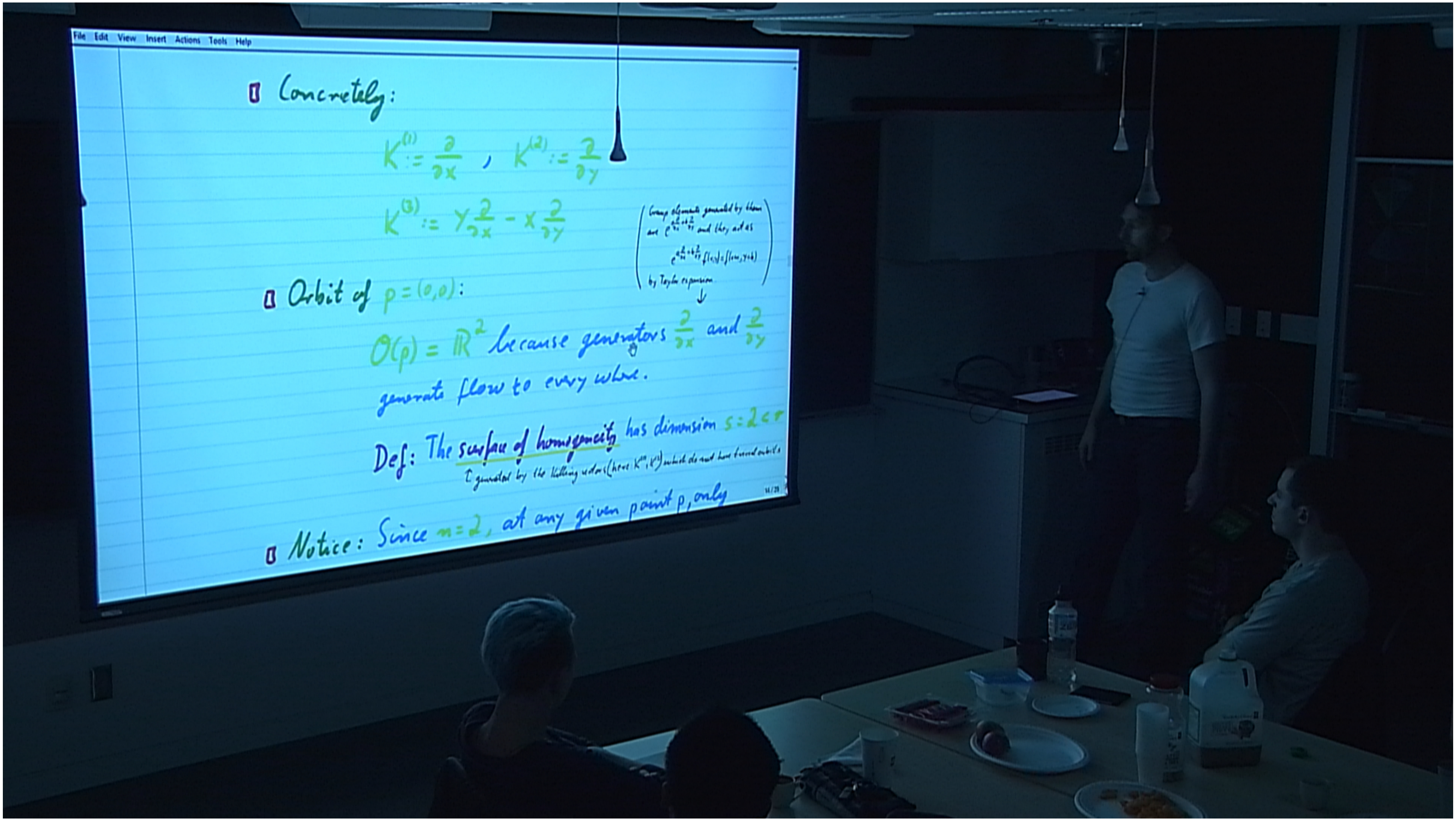
Summary of the big picture:

1. The symmetries of any $(\mathcal{M}, g)$ form a group: they can be concatenated associatively, and all possess an inverse. Some symmetries (Recall: there can be discrete symmetries too.) are differentiable, parametrized by the flow $\Rightarrow$ the symmetries form a Lie group.

2. Each Killing vector field is the infinitesimal generator of a flow of isometric diffeomorphisms, i.e., of a symmetry.

3. We see here that the Killing vector fields indeed form a Lie algebra.

11 / 29


File Edit View Insert Actions Tools Help
Concretely:
K(1) := ∂/∂x , K(2) := ∂/∂y
K(3) := y ∂/∂x − x ∂/∂y
Orbit of p = (0,0):
O(p) = ℝ2 because generators ∂/∂x and ∂/∂y
generate flow to everywhere.
Def: The surface of homogeneity has dimension s = 2 < r
Notice: Since n = 2, at any given point p, only

Wainwright & Ellis, Dyn. systems in cosmology, Cambridge Univ. Press (1997)

□ Examples:

	homogeneity s	isotropy d	
□	4	3	Einstein's static model
	4	1	Gödel's model
	4	0	Oszvath-Kerr models
	3	3	Friedmann Lemaître models
	3	1	spatially hom & locally one rot. sym axis
	3	0	Bianchi models
	⋮	⋮	

Powerful alternative classification approach:

File Edit View Insert Actions Tools Help

Idea: Classify the possible $T_{\mu\nu}$, then use Einstein equation to obtain classification of curvature.

Proposition:

For every physical energy momentum tensor $T_{\mu\nu}$ there exists a unique timelike vector field u so that $T_{\mu\nu}$ takes this standard form:

scalar, vector, scalar, tensor

$$T_{ab} = \mu u_a u_b + q_a u_b + q_b u_a + p(g_{ab} + u_a u_b) + \pi_{ab}$$

where q and π are a vector field and a tensor field obeying:

$$q_a u^a = 0, \quad \pi_{ab} u^b = 0, \quad \pi_a{}^a = 0, \quad \pi_{ab} = \pi_{ba}$$

18 / 29

File Edit View Insert Actions Tools Help

For every physical energy momentum tensor $T_{\mu\nu}$ there exists a unique timelike vector field u so that $T_{\mu\nu}$ takes this standard form:

$$T_{ab} = \overset{\text{scalar}}{\mu}\, u_a u_b + \overset{\text{vector}}{q_a}\, u_b + q_b u_a + \overset{\text{scalar}}{p}\,(g_{ab} + u_a u_b) + \overset{\text{tensor}}{\pi_{ab}}$$

where q and π are a vector field and a tensor field obeying:

$$q_a u^a = 0, \quad \pi_{ab} u^b = 0, \quad \pi_a{}^a = 0, \quad \pi_{ab} = \pi_{ba}$$

Definition: u is called the "fundamental 4-velocity field"

18 / 29

Definition: u is called the "fundamental 4-velocity field"

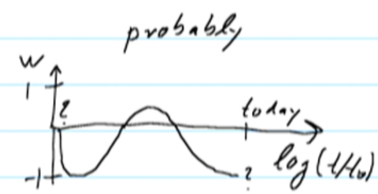
Note: E.g., for a perfect fluid this is the fluid velocity:

$$T_{ab} = \mu u_a u_b + p(g_{ab} + u_a u_b), \quad u_a u^a = -1$$

Recall: equation of state is

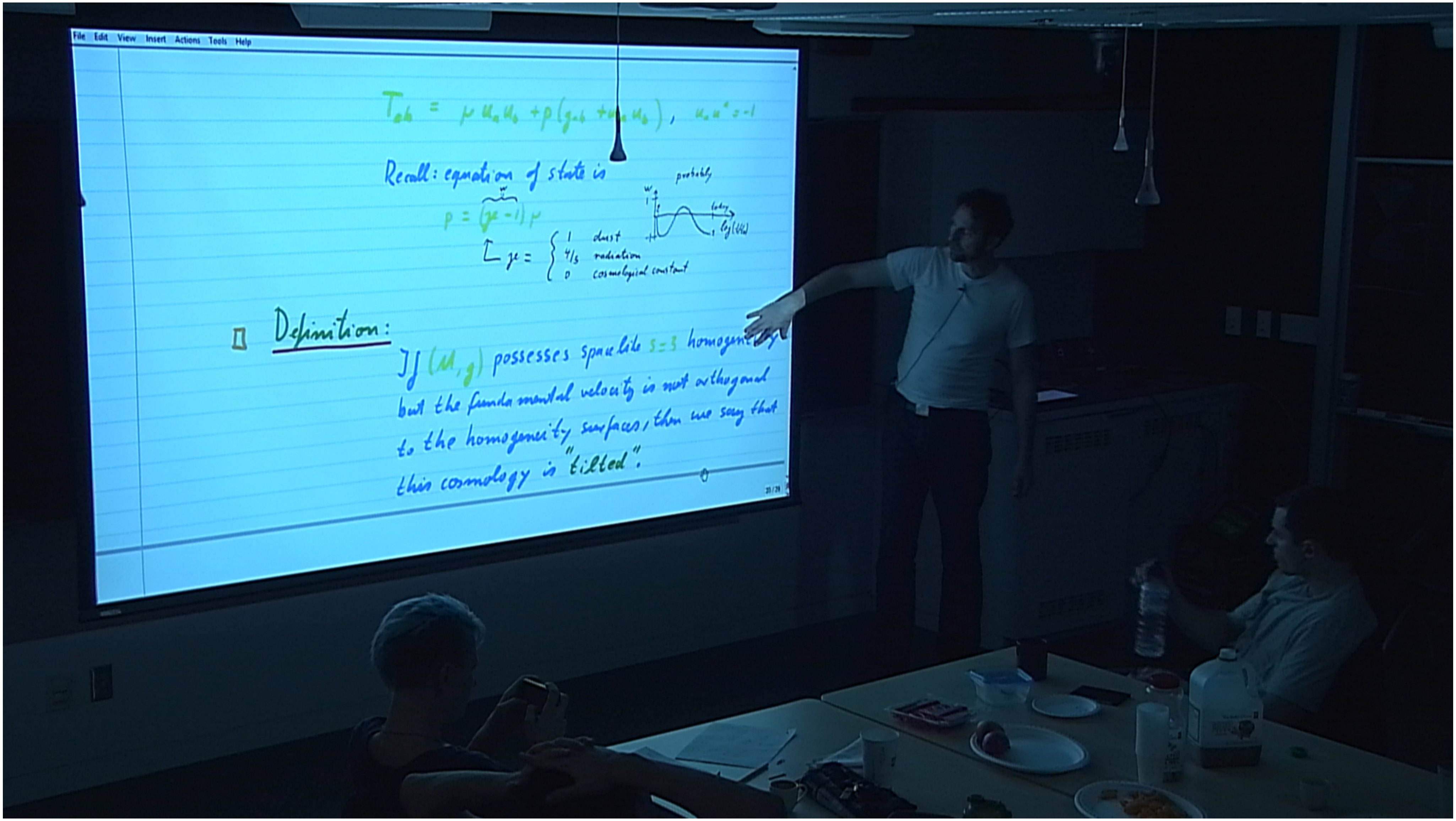
$$p = \overbrace{(\gamma - 1)}^{w} \mu$$

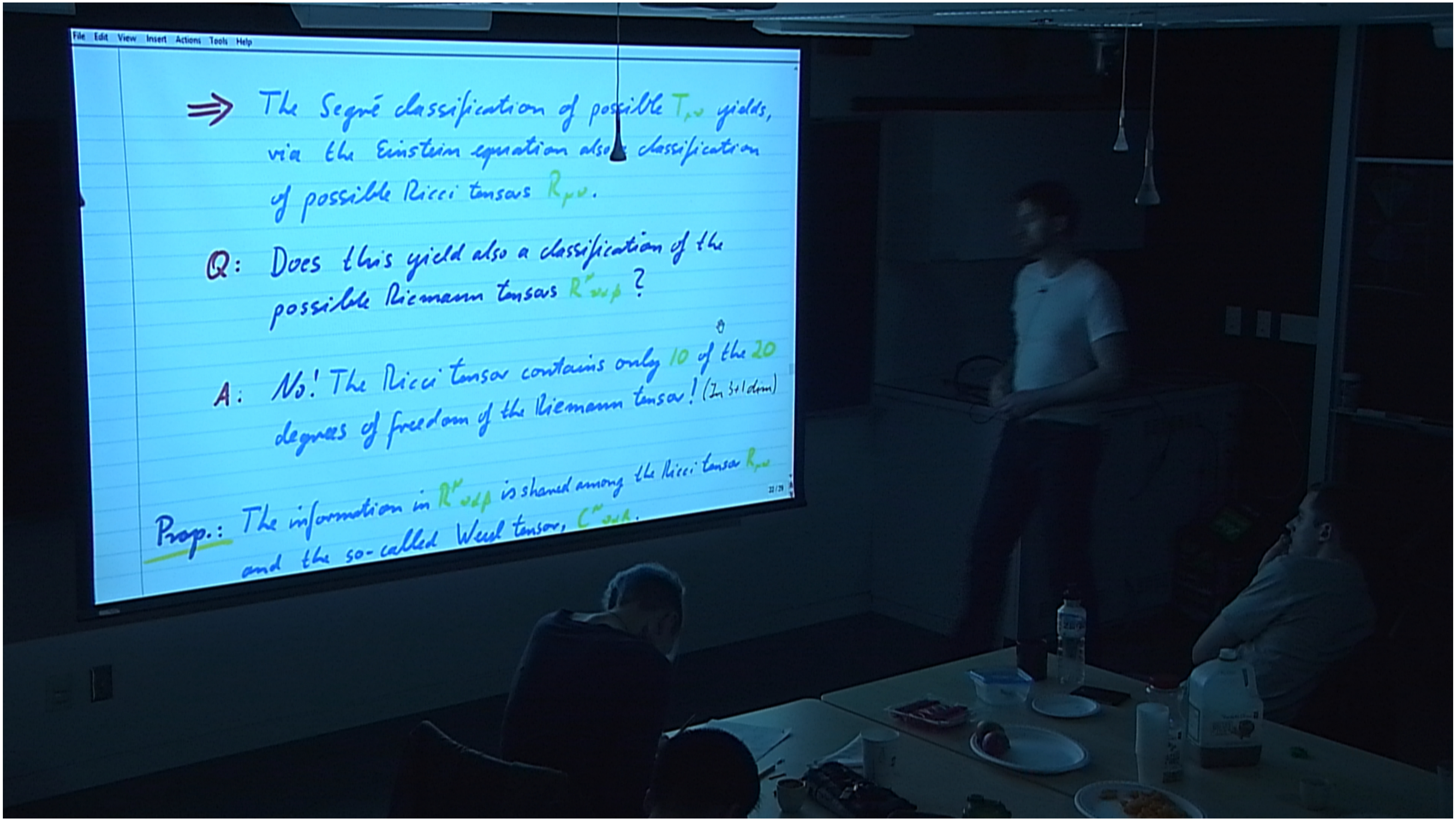
$$\gamma = \begin{cases} 1 & \text{dust} \\ 4/3 & \text{radiation} \\ 0 & \text{cosmological constant} \end{cases}$$



Definition:

If (M, g) possesses spacelike $s=3$ homogeneity but the fundamental velocity is not orthogonal


File Edit View Insert Actions Tools Help
Recall: equation of state is
probably
dust
radiation
cosmological constant
Definition:
possesses spacelike s=3
but the fundamental velocity is not orthogonal
to the homogeneity surfaces, then we say that
this cosmology is "tilted".


File Edit View Insert Actions Tools Help
⇒ The Segré classification of possible $T_{\mu\nu}$ yields, via the Einstein equation also a classification of possible Ricci tensors $R_{\mu\nu}$.
Q: Does this yield also a classification of the possible Riemann tensors $R^{\mu}{}_{\nu\alpha\beta}$?
A: No! The Ricci tensor contains only 10 of the 20 degrees of freedom of the Riemann tensor! (In 3+1 dim)
Prop.: The information in $R^{\mu}{}_{\nu\alpha\beta}$ is shared among the Ricci tensor $R_{\mu\nu}$ and the so-called Weyl tensor, $C^{\mu}{}_{\nu\alpha\beta}$.

File Edit View Insert Actions Tools Help

The Weyl tensor, $C^{am}{}_{sq}$:

$$C^{am}{}_{sq} := R^{am}{}_{sq} - \frac{1}{2}\left(g^a_s R^m_q + g^m_q R^a_s - g^m_s R^a_q - g^a_q R^m_s\right) + \frac{1}{6}\left(g^a_s g^m_q - g^a_q g^m_s\right) R$$

Notice: If $R^a{}_b$ and $C^{am}{}_{sq}$ are given, they determine $R^{am}{}_{sq}$ fully:

$$R^{am}{}_{sq} = C^{am}{}_{sq} + \frac{1}{2}\left(g^a_s R^m_q + g^m_q R^a_s - g^m_s R^a_q - g^a_q R^m_s\right) - \frac{1}{6}\left(g^a_s g^m_q - g^a_q g^m_s\right) R$$

$\leadsto$ $\underset{\uparrow \text{ 20 indep. components}}{R^{am}{}_{sq}}$ is expressed through $\underset{\uparrow \text{ 10 indep. comp.}}{C^{am}{}_{sq}}$ and $\underset{\uparrow \text{ 10 indep. comp.}}{R^{ab}}$

$\Rightarrow$ The Weyl tensor $C^{am}{}_{sq}$ indeed contains all that information

23 / 29

File Edit View Insert Actions Tools Help

$\Rightarrow$ $\boxed{C^{a\ m}{}_{sq} \text{ describes all that curvature which can exist even where there is no matter! (e.g.: gravity waves)}}$

also e.g. sun's gravity away from the sun in empty space

Proposition

- Assume (M, g) is a 3+1 dimensional Lorentzian manifold.
- Choose any smooth positive scalar function ϕ on M.
- Define $(M, \tilde{g})$ with the new metric $\tilde{g}$ obtained through the "conformal transformation":

$$g_{\mu\nu}(x) \longrightarrow \tilde{g}_{\mu\nu}(x) := \phi(x)\, g_{\mu\nu}(x)$$

Intuition:
Weyl curvature distorts (o o 0)
but only Ricci curvature
shrinks or expands overall: (◦OO)

Then: $\tilde{C}^{\mu}{}_{\nu\alpha\beta}(x) = C^{\mu}{}_{\nu\alpha\beta}(x) \quad \forall x \in M$ (Exercise: what would be a proof strategy?)

File Edit View Insert Actions Tools Help

- Consider the equivalence class of spacetimes $(\mathcal{M}, \tilde{g})$ that are conformally equivalent to Minkowski space:

$$g_{\mu\nu}(x) = \phi^2(x)\,\eta_{\mu\nu}$$

- Einstein and Fokker initially considered a theory in which the metric possesses only this conformal degree of freedom ϕ (to play rôle of Newton's gravitational potential).

Newton gravity does come out correctly as a limiting case!

- Then, $S' = \int_{\mathcal{M}} R\sqrt{-g}\,d^4x + \int_{\mathcal{M}} \mathcal{L}_{matter}\sqrt{-g}\,d^4x$ and $\frac{\delta S'}{\delta\phi} = 0$ yield:

$$R = 8\pi G\, T^{\mu}{}_{\mu}$$

In electromagnetism $T^{(EM)\mu}{}_{\mu} = 0$ i.e. EM fields would not gravitate.

No gravity waves here because $C^{ab}{}_{cd} = C^{ab\,(\text{Minkowski})}{}_{cd} = 0$

- Equivalence principle ok.
- Light bending & Mercury perihelion shift wr

John Klauder
is online

File Edit View Insert Actions Tools Help

Recall: via the Einstein equation the Segré classification implies a classification of properties of the Ricci tensor $R_{\mu\nu}$.

It remains to classify the Weyl tensor:

Petrov classification:

This is a classification of the Weyl tensor $C^{\mu\nu}{}_{\rho\varepsilon}$, which possesses the 10 remaining degrees of freedom of $R^{\alpha}{}_{\beta\gamma\delta}$.

- $C^{\mu\nu}{}_{\rho\varepsilon}$, just like the Riemann tensor, is antisymmetric in $\mu\leftrightarrow\nu$ and in $\rho\leftrightarrow\varepsilon$, and symmetric in $\mu\nu\leftrightarrow\rho\varepsilon$.

File Edit View Insert Actions Tools Help

Type I: Longitudinal gravitational waves

These waves cause a shear effect.

However, they decay fast: $\sim \frac{1}{r^2}$

Why? Gravitational waves, when small enough, travel with speed of light. Like light, they then cannot oscillate longitudinally.

Types II, III: Mixtures of the above.

Potential problem: (with symmetry assumptions):

29 / 29

File Edit View Insert Actions Tools Help

Potential problem: (with symmetry assumptions):

(E.g. recall that flatness in FL spacetimes is unstable)

- The so-obtained highly symmetric solutions, e.g. Friedman-Lemaître, may possess properties that are peculiar to high symmetry.

(E.g.: In Newtonian gravity, a slightly non-symmetric collapse of a star would not lead to a singularity but to a bounce - think figure skater.)

- **E.g.:** When a Friedmann-Lemaître solution, or a Schwarzschild solution exhibits a singularity: Is it due to symmetry, or realistic?

- Singularity theorems (see later) confirm the robustness under certain conditions (such as strong energy condition).

29 / 29

File Edit View Insert Actions Tools Help

(E.g. recall that flatness in FL spacetimes is unstable)

- The so-obtained highly symmetric solutions, e.g. Friedman-Lemaître, may possess properties that are peculiar to high symmetry.

(E.g.: In Newtonian gravity, a slightly non-symmetric collapse of a star would not lead to a singularity but to a bounce - think figure skater.)

- E.g.: When a Friedmann-Lemaître solution, or a Schwarzschild solution exhibits a singularity: Is it due to symmetry, or realistic?

- Singularity theorems (see later) confirm the robustness under certain conditions (such as strong energy condition).

→ More confidence in significance of the properties of highly symmetric solutions.

29 / 29