Title: AMATH 875/PHYS 786 - Fall 2015 - Lecture 22

Date: Nov 16, 2015 01:45 PM

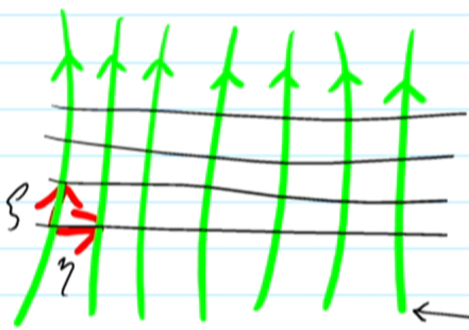
URL: http://pirsa.org/15110017

Abstract:

- We consider a one-parameter sub-family of these geodesics:

$$\gamma(\tau, s)$$

$\uparrow$ eigentime; $\uparrow$ parameter of family of neighboring geodesics.



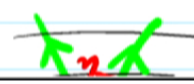
← a line of constant τ value

← a geodesic, i.e., a line of constant s value

- Then, we define the deviation vector to a neighboring geodesic:

$$\eta := \frac{d}{ds}$$

- The singularity theorem claims that this happened in the past:



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How does η change along a past-directed timelike geodesic with tangent ξ?

We showed:

$$\boxed{\xi^\mu \eta^\nu{}_{;\mu} = \eta^\alpha B^\nu{}_\alpha \quad \text{where} \quad B^\nu{}_\mu := \xi^\nu{}_{;\mu}}$$

$\Rightarrow$ Along the geodesic, ξ, the deviation vector η^μ changes its direction and length by $B^\nu{}_\mu \eta^\mu$.

The tensor $B^\nu{}_\mu$ can be decomposed covariantly and uniquely:

Symmetric and trace = 0

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$\Rightarrow$ Along the geodesic, ξ, the deviation vector η^μ changes its direction and length by $B^\nu{}_\mu \eta^\mu$.

The tensor $B^\nu{}_\mu$ can be decomposed covariantly and uniquely:

$$B_{\mu\nu} = \underbrace{\omega_{\mu\nu}}_{\text{antisymmetric}} + \overbrace{\sigma_{\mu\nu}}^{\text{Symmetric and trace}=0} + \underbrace{t_{\mu\nu}}_{\text{rest}}$$

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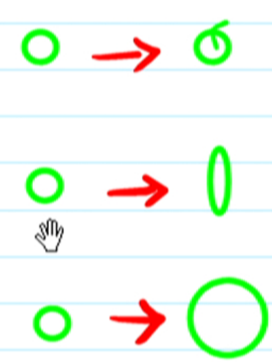
Explicitly:

Volume preserving →

$$\omega_{\mu\nu} = \frac{1}{2}\left(B_{\mu\nu} - B_{\nu\mu}\right)$$ Twist:

$$\sigma_{\mu\nu} = \frac{1}{2}\left(B_{\mu\nu} + B_{\nu\mu}\right) - \frac{1}{3}\theta h_{\mu\nu}$$ Shear:

Volume changing: $$t_{\mu\nu} = \frac{1}{3}\theta h_{\mu\nu}$$ Expansion:



Here, we defined: $\theta := B^{\mu\nu} g_{\mu\nu}$ and $h_{\mu\nu} := g_{\mu\nu} + \xi_\mu \xi_\nu$

I.e., the Expansion, θ, is the trace of B, which we showed is also equal to the magnitude of the spatial part of B: $\theta = B^{\mu\nu} h_{\mu\nu}$.

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Key question: $\boxed{\text{What is the dynamics of } \theta \text{ ?}}$

The Raychaudhuri equation

For the derivation, we will use:

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The Raychaudhuri equation

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A) Definition of B is: $B_{\mu\nu} := \xi_{\mu;\nu}$

B) The curvature tensor obeys the Ricci equation:

$$\boxed{\xi^a{}_{;bc} - \xi^a{}_{;cb} = R^a{}_{bcd}\,\xi^d}$$

C) ξ is tangent to a geodesic, i.e., it obeys: $\nabla_\xi \xi = 0$

i.e.: $0 = \nabla_{\xi} \xi^b e_b = \xi^a \nabla_a \xi^b e_b = \xi^a \xi^b{}_{;a} e_b$

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True for all e_a, thus: $\boxed{\xi^a \xi^b{}_{;a} = 0}$

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Now calculate the rate of change of B along the geodesic:

$$\underbrace{\xi^c B_{ab;c}}_{\nabla_\xi B} \overset{(A)}{=} \xi^c \xi_{a;bc}$$

$$\overset{(B)}{=} \xi^c \xi_{a;cb} + \xi^c R_{abcd} \xi^d$$

$$\underset{\text{rule}}{\overset{\text{Leibnitz}}{=}} \underbrace{\left(\xi^c \xi_{a;c}\right)_{;b}}_{=0} - \xi^c{}_{;b}\, \xi_{a;c} + R_{abcd}\, \xi^c \xi^d$$

$$\overset{(C)}{=} -\xi^c{}_{;b}\, \xi_{a;c} + R_{abcd}\, \xi^c \xi^d$$

$$\overset{(A)}{=} -B^c{}_b B_{\ldots} + R_{\ldots} \xi^c \xi^d$$

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$$\xi^c B_{ab;c} = \xi^c \xi_{a;bc}$$

$\nabla_\xi B$

$$\overset{\textbf{(B)}}{=} \xi^c \xi_{a;cb} + \xi^c R_{abcd} \xi^d$$

$$\overset{\text{Leibnitz}}{\underset{\text{rule}}{=}} \underbrace{(\xi^c \xi_{a;c})_{;b}}_{=0} - \xi^c{}_{;b} \xi_{a;c} + R_{abcd} \xi^c \xi^d$$

$$\overset{\textbf{(C)}}{=} - \xi^c{}_{;b} \xi_{a;c} + R_{abcd} \xi^c \xi^d$$

$$\overset{\textbf{(A)}}{=} - B^c{}_b B_{ac} + R_{abcd} \xi^c \xi^d$$

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In summary, we derived:

$$\xi^c B_{ab;c} = -B^c{}_b B_{ac} + R_{abcd}\xi^c\xi^d \qquad (*)$$

The trace of (*) will be the Raychaudhuri equation.

But first, we recall:

$\square$ $\xi = \frac{d}{d\tau}$

$\square$ $\mathrm{Tr}\, B = B_{\mu\nu} g^{\mu\nu} = \theta$

$\Rightarrow$ Trace(LHS) of (*) reads $\frac{d}{d\tau}\theta$!

Now on the RHS of (*) use the decomposition

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Now on the RHS of (*) use the decomposition

$B_{\mu\nu} = \omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{3}\theta h_{\mu\nu}$ to express $B^c{}_b B_{ac}$:

$$B^c{}_b B_{ac} = \omega^c{}_b \left(\underline{\omega_{ac}} + \sigma_{ac} + \frac{1}{3}\theta h_{ac} \right)$$

$$+ \sigma^c{}_b \left(\omega_{ac} + \underline{\sigma_{ac}} + \frac{1}{3}\theta h_{ac} \right)$$

$$+ \frac{1}{3}\theta h^c{}_b \left(\omega_{ac} + \sigma_{ac} + \underline{\frac{1}{3}\theta h_{ac}} \right)$$

When taking the trace, $g^{ab}B^c{}_b B_{ac}$, only the diagonal terms survive:

Exercise: show it is 3

$$Tr(BB) = g^{ab}B^c{}_b B_{ac} = \omega_{ab}\omega^{ab} + \sigma_{ab}\sigma^{ab} + \frac{1}{9}\theta^2 \overbrace{h_{ab}h^{ab}}^{=3}$$

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The Raychaudhuri equation is then the trace of Eq. (*):

recall: Ricci tensor is $R_{cd} = R_{cda}{}^{a}$

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$$+\frac{1}{3}\theta h^a{}_b\left(\omega_{ac}+\sigma_{ac}+\underline{\frac{1}{3}\theta h_{ac}}\right)$$

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$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \underbrace{\sigma_{ab}\sigma^{ab}}_{\text{always positive}} - \underbrace{\omega_{ab}\omega^{ab}}_{\text{always positive (and vanishes if choose congruence }\perp\Sigma)} - \underbrace{R_{cd}\xi^c\xi^d}_{\text{pos. or neg?}}$$

?

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Thus, assuming the strong energy condition:

$$\frac{d\theta}{d\tau} + \frac{1}{3}\theta^2 \leq 0$$

i.e., $-\frac{1}{\theta^2}\frac{d\theta}{d\tau} - \frac{1}{3} \geq 0$

i.e., $\boxed{\frac{d}{d\tau}\theta^{-1} \geq \frac{1}{3}}$ (+)

Consider the cases when the geodesics are initially all either

a.) diverging, i.e., $\theta(\tau_0) > 0$ (expanding universe) or

b.) converging, i.e., $\theta(\tau_0) < 0$ (contracting universe)

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i.e., $\boxed{\frac{d}{d\tau}\,\theta^{-1} \geqslant \frac{1}{3}}$ $(\dagger)$

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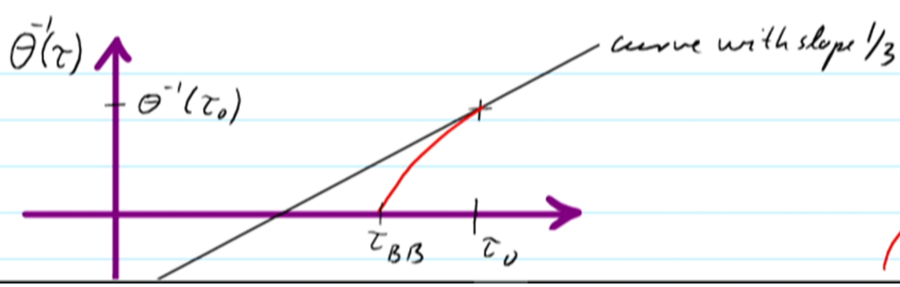
(This is reformulating the theorem's assumption that the extrinsic curvature (i.e. the expansion or contraction at some time exceeds a certain finite value everywhere)

a.)



τ_0 = e.g. today

= curve $\theta^{-1}(\tau)$ of slope $\geqslant \frac{1}{3}$

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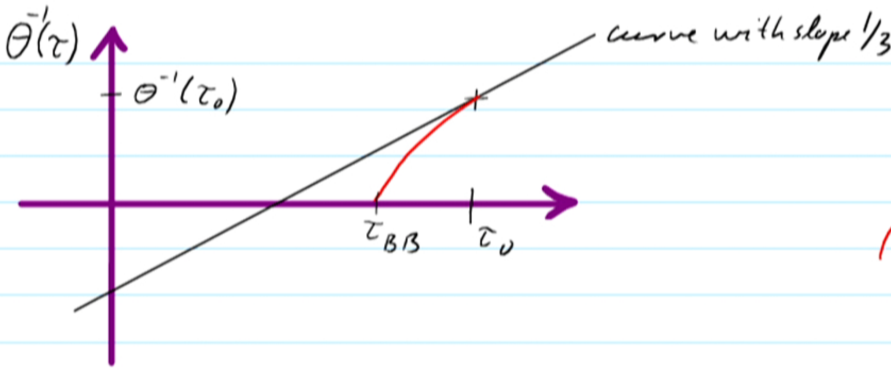

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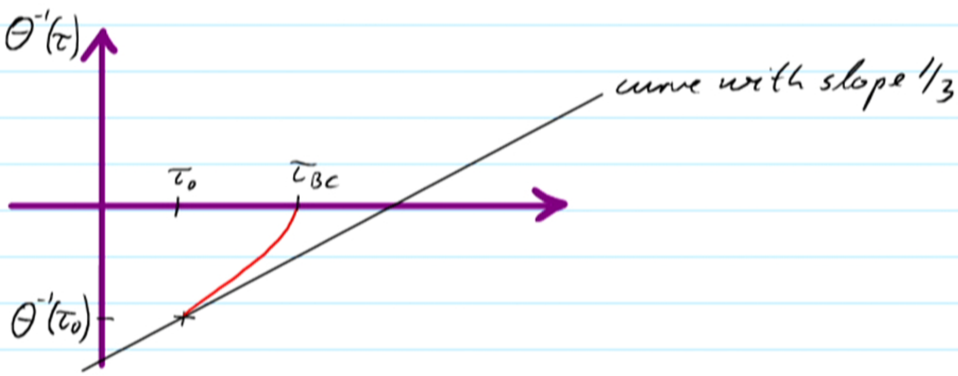
We see that $\theta^{-1}(\tau)$ must have hit $\theta^{-1}(\tau)=0$ at a finite time τ_{BB} (Big Bang).

b)



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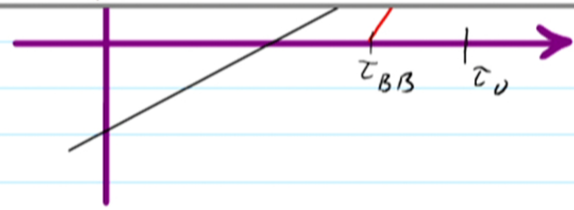

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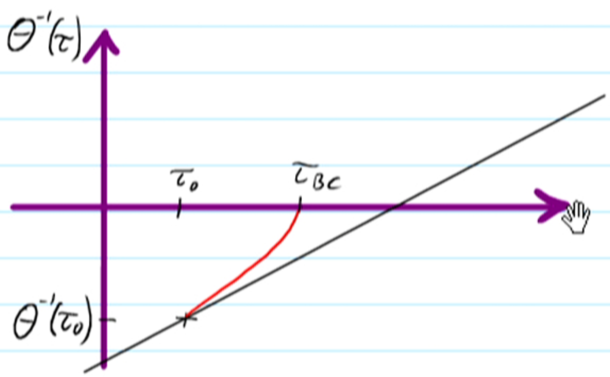

τ_0 = e.g. today

$=$ curve of slope $\geq 1/3$

We see that $\Theta^{-1}(\tau)$ will hit $\Theta^{-1}(\tau)=0$ at a finite time τ_{BC} (Big Crunch)

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(Big Crunch)

Conclusion:

Eq. (+) implies that $\theta^{-1}(\tau)$ must go through 0, i.e.:

a.) for sufficiently early τ, have $\theta \to +\infty$, i.e.: Big Bang

b.) for sufficiently late τ, have $\theta \to -\infty$, i.e.: Big Crunch

Note: This type of reasoning leads also to further cosmological singularity theorems.

E.g., another cosmological singularity theorem does not

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Note: This type of reasoning leads also to further cosmological singularity theorems.

E.g., another cosmological singularity theorem does not assume global hyperbolicity, and its conclusion is weaker:

There is at least one incomplete timelike geodesic.

Results, e.g., regarding types of cosmic singularity?

- Assume a set of symmetries of matter and spacetime has been chosen.

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Results, e.g., regarding types of cosmic singularity?

- Assume a set of symmetries of matter and spacetime has been chosen.
- Assume an exact solution or at least its asymptotic properties at early times have been found.
- Assume, we choose a timelike congruence e.g. of geodesics.

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The Hubble functions:

In particular, we can see how the expansion or contraction of the universe behaves dynamically e.g. when the condition of perfect isotropy is relaxed:

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- Recall:

  The expansion in one direction can be say enhanced by shear, as long as shear shrinks other directions.

- Definition:

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- Recall:

The expansion in one direction can be say enhanced by shear, as long as shear shrinks other directions.

- Definition:

We define a rate of expansion tensor that includes shear:

shear — projector $\perp$ to the timelike u-field

$$\Theta_{\mu\nu} := \sigma_{\mu\nu} + \frac{1}{3}\,\theta\, h_{\mu\nu}$$

symmetric part of $B_{\mu\nu}$ — expansion scalar.

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symmetric part of $B_{\mu\nu}$

$\Theta_{\mu\nu}$ is fully spacelike and symmetric $\Rightarrow$ $\Theta_{\mu\nu}$ can be diagonalized in suitable ON frame $\{e_0, e_1, e_2, e_3\}$:

$$\Theta_{\mu\nu} = \begin{pmatrix} 0 & & & 0 \\ & \theta_1 & & \\ & & \theta_2 & \\ 0 & & & \theta_3 \end{pmatrix}$$

3 space-like directions.

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$$\theta_{\mu\nu} = \begin{pmatrix} 0 & & & \\ & \theta_1 & & 0 \\ 0 & & \theta_2 & \\ & & & \theta_3 \end{pmatrix}$$ 3 space-like directions.

with the traditional expansion being the trace (because $\sigma_{\mu\nu}$ is traceless):

$$\theta = \theta_1 + \theta_2 + \theta_3$$

why $\frac{1}{3}$? Recall that $\underbrace{Tr(h_{\mu\nu})}_{\Rightarrow \text{ is not quite projector}} = 3$

▫ Definition: $H_i := \frac{1}{3}\theta_i$ Local Hubble expansion function in direction e_i.

$H := \frac{1}{3}\theta$ Overall local Hubble expansion function.

▫ Definition :

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Definition:

We use H_i, H to define local directional and general scale factors l_i, l:

The l_i, l are defined as the solutions to:

$$\frac{\dot{l}_i}{l_i} = H_i$$

$$\frac{\dot{l}}{l} = H$$

Here, the time derivative is defined as:

recall: u is timelike.

$$\dot{l} = u(l) = u^\mu \frac{\partial}{\partial x^\mu} l$$

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What behavior can occur in the far past?

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Full set of cases not yet known.

But:

Explicit examples are known where e.g.:

- All $\ell_i \to 0$ as in FL cosmologies
- $\ell_1, \ell_2 \to 0$, $\ell_3 \to \infty$ "cigar singularity"
- $\ell_1, \ell_2 \to 0$, $\ell_3 \to$ const "barrel singularity"
- $\ell_1, \ell_2 \to$ const, $\ell_3 \to 0$ "pancake singularity"

- <u>Note:</u> For homogeneous, isotropic FL models, H is the regular Hubble parameter and ℓ is

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Regarding black holes:

Recall: Singularity theorems suitable for black holes involve the concept and assumption of a trapped surface:

Def:

- Let Σ be a spacelike hypersurface. (Note: Σ is 3-dimensional)
- Let $T \subset \Sigma$ be a compact, 2-dimensional smooth spacelike submanifold of Σ. Consider the ingoing and the outgoing future-directed null geodesics that are orthogonal to T.
- If all these geodesics possess negative expansion, $\theta < 0$, then T is called a trapped surface

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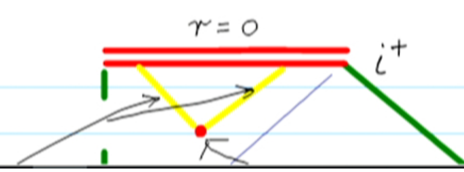
- If all these geodesics possess negative expansion, $\theta < 0$, then T is called a trapped surface.

Examples: All concentric spheres inside a Schwarzschild black hole.



Note: For $r < 2m$, r is timelike and t is spacelike.

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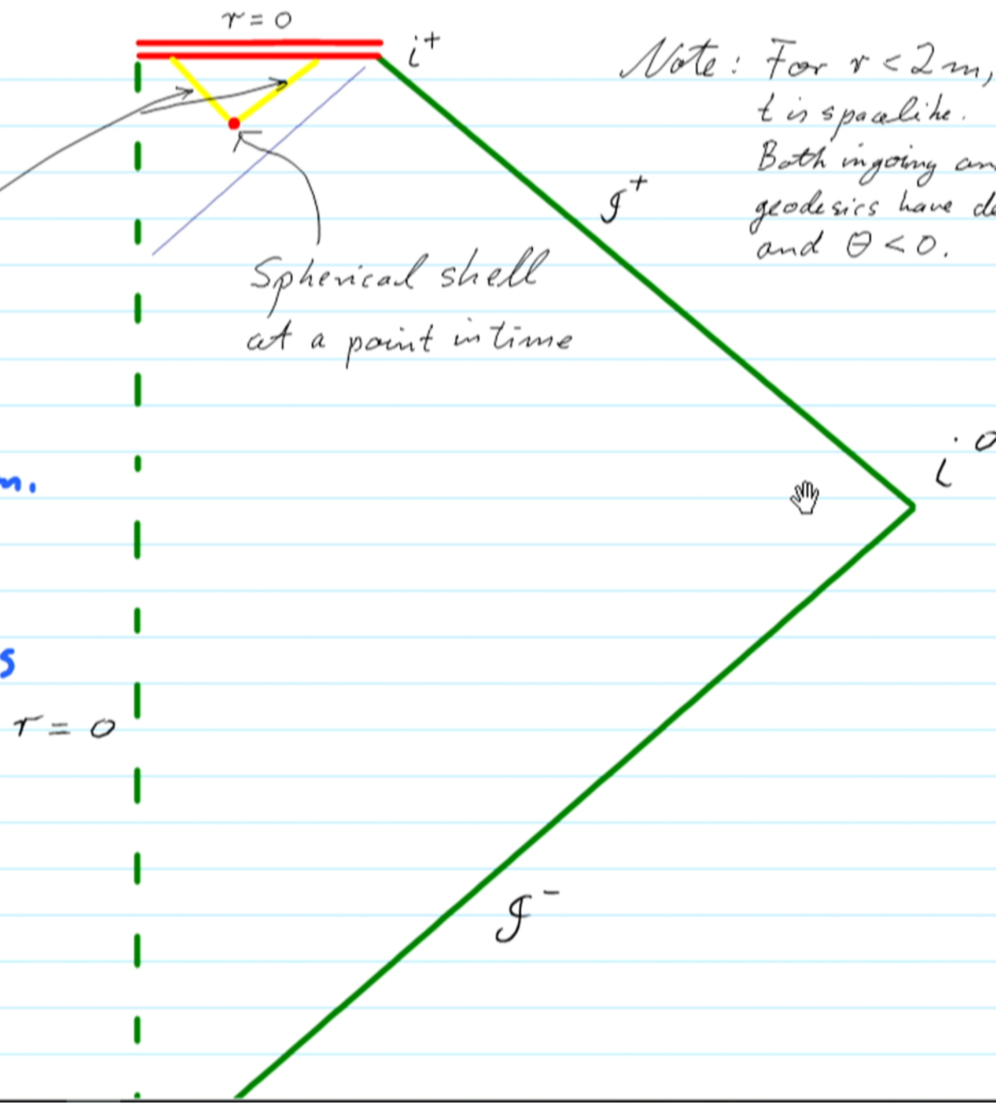

Note: For $r < 2m$, r is timelike and t is spacelike.
Both ingoing and outgoing null geodesics have decreasing r, and $\Theta < 0$.

The in- and outgoing null geodesics both have negative expansion. Can't see it here b/c the neighboring geodesics are neighbors in the suppressed angular directions.

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Def: Let Σ be a spacelike hypersurface.

Then, the (3-dim. spacelike) union, $\mathcal{T}$, of all trapped surfaces $T \subset \Sigma$ is called the **trapped region** of Σ.

Def: The boundary $\partial\mathcal{T} \subset \Sigma$ is called the **apparent horizon** of the spacelike hypersurface Σ.

Note: $\partial\mathcal{T}$ is 2-dimensional and spacelike.

Def: If we foliate spacetime into spacelike hypersurfaces

$$\Sigma_\alpha, \alpha \in I \subset \mathbb{R}$$

each with its apparent horizon, $\mathcal{T}_\alpha$, then their union

$$\mathcal{A} := \bigcup \mathcal{T}_\alpha$$

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each with its apparent horizon, $\mathcal{T}_\alpha$, then their union

$$\mathcal{A} := \bigcup_\alpha \mathcal{T}_\alpha$$

is called the Trapping horizon of the spacetime.

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Remarks:

- To check for the existence of an event horizon

  $\mathscr{J}^-$ (worldline to i^+)

  in principle requires knowledge of the full future.

- But one can check for the existence of an apparent horizon in any spacelike hypersurface by calculating the expansions only at that time!

- For static Schwarzschild black holes the event and apparent horizons coincide.

- For general black holes, apparent horizons are on or inside

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[illegible]

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