Title: AMATH 875/PHYS 786 - Fall 2015 - Lecture 16

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Abstract:

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GR for Cosmology, Achim Kempf, Fall 2015, Lecture 16

Note Title

Fridmann-Lemaître cosmological solutions

Experimental evidence:

Hubble, Humason 1929

- The universe is and has been expanding.
- It appears to be spatially essentially isotropic and homogeneous on scales longer than a few (3-4) billion light years.

(see e.g. Sloan Digital Sky Survey (SDSS) at www.sdss.org)

Idealizing models:

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Idealizing models:

- Assume perfect spatial isotropy and homogeneity:
- ⇝ "Friedmann & Lemaître" (later Robertson & Walker) spacetimes

Concretely:

We assume we can model spacetime as a manifold (M, g) with:

$$M = J \times \Sigma$$

$$g = -dt^2 + a^2(t)\,\bar{g}$$

In the basis $\{dx^\mu\}$ which comes with the coordinate system.

(we will later use an ON frame so that $g_{\mu\nu} = \eta_{\mu\nu}$)

Here:

- J is an interval, $J \subset \mathbb{R}$, and $t \in J$ is called

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In the basis $\{dx^\mu\}$ which comes with the coordinate system.

Here:

- J is an interval, $J \subset \mathbb{R}$, and $t \in J$ is called "cosmic time". $a(t)$ is called the "scale factor".

- $(\Sigma, \bar{g})$ is a fully isotropic & homogeneous Riemannian manifold, i.e., a manifold of constant curvature, providing a 3-dim. surface of

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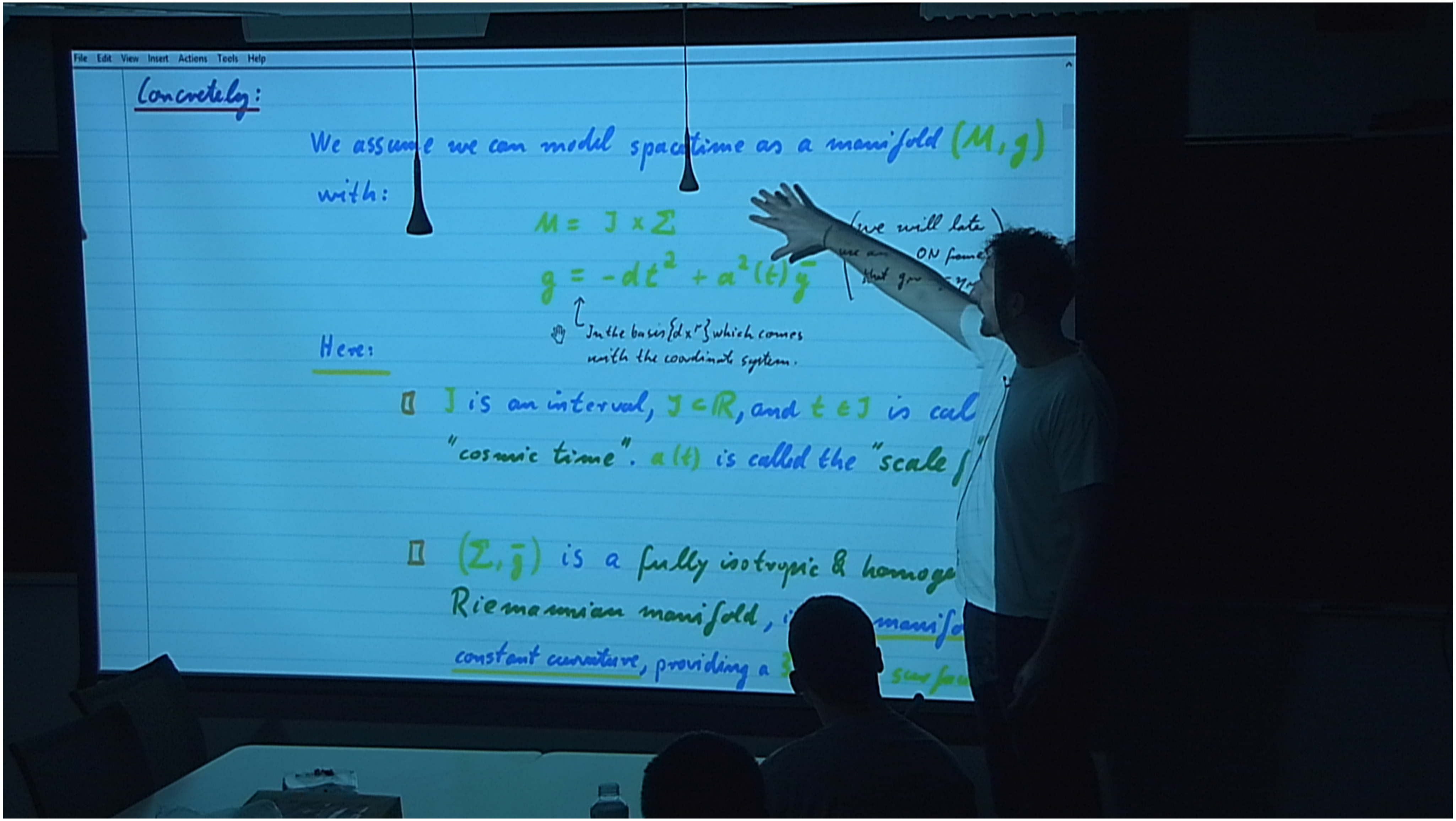
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Concretely:
We assume we can model spacetime as a manifold (M, g)
with:
M = I × Σ
g = −dt² + a²(t) ḡ
In the basis {dx^μ} which comes with the coordinate system.
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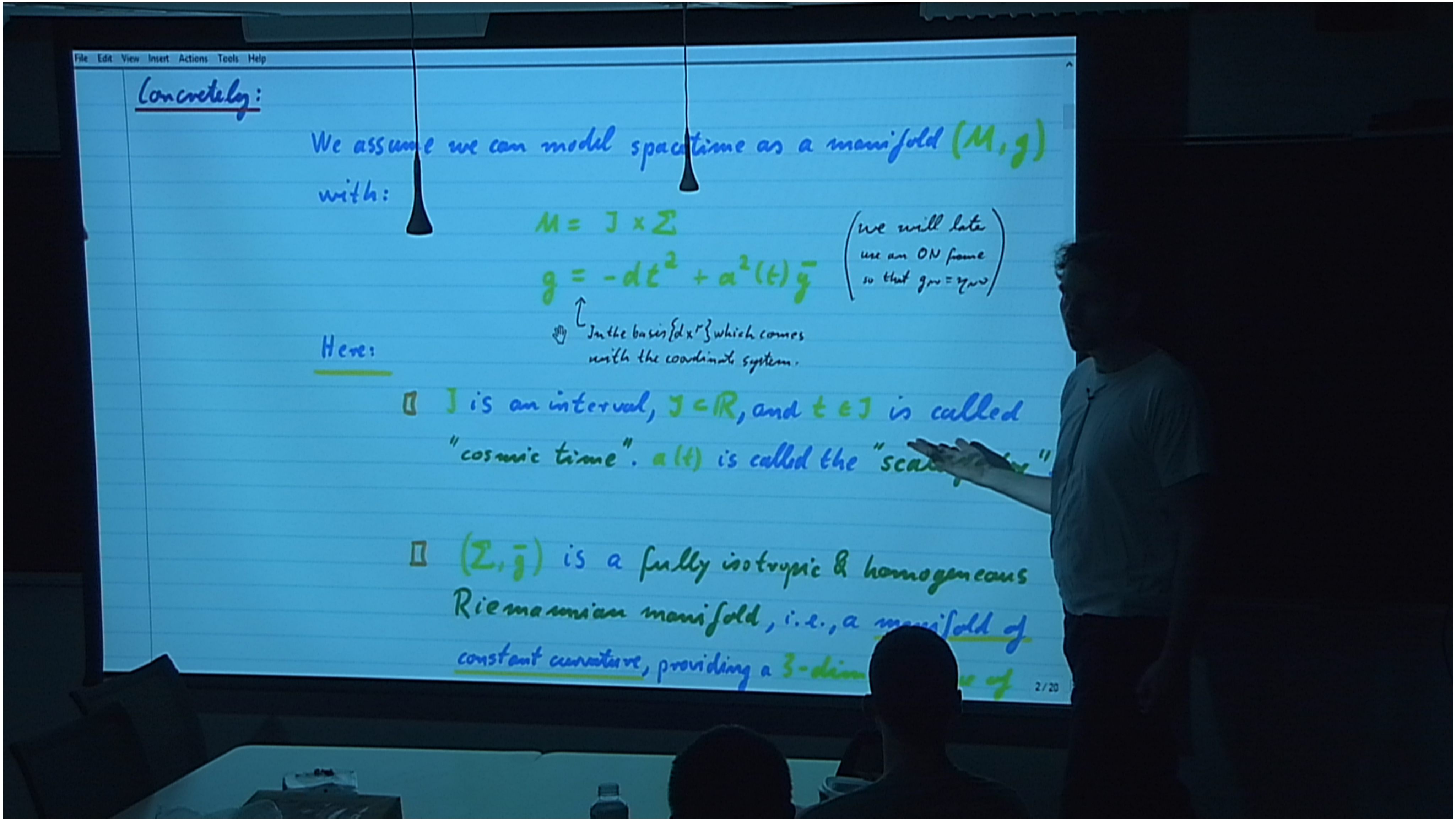
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⇒ Using a "Triad" $\{\bar{\theta}^i\}$:

(ON bases of $T_p(\Sigma)$, $\forall p$)

$$\boxed{\bar{\Omega}_{ij} \underset{\text{Def.}}{\overset{\text{by}}{=}} \frac{1}{2} \bar{R}_{ijkl} \, \bar{\theta}^k \wedge \bar{\theta}^l \underset{(*)}{\overset{\text{use}}{=}} K \, \bar{\theta}_i \wedge \bar{\theta}_j}$$

Note: 4-dim pseudo-Riemannian manifolds with constant curvature are called "de Sitter universes".

Rôle of the signature of K:

$K > 0$: ⇒ Σ is a 3-dim. sphere (that can be embedded e.g. in a 4 dim euclidean (i.e. flat) space: closed universe

$K = 0$: ⇒ Σ is euclidean $\mathbb{R}^3$. flat, infinite universe

$K < 0$ ⇒ Σ is a 3-dim. "pseudo sphere". The constant negative curvature means it is everywhere "like a saddle". These Σ also possess ∞ volume.

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Determine the 4-connection $\omega^\mu{}_\nu$: (in spatially isotropic & homogeneous case)

Strategy: Calculate $d\theta^i$ in two ways:

1.) $$d\theta^i = d(a\bar\theta^i) = (da)\wedge\bar\theta^i + \underbrace{a\, d\bar\theta^i}_{\text{use Eq. } \Sigma 1}$$

$$= \left(\frac{da}{dt}dt\right)\wedge\bar\theta^i - a\,\bar\omega^i{}_j\wedge\bar\theta^j$$

(use $a\bar\theta^i = \theta^i$) $\Rightarrow$ $$= \dot a \overset{dt}{\overset{\shortparallel}{\theta^0}}\wedge\bar\theta^i - \bar\omega^i{}_j\wedge\theta^j$$

(use $\bar\theta^i = \frac{1}{a}\theta^i$ and $\theta^i\wedge\theta^0 = -\theta^0\wedge\theta^i$) $\Rightarrow$ $$= -\frac{\dot a}{a}\theta^i\wedge\theta^0 - \bar\omega^i{}_j\wedge\theta^j \qquad (A)$$

2.) $$d\theta^i \overset{(M1)}{=} -\omega^i{}_\nu\wedge\theta^\nu = -\omega^i{}_0\wedge\theta^0 - \omega^i{}_j\wedge\theta^j \qquad (B)$$

What is $\omega^0{}_0$? Recall

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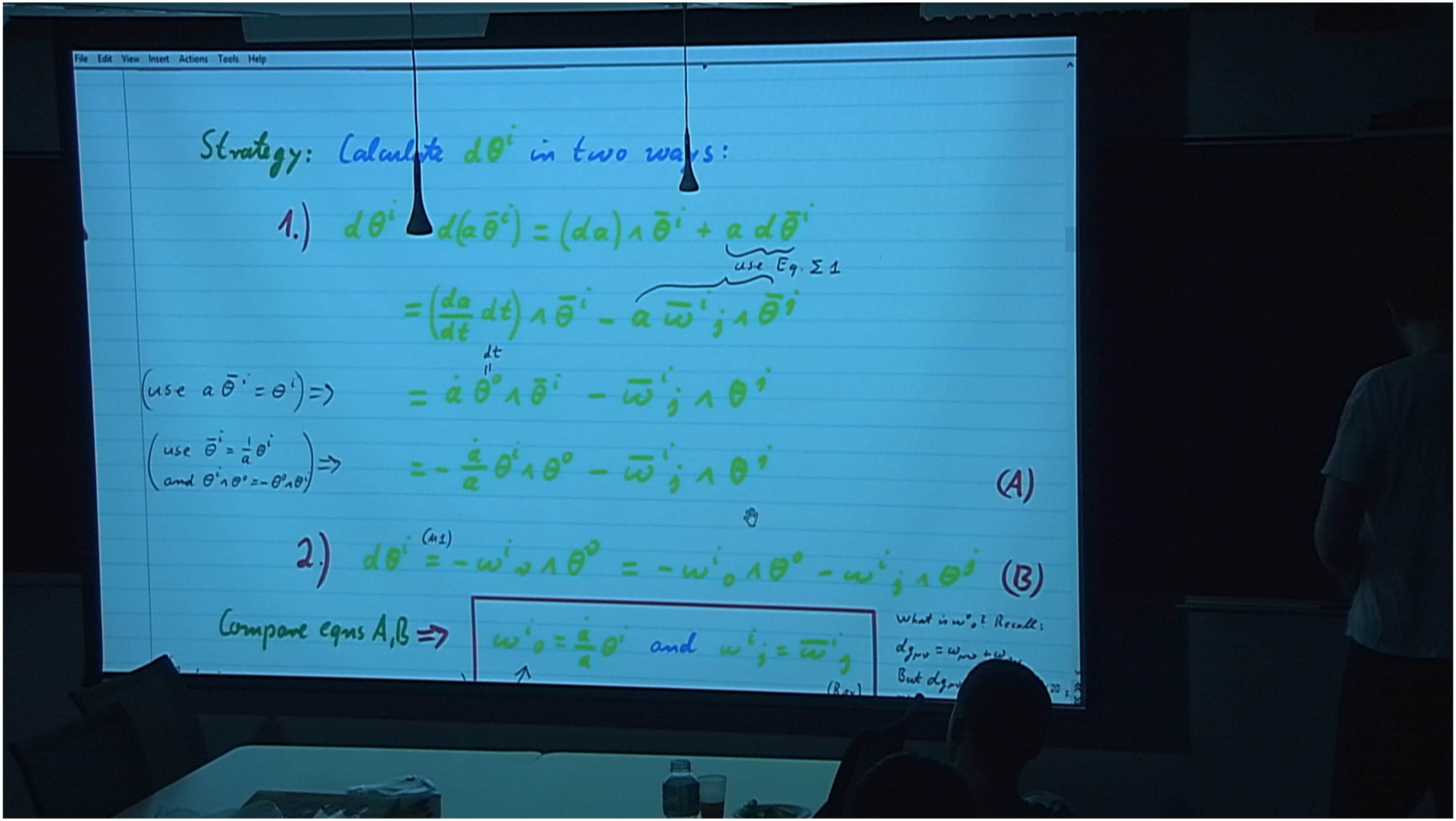
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Compare eqns A,B ⇒ $\boxed{\omega^i{}_0 = \frac{\dot a}{a}\theta^i \text{ and } \omega^i{}_j = \bar\omega^i{}_j}$ (R_{ex})
What is $\omega^0{}_i$? Recall:
$dg_{\mu\nu} = \omega_{\mu\nu} + \omega$
But $dg_{\mu\nu}$

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The Cartan structure equations express the torsion and curvature forms in terms of the connection form.
(2. eqn: $\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$)

$$d\theta^i + \omega^i{}_j \wedge \theta^j = 0 \qquad (\Sigma 1)$$

* 1st structure equation on M: $(\mu, \nu = 0, 1, 2, 3)$

$$d\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu = 0 \qquad (M1)$$

Determine the 4-connection $\omega^\mu{}_\nu$: (in spatially isotropic & homogenous case)

Strategy: Calculate $d\theta^i$ in two ways:

1.) $d\theta^i = d(a\bar{\theta}^i) = (da) \wedge \bar{\theta}^i + a\, d\bar{\theta}^i$

use Eq. $\Sigma 1$

$$= \left(\frac{da}{dt}\, dt\right) \wedge \bar{\theta}^i - a\, \bar{\omega}^i{}_j \wedge \bar{\theta}^j$$

$\frac{dt}{}$

(use $a\bar{\theta}^i = \theta^i$) $\Rightarrow$

$$= \dot{a}\, \theta^0 \wedge \bar{\theta}^i = \bar{\omega}^i{}_j \wedge \theta^j$$

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Calculate the Einstein tensor:

Recall:

$$\Omega_{\mu\nu} = \frac{1}{2} R_{\mu\nu\rho\varepsilon}\, \theta^{\rho} \wedge \theta^{\varepsilon}$$

$\Rightarrow$ We can read off $R_{\mu\nu\rho\varepsilon}$.

$\Rightarrow$ We obtain the Ricci tensor $R_{\mu\nu}$ and the curvature scalar R.

$\Rightarrow$ We obtain the Einstein tensor:

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⇒ We obtain the Ricci tensor $R_{\mu\nu}$ and the curvature scalar R.

⇒ We obtain the Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

Result:

$$\boxed{\begin{aligned} G_{00} &= 3\left(\frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right) \\ G_{ii} &= -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{K}{a^2} \\ G_{\mu\nu} &= 0 \quad \text{for } \mu \neq \nu \end{aligned}}$$

Exercise: verify

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Exercise: verify

i.e., $G_{\mu\nu}$ is diagonal in this frame.

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The energy-momentum tensor:

- From $G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$

  we obtain that $T_{\mu\nu}$ must also be diagonal.

- We identify the diagonal entries as usual as matter energy density ρ, matter pressure p and cosmological constant Λ:

(Why this factor here? Because Λ was traditionally put on the LHS, with the curvature)

Any diagonal $T_{\mu\nu}$ can be expanded uniquely this way.

$$T_{\mu\nu} = \begin{pmatrix} \rho & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{pmatrix} - \frac{1}{8\pi G}\Lambda \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

$\Rightarrow$ The only nontrivial dynamics of matter is here its

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$\Rightarrow$ The only nontrivial dynamics of matter is here its equation of state:

$$\boxed{\rho = \rho(p) \quad \text{or} \quad p = p(\rho)}\ !$$

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What kind of matter causes such a $T_{\mu\nu}$?

Proposition:

The $T_{\mu\nu}$ of any F.L. spacetime is always of the form of that of a perfect fluid.

* The matter doesn't have to be fluid - it could also be e.g. a suitable quantum field.

* But the high symmetry of a F.L. spacetime requires that the matter's $T_{\mu\nu}$ matches that of a perfect fluid

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The Einstein equation:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

now consists of merely 2 equations: Exercise: verify

$G_{00} = 8\pi G\, T_{00} \Rightarrow$

$$\boxed{3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) = 8\pi G \rho + \Lambda}$$

"Friedmann equation" (A)

$G_{ii} = 8\pi G\, T_{ii} \Rightarrow$

$$\boxed{-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} = 8\pi G p - \Lambda}$$

(B)

- Notice that Λ contributes
  - positively to the energy but

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Observation: K/a^2 occurs in (A) and (B), i.e., we can eliminate it:

$$-\frac{1}{2}a\left(\text{Eqn}(B) + \frac{1}{3}\text{Eqn}(A)\right) \text{ yields:}$$

$$\ddot{a} = -\frac{1}{2}a\, 8\pi G\left(\frac{\rho}{3}+p\right) - \frac{1}{2}a\Lambda\left(-1+\frac{1}{3}\right)$$

$$\Rightarrow \boxed{\ddot{a} = -\frac{4\pi G a}{3}(\rho + 3p) + \frac{1}{3}a\Lambda}$$

Thus for all k: For ordinary matter must have deceleration, i.e., $\ddot{a} < 0$, but a positive cosm. constant Λ

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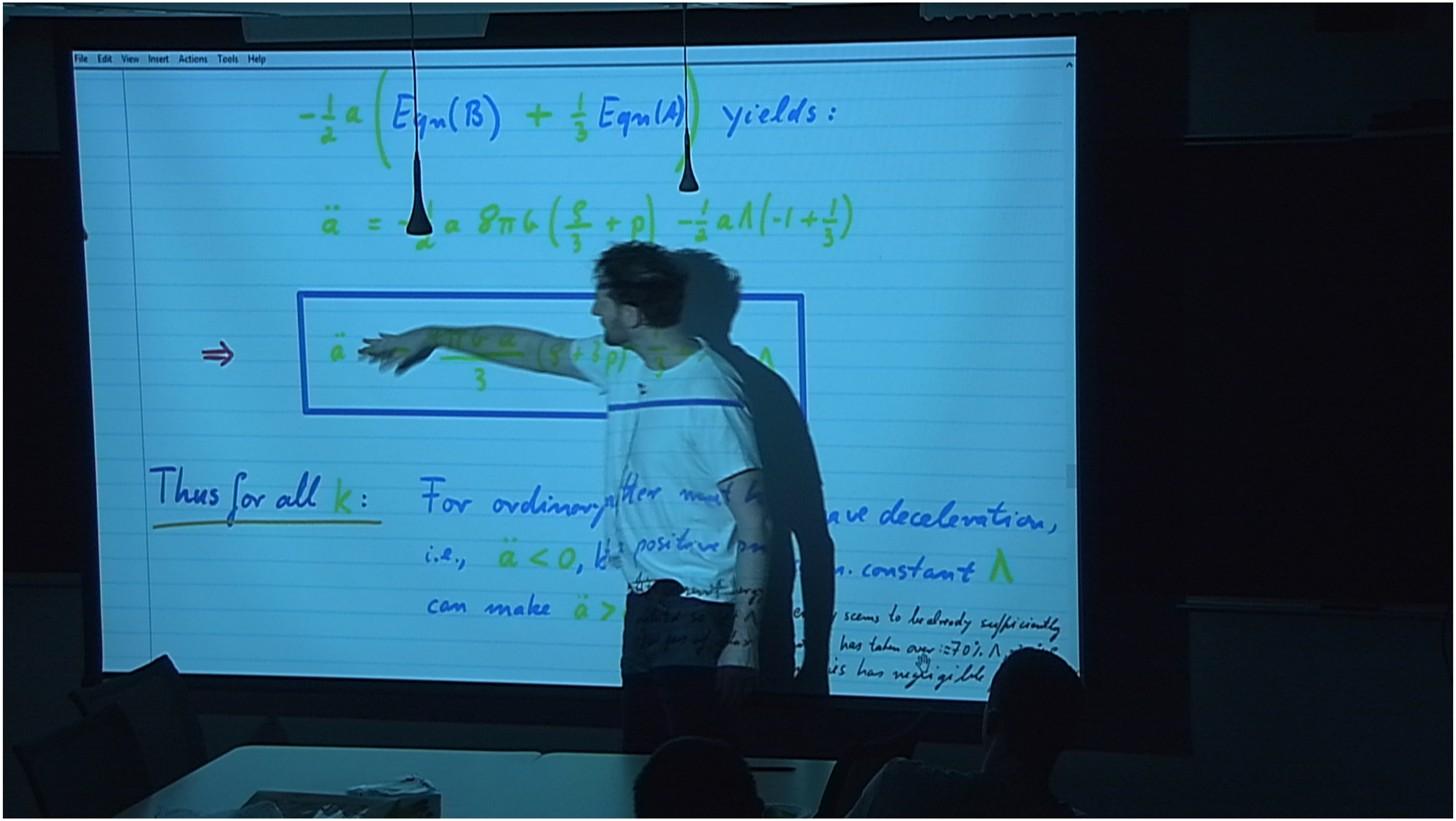
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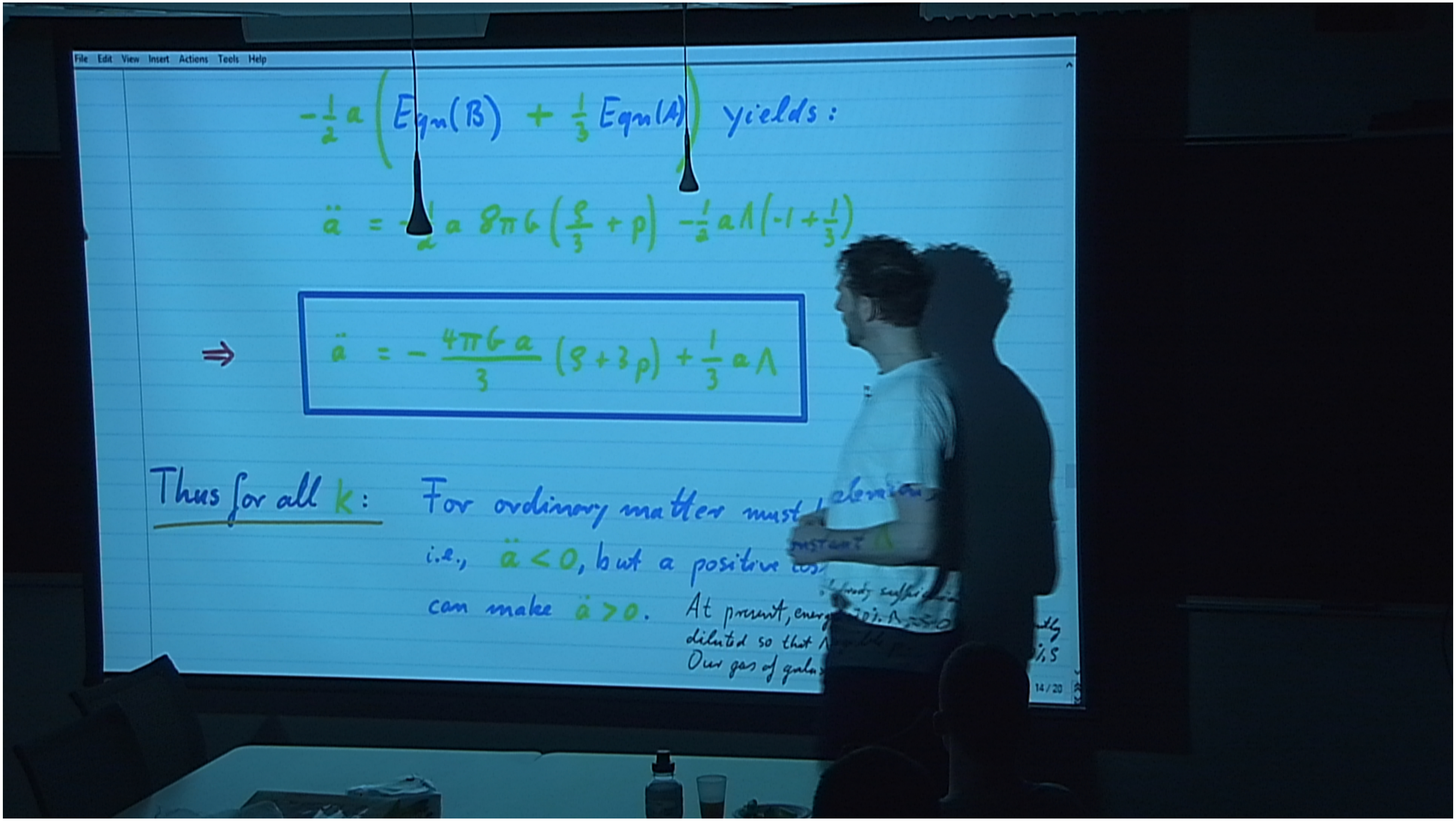
-1/2 a (Eqn(B) + 1/3 Eqn(A)) yields:
Thus for all k:
ave deceleration,
i.e., ä < 0,
constant Λ
can make ä >
seems to be already sufficiently
has taken over ≈70% Λ
has negligible



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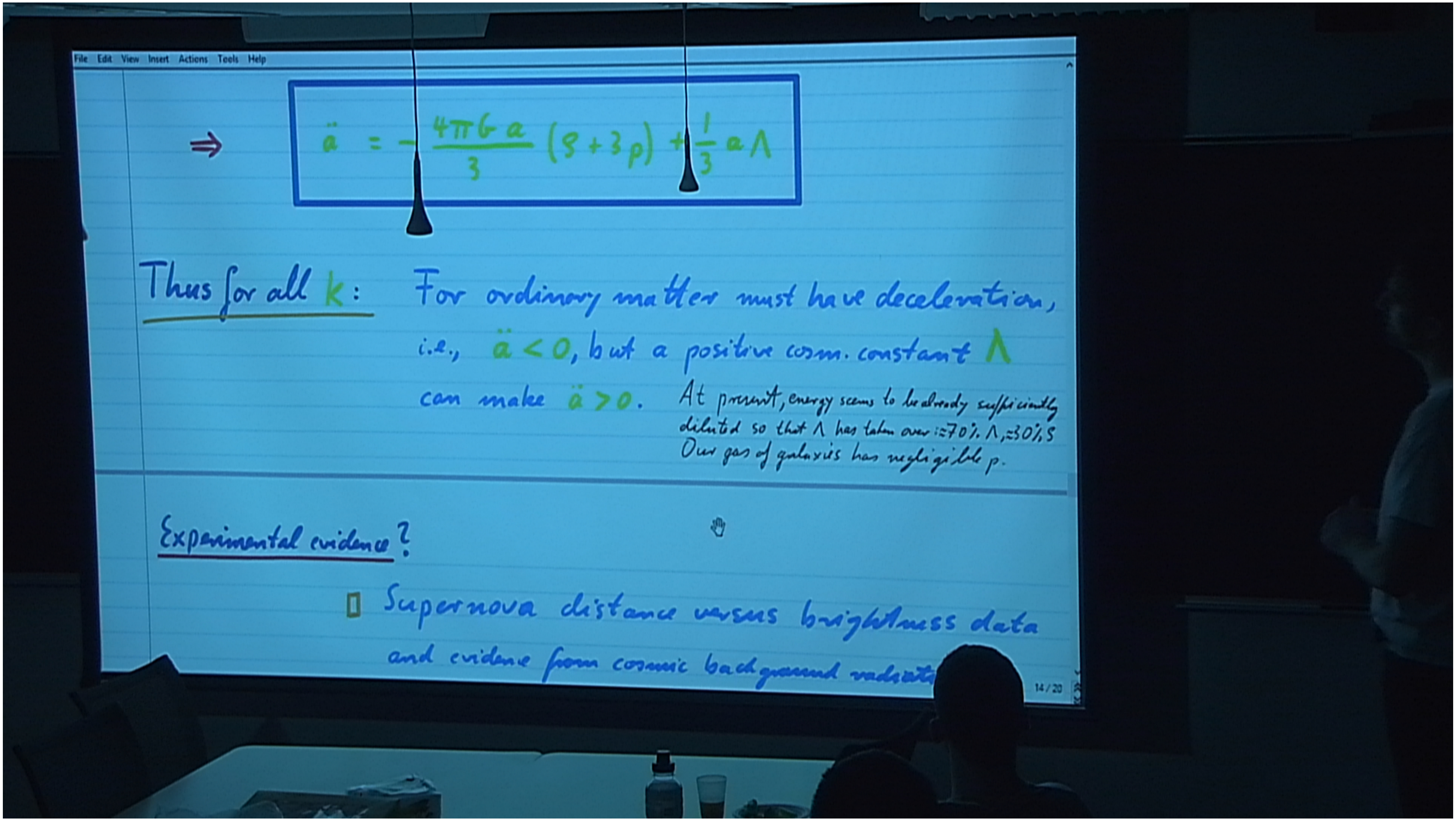
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Thus for all k:
For ordinary matter must have deceleration,
i.e., $\ddot{a} < 0$, but a positive cosm. constant Λ
can make $\ddot{a} > 0$.
At present, energy seems to be already sufficiently diluted so that Λ has taken over: $\approx 70\%\ \Lambda$, $\approx 30\%\ \rho$
Our gas of galaxies has negligible p.
Experimental evidence?
Supernova distance versus brightness data
and evidence from cosmic background
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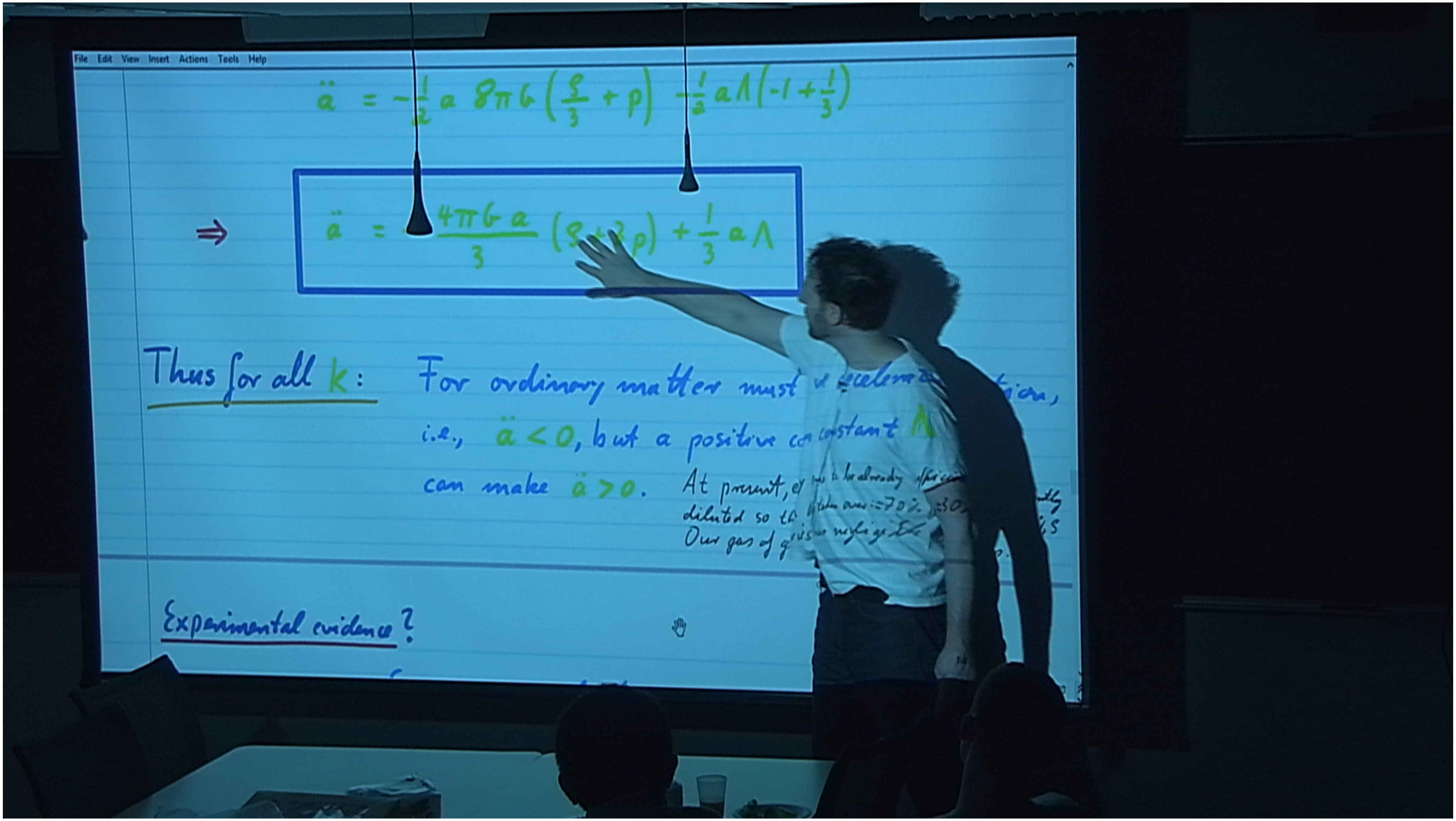
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$\ddot{a} = -\frac{1}{2} a\, 8\pi G \left(\frac{\rho}{3} + p\right) - \frac{1}{2} a \Lambda \left(-1 + \frac{1}{3}\right)$
$\Rightarrow$ $\boxed{\ddot{a} = \frac{4\pi G a}{3} (\rho + 3p) + \frac{1}{3} a \Lambda}$
Thus for all k:
For ordinary matter must
i.e., $\ddot{a} < 0$, but a positive constant Λ
can make $\ddot{a} > 0$.
Experimental evidence?

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$\ddot{a} > 0$ now!

$\Rightarrow$ At present, energy is already sufficiently diluted so that Λ dominates over ρ: $\approx 70\%$ Λ and $\approx 30\%$ ρ^{matter} (dark + visible)

Note: p of a gas of galaxies is negligible.
Note: ρ includes dark matter.
Visible matter is only $\approx 3\%$.

In the far future, ρ & p will have diluted $\to 0$, leaving only Λ. Then, the Friedmann eqn reads:

$$3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) = \Lambda$$

Or something other than Λ will dominate $T_{\mu\nu}$ then. Experiments indicate that indeed a faster than exponential expansion may be under way. That cannot come from just Λ dominance alone. See essay topic!

Solutions:

$$a(t) = \begin{cases} \cosh\left(t\sqrt{\frac{\Lambda}{3}}\right) & \text{for } k=1 \\ \exp\left(t\sqrt{\frac{\Lambda}{3}}\right) & \text{for } k=0 \\ \sinh\left(t\sqrt{\frac{\Lambda}{3}}\right) & \text{for } k=-1 \end{cases}$$

$\Rightarrow$ Exponential expansion is predicted!

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General solution strategy with cosm. constant and matter:

- We have 3 unknown functions of time

$$a(t),\ \rho(t),\ p(t)$$

and we have 3 equations that they obey:

Eqs. A, B and an equation of state $p = p(\rho)$ that depends on the "matter":

$p_\Lambda(\rho_\Lambda) = -\rho_\Lambda$ for pure vacuum energy (e.g., in very early universe)

$p(\rho) = \frac{1}{3}\rho$ for pure radiation (e.g., in the early universe)

$p(\rho) = 0$ for pure dust (e.g., middle aged universe before Λ took over)

- Observation:

(Eqn. A)

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$$\boxed{\frac{d}{da}(\rho a^3) = -3pa^2} \qquad (P)$$

Indeed, when the parameter w in $p = w\rho$ is known, (P) yields $\rho(a)$:

- For dust, $p = 0$ $\Rightarrow \rho \sim a^{-3}$
- For radiation, $p = \rho/3$ $\Rightarrow \rho \sim a^{-4}$

→ ρ of radiation decays quicker than ρ of dust because radiation is not only diluted, its wavelengths are also stretched, which reduces the energy too.

- For pure Λ: $p = -\rho$ $\Rightarrow \rho = \text{const}$ → ρ of vacuum energy does not dilute!

Intuitive meaning of (P)?

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Intuitive meaning of (P)?

- (P) is the GR version of the continuity equation for (i.e., without heat exchange with an environment) → adiabatic expansion: $dE = -p\,dV$
- With $V := a^3$, $E := \rho V$ it yields:

$$d(a^3\rho) = -p\,d(a^3) = -3pa^2\,da$$

- Thus: $\frac{d}{da}(a^3\rho) = -3pa^2$ which is indeed (P).

Exact proof of proposition (P):

- The Einstein equation $G^{\mu\nu} = 8\pi G T^{\mu\nu}$ and $G^{\mu\nu}{}_{;\nu} = 0$ imply $T^{\mu\nu}{}_{;\nu} = 0$
- Here: $T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p\,g^{\mu\nu}$

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$$= \theta^{\lambda}\left(\omega^{c}{}_{0}(e_\lambda)\, e_c\right) = \underset{\text{used } \theta^{\lambda}(e_c) = \delta^{\lambda}_{c}}{\omega^{\lambda}{}_{0}(e_\lambda)} = \omega^{i}{}_{0}(e_i)$$

Recall that $\omega^{i}{}_{0} = \frac{\dot a}{a}\theta^{i} \Rightarrow$

$$= \frac{\dot a}{a}\theta^{i}(e_i) = 3\frac{\dot a}{a}$$

since $\omega^{0}{}_{0} = 0$

$\Rightarrow$ Eqn. (X) becomes:

Thus:

$$\underbrace{\dot\rho}_{=\nabla_u \rho} + 3\frac{\dot a}{a}(\rho + p) = 0$$

(Recall: $u = \frac{d}{dt}$)

$$\dot\rho\frac{a^3}{\dot a} + 3(\rho + p)a^2 = 0$$

$$\frac{d\rho}{dt}\frac{dt}{da}a^3 + 3\rho a^2 = -3pa^2$$

$$\frac{d\rho}{da}a^3 + \rho\, 3a^2 = -3pa^2$$

$$\Rightarrow \boxed{\frac{d}{da}(\rho a^3) = -3pa^2}$$

this is Eqn (P) ✓

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