Title: AMATH 875/PHYS 786 - Fall 2015 - Lecture 23

Date: Nov 20, 2015 01:30 PM

URL: http://pirsa.org/15110005

Abstract: <p>Course Description coming soon.</p>

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GR for Cosmology, Achim Kempf, Fall 2015, Lecture 23

Note Title

Problem: In general relativity, what are the effective generalizations of the conservation laws for energy, momentum and angular momentum, e.g., to calculate collisions of galaxies?

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Source: NA

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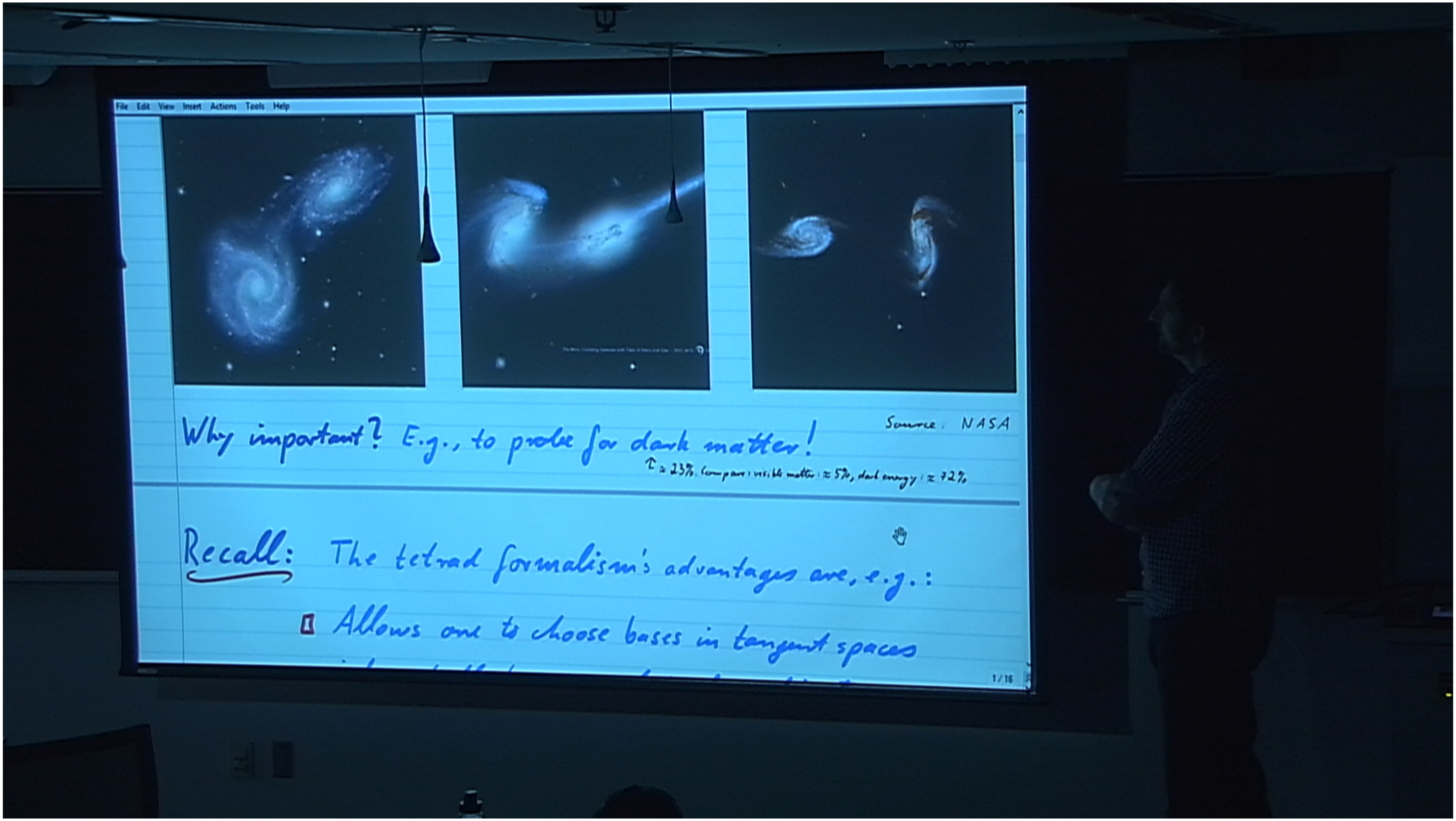
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Source: NASA
Why important? E.g., to probe for dark matter!
≈ 23%. Compare: visible matter: ≈ 5%, dark energy: ≈ 72%
Recall: The tetrad formalism's advantages are, e.g.:
Allows one to choose bases in tangent spaces
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Recall: The tetrad formalism's advantages are, e.g.:

- Allows one to choose bases in tangent spaces independently from any choice of coordinates

- Can have $g_{\mu\nu}(x) = \eta_{\mu\nu}$, which allows one to use the usual γ matrices $\{\gamma^\mu, \gamma^\nu\} = 2\,\eta^{\mu\nu}$ and obtain the Dirac equation.

- Re-express GR in terms of tetrads e^a_μ as a gauge theory
  $\rightsquigarrow$ Starting point for quantum gravity, e.g. Loop Quantum Gravity

Also: Tetrad formalism of tensor-valued forms lends itself to issues that require integration, such as the question

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Global conservation laws:

Recall:

- In special relativity, energy and momentum conservation etc express the fact that the translation in time or space (or rotation etc) of a solution is a solution too.
  This doesn't hold in curved space-time, of course.

- It is always true that

$$T^{\mu\nu}{}_{;\nu} = 0$$

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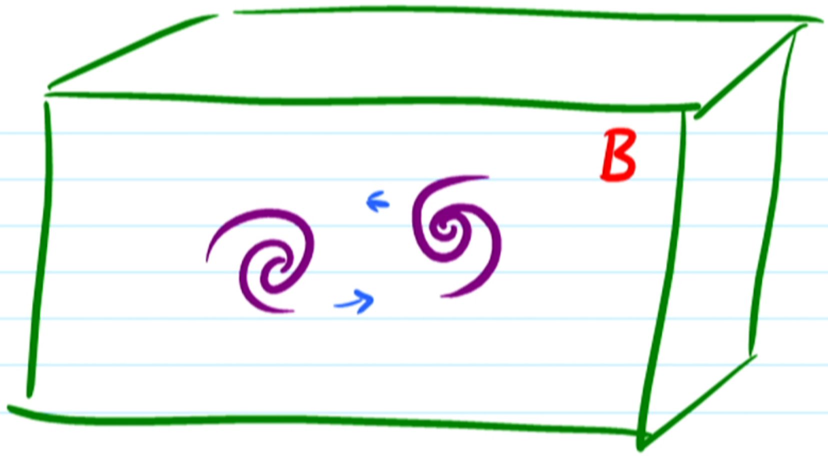
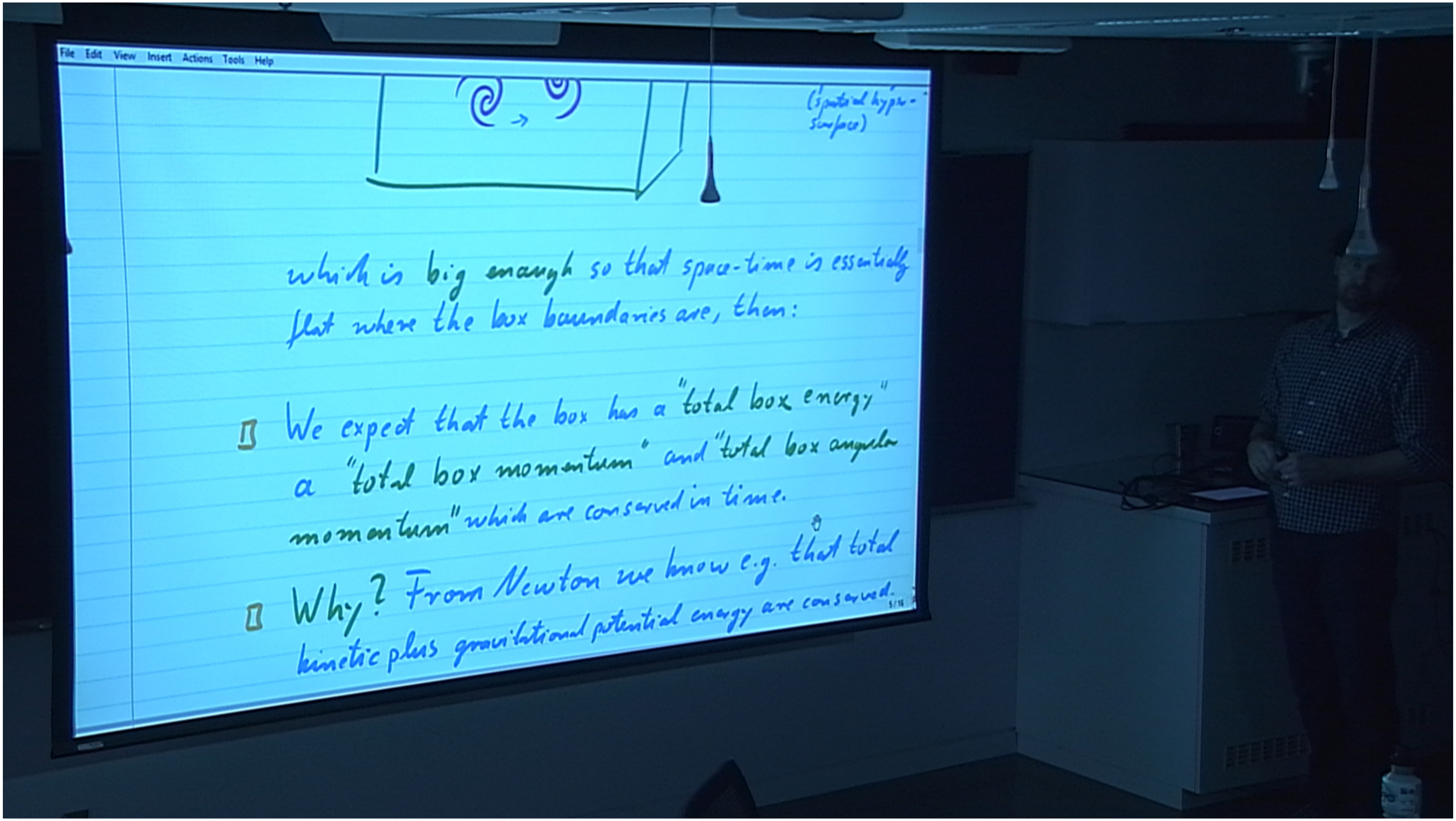
- But, if we consider the system in a box,



Note: The box **B** is a 3-dim spatial region (spatial hyper-surface)

which is big enough so that space-time is essentially flat where the box boundaries are there: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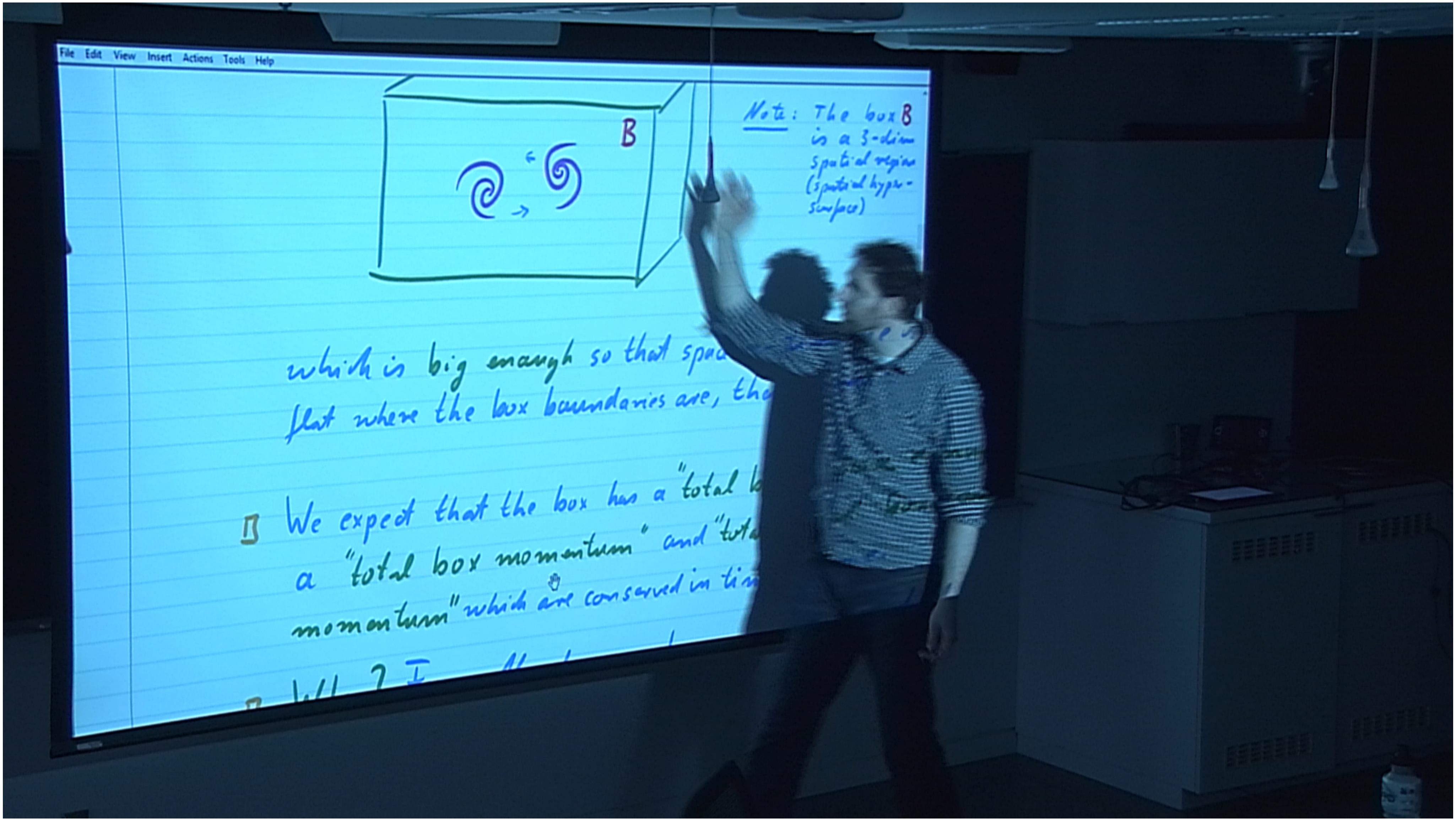
(spatial hyper-surface)
which is big enough so that space-time is essentially flat where the box boundaries are, then:
We expect that the box has a "total box energy"
a "total box momentum" and "total box angular momentum" which are conserved in time.
Why? From Newton we know e.g. that total kinetic plus gravitational potential energy are conserved.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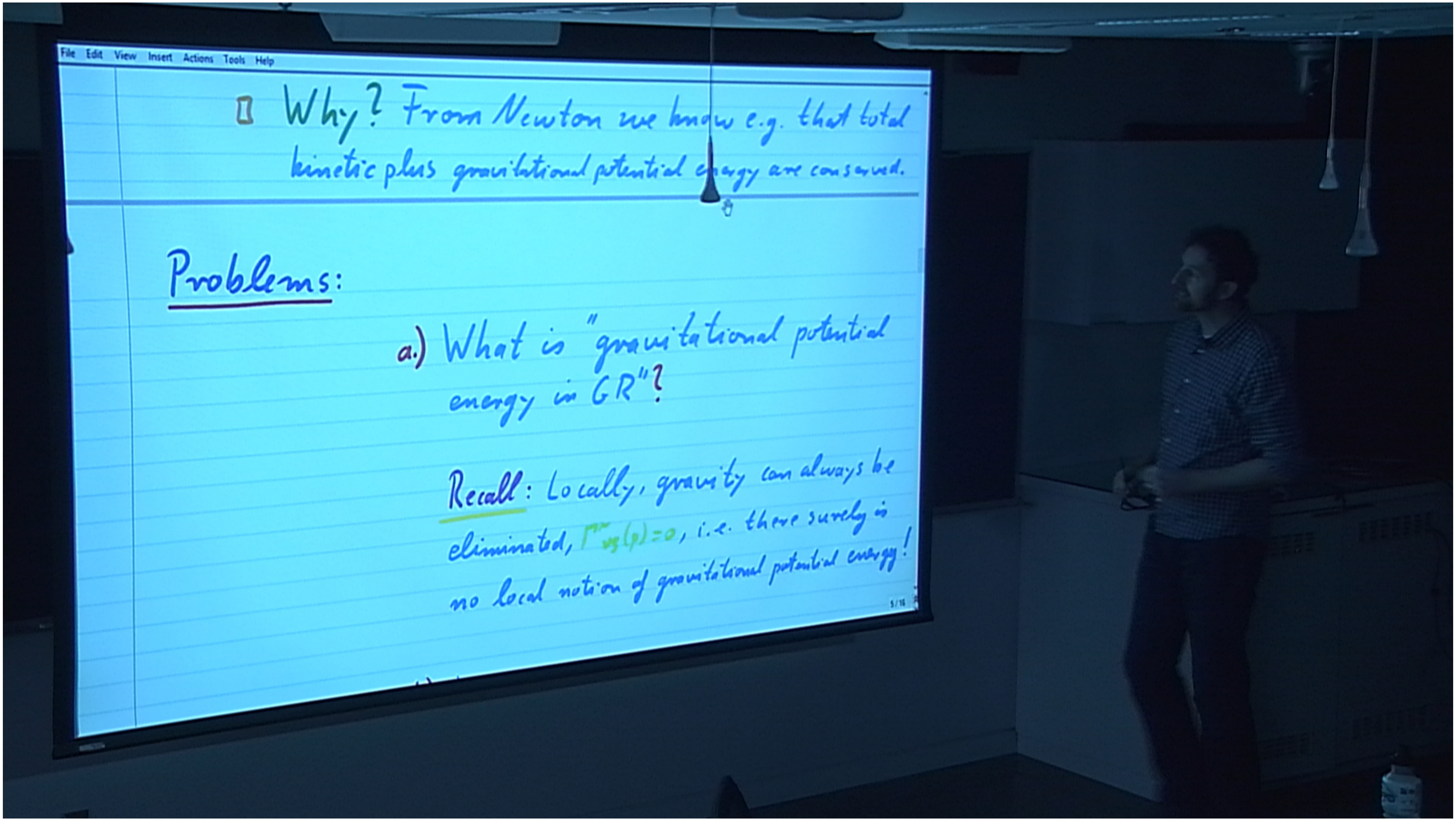
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Why? From Newton we know e.g. that total
kinetic plus gravitational potential energy are conserved.
Problems:
a) What is "gravitational potential
energy in GR"?
Recall: Locally, gravity can always be
eliminated, $\Gamma^{\mu}_{\nu\rho}(p)=0$, i.e. there surely is
no local notion of gravitational potential energy!
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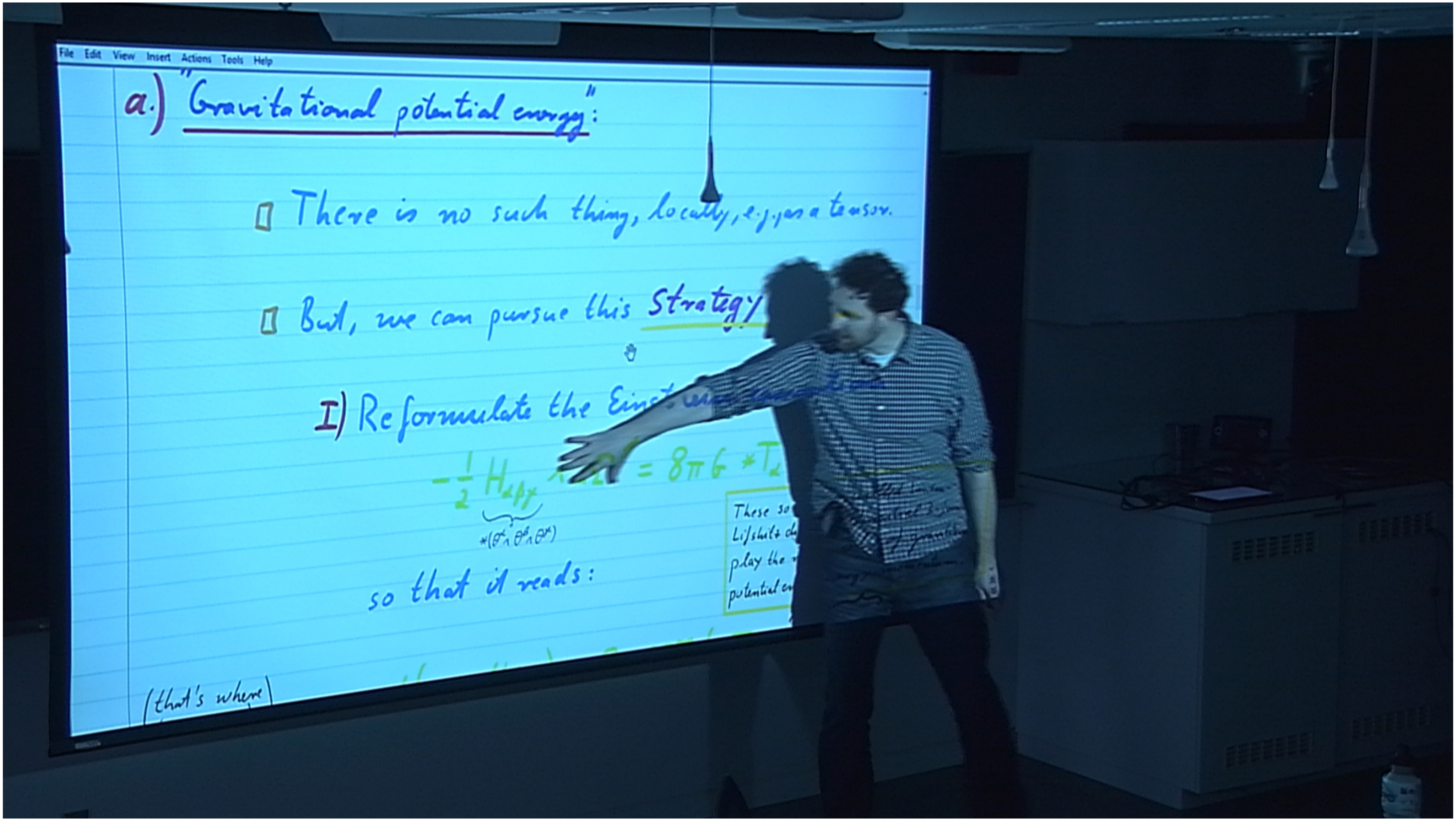
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a.) "Gravitational potential energy":
There is no such thing, locally, e.g., as a tensor.
But, we can pursue this Strategy
I) Reformulate the Ein
= 8πG *T
These so
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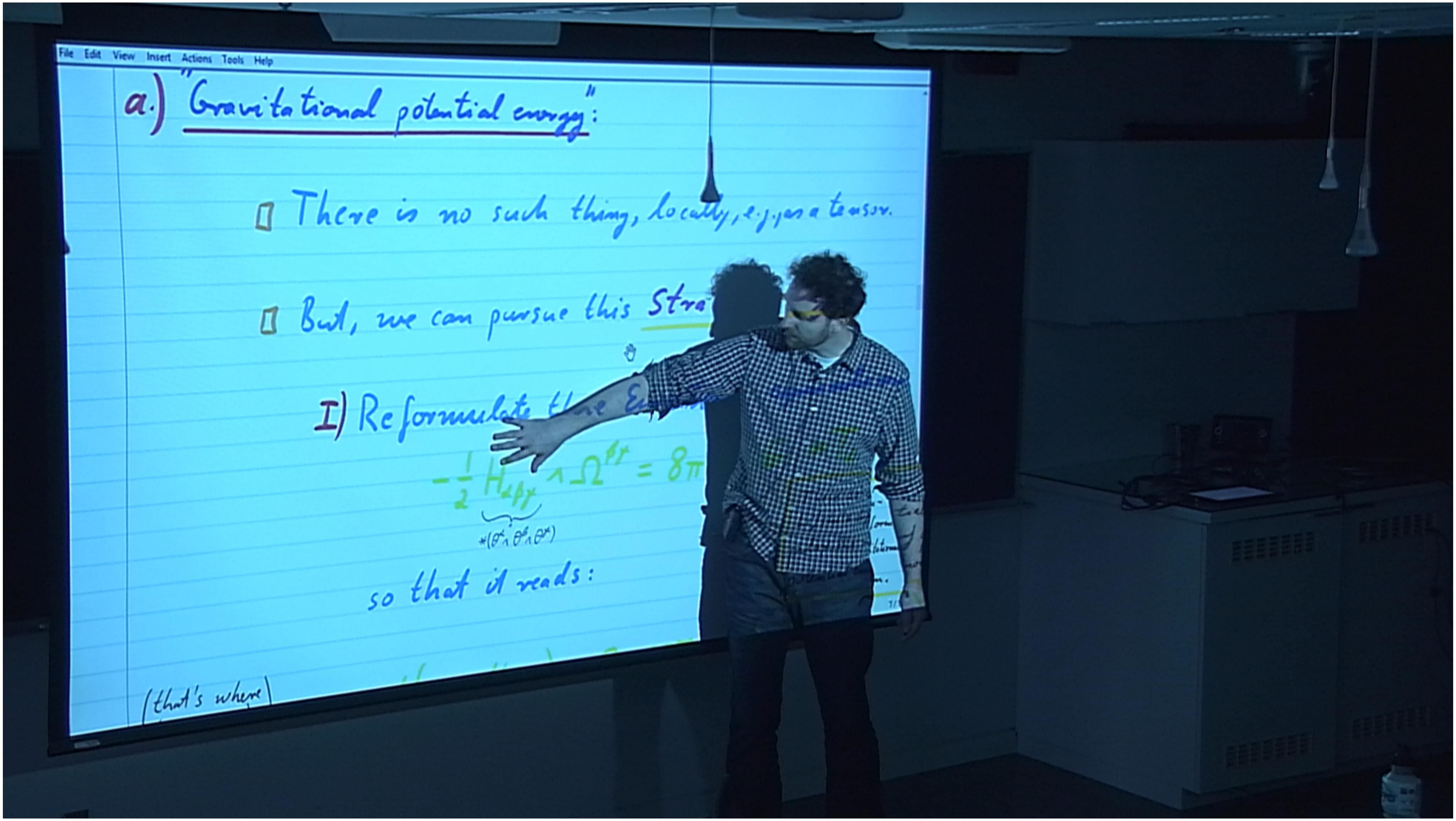
so that it reads:
(that's where)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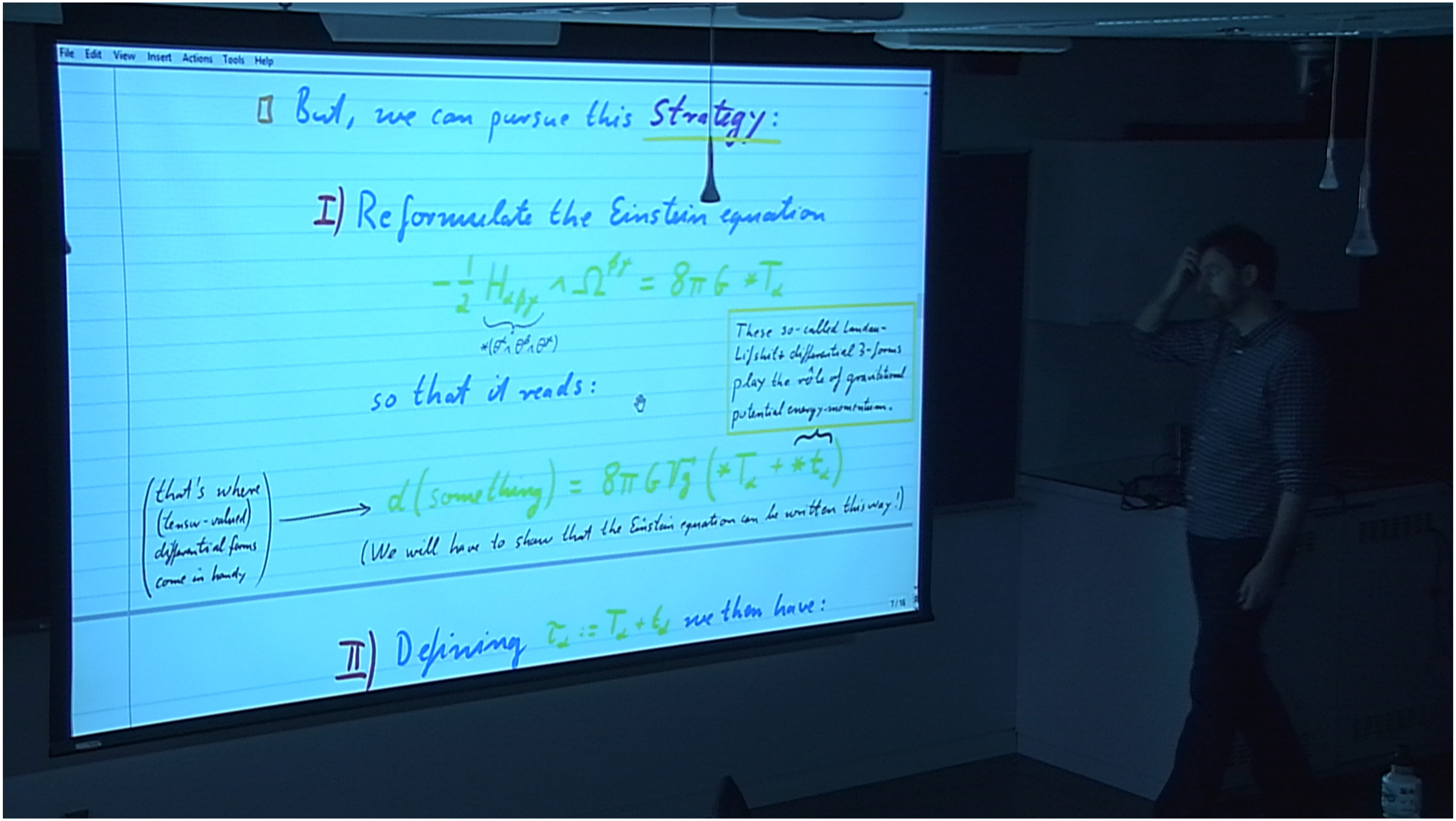
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But, we can pursue this Strategy:
I) Reformulate the Einstein equation
$-\frac{1}{2} \underbrace{H_{\alpha\beta\gamma}}_{*(\vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma)} \wedge \Omega^{\beta\gamma} = 8\pi G \, {*T_\alpha}$
These so-called Landau-Lifshitz differential 3-forms play the rôle of gravitational potential energy-momentum.
so that it reads:
(that's where (tensor-valued) differential forms come in handy)
$d(\text{something}) = 8\pi G \sqrt{g}\,(*T_\alpha + *t_\alpha)$
(We will have to show that the Einstein equation can be written this way!)
II) Defining $\tau_\alpha := T_\alpha + t_\alpha$ we then have:

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play the role of gravitational potential energy-momentum.

(that's where (tensor-valued) differential forms come in handy) $\longrightarrow$ $$d(\text{something}) = 8\pi G\sqrt{g}\,(*T_\alpha + \overbrace{*t_\alpha})$$

(We will have to show that the Einstein equation can be written this way!)

II) Defining $\tau_\alpha := T_\alpha + t_\alpha$ we then have:

$$d(\text{something}) = 8\pi G\,\sqrt{g}\,*\tau_\alpha$$

III) Then, from $d^2=0$ we obtain:

$$d\left(\sqrt{g}\,*\tau_\alpha\right) = 0$$

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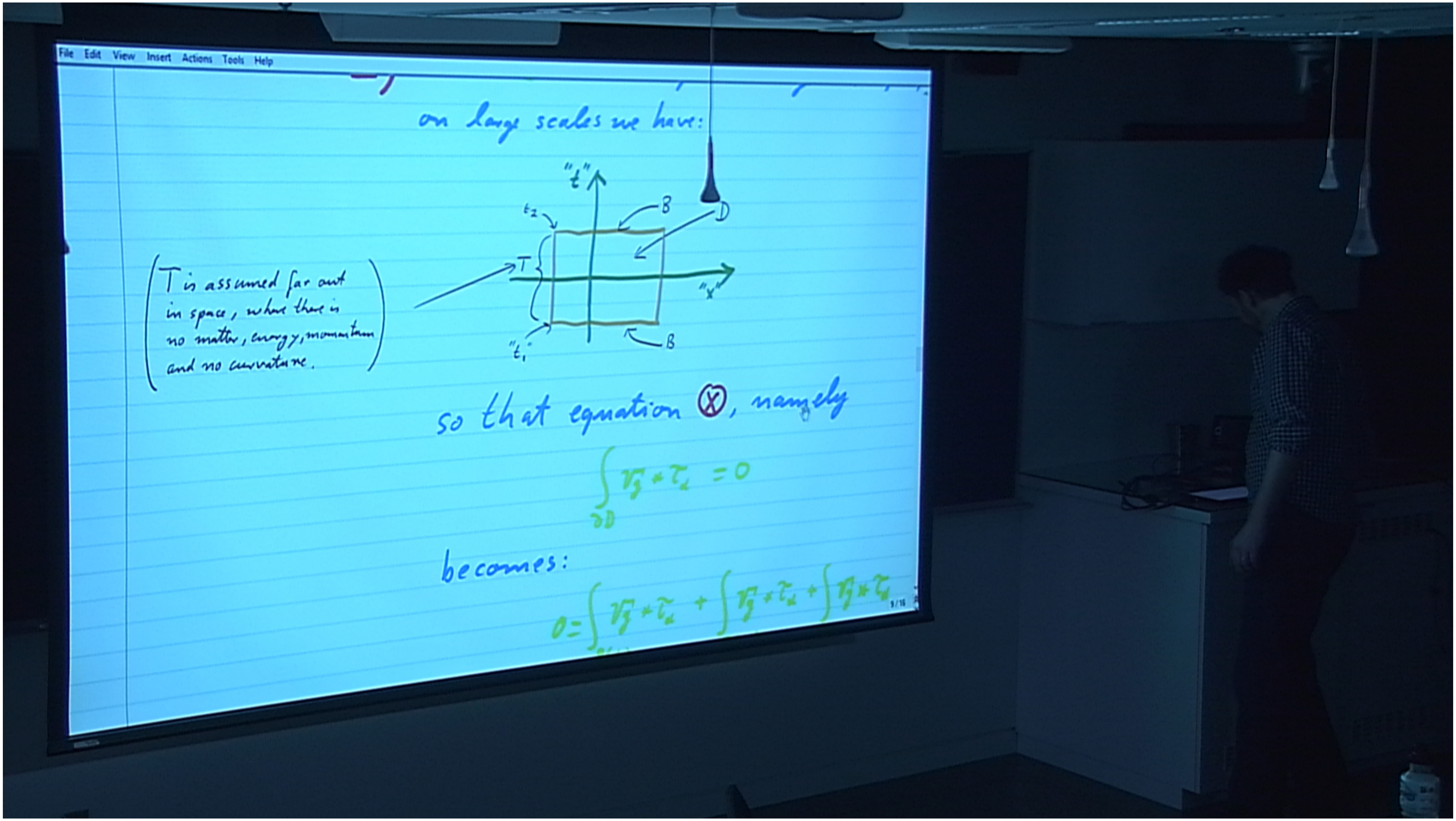
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on large scales we have:
"t"
"x"
B
D
T
T is assumed far out in space, where there is no matter, energy, momentum and no curvature.
so that equation ⊗, namely
= 0
becomes:
0 =

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VIII) Define the total "ADM 4-momentum":

Arnowitt, Deser & Misner

$$P_\mu := \int_B \sqrt{g} * \tau_\mu$$

big box

It is conserved in time:

$$\boxed{P_\mu(t_1) = P_\mu(t_2)}$$

Note: It is a Minkowski tensor with respect to local Lorentz transformations that approach a constant Lorentz transformation far out in space.

(Because under
$\theta(x)^\mu \to A^\mu{}_\nu(x)\,\theta^\nu(x)$
we have generally
$\omega(x) \to A(x)\,\omega(x)\,A^{-1}(x) - (dA)A$
but far out in space we now have:
$A(x) \to$ const. i.e.
$\omega(x) \to A(x)\,\omega(x)\,A^{-1}(x) - 0$)

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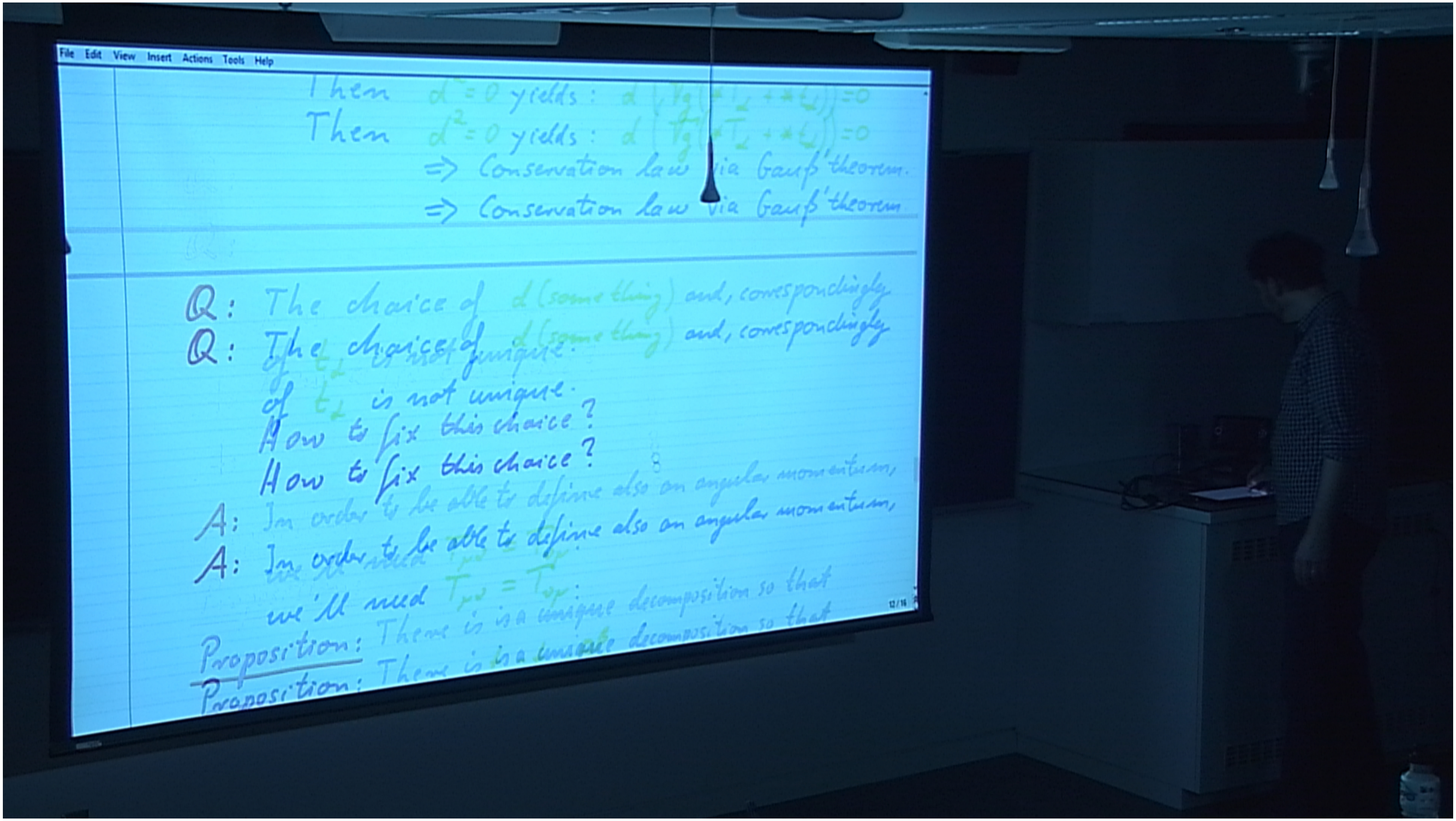
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Then $d^2=0$ yields: $d(\sqrt{g}(*T_\alpha + *t_\alpha))=0$
⇒ Conservation law via Gauß' theorem.
Q: The choice of d(something) and, correspondingly
of t_α is not unique.
How to fix this choice?
A: In order to be able to define also an angular momentum,
we'll need $T_{\mu\nu} = T_{\nu\mu}$:
Proposition: There is a unique decomposition so that
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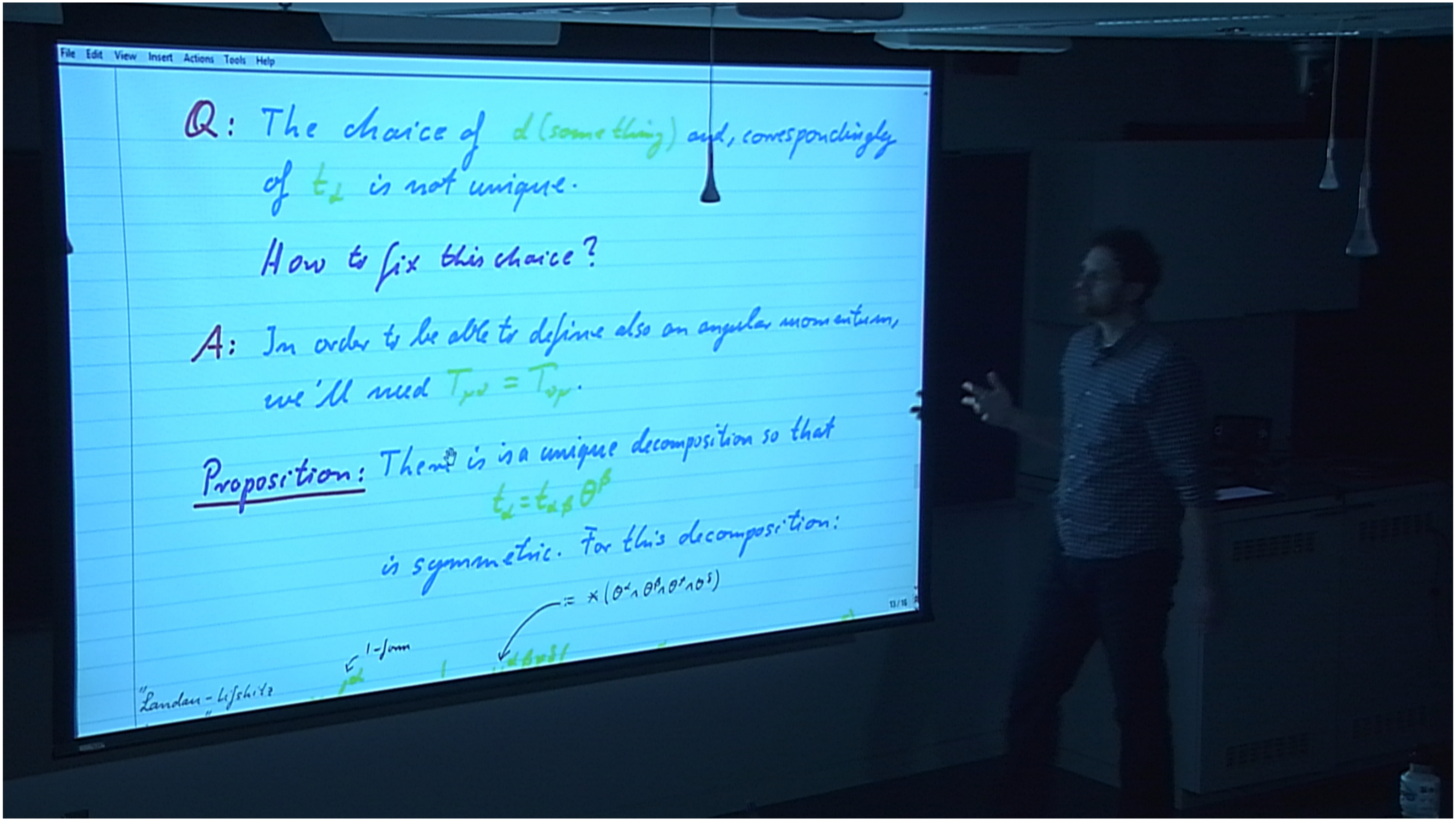
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A: In order to be able to define also an angular momentum, we'll need $T_{\mu\nu} = T_{\nu\mu}$.
Proposition: There is is a unique decomposition so that
$t_\alpha = t_{\alpha\beta}\,\theta^\beta$
is symmetric. For this decomposition:
$:= *(\theta^\alpha \wedge \theta^\beta \wedge \theta^\gamma \wedge \theta^\delta)$
1-form
"Landau-Lifshitz"
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Sketch of proof:

Einstein equation: $-\frac{1}{2}\Omega_{\beta\gamma}\wedge H^{\beta\gamma}{}_{\alpha} = 8\pi G * T_{\alpha}$

2nd structure equation: $\Omega_{\beta\gamma} = d\omega_{\beta\gamma} - \omega_{\sigma\beta}\wedge\omega^{\sigma}{}_{\gamma}$

$$\Rightarrow \underbrace{-\frac{1}{2} d\omega_{\beta\gamma}\wedge H^{\beta\gamma}}_{=\; -\frac{1}{2} d(\omega_{\beta\gamma}\wedge H^{\beta\gamma}) - \frac{1}{2}\omega_{\beta\gamma}\wedge \underbrace{d H^{\beta\gamma}{}_{\alpha}}} + \frac{1}{2}\omega_{\sigma\beta}\wedge\omega^{\sigma}{}_{\gamma} = 8\pi G * T_{\alpha} \qquad (*)$$

Re-write, using (recall):

$$0 = DH^{\beta\gamma}{}_{\alpha} = dH^{\beta\gamma}{}_{\alpha} + \omega^{\beta}{}_{\sigma}\wedge H^{\sigma\gamma}{}_{\alpha} + \omega^{\gamma}{}_{\sigma}\wedge H^{\beta\sigma}{}_{\alpha} - \omega^{\sigma}{}_{\alpha}\wedge H^{\beta\gamma}{}_{\sigma}$$

$\Rightarrow$ Einstein equation: $-\frac{1}{2} d(\omega_{\beta\gamma}\wedge H^{\beta\gamma}) + \frac{1}{2}\omega_{\beta\gamma}\wedge\left(\omega^{\beta}{}_{\sigma}\wedge H^{\sigma\gamma}{}_{\alpha} + \omega^{\gamma}{}_{\sigma}\wedge H^{\beta\sigma}{}_{\alpha} - \omega^{\sigma}{}_{\alpha}\wedge H^{\beta\gamma}{}_{\sigma}\right)$
$+\frac{1}{2}\omega_{\sigma\beta}\wedge\omega^{\sigma}{}_{\gamma} = 8\pi G * T_{\alpha}$

Notice: It is of the form $d(\text{something}) = 8\pi G(*T_{\alpha} + *$ ...

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⇒ We now have all ingredients to calculate the conserved ADM energy-momentum vector

$$P_\mu := \int_B \sqrt{g} * \tau_\mu$$

B ← big box

(The "positive energy theorem") with $\tau_\mu = T_\mu + t_\mu$

T from matter; t from gravity using Eqn. (G) above.

Theorem: If the dominant energy condition holds, then $P_0 \geq 0$

(i.e. the ADM 4-vector is future-directed timelike or lightlike: $p_\mu p^\mu \leq 0$ and $p_0 \geq 0$)

Angular momentum?

- Choose coordinates that become cartesian Minkowski far out.
- Define: $*M^{\alpha\beta} := x^\alpha * \tau^\beta - x^\beta * \tau^\alpha$
- Proposition: $d(\sqrt{g} * M^{\alpha\beta}) = 0$

→ Note: For this it is necessary to have chosen the definition of t which has $t_{\alpha\beta}$, and therefore

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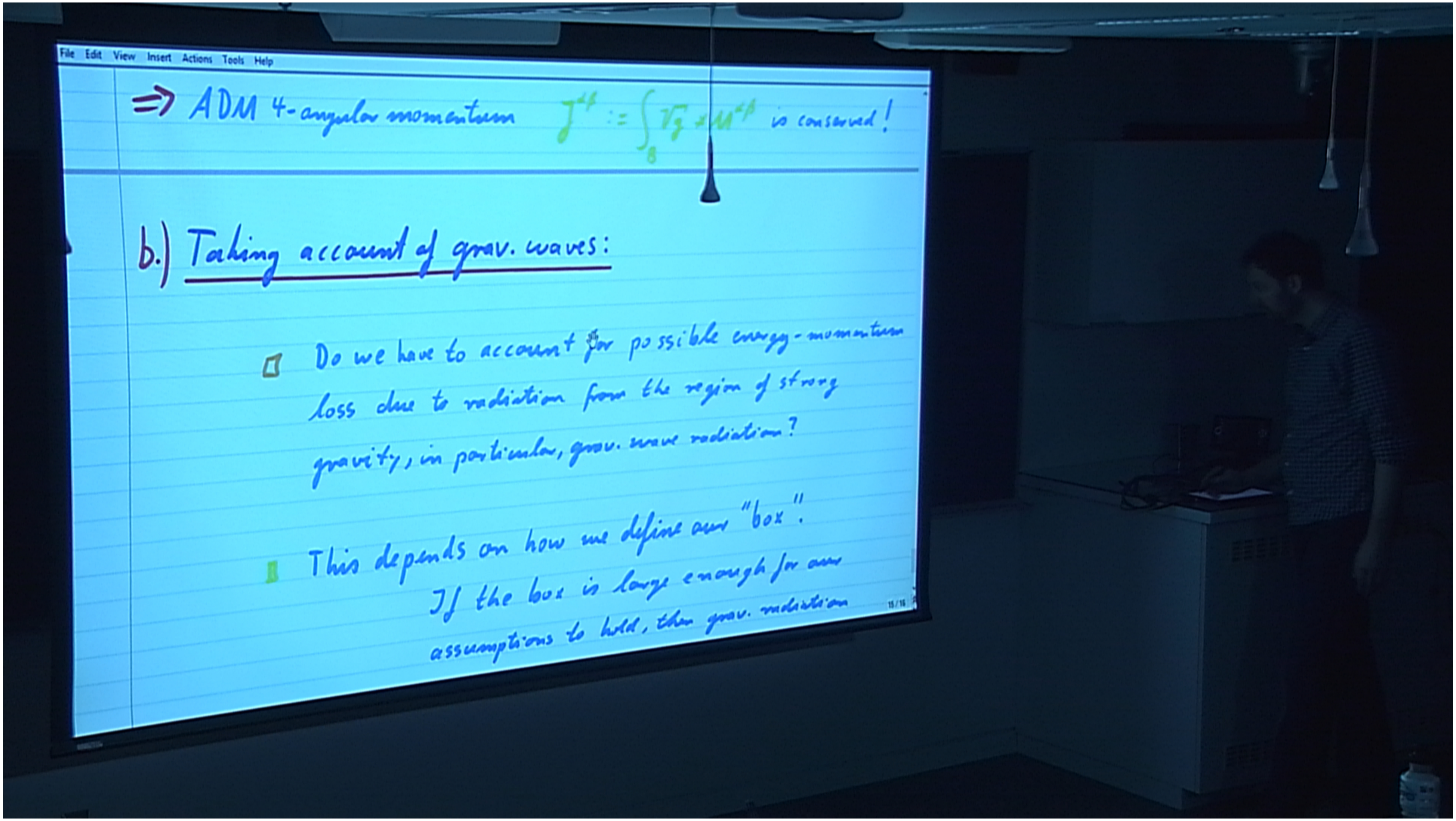
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⇒ ADM 4-angular momentum $J^{\alpha\beta} := \int_B \sqrt{g} * M^{\alpha\beta}$ is conserved!
b.) Taking account of grav. waves:
Do we have to account for possible energy-momentum loss due to radiation from the region of strong gravity, in particular, grav. wave radiation?
This depends on how we define our "box".
If the box is large enough for our assumptions to hold, then grav. radiation
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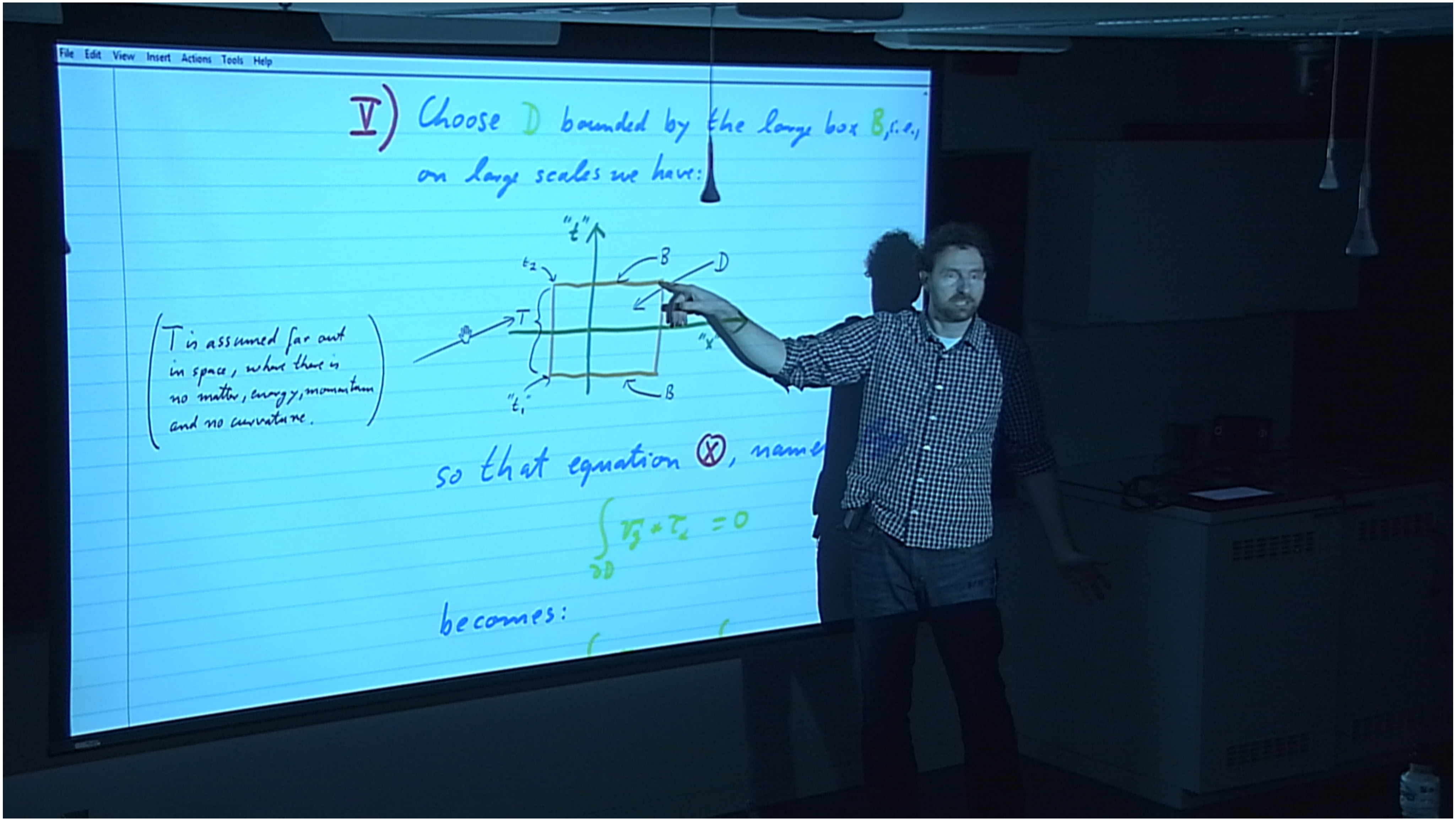
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V) Choose D bounded by the large box B, i.e.,
on large scales we have:
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B
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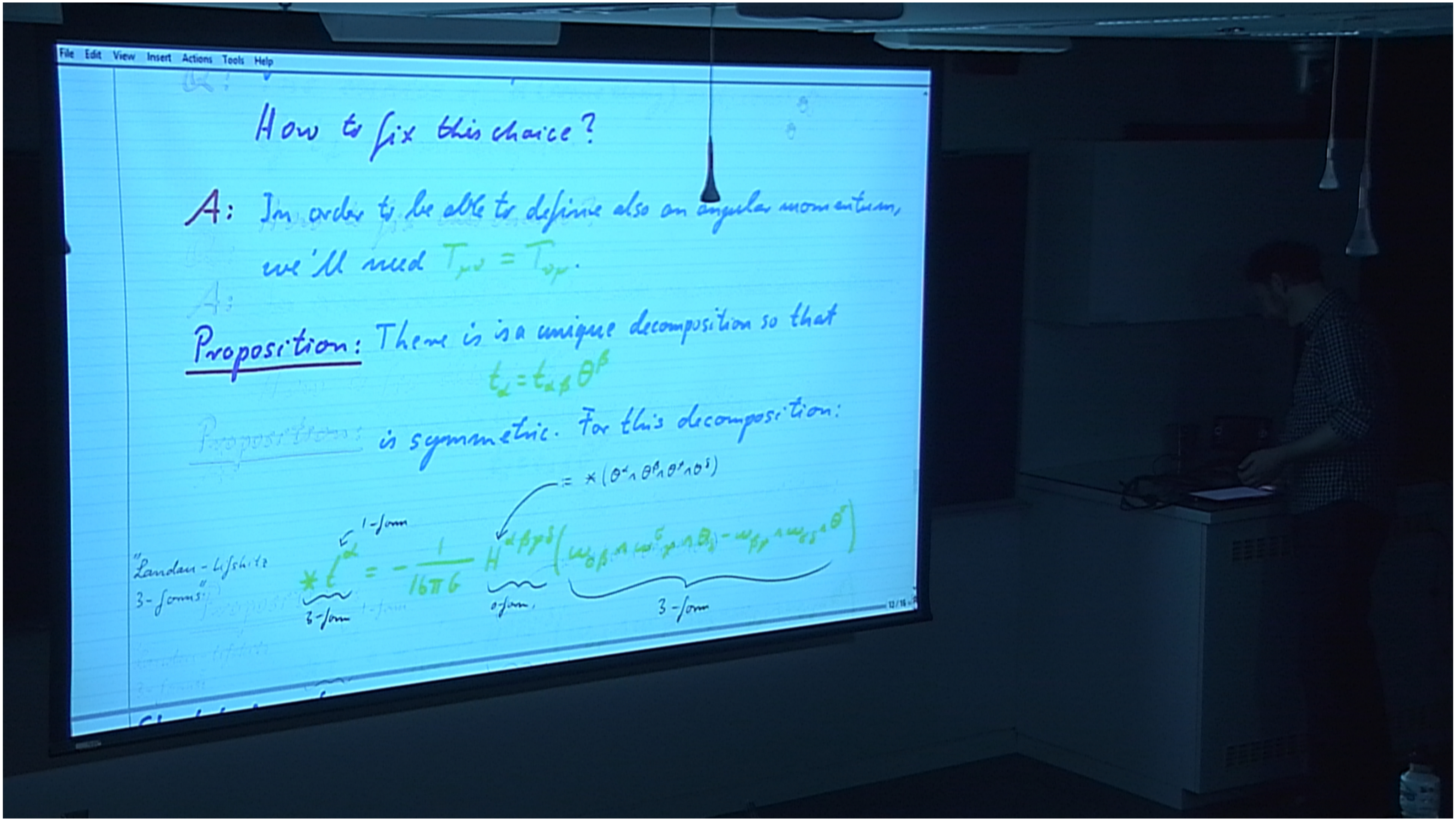
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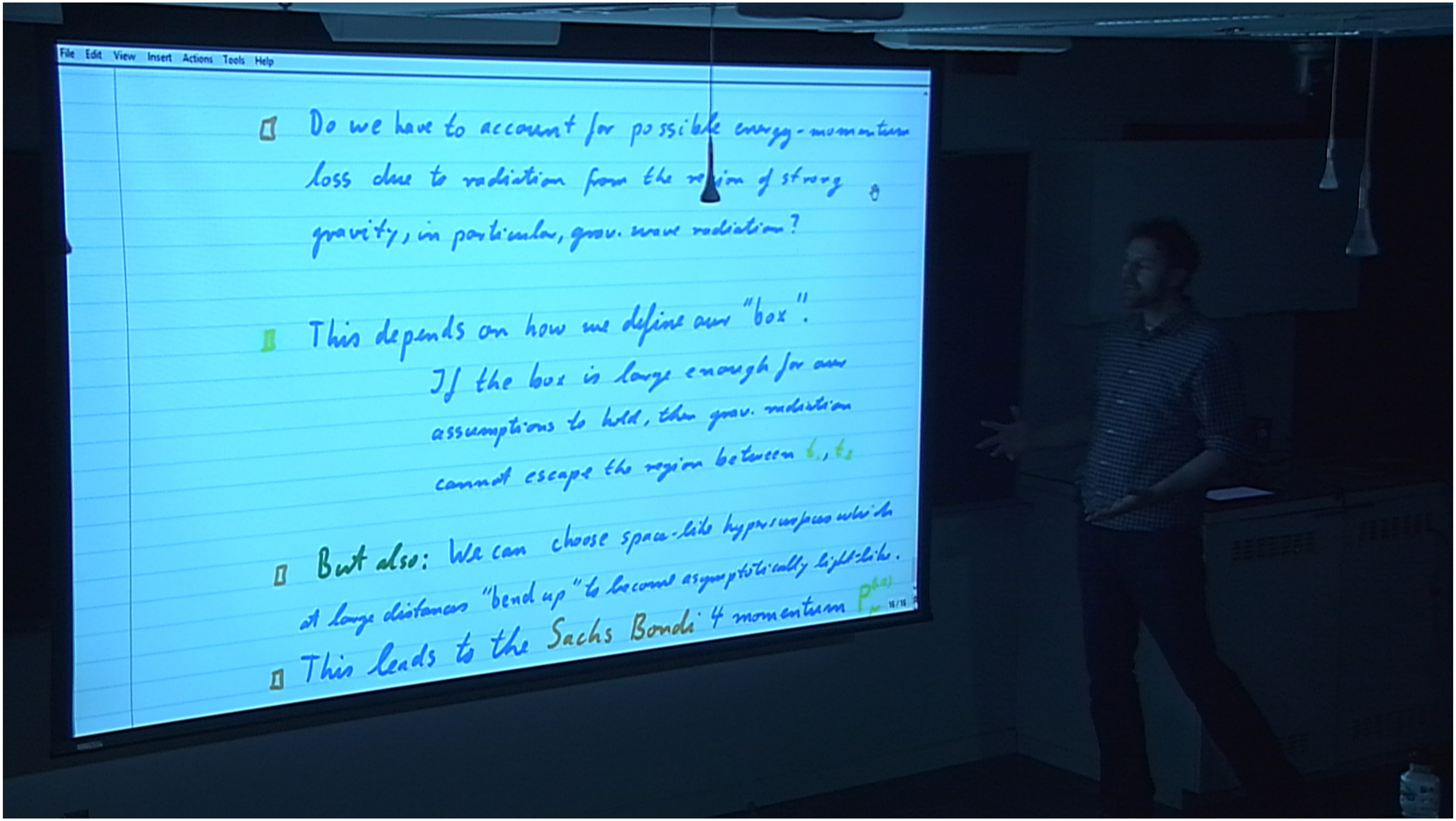
"Landau-Lifshitz 3-forms"
$\underbrace{*t^\alpha}_{3\text{-form}} = -\frac{1}{16\pi G}\underbrace{H^{\alpha\beta\gamma\delta}}_{0\text{-form}}\underbrace{(\omega_{\delta\beta}\wedge\omega^\sigma{}_\gamma\wedge\theta_\sigma - \omega_{\beta\gamma}\wedge\omega_{\delta\sigma}\wedge\theta^\sigma)}_{3\text{-form}}$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If the box is large enough for our assumptions to hold, then grav. radiation cannot escape the region between t_1, t_2
But also: We can choose space-like hypersurfaces which at large distances "bend up" to become asymptotically light-like.
This leads to the Sachs Bondi 4 momentum P_μ
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- But also: We can choose space-like hypersurfaces which at large distances "bend up" to become asymptotically light-like.

- This leads to the Sachs Bondi 4 momentum $P_\mu^{(SB)}$

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