Title: AMATH 875/PHYS 786 - Fall 2015 - Lecture 21

Date: Nov 16, 2015 12:00 PM

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Abstract: <p>Course Description coming soon.</p>

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Note Title

GR for Cosmology, Achim Kempf, Fall 2015, Lecture 21

Recall: A key prediction of GR is its own downfall in singularities. Or is it? Can one prove that GR has generic situations that must lead to a singularity, even in the absence of any symmetry?

The plan:

1. Define and study suitable notions of:

- Causality
- Horizons (next to define: "Cauchy horizons")
- Singularities

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in singularities. Or is it? Can one prove that GR has generic situations that must lead to a singularity, even in the absence of any symmetry?

The plan:

1. Define and study suitable notions of:

   - Causality
   - Horizons (next to define: "Cauchy horizons")
   - Singularities

2. Develop singularity theorems.

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i.e. paths without endpoint $p \in M$

Intuition: Therefore, inextendible paths either:

a.) go to ∞, or

b.) end in a singularity

⇝ Continue to study inextendible curves

⇝ Arrive at key concepts of Cauchy horizon and global hyperbolicity.

Recall:

"the causal future"

- We considered the set of points $J^+(S)$ that can somehow be reached from a set S. (i.e. the set of points that are affected by S)
- Now consider set of points that can only be reached from S: (i.e. the set of events that depend on S and only S)

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i.e., a set of events among which no object could travel

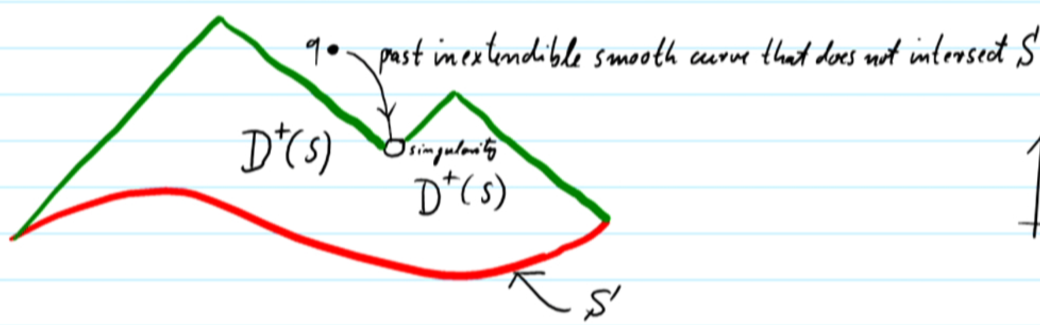
Definition:

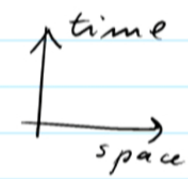
Assume $S \subset M$ is a closed achronal set.

Then, the "future domain of dependence of S" is defined as:

$$D^+(S) := \left\{ p \in M \;\middle|\; \begin{array}{l}\text{Every past inextendible causal} \\ \text{curve through } p \text{ intersects } S\end{array} \right\}$$

Example:





Why $q \notin D^+(S)$? Some of its past inextendible

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Definition:

Analogously, the "past domain of dependence of S′" is:

$$D^-(S') := \left\{ p \in M \;\middle|\; \begin{array}{l}\text{Every future inextendible causal}\\ \text{curve through } p \text{ intersects } S\end{array} \right\}$$

(the set of events p that affect only S′)

Definition:

The "full domain of dependence of S′" is:

$$D(S') := D^+(S') \cup D^-(S')$$

Definition: (set of latest events that are affected only by S′? How far have initial conditions on S full predictive power?

The "future Cauchy horizon of S′", denoted H(...)

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The "full domain of dependence of S'" is:

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Definition: (set of latest events that are affected only by S'? How far have initial conditions on S' full predictive power?

The "future Cauchy horizon of S'", denoted $H^+(S)$ is:

chronological past

$$H^+(S') := \overline{D^+(S)} - I^-(D^+(S))$$

(Note: $\Rightarrow H^+(S')$ is achronal. Why?)

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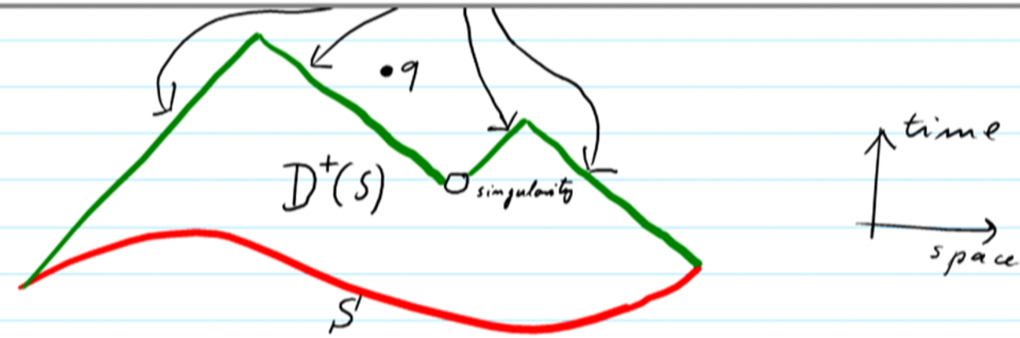

analogously:

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The "past Cauchy horizon of S", denoted $H^-(S)$

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Proposition:

$$H(S') = \dot{D}(S')$$

Definition:

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Proposition:

$$H(S') = \dot{D}(S')$$

Definition:

A closed, achronal set S' is called a "Cauchy surface", if its full Cauchy horizon vanishes, i.e., if

a.) $H(S) = \phi$ ← empty set   or equivalently if

b.) $\dot{D}(S) = \phi$ or equivalently if

c.) $D(S') = \mathcal{M}$

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Definition:

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c.) $D(S') = M$

Note: This follows Wald. The definitions by others (Hawking, Ellis, Geroch et al.) are equivalent, but more tedious.

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Remarks:

- Cauchy surfaces are important because if the conditions on a Cauchy surface are known, then everything on $\mathcal{M}$ can be predicted and retrodicted.

  Note: E.g., anti de Sitter space has no Cauchy surfaces!

- Since a Cauchy surface is achronal, it can be viewed as an "instant in time".

- The term "surface" is motivated by a theorem:

  Every Cauchy surface, Σ, is a 3-dimensional

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- The term "surface" is motivated by a <u>theorem:</u>

  Every Cauchy surface, Σ, is a 3-dimensional C^0 submanifold of M.

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Definition:

If (M, g) possesses a Cauchy surface then it is called "globally hyperbolic".

Remark: We'll need this notion later for a cosmological singularity theorem.

Proposition:

If (M, g) is globally hyperbolic, then:

- There exists a "global time function f" so that every surface of constant f is a Cauchy surface.

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Proposition:

If (M,g) is globally hyperbolic, then:

- There exists a "global time function f" so that every surface of constant f is a Cauchy surface.

- (M,g) is stably (and therefore also strongly) causal.

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Recall: Plan is to study inextendible geodesics in order to detect singularities.

Now: How to identify these geodesics which are inextendible because they end at a singularity in the manifold?

First: Avoid trivial cases where manifold is ending but could be extended.

Definition:

We say that (M, g) is inextendible, if it is not isometric to a proper subset of another spacetime (M', g')

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Definition:

A geodesic which is inextendible but possesses a finite range of its affine parameter is called "incomplete".

Note: This is to exclude inextendible geodesics which keep going to ∞.

Definition:

We say that (M, g) possesses a "singularity" if it possesses an incomplete geodesic.

⇝ We distinguish singularities of null, spacelike and timelike type.

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Definition:

- We say that $(\mathcal{M}, g)$ possesses a "**singularity**" if it possesses an incomplete geodesic.

⇝ We distinguish singularities of null, spacelike and timelike type.

- When going along an incomplete geodesic towards a "singularity", 3 things can happen:

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I) A scalar constructed from $R^{\mu\nu}{}_{\rho\sigma}$,

e.g. R, $R^{\mu\nu}R_{\mu\nu}$, etc diverges.

$\rightsquigarrow$ We say it is a "scalar curvature singularity".

II) In a parallel transported tetrad frame, a scalar component of $R^{\mu\nu}{}_{\rho\sigma}$ or its covariant derivatives diverge.

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III) None of the above. Example: "Conical singularity".

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derivatives diverge.

⤳ We say it is a "parallel-propagated curvature singularity"

III) None of the above. Example: "Conical singularity".
(cut out a suitable piece and identify the boundaries of the cut)

(This way mfld can be diffable while some paths cannot.)

⤳ We say it is a non-curvature singularity

Fundamental problem:

- In concrete solutions, such as Schwarzschild or FRW cosmologies, curvature singularities a…

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Fundamental problem:

- In concrete solutions, such as Schwarzschild or FRW cosmologies, curvature singularities are obviously present.

- But these spacetimes are highly symmetric.

  Do more realistic, i.e. perturbed spacetimes also show these singularities?

- Example:

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Example:

Spherically symmetric dust shell infall.

In Newton gravity: Use catastrophe theory

$\Rightarrow$ e.g., predict ∞ mass density to occur,
but not if symmetry perturbed!

In Einstein gravity: Use singularity theorems

$\Rightarrow$ e.g., predict black hole singularity to occur,
even if symmetry is perturbed,
(if assuming e.g. dominant energy cond. etc.)

Remark:
Black holes provide finite energy endpoint of grav. collaps, thus stabilizing GR energetically.
Note: In QM, charge driven collaps is bounded at finite energy

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In Einstein gravity: Use singularity theorems

Remark:
Black holes provide finite energy endpoint of grav. collapse, thus stabilizing GR energetically.
Note: In QM, charge driven collapse is bounded at finite energy by uncertainty principle.

⇒ e.g., predict black hole singularity to occur, even if symmetry is perturbed, (if assuming e.g. dominant energy cond. etc.)

or also: post-dict a cosmological singularity

Remark:

Singularity theorems ⇒ prediction of singularities is robust.

Thus: If quantum gravity is to resolve singularities, it will have to overcome this robustness!

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Remark:

Singularity theorems $\Rightarrow$ prediction of singularities is robust.

Thus: If quantum gravity is to resolve singularities, it will have to overcome this robustness!

Strategy for singularity theorems:

a) Focus attention on singularities that can be identified by the existence of incomplete inextendible timelike (or null) geodesics.

Why? It is clear that these are important singularities

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Strategy for singularity theorems:

a) Focus attention on singularities that can be identified by the existence of incomplete inextendible timelike (or null) geodesics.

Why? It is clear that these are important singularities because observers travelling such a geodesic have their eigentime bounded above and/or below.

Other singularities? (e.g. singularities identified through incomplete spacelike geodesics or singularities identified by some other criterion.)

May well exist in addition but the

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Why? It is clear that these are important singularities because observers travelling such a geodesic have their eigentime bounded above and/or below.

Other singularities? (e.g. singularities identified through incomplete spacelike geodesics or singularities identified by some other criterion.)

May well exist in addition but the standard singularity theorems do not attempt to predict them too.

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b.) Basic idea:

Singularities can be in the way of geodesics.

$\Rightarrow$

The presence of singularities interferes with the property of geodesics of being extremal length curves.

c.) Recall:

(Euler Lagrange)

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c.) Recall:

Extremizing curve length $\xRightarrow{\text{(Euler Lagrange equation)}}$ geodesic equation

The geodesic equation is a differential equation.

Thus:

At least locally, geodesics are paths of extremal length:

- Space-like geodesics are curves of shortest proper distance.

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(equation)

Extremizing curve length $\Longrightarrow$ geodesic equation

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At least locally, geodesics are paths of extremal length:

- Space-like geodesics are curves of shortest proper distance.

- Time-like geodesics are curves of maximal proper time (i.e. of maximal eigentime).

Why maximal?

If there is a timelike curve between two events p, q, then there are timelike curves with shorter eigentime: just take a spatially longer path and travel it faster.

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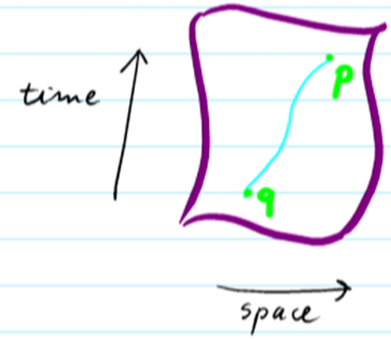
d.) Prove that, even in generic spacetimes:

There always exist curves of maximal length between two events.



What assumptions are needed?

E.g., the assumption that spacetime is globally hyperbolic suffices.

e.) Further assume that matter obeys a suitable energy condition,

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e) Further assume that matter obeys a suitable energy condition, (usually the so-called strong energy condition) and use it to prove that geodesics meet a divergence of a quantity called expansion, θ, in finite proper time.

$\Rightarrow$ these extremal length curves cannot be geodesics with eigentime larger than a certain finite amount either into the past or future.

f) Conclude that there are incomplete geodesics, i.e., that we have a singularity in the past (or future).

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f.) Conclude that there are incomplete geodesics, i.e., that we have a singularity in the past (or future).

A singularity theorem:

Assume that: ▫ (M, g) is a globally hyperbolic spacetime

▫ The energy-momentum tensor of matter obeys the "Strong energy condition":

Notice: Since the Einstein equation

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Notice: Since the Einstein equation can be brought in the form $kR_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}$, the strong energy condition is a condition on the Ricci tensor too. This will be the use of the strong energy condition.

- The energy-momentum tensor of matter obeys the "**Strong energy condition**":

$$\left(T_{\mu\nu} - \frac{1}{2} T^{\rho}{}_{\rho}\, g_{\mu\nu}\right) \xi^{\mu} \xi^{\nu} \geq 0 \qquad \forall \text{ timelike } \xi.$$

- There exists a C^2 spacelike Cauchy surface Σ, on which the trace of the extrinsic curvature, K, is bounded from above by a negative constant C:

$$K(p) \leq C < 0 \quad \text{for all } p \in \Sigma$$

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Then:

No past-directed timelike curve from a spacelike hypersurface Σ can have eigentim…

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Then:

No past-directed timelike curve from a spacelike hypersurface Σ can have eigentime, i.e., proper length, larger than $\frac{3}{c}$.

I.e.: All past-directed timelike geodesics are incomplete.

because all past-directed paths end on it.

$\Longrightarrow$ There is a cosmological singularity in the finite past!

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Extrinsic curvature?

later more on this

- The extrinsic curvature of a spacelike hypersurface describes how much curvature there is in between the spacelike hypersurface and the time dimension.

Intuitively: it is the rate of the expansion of spacetime, more precisely its negative, the rate of contraction.

$\forall p \in \Sigma$

Thus: Assuming $K(p) \leq C < 0$ meant that spacetime has a finite minimum expansion rate everywhere on Σ.

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Thus: Assuming $K(p) \leq C < 0$ meant

$\forall p \in \Sigma$

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$\rightsquigarrow$ We'll define expansion below in detail.

The strong energy condition?

Recall: □ The "weak energy condition":

$T_{\mu\nu} v^\mu v^\nu \geq 0$ for all timelike v: $g(v,v) < 0$

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The strong energy condition?

Recall: ☐ The "weak energy condition":

$$T_{\mu\nu} v^\mu v^\nu \geq 0 \quad \text{for all timelike } v: \; g(v,v) < 0$$

Meaning? For an observer with unit tangent v the local energy density is: $T_{\mu\nu} v^\mu v^\nu \geq 0$

☐ The "dominant energy condition":

$$\underbrace{T_{\mu\nu} v^\mu v^\nu \geq 0}_{\text{weak energy condition}} \quad \text{and} \quad K_\mu K^\mu \leq 0$$

[i.e. $T_{\mu\nu} v^\nu$ is non-space-like.

where v is any timelike vector and $K_\mu := T_{\mu\nu} v^\nu$

Meaning? The local energy-momentum flow

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- The "dominant energy condition":

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i.e. $T_{\mu\nu} v^\nu$ is non-space-like.

where v is any timelike vector and $K_\mu := T_{\mu\nu} v^\nu$

Meaning? The local energy-momentum flow vector K may not be conserved but has to be non-space-like: Flow should be into the future ← need for causality.

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The "strong energy condition"

Matter is said to obey the strong energy condition iff:

$$\left(T_{\mu\nu} - \frac{1}{2} T^{\rho}{}_{\rho}\, g_{\mu\nu}\right) \xi^{\mu}\xi^{\nu} \geq 0 \quad \forall \text{ timelike } \xi.$$

- Intuition? Excludes matter that causes accelerated expansion. (as we will discuss below)
- Plausible? Yes, obeyed by known matter. (but not by dark energy)
- Relationship? Independent of weak and dominan

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- **Relationship?** Independent of weak and dominant energy condition.

Concretely: For known matter, $T_{\mu\nu}$ is diagonalizable to obtain:

energy density observed by comoving observer

$$T_{\mu\nu} = \begin{pmatrix} \rho & & & 0 \\ & p_1 & & \\ & & p_2 & \\ 0 & & & p_3 \end{pmatrix}$$

principal pressures

The energy conditions then read:

- **Weak:** $\rho \geq 0$ and $\rho + p_i \geq 0$ for $i \in \{1,2,3\}$
- **Dominant:** $\rho \geq |p_i|$ for $i \in \{1,2,3\}$

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The energy conditions then read:

- Weak: $\rho \geq 0$ and $\rho + p_i \geq 0$ for $i \in \{1,2,3\}$

- Dominant: $\rho \geq |p_i|$ for $i \in \{1,2,3\}$

Exercise: Show this →
- Strong: $\rho + \sum_{i=1}^{3} p_i \geq 0$ and $\rho + p_i \geq 0$ for $i \in \{1,2,3\}$

Note: could possibly be also negative.

Recall: A cosmological constant Λ can be viewed as a contribution to $T_{\mu\nu}$.

Indeed, there is no big bang singularity, e.g., if $w = -1\ \forall t$, i.e., in de Sitter spacetime inflation $a(t) = e^{Ht}$.

Exercise: Show that the strong energy condition is violated in cosmology iff $w < -\frac{1}{3}$, i.e., iff the expansion is accelerating: $\ddot{a}(t) > 0$.

Essence of point e):

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Given, in particular, the strong energy condition, one can show that geodesics meet a divergence of a quantity called expansion, θ, in finite proper time:

important notion also e.g. in study of grav. collaps of stars.

The "expansion", θ:

- Consider a "congruence of timelike geodesics"

e.g., freely falling dust.

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Given, in particular, the strong energy condition, one can show that geodesics meet a divergence of a quantity called expansion, θ, in finite proper time:

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The "expansion", θ:

- Consider a "congruence of timelike geodesics" through Σ, i.e., a smooth family of timelike geodesics, exactly one through each $p \in \Sigma$. If parametrized by proper

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Consider a "congruence of timelike geodesics"

e.g., freely falling dust.

through Σ, i.e., a smooth family of timelike geodesics, exactly one through each $p \in \Sigma$. If parametrized by proper time, their tangent vector field ξ, namely

$$\xi := \frac{d}{d\tau}$$

proper time

will obey : $g(\xi,\xi) = -1 \ \forall p$.

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$$\xi := \frac{d}{d\tau}$$ ← proper time

will obey : $g(\xi,\xi) = -1 \;\; \forall p$.

Consider now a one-parameter sub family of these geodesics :

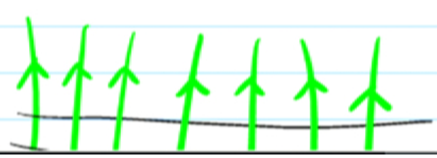
$$\gamma(\tau, s)$$

↑ parameter of family of neighboring geodesics.

↓ a "connecting vector field"

Then, we define the deviation vector :

$$\eta := \frac{d}{ds}$$



a line of constant τ value

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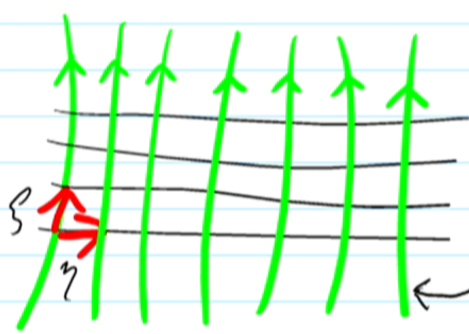

a line of constant τ value

a geodesic, i.e., a line of constant s value

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- How does η change along a geodesic?

τ, s are Riemann normal coordinates for a geodesic traveller.

$$\Rightarrow \frac{d}{d\tau}\frac{d}{ds} = \frac{d}{ds}\frac{d}{d\tau}, \text{ i.e., } [\xi, \eta] = 0$$

- Since the torsion vanishes: $0 = T(\xi, \eta) = \nabla_\xi \eta - \nabla_\eta \xi - [\xi, \eta]$

$$\Longrightarrow \quad \nabla_\xi \eta = \nabla_\eta \xi$$

$$\Rightarrow \quad \xi^\mu \nabla_{e_\mu} \eta^\nu e_\nu = \eta^\alpha \nabla_{e_\alpha} \xi^\beta e_\beta$$

$$\Rightarrow \quad \xi^\mu \eta^\nu{}_{;\mu} e_\nu = \eta^\alpha \xi^\beta{}_{;\alpha} e_\beta$$

$$\Rightarrow \quad \xi^\mu \eta^\nu{}_{;\mu} = \eta^\alpha \xi^\nu{}_{;\alpha} = \eta^\alpha B^\nu{}_\alpha \text{ for } \boxed{B^\nu{}_\mu := \xi^\nu{}_{;\mu}}$$

$\Rightarrow$ Along the geodesic's direction, ξ, the deviation vector η^μ changes its direction and length by $B^\nu{}_\mu \eta^\mu$.

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$\Rightarrow$ Along the geodesic's direction, ξ, the deviation vector η^μ changes its direction and length by $B^\nu{}_\mu \eta^\mu$.

□ The tensor $B^\nu{}_\mu$ can be decomposed covariantly and uniquely into:

Symmetric and trace = 0

$$B_{\mu\nu} = \omega_{\mu\nu} + \sigma_{\mu\nu} + t_{\mu\nu}$$

antisymmetric

rest

Cosmic ballet tensor field.

(all 3 terms are tensors because the split is covariant)

We have: $\omega_{\mu\nu} = \frac{1}{2}\left(B_{\mu\nu} - B_{\nu\mu}\right)$, clearly.

But $\sigma_{\mu\nu}, t_{\mu\nu} = ?$

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We have: $\omega_{\mu\nu} = \frac{1}{2}(B_{\mu\nu} - B_{\nu\mu})$, clearly.

But $\sigma_{\mu\nu}, t_{\mu\nu} = ?$

In preparation: define the projector $h_{\mu\nu}$ onto $(\mathbb{R}\xi)^{\perp}$ i.e. onto the spatial components:

is timelike

$$h_{\mu\nu} := g_{\mu\nu} + \xi_{\mu}\xi_{\nu}$$

Check: is $h_{\mu\nu} w^{\nu}$ really always $\perp$ to ξ ?

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But $G_{\mu\nu}, t_{\mu\nu} = ?$

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$$h_{\mu\nu} := g_{\mu\nu} + \xi_\mu \xi_\nu$$

Check: is $h_{\mu\nu} w^\nu$ really always $\perp$ to ξ ?

Indeed: $\xi^\mu h_{\mu\nu} w^\nu = (\xi, w) + \underbrace{(\xi,\xi)}_{=-1}(\xi, w) = 0$

Define: The "expansion" θ is defined as the

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Define: The "expansion", θ, is defined as the magnitude of the spatial part of B:

$$\theta := B^{\mu\nu} h_{\mu\nu}$$

Claim: $Tr(B) = \theta$

Indeed:

$$\theta = B^{\mu\nu} h_{\mu\nu} = B^{\mu\nu} g_{\mu\nu} + \xi^\mu \xi_\nu B_\mu{}^\nu$$

$$= Tr(B) + \xi^\mu \underbrace{\xi_\nu \nabla_\mu \xi^\nu}$$

($= 0$ because $\nabla_\xi \xi = 0$ for geodesics.)

Therefore:

$$\sigma_{\mu\nu} = \frac{1}{2}\left(B_{\mu\nu} + B_{\nu\mu}\right) - \frac{1}{3}\theta h_{\mu\nu}$$

(because: $Tr(h_{\mu\nu}) = g^{\mu\nu} h_{\mu\nu}$ $= g^{\mu\nu}(g_{\mu\nu} + \xi_\mu \xi_\nu)$ $= 4 - 1$)

the part of $B_{\mu\nu}$ which is symmetric and traceless.

and:

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Therefore:
$$\sigma_{\mu\nu} = \frac{1}{2}(B_{\mu\nu} + B_{\nu\mu}) - \frac{1}{3} \Theta h_{\mu\nu}$$

(because: $\mathrm{Tr}(h_{\mu\nu}) = g^{\mu\nu} h_{\mu\nu}$
$= g^{\mu\nu}(g_{\mu\nu} + \xi_\mu \xi_\nu)$
$= 4 - 1$)

the part of $B_{\mu\nu}$ which is symmetric and traceless.

and:
$$t_{\mu\nu} = \frac{1}{3} \Theta h_{\mu\nu}$$
← the "rest term".

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Interpretation:

a) $\omega_{\mu\nu}$ is antisymmetric: $\omega_{\mu\nu} = -\omega_{\nu\mu}$

$\Rightarrow$ it generates Lorentz transformation for η.

but all η are $\perp$ to the time direction

$\Rightarrow$ $\omega_{\mu\nu}$ generates spatial rotations of neighboring geodesics around another. So, $\omega_{\mu\nu}$ is called

$\omega =$ "Twists tensor"

One can prove: (nontrivial)

If one chooses the convenience ...

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$\omega =$ "Twists tensor"

One can prove: (nontrivial)

If one chooses the congruence of geodesics $\perp$ to Σ then $\omega_{\mu\nu} = 0$.

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b.) $\sigma_{\mu\nu}$ is symmetric, $\sigma_{\mu\nu} = \sigma_{\nu\mu}$. (i.e. hermitean)

Consider "diagonalized", by suitable choice of cd basis.

$\Rightarrow$ $\sigma_{\mu\nu}$ changes the relative lengths of the basis vectors, by multiplying them with its eigenvalues.

i.e. points on a sphere will under geodesic flow $\rightarrow$ become points on an ellipsoid.

Note: Since $Tr(\underbrace{\sigma}_{\text{infinitesimal transport along geodesics}}) = 0$ we have $\det(\underbrace{e^{\alpha\sigma}}_{\text{finite transport}}) = 1$

$\Rightarrow$ The volume spanned by basis vectors stays the same under the action of σ.

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infinitesimal transport along geodesics

Note: Since $\mathrm{Tr}(\sigma) = 0$ we have $\det(\underbrace{e^{\alpha\sigma}}_{\text{finite transport}}) = 1$

⇒ The volume spanned by basis vectors stays the same under the action of σ.

⇝ **Definition:** $\sigma_{\mu\nu} =:$ "Shear tensor" 

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c.) While the twist and shear tensors are both traceless and therefore volume-preserving, we see that the trace part, θ, i.e., more precisely

$$t_{\mu\nu} = \frac{1}{3}\,\theta\, h_{\mu\nu} =: \text{"Expansion tensor"}$$

recall: is projector on spatial part.

is indeed generating the spatial expansion or contraction of nearby geodesics!

Evolution of θ along a geodesic?

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