Title: AMATH 875/PHYS 786 - Fall 2015 - Lecture 20

Date: Nov 13, 2015 01:30 PM

URL: http://pirsa.org/15110003

Abstract: <p>Course Description coming soon.</p>

File Edit View Insert Actions Tools Help

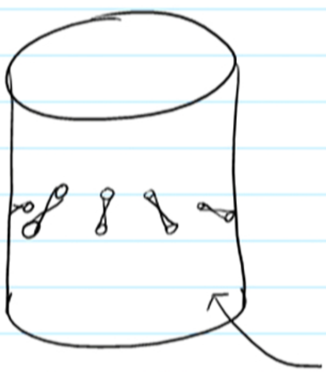
Causal Structure & "Singularities"

Definition:

We say that (M, g) is <u>time-orientable</u> if at each point $p \in M$ we can separate the non-spacelike vectors

$$\xi \in T_p(M), \; g(\xi, \xi) \leq 0$$

into two classes, which will be called future-directed and past-directed so that this separation is continuous in M.



Consider e.g. such a spacetime, which is the outside of a cylinder.

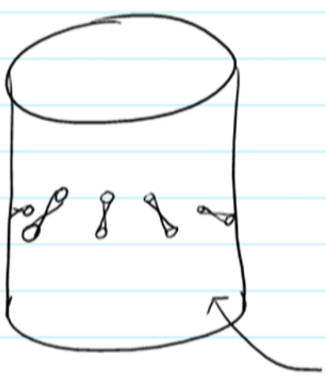
1 / 22

File Edit View Insert Actions Tools Help

We say that (M, g) is time-orientable if at each point $p \in M$ we can separate the non-spacelike vectors

$$\xi \in T_p(M), \; g(\xi, \xi) \leq 0$$

into two classes, which will be called future-directed and past-directed so that this is separation is continuous in M.



Consider e.g. such a spacetime, which is the outside of a cylinder. If the metric has the drawn light cones, M is not (continuously) time-orientable.

1 / 22

File Edit View Insert Actions Tools Help

Lemma:

Iff (M, g) is time-orientable, then there exist smooth nonvanishing timelike vector fields, v, on M.

Note:

If such a v is given, we can define at each $p \in M$ the future-directed vectors ξ as those non-spacelike vectors for which $g(v, \xi) > 0$, i.e. the past-directed non-spacelike ξ obey $g(v, \xi) < 0$. (Alternatively the "future" and "past" designations can be globally swapped.)

Why? Consider basis in $T_p(M)$ so that $g_{\mu\nu} = \eta_{\mu\nu}$

2 / 22

File Edit View Insert Actions Tools Help

exist smooth nonvanishing timelike vector fields, v, on M.

Note:

If such a v is given, we can define at each $p \in M$ the future-directed vectors ξ as those non-spacelike vectors for which $g(v,\xi) > 0$, i.e. the past-directed non-spacelike ξ obey $g(v,\xi) < 0$. (Alternatively the "future" and "past" designations can be globally swapped.)

Why? Consider basis in $T_p(M)$ so that $g_{\mu\nu} = \eta_{\mu\nu}$ and $v = \begin{pmatrix} v_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. Then $g(v,\xi) = -v_0 \xi_0$, i.e., "sign" of ξ is well-defined relative to v.

2 / 22

File Edit View Insert Actions Tools Help

Assumption: We will henceforth only consider time-orientable space-times, and we will always assume a time orientation has been chosen.

Time-like curves

Definition: A curve (or "path") $\gamma : \underset{\mathbb{R}}{\cup} J \to M$ is called a "future-directed timelike curve"

if each $\dot{\gamma}(t) \in T_{\gamma(t)}(M)$ is a future-directed timelike vector.

Note: there is no restriction to geodesics. These are *all* paths that material objects can travel, given suitable engine or outside

3 / 22

File Edit View Insert Actions Tools Help

"future-directed causal curve"

if each $\dot{\gamma}(t) \in T_{\gamma(t)}(\mathcal{M})$ is a future-directed non-spacelike vector.

Note: These are all paths that matter or massless particles (light, gluons etc) can travel.

Note: Here, $\dot{\gamma}(t) = 0$ is allowed because non-spacelikeness merely requires $g(\dot{\gamma}, \dot{\gamma}) \leq 0$.

Definition:

Past-directed timelike or causal curves are defined analogously.

4 / 22

File Edit View Insert Actions Tools Help

The past & future of an event

Definition:

The "chronological future" of a $p \in M$ is the set

$$I^+(p)$$

of events that can be reached from p on a future-directed timelike curve.

Mnemonic help: $I^+(p)$ is the set of events an actual chronometer could reach from p.

Note: $I^+(p)$ is always an open subset of M (because if γ is a timelike future-directed curve from p then any sufficiently small perturbation of γ is

5 / 22

File Edit View Insert Actions Tools Help

We will assume that there are no closed timelike curves. For travellers on such curves, life would repeat itself but we know that everybody ages, so it can't. I.e. on such spacetimes there'd be no thermodynamic arrow of time in whose direction entropy increases. →

Note: Since $\mathfrak{j}e = 0$ is excluded we have

$$p \notin I^+(p)$$

except if there is a closed timelike curve through p.

Definition:

The "causal future" of a $p \in M$ is the set

$$J^+(p)$$

In $J^+(p)$ is any event that could be causally affected by what happened at p.

6 / 22

File Edit View Insert Actions Tools Help

Definition:

The "causal future" of a $p \in M$

is the set

$$J^+(p)$$

of events that can be reached from p on a future-directed causal curve.

In $J^+(p)$ is any event that could be causally affected by what happened at p.

Note: Clearly, we always have $p \in J^+(p)$ because $\dot{x} = 0$ is allowed.

Definitions:

- Analogously, one defines the chronological

6 / 22

File Edit View Insert Actions Tools Help

Definitions:

- Analogously, one defines the chronological past $I^-(p)$ and the causal past $J^-(p)$.

- One defines the pasts and futures of a set S of events through:

$$I^{\pm}(S) := \bigcup_{p\in S} I^{\pm}(p), \quad J^{\pm}(S) := \bigcup_{p\in S} J^{\pm}(p)$$

- Their boundaries are denoted: $\dot{I}^{\pm}(S), \dot{J}^{\pm}(S)$

Example: Assume $S := \{$events travelled by immortal observer$\}$

Then $\dot{J}^-(S) =$ event horizon of this observer.

7 / 22

File Edit View Insert Actions Tools Help

Relation to geodesics:

- The above definitions do not refer to geodesics.
- For Minkowski space, clear:

  $I^+(p)$ = set of events reachable by future-directed timelike geodesics from p.

  $\dot{I}^+(p)$ = set of events reachable by future-directed null geodesics from p.

  But this is not true in general spacetimes!

<u>Intuition:</u> E.g. singularities can be in the

8 / 22

File Edit View Insert Actions Tools Help

The above definitions do not refer to geodesics.

- For Minkowski space, clear:

  $I^+(p)$ = set of events reachable by future-directed timelike geodesics from p.

  $\dot{I}^+(p)$ = set of events reachable by future-directed null geodesics from p.

But this is not true in general spacetimes!

Intuition: E.g. singularities can be in the way of a geodesic.

8 / 22

File Edit View Insert Actions Tools Help

Theorem:

Assume $p \in M$ and $U \subset M$ is a convex normal neighborhood of p.

(Any 2 pts in U are connected by a unique geodesic.)

Then: a) $I^+(p)\big|_U$ = set of events reachable by future-directed timelike geodesics from p.

(means "restricted to")

b) $\dot{I}^+(p)\big|_U$ = set of events reachable by future-directed null geodesics from p.

c) If $q \in J^+(p) - I^+(p)$ then any causal curve between p and q is a null geodesic.

9 / 22

File Edit View Insert Actions Tools Help

Recall:

- The "chronological future" of a set S is the set

$$I^+(S)$$

of events that can be reached from S on a future-directed timelike curve.

- The "causal future" of a set S is the set

$$J^+(S)$$

of events that can be reached from S on a future-directed causal curve.

recall: includes null curves.

10 / 22

File Edit View Insert Actions Tools Help

- In Minkowski space: $J^+(S) = I^+(S) \cup \dot{I}^+(S)$

Recall: $\dot{I}^+(S)$ is the boundary of $I(S)$

Properties of $\dot{I}^+(S)$, in general?

- **Definition:**

A subset $Q \subset M$ is called

"achronal"

if no two points in Q can be

10 / 22

File Edit View Insert Actions Tools Help

Definition:

A subset $Q \subset M$ is called

"achronal"

if no two points in Q can be connected by a future-directed time-like curve, i.e., by a curve that, e.g., a clock (with mass) could travel. Thus, $Q \subset M$ is achronal iff:

$$I^+(Q) \cap Q = \emptyset$$

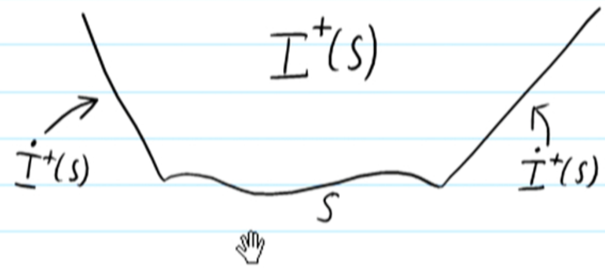
← empty set

Theorem:

For all $S \subset M$, the set

$$\dot{I}^+(S)$$

(if not empty) is an achronal 3-dimensional submanifold of M.



Example: In Minkowski space, if S is a point p, then $\dot{I}^+(p)$ is the boundary of the light cone.

Indeed, no two points of the boundary

File Edit View Insert Actions Tools Help

Example: In Minkowski space, if S is a point p, then $\dot{I}^+(p)$ is the boundary of the light cone.

Indeed, no two points of the boundary of the lightcone are connected by time-like paths.

In general spacetimes, however:

It is clear that

$$\overline{J^+(S)} = \overline{I^+(S)}$$

← bar denotes closure of the set

but we notice that generally:

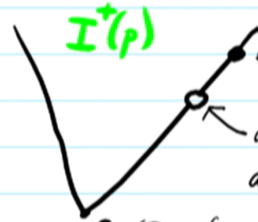
12 / 22

File Edit View Insert Actions Tools Help

but we notice that, generally:

$$J^+(S) \neq \overline{I^+(S)}$$

Example:



⇒ here, $q \in \overline{I^+(p)}$, but $q \notin J^+(p)$ because there is no nonspacelike curve between p and q.

⇝ Idea:

Let us use the ...

13 / 22

File Edit View Insert Actions Tools Help

→ **Idea:**

Let us use the extendibility or nonextendibility of curves (or especially of geodesics) as indicator for the absence or existence of a singularity.

Definition:

- We say that a point $p \in M$ is future (past) endpoint of a curve γ if

  $\forall$ neighborhoods U of p there exists a $t_0 \in \mathbb{R}$ so that

$\gamma(t) \subset U \quad \forall\, t > t_0 \qquad (t < t_0$ for past)

14 / 22

File Edit View Insert Actions Tools Help

Definition:

- We say that a curve γ is future (past) inextendible if it does not possess a future (past) endpoint.
- Intuition: If γ future inextendible, then:

  a.) γ runs to ∞, or

  b.) γ runs around forever, or

  c.) γ hits a hole, i.e., singularity, thus can't be continuously extended:

Theorem:

Assume $\zeta \subset M$ is closed and assume:

15 / 22

File Edit View Insert Actions Tools Help

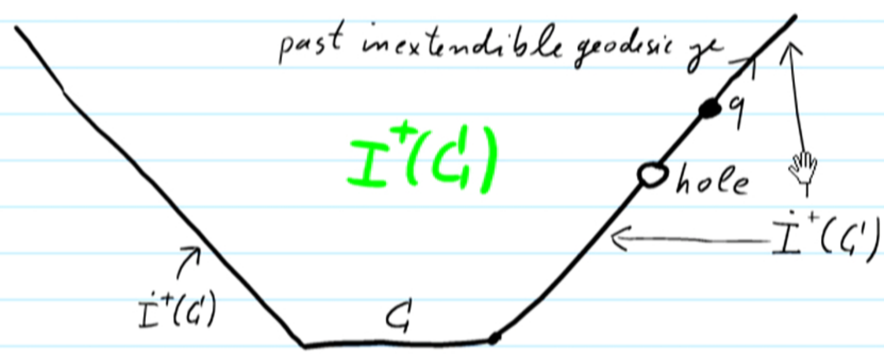
$$q \in \dot{I}^+(G') \quad , \quad q \notin G'$$

Then, q lies on a null geodesic γ inside $\dot{I}^+(G')$ and the curve γ either:

a.) has past endpoint on G'

or b.) is past inextendible (because meets hole)

Example for b:



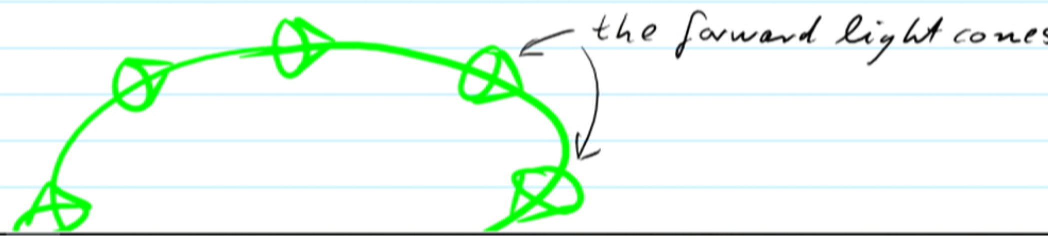
File Edit View Insert Actions Tools Help

Strategy:

- Study inextendible curves!
- This includes these cases:
  - a.) γ hits singularity – will be main interest!
  - b.) γ running off to ∞.
  - c.) γ going round and round forever.

⇝ Must address potential causality problems of case c)

Example:

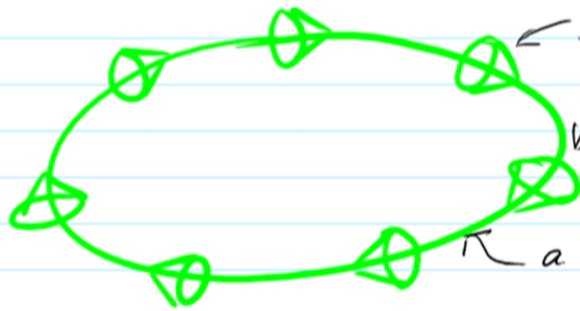


17 / 22

File Edit View Insert Actions Tools Help

⇝ Must address potential causality problems of case c)

Example:



Note: This is not a problem with time-orientability here!

Causality conditions:

- We say that (M, g) is "causal" if it does not contain closed causal (i.e. time or null) curves.

Problem: (M, g) may nevertheless be arbitrarily close

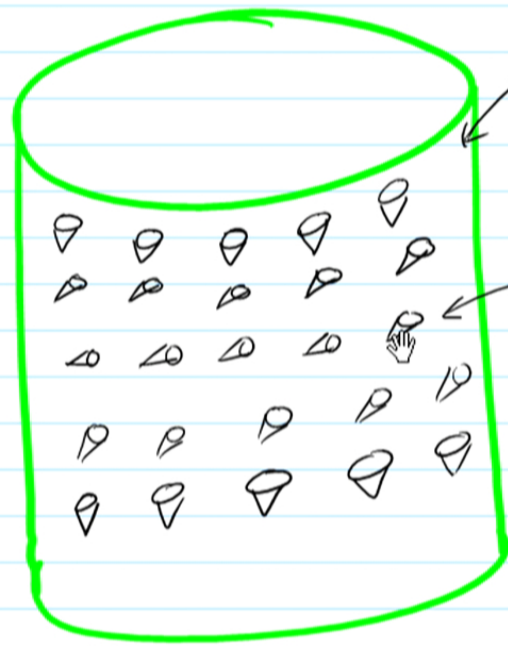
17 / 22

File Edit View Insert Actions Tools Help

contain closed causal (i.e. time or null) curves.

Problem: (M,g) may nevertheless be arbitrarily close to being acausal:

(M,g) is the boundary of a cylinder.



those light cones could be arbitrarily close to the picture above i.e. to allowing a closed timelike curve!

18 / 22

File Edit View Insert Actions Tools Help

→ ▫ We say that a spacetime $(\mathcal{M},g)$ is "strongly causal", if

$\forall p$ and $\forall$ neighborhoods $\mathcal{U}$ of p there is a neighborhood $V \subset \mathcal{U}$ so that:

No causal curve γ intersects V more than once.

▫ Indeed:

If $(\mathcal{M},g)$ is not strongly causal $\Rightarrow$ there exists a causal curve γ which comes arbitrarily close to intersecting itself.

▫ → We require strong causality to keep causal curves

19 / 22

File Edit View Insert Actions Tools Help

Problem:

Still, arbitrarily small perturbations in the metric, somewhere, could allow causal curves to self-intersect!

Solution:

a) Consider perturbing the metric g through

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = g_{\mu\nu} - \omega_\mu \omega_\nu$$

with a time-like cotangent vector field.

↑ needed for theorem 2 below.

b) Notation:

20 / 22

File Edit View Insert Actions Tools Help

Indeed:

If $(\mathcal{M}, g)$ is not strongly causal $\Rightarrow$ there exists a causal curve γ which comes arbitrarily close to intersecting itself.

$\rightsquigarrow$ We require strong causality to keep causal curves at least a finite distance from intersecting themselves.

Problem:

Still, arbitrarily small perturbations in the metric, somewhere, could allow causal curves to self-intersect!

19 / 22

Solution:

a) Consider perturbing the metric g through

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = g_{\mu\nu} - \omega_\mu \omega_\nu$$

with a time-like cotangent vector field.
↑ needed for theorem 2 below.

b) Notice: $\tilde{g}_{\mu\nu}$ still has same signature but light cones are now "wider":

Compare $v^\mu v^\nu \tilde{g}_{\mu\nu}$ and $v^\mu v^\nu g_{\mu\nu}$:

$$v^\mu v^\nu \tilde{g}_{\mu\nu} = v^\mu v^\nu g_{\mu\nu} - v^\mu \omega_\mu v^\nu \omega_\nu < v^\mu v^\nu g_{\mu\nu}$$

File Edit View Insert Actions Tools Help

Compare $v^\mu v^\nu \tilde{g}_{\mu\nu}$ and $v^\mu v^\nu g_{\mu\nu}$:

$$v^\mu v^\nu \tilde{g}_{\mu\nu} = v^\mu v^\nu g_{\mu\nu} \underbrace{- v^\mu \omega_\mu v^\nu \omega_\nu}_{\text{always } < 0} < v^\mu v^\nu g_{\mu\nu}$$

Thus, it is easier for vectors v to be timelike or null for $\tilde{g}$ than for g.

$\Longrightarrow$

$(M, \tilde{g})$ has all the causal curves of (M, g), and more!

c) Define:

(M, g) is called "stably causal", if

21 / 22

File Edit View Insert Actions Tools Help

always < 0

Thus, it is easier for vectors v to be timelike or null for $\tilde{g}$ than for g.

$\Rightarrow$

$(M, \tilde{g})$ has all the causal curves of (M, g), and more!

c) Define:

(M, g) is called "stably causal", if there exists a ω so that even $(M, \tilde{g})$ is causal.

21 / 22

File Edit View Insert Actions Tools Help

Theorem 2: (M, g) stably causal

$\Updownarrow$

There exists a differentiable function $f \in \mathcal{F}(M)$ so that ∇f is a past-directed time-like vector field.

Remark: This means that f can be viewed as a cosmic "clock". (It is not unique, however)

Recall: Time-orientability $\Leftrightarrow$ $\exists$ past-pointing smooth timelike vector field. (which need not be a gradient field)

22 / 22