Title: Localization on twisted spheres and supersymmetric GLSM in 2d

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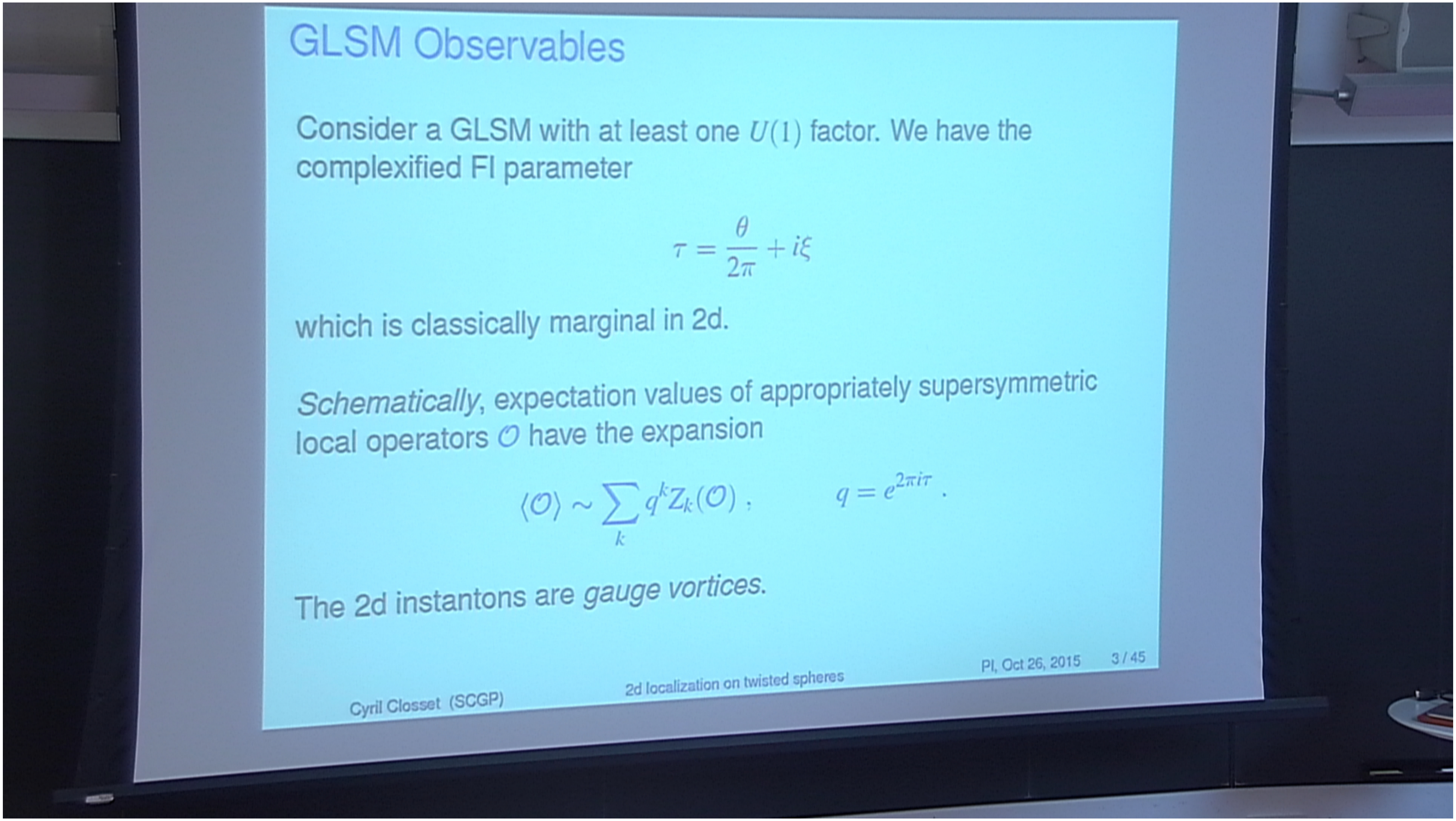
Abstract: <p>I will revisit the A-twisted gauged linear sigma model (GLSM) in the case of (2,2) supersymmetry in two dimensions, and its Omega-background deformation. Exact results for correlation functions on the sphere can be obtained in terms of Jeffrey-Kirwan residues on the Coulomb branch, which has a number of interesting applications. I will also explain an interesting generalization to (0,2) supersymmetric GLSMs of a special type.</p>


GLSM Observables
Consider a GLSM with at least one $U(1)$ factor. We have the complexified FI parameter
$$\tau = \frac{\theta}{2\pi} + i\xi$$
which is classically marginal in 2d.
Schematically, expectation values of appropriately supersymmetric local operators $\mathcal{O}$ have the expansion
$$\langle \mathcal{O} \rangle \sim \sum_k q^k Z_k(\mathcal{O}) \,, \qquad q = e^{2\pi i \tau} \,.$$
The 2d instantons are *gauge vortices*.
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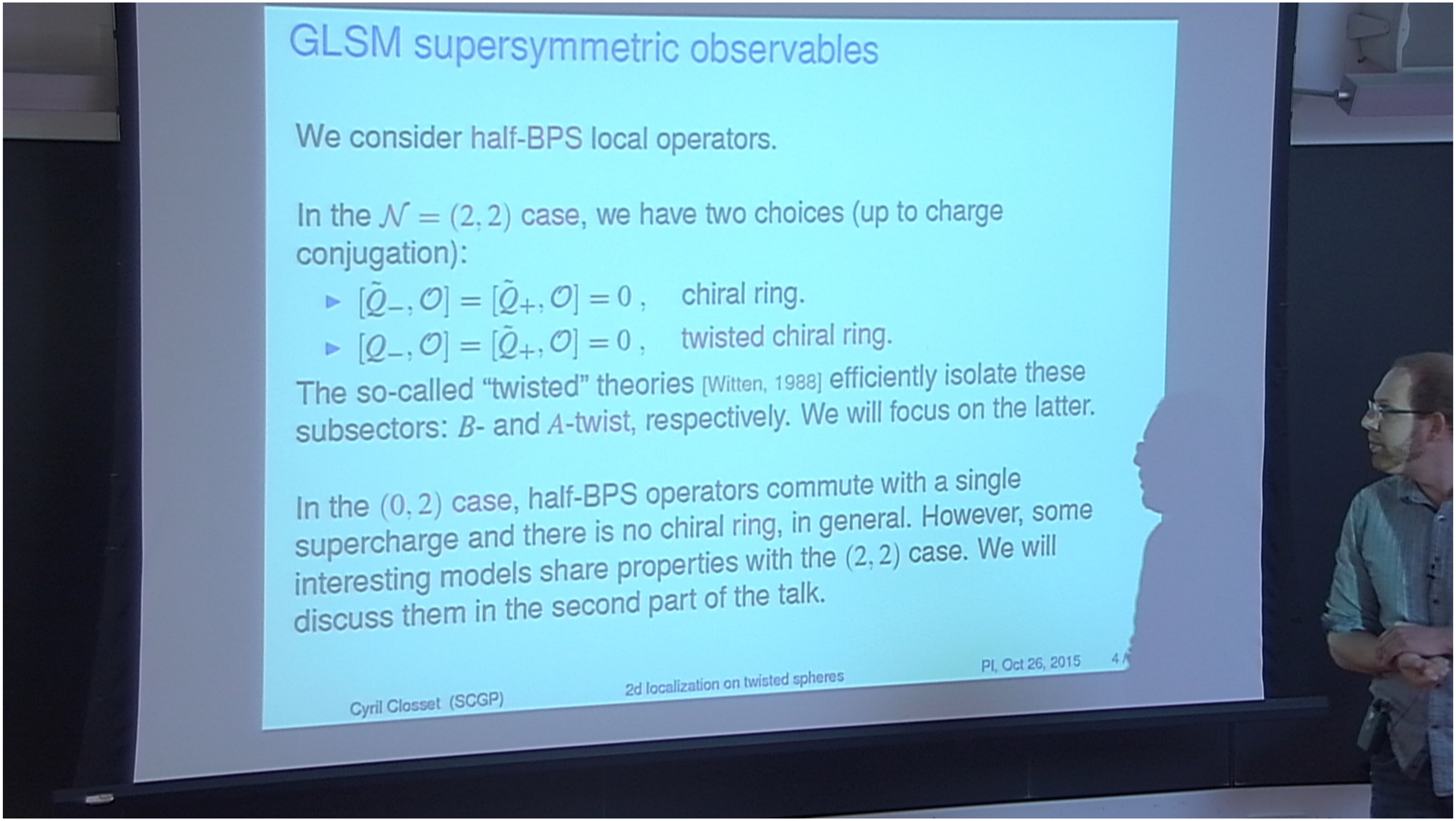
2d localization on twisted spheres
PI, Oct 26, 2015
3 / 45


GLSM supersymmetric observables
We consider half-BPS local operators.
In the $\mathcal{N} = (2, 2)$ case, we have two choices (up to charge conjugation):
$[\tilde{Q}_-, \mathcal{O}] = [\tilde{Q}_+, \mathcal{O}] = 0$, chiral ring.
$[Q_-, \mathcal{O}] = [\tilde{Q}_+, \mathcal{O}] = 0$, twisted chiral ring.
The so-called "twisted" theories [Witten, 1988] efficiently isolate these subsectors: B- and A-twist, respectively. We will focus on the latter.
In the $(0, 2)$ case, half-BPS operators commute with a single supercharge and there is no chiral ring, in general. However, some interesting models share properties with the $(2, 2)$ case. We will discuss them in the second part of the talk.
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PI, Oct 26, 2015
4

$S^2_{\epsilon_\Omega}$ correlators for $(2,2)$ theories

We will consider correlations of twisted chiral ring operators on the Ω-deformed sphere,

$$\langle \mathcal{O} \rangle_{S^2_\Omega} .$$

This Ω-background constitutes a one-parameter deformation of the A-twist at genus zero.

We will derive a formula for GLSM supersymmetric observables on S^2_Ω of the schematic form:

$$\langle \mathcal{O} \rangle = \sum_k q^k \oint_C d^r\sigma \, Z_k^{\text{1-loop}}(\sigma) \, \mathcal{O}(\sigma) ,$$

valid for any standard GLSM. This results simplifies previous computations [Morrison, Plesser, 1994; Szenes, Vergne, 2003] and generalizes them to non-Abelian theories.

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Some further motivations

In field theory:

- These 2d $\mathcal{N} = (2,2)$ theories appear on the worldvolume of *surface operators* in 4d $\mathcal{N} = 2$ theories.
- Our 2d setup can also be uplifted to 4d $\mathcal{N} = 1$ on $S^2 \times T^2$.
  [C.C., Shamir, 2013, Benini, Zaffaroni, 2015, Gadde, Razamat, Willett, 2015]

In string theory or "quantum geometry":

- Think in terms of a target space X_d with $\xi \sim \mathrm{vol}(X_d)$. New localization results can give new tools for enumerative geometry.
  [Jockers, Kumar, Lapan, Morrison, Romo, 2012]
- The $(0,2)$ results are relevant for heterotic string compactifications.

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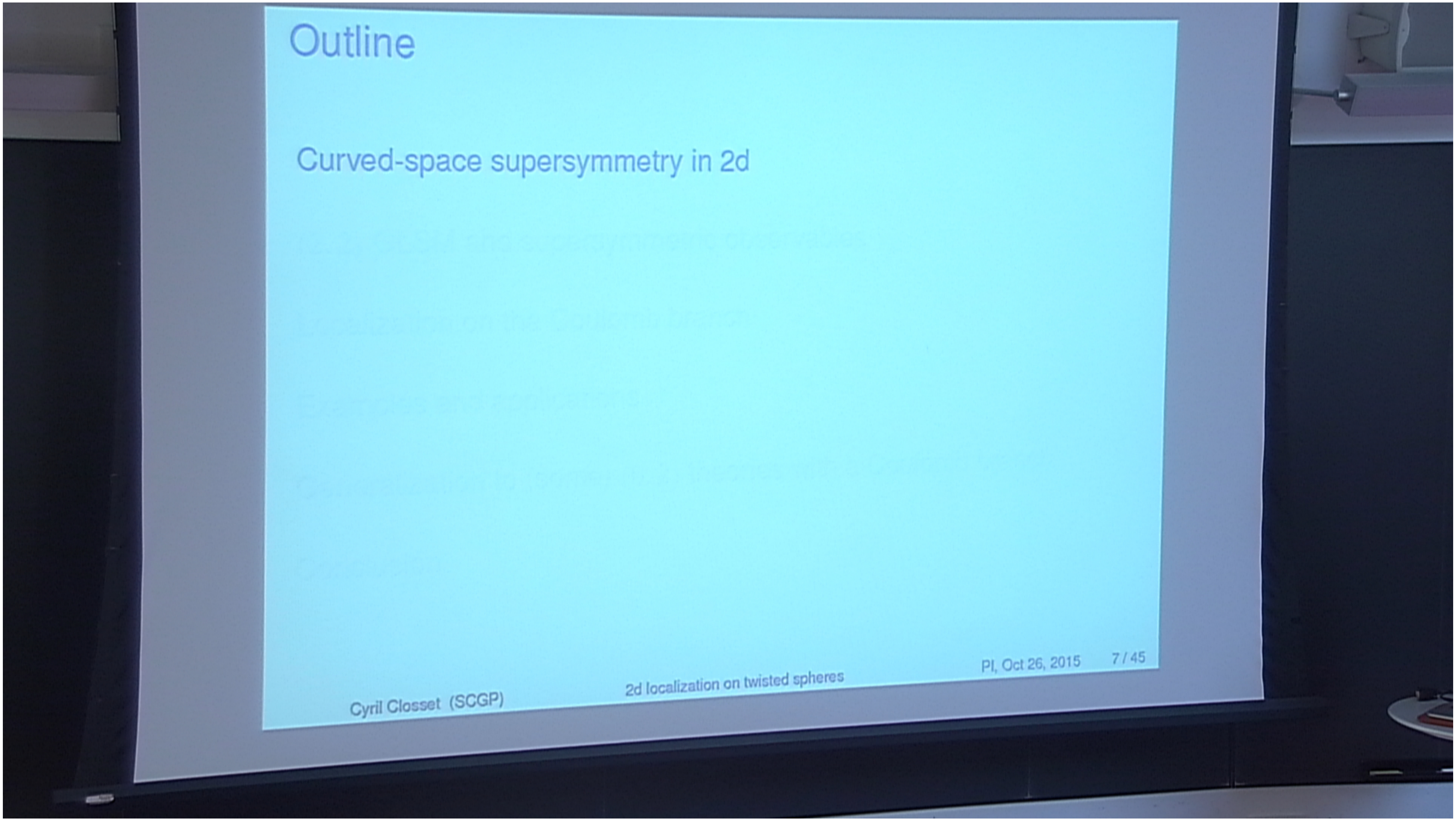
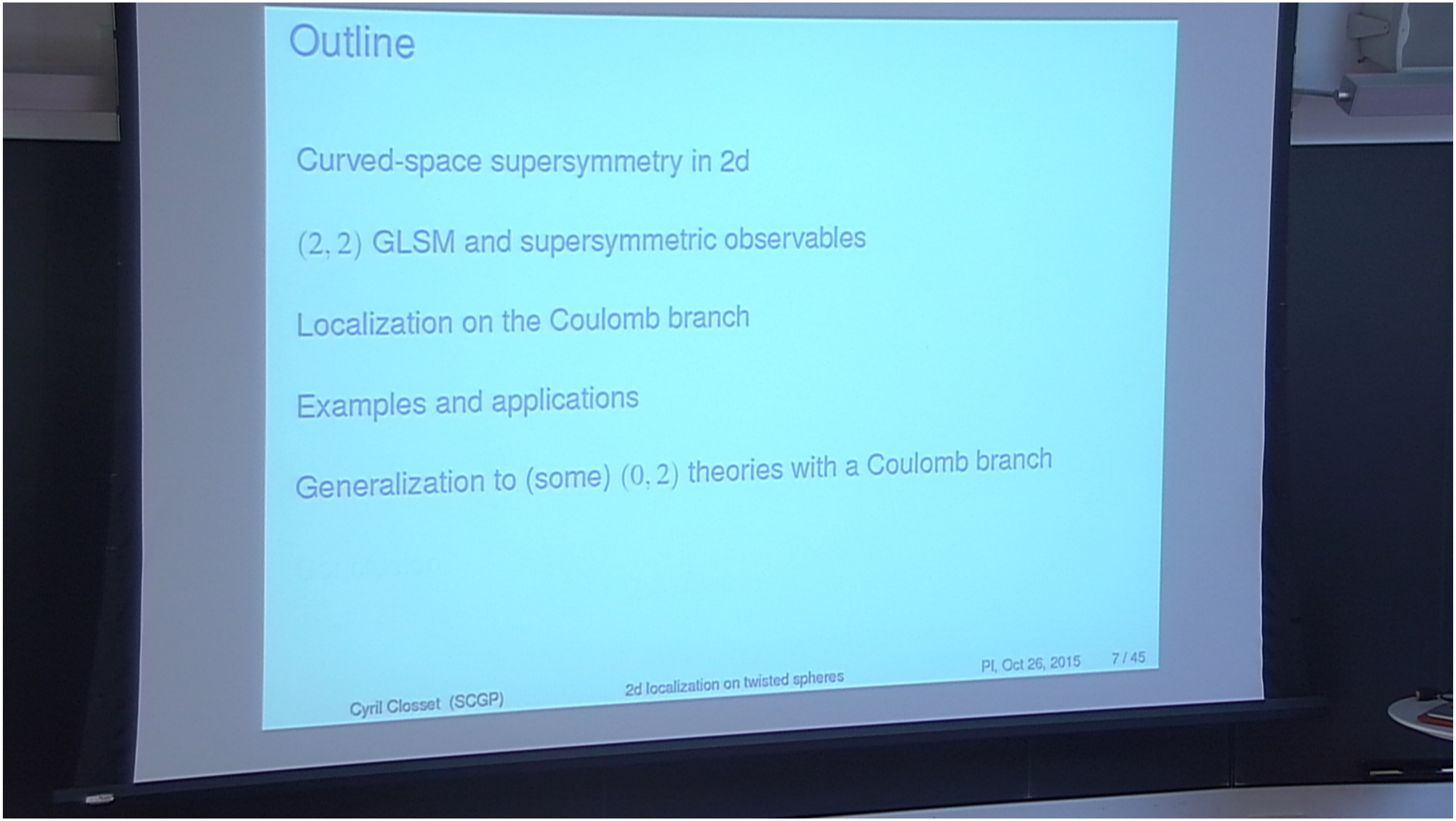

Outline
Curved-space supersymmetry in 2d
Cyril Closset (SCGP)
2d localization on twisted spheres
PI, Oct 26, 2015
7 / 45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Curved-space supersymmetry in 2d
$(2,2)$ GLSM and supersymmetric observables
Localization on the Coulomb branch
Examples and applications
Generalization to (some) $(0,2)$ theories with a Coulomb branch
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PI, Oct 26, 2015
7 / 45

Curved-space $(2,2)$ supersymmetry

The first step is to define the theory of interest in *curved space*, while preserving some supersymmetry. A systematic way to do this is by coupling to background supergravity. [Festuccia, Seiberg, 2011]

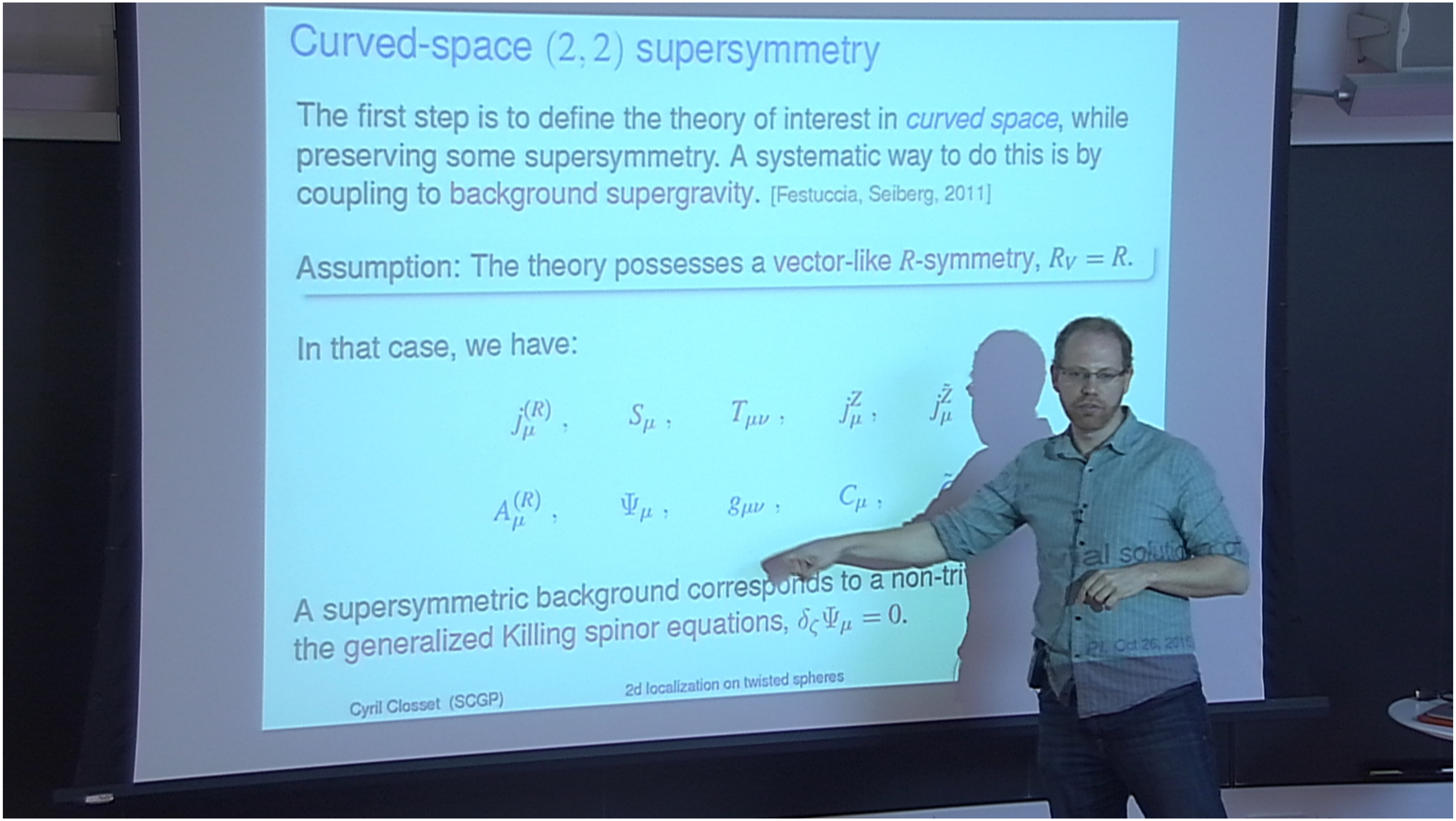
Assumption: The theory possesses a vector-like R-symmetry, $R_V = R$.

In that case, we have:

$$j_\mu^{(R)}\,, \qquad S_\mu\,, \qquad T_{\mu\nu}\,, \qquad j_\mu^{Z}\,, \qquad j_\mu^{\tilde Z}$$

$$A_\mu^{(R)}\,, \qquad \Psi_\mu\,, \qquad g_{\mu\nu}\,, \qquad C_\mu\,, \qquad \tilde C_\mu$$

A supersymmetric background corresponds to a non-trivial solution of the generalized Killing spinor equations, $\delta_\zeta \Psi_\mu = 0$.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PI, Oct 26, 2015

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Supersymmetric backgrounds on S^2

On the sphere, we can have:

$$\frac{1}{2\pi}\int_{S^2} dA^R = 0\,, \qquad \frac{1}{2\pi}\int_{S^2} dC = \frac{1}{2\pi}\int_{S^2} d\tilde{C} = 1$$

This was studied in detail in [Doroud, Le Floch, Gomis, Lee, 2012; Benini, Cremonesi, 2012]. In this case, the R-charge can be arbitrary but the real part of the central charge, $Z+\tilde{Z}$, is constrained by Dirac quantization.

The second possibility is

$$\frac{1}{2\pi}\int_{S^2} dA^R = 1\,, \qquad \frac{1}{2\pi}\int_{S^2} dC = \frac{1}{2\pi}\int_{S^2} d\tilde{C} = 0$$

This is the case of interest to us. Note that the R-charges must be integers, while $Z, \tilde{Z}$ can be arbitrary.

Equivariant A-twist, a.k.a. Ω-deformation

Consider this latter case. We preserve two supercharges if the metric on S^2 has a $U(1)$ isometry with Killing vector V^μ. This gives a one-parameter deformation of the A-twist:

$$\mathcal{Q}^2 = 0\,, \qquad \tilde{\mathcal{Q}}^2 = 0\,, \qquad \{\mathcal{Q}, \tilde{\mathcal{Q}}\} = Z + \epsilon_\Omega \mathcal{L}_V\,.$$

The supergravity background reads:

$$ds^2 = \sqrt{g}(|z|^2) dz d\bar{z}\,, \qquad A_\mu^{(R)} = \frac{1}{2}\omega_\mu\,, \qquad C_\mu = \frac{1}{2}\epsilon_\Omega V_\mu\,, \qquad \tilde{C}_\mu = 0\,.$$

Using the general results of [C.C., Cremonesi, 2014], we can write down any supersymmetric Lagrangian we want.

GLSMs: Lightning review

Let us consider 2d $\mathcal{N} = (2,2)$ supersymmetric GLSM on this S^2_Ω.

We have the following field content:

- Vector multiplets $\mathcal{V}_a$ for a gauge group G, with Lie algebra $\mathfrak{g}$.
- Chiral multiplets Φ_i in representations $\mathfrak{R}_i$ of $\mathfrak{g}$.

We also have interactions dictated by:

- A superpotential $W(\Phi)$
- A twisted superpotential $\hat{W}(\sigma)$, where $\sigma \subset \mathcal{V}$.

Assumption: The classical twisted superpotential is linear in σ:

$$\hat{W} = \tau^I \, \mathrm{Tr}_I(\sigma) \, .$$

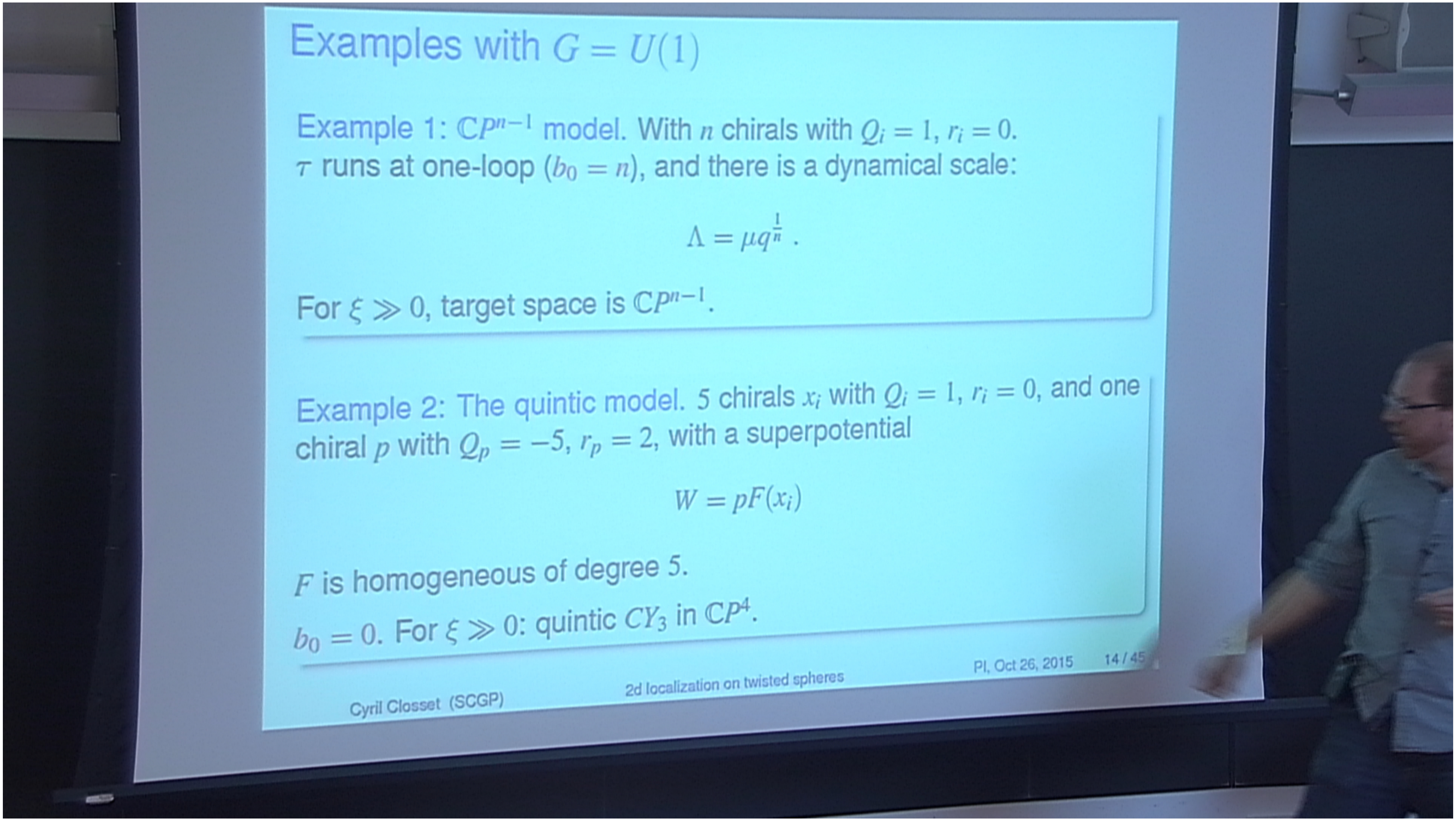
That is, we turn on one FI parameter for each $U(1)_I$ factor in G.

The FI term often runs at one-loop:

$$\tau(\mu) = \tau(\mu_0) - \frac{b_0}{2\pi i} \log\left(\frac{\mu}{\mu_0}\right) \, .$$

If $b_0 = 0$, we expect an SCFT in infrared.

This $\hat{W}$ preserves a $U(1)_A$ axial R-symmetry, broken to $\mathbb{Z}_{2b_0}$ by an anomaly if $b_0 \neq 0$.


Examples with $G = U(1)$
Example 1: $\mathbb{C}P^{n-1}$ model. With n chirals with $Q_i = 1$, $r_i = 0$.
τ runs at one-loop ($b_0 = n$), and there is a dynamical scale:
$\Lambda = \mu q^{\frac{1}{n}}$.
For $\xi \gg 0$, target space is $\mathbb{C}P^{n-1}$.
Example 2: The quintic model. 5 chirals x_i with $Q_i = 1$, $r_i = 0$, and one chiral p with $Q_p = -5$, $r_p = 2$, with a superpotential
$W = pF(x_i)$
F is homogeneous of degree 5.
$b_0 = 0$. For $\xi \gg 0$: quintic CY_3 in $\mathbb{C}P^4$.
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PI, Oct 26, 2015
14 / 45

Non-Abelian examples

Example 3: Grassmanian models. Consider a $U(N_c)$ vector multiplet with N_f chirals in the fundamental.

This non-Abelian GLSM flows to the NLσM on the Grassmanian $Gr(N_c, N_f)$.

The Grassmanian duality

$$Gr(N_c, N_f) \cong Gr(N_f - N_c, N_f)$$

corresponds to a Seiberg-like duality of the GLSMs.

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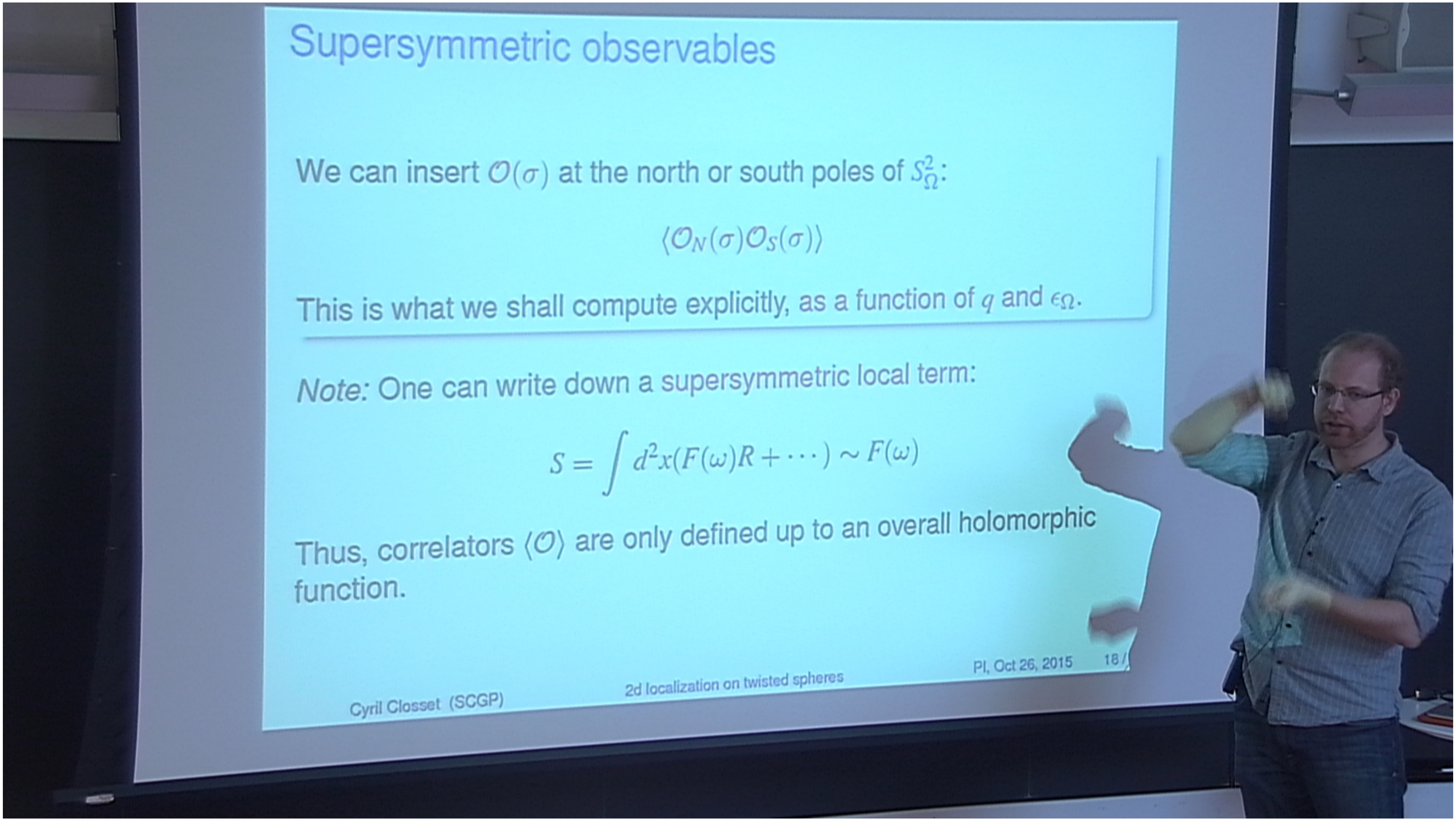
Supersymmetric observables
We can insert $\mathcal{O}(\sigma)$ at the north or south poles of S^2_Ω:
$\langle \mathcal{O}_N(\sigma)\mathcal{O}_S(\sigma)\rangle$
This is what we shall compute explicitly, as a function of q and ϵ_Ω.
Note: One can write down a supersymmetric local term:
$S = \int d^2x (F(\omega)R + \cdots) \sim F(\omega)$
Thus, correlators $\langle \mathcal{O} \rangle$ are only defined up to an overall holomorphic function.
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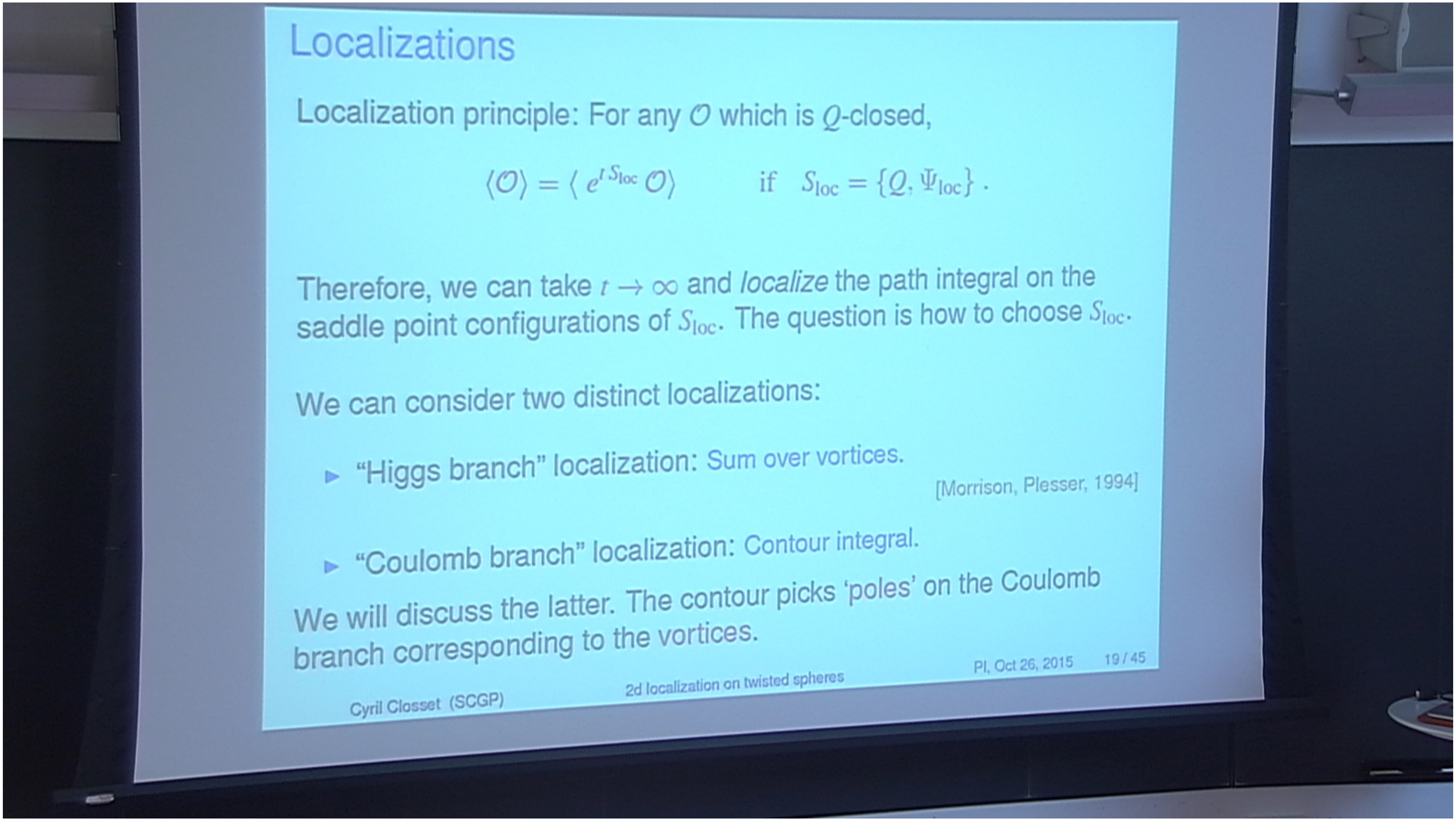
PI, Oct 26, 2015
18 /


Localizations
Localization principle: For any $\mathcal{O}$ which is Q-closed,
$\langle \mathcal{O} \rangle = \langle e^{t S_{\rm loc}} \mathcal{O} \rangle \qquad \text{if} \quad S_{\rm loc} = \{Q, \Psi_{\rm loc}\}$.
Therefore, we can take $t \to \infty$ and *localize* the path integral on the saddle point configurations of $S_{\rm loc}$. The question is how to choose $S_{\rm loc}$.
We can consider two distinct localizations:
▶ "Higgs branch" localization: Sum over vortices.
[Morrison, Plesser, 1994]
▶ "Coulomb branch" localization: Contour integral.
We will discuss the latter. The contour picks 'poles' on the Coulomb branch corresponding to the vortices.
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19 / 45

"Coulomb branch" localization

Choose:

$$\mathscr{L}_{\text{loc}} = \mathscr{L}_{\text{YM}} \, .$$

Note: We also localize the matter sector with its standard kinetic term.

The saddles are on the Coulomb branch:

$$\sigma = \text{diag}(\sigma_a) \, , \qquad G \to H = \prod_{a=1}^{\text{rank}(G)} U(1)_a$$

There is a family of gauge field saddles for each allowed (GNO) flux:

$$k = (k_a) \in \Gamma_{\mathbf{G}^\vee}$$

In this localization scheme, we also have gaugino zero modes, $\lambda, \tilde{\lambda} = \text{constant}$.

The path integral reduces to a supersymmetric ordinary integral:

$$\langle \mathcal{O}_{N,S}(\sigma) \rangle \sim \sum_k \int d\lambda d\tilde{\lambda} \int dD \int d^2\hat{\sigma}\, \mathcal{Z}_k(\hat{\sigma}, \hat{\tilde{\sigma}}, \lambda, \tilde{\lambda}, D)\, \mathcal{O}_{N,S}(\sigma_{N,S})$$

We refrained from integrating over the constant mode of the auxiliary field D in the vector multiplet.

We have

$$\mathcal{Z}_k = e^{-S_{\rm cl}}\, \mathcal{Z}_k^{1-\text{loop}}\,.$$

The one-loop term results from integrating out the chiral multiplets and the W-bosons. It can be computed explicitly by standard techniques.

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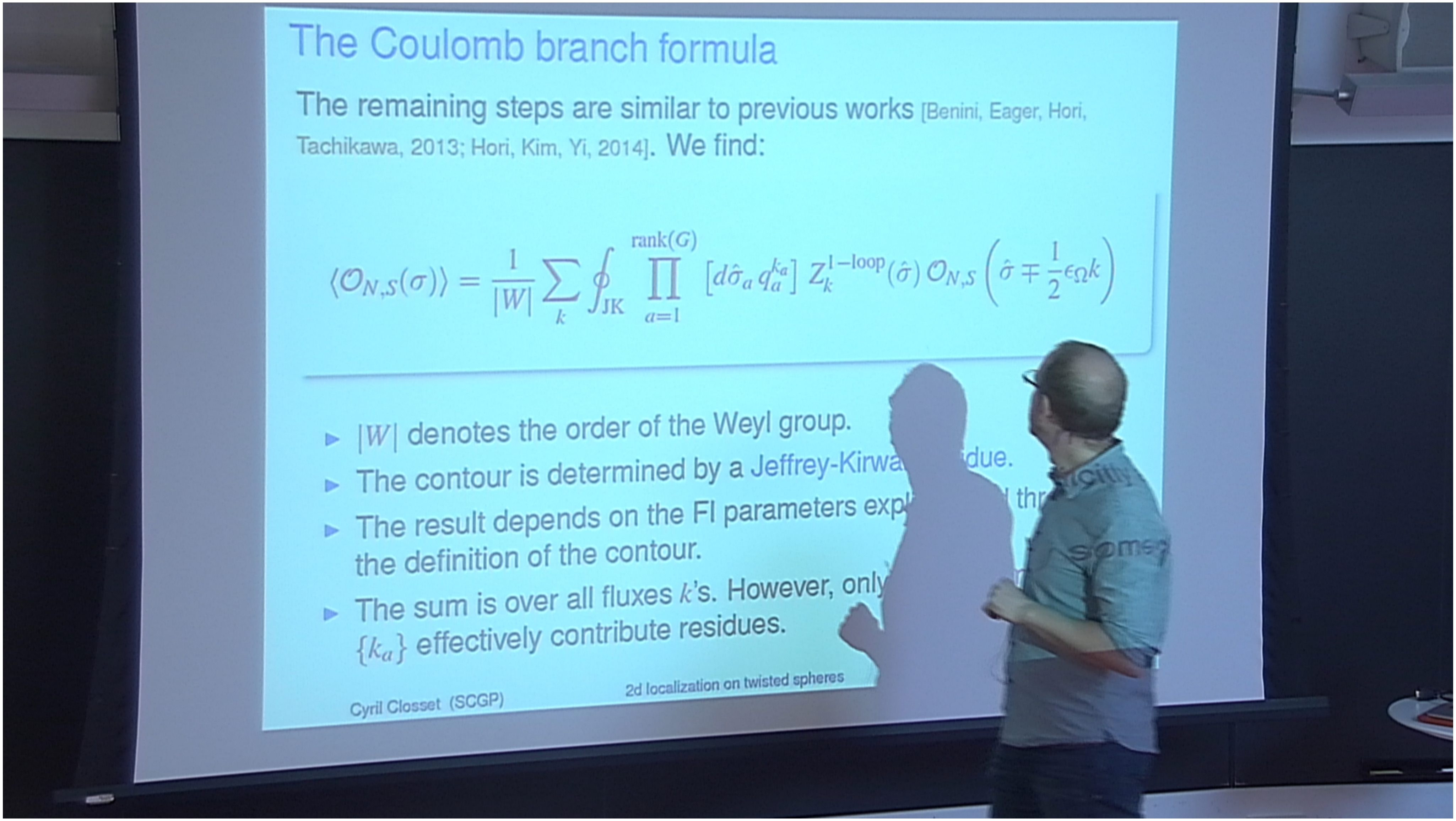
The Coulomb branch formula

The remaining steps are similar to previous works [Benini, Eager, Hori, Tachikawa, 2013; Hori, Kim, Yi, 2014]. We find:

$$\langle \mathcal{O}_{N,S}(\sigma) \rangle = \frac{1}{|W|} \sum_k \oint_{\mathrm{JK}} \prod_{a=1}^{\mathrm{rank}(G)} \left[d\hat{\sigma}_a \, q_a^{k_a} \right] Z_k^{1-\mathrm{loop}}(\hat{\sigma}) \, \mathcal{O}_{N,S}\left(\hat{\sigma} \mp \frac{1}{2} \epsilon_\Omega k \right)$$

- $|W|$ denotes the order of the Weyl group.
- The contour is determined by a Jeffrey-Kirwan residue.
- The result depends on the FI parameters explicitly and through the definition of the contour.
- The sum is over all fluxes k's. However, only some chambers in $\{k_a\}$ effectively contribute residues.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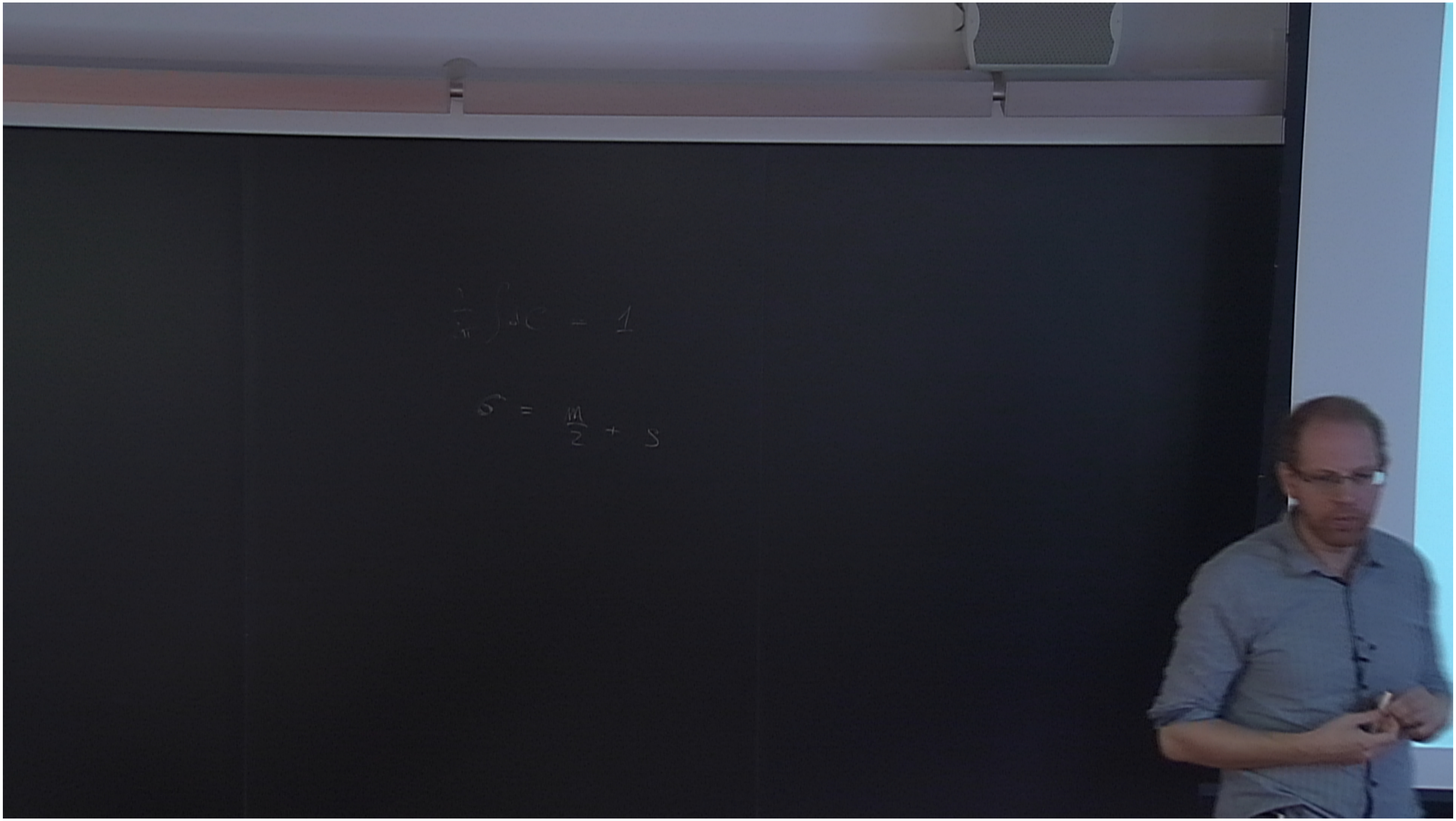
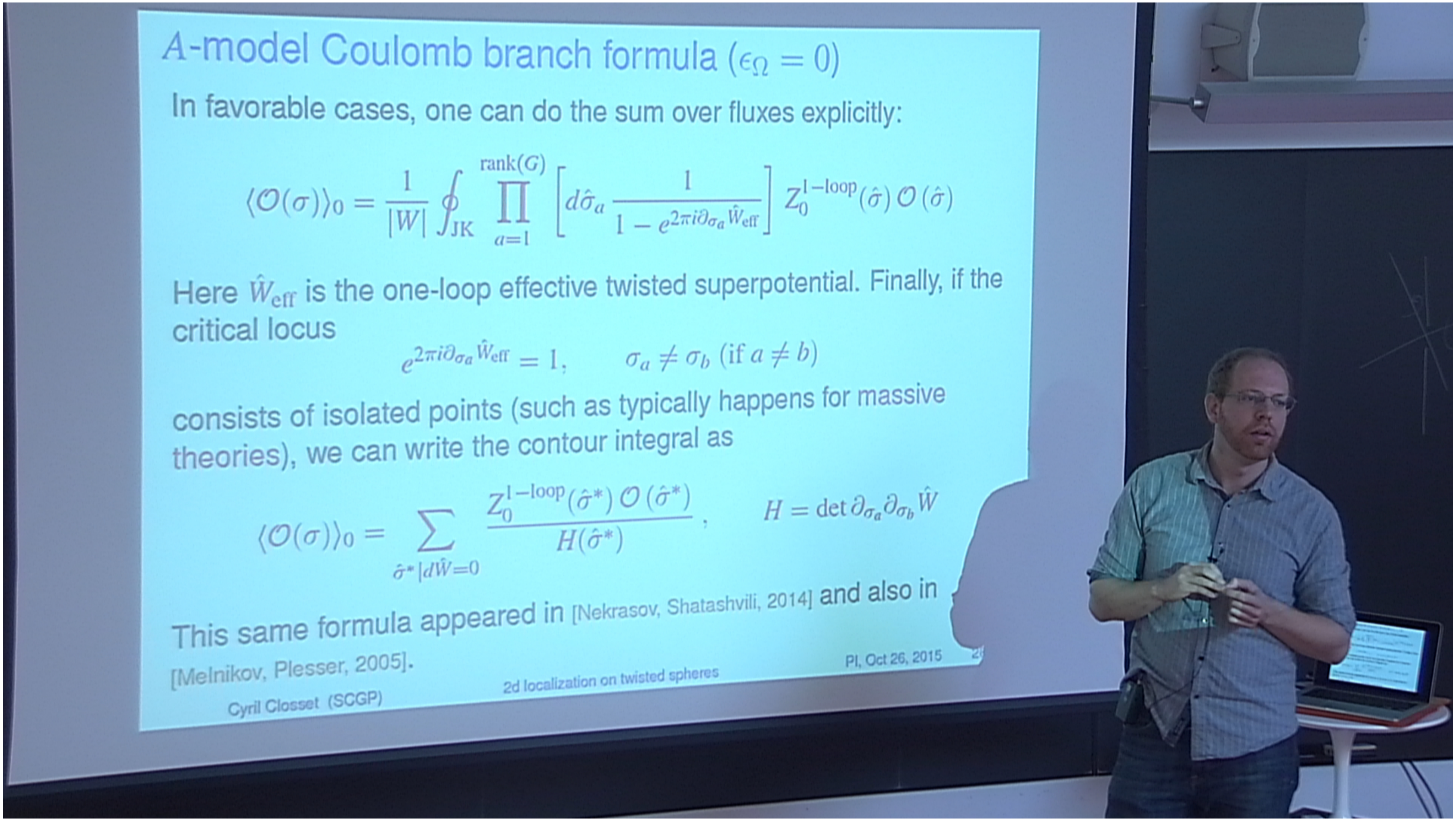

$\frac{1}{2\pi}\int dC = 1$
$\sigma = \frac{m}{2} + s$


A-model Coulomb branch formula ($\epsilon_\Omega = 0$)
In favorable cases, one can do the sum over fluxes explicitly:
$$\langle \mathcal{O}(\sigma)\rangle_0 = \frac{1}{|W|}\oint_{\mathrm{JK}} \prod_{a=1}^{\mathrm{rank}(G)} \left[d\hat\sigma_a \, \frac{1}{1 - e^{2\pi i \partial_{\sigma_a} \hat W_{\mathrm{eff}}}} \right] Z_0^{1-\mathrm{loop}}(\hat\sigma)\, \mathcal{O}(\hat\sigma)$$
Here $\hat W_{\mathrm{eff}}$ is the one-loop effective twisted superpotential. Finally, if the critical locus
$$e^{2\pi i \partial_{\sigma_a} \hat W_{\mathrm{eff}}} = 1, \qquad \sigma_a \neq \sigma_b \ (\text{if } a \neq b)$$
consists of isolated points (such as typically happens for massive theories), we can write the contour integral as
$$\langle \mathcal{O}(\sigma)\rangle_0 = \sum_{\hat\sigma^* | d\hat W = 0} \frac{Z_0^{1-\mathrm{loop}}(\hat\sigma^*)\, \mathcal{O}(\hat\sigma^*)}{H(\hat\sigma^*)}, \qquad H = \det \partial_{\sigma_a}\partial_{\sigma_b} \hat W$$
This same formula appeared in [Nekrasov, Shatashvili, 2014] and also in [Melnikov, Plesser, 2005].
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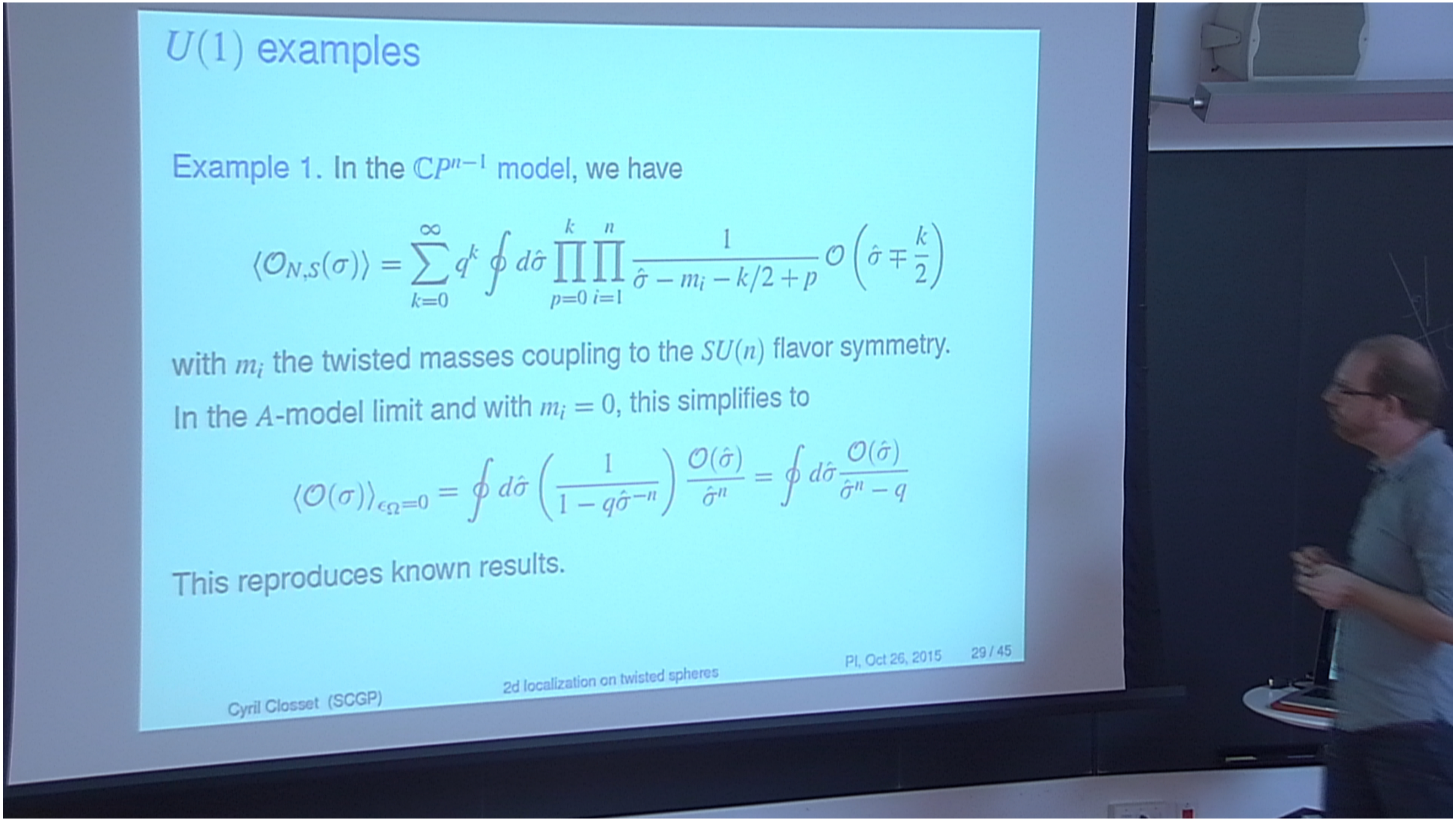
PI, Oct 26, 2015



$U(1)$ examples

Example 1. In the $\mathbb{C}P^{n-1}$ model, we have

$$\langle \mathcal{O}_{N,S}(\sigma)\rangle = \sum_{k=0}^{\infty} q^k \oint d\hat{\sigma} \prod_{p=0}^{k}\prod_{i=1}^{n} \frac{1}{\hat{\sigma} - m_i - k/2 + p}\, \mathcal{O}\left(\hat{\sigma} \mp \frac{k}{2}\right)$$

with m_i the twisted masses coupling to the $SU(n)$ flavor symmetry.

In the A-model limit and with $m_i = 0$, this simplifies to

$$\langle \mathcal{O}(\sigma)\rangle_{\epsilon_\Omega=0} = \oint d\hat{\sigma} \left(\frac{1}{1 - q\hat{\sigma}^{-n}}\right) \frac{\mathcal{O}(\hat{\sigma})}{\hat{\sigma}^n} = \oint d\hat{\sigma} \frac{\mathcal{O}(\hat{\sigma})}{\hat{\sigma}^n - q}$$

This reproduces known results.

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PI, Oct 26, 2015
29 / 45

Example 2. For the quintic model, we have

$$\langle \mathcal{O}_N(\sigma) \rangle = \frac{1}{\epsilon_\Omega{}^3} \sum_{k=0}^{\infty} q^k \oint ds \frac{\prod_{l=0}^{5k}(-5s-l)}{\prod_{p=0}^{k}(s+p)^5} \mathcal{O}(\epsilon_\Omega s)$$

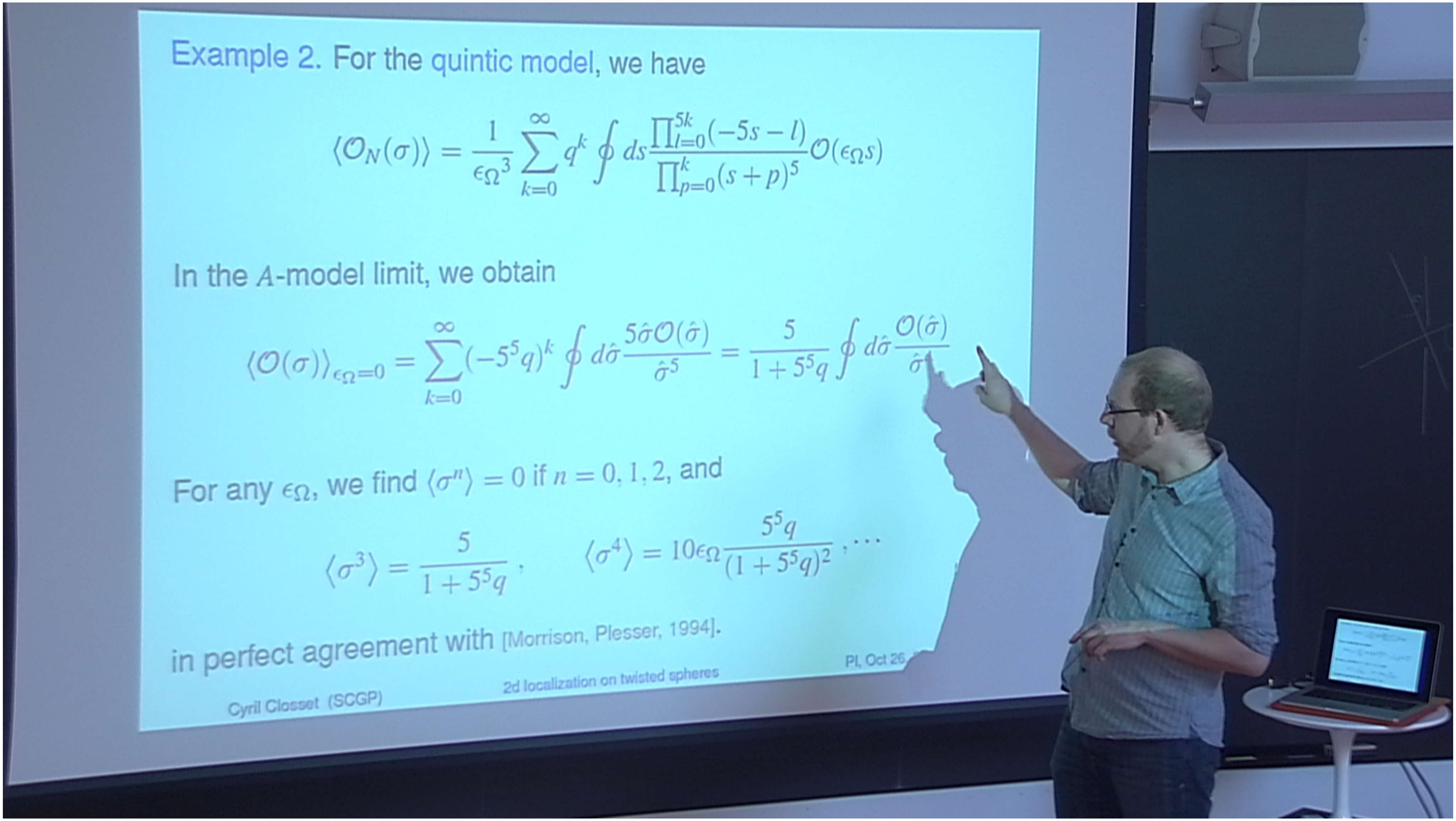
In the A-model limit, we obtain

$$\langle \mathcal{O}(\sigma) \rangle_{\epsilon_\Omega = 0} = \sum_{k=0}^{\infty} (-5^5 q)^k \oint d\hat{\sigma} \frac{5\hat{\sigma}\mathcal{O}(\hat{\sigma})}{\hat{\sigma}^5} = \frac{5}{1+5^5 q} \oint d\hat{\sigma} \frac{\mathcal{O}(\hat{\sigma})}{\hat{\sigma}^4}$$

For any ϵ_Ω, we find $\langle \sigma^n \rangle = 0$ if $n = 0, 1, 2$, and

$$\langle \sigma^3 \rangle = \frac{5}{1+5^5 q}\,, \qquad \langle \sigma^4 \rangle = 10\epsilon_\Omega \frac{5^5 q}{(1+5^5 q)^2}\,, \cdots$$

in perfect agreement with [Morrison, Plesser, 1994].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2d localization on twisted spheres
PI, Oct 26,

Non-Abelian examples

For simplicitly, let us focus on $\epsilon_\Omega = 0$, the A-model.

Example 3. For the Grassmanian model, the residue formula gives

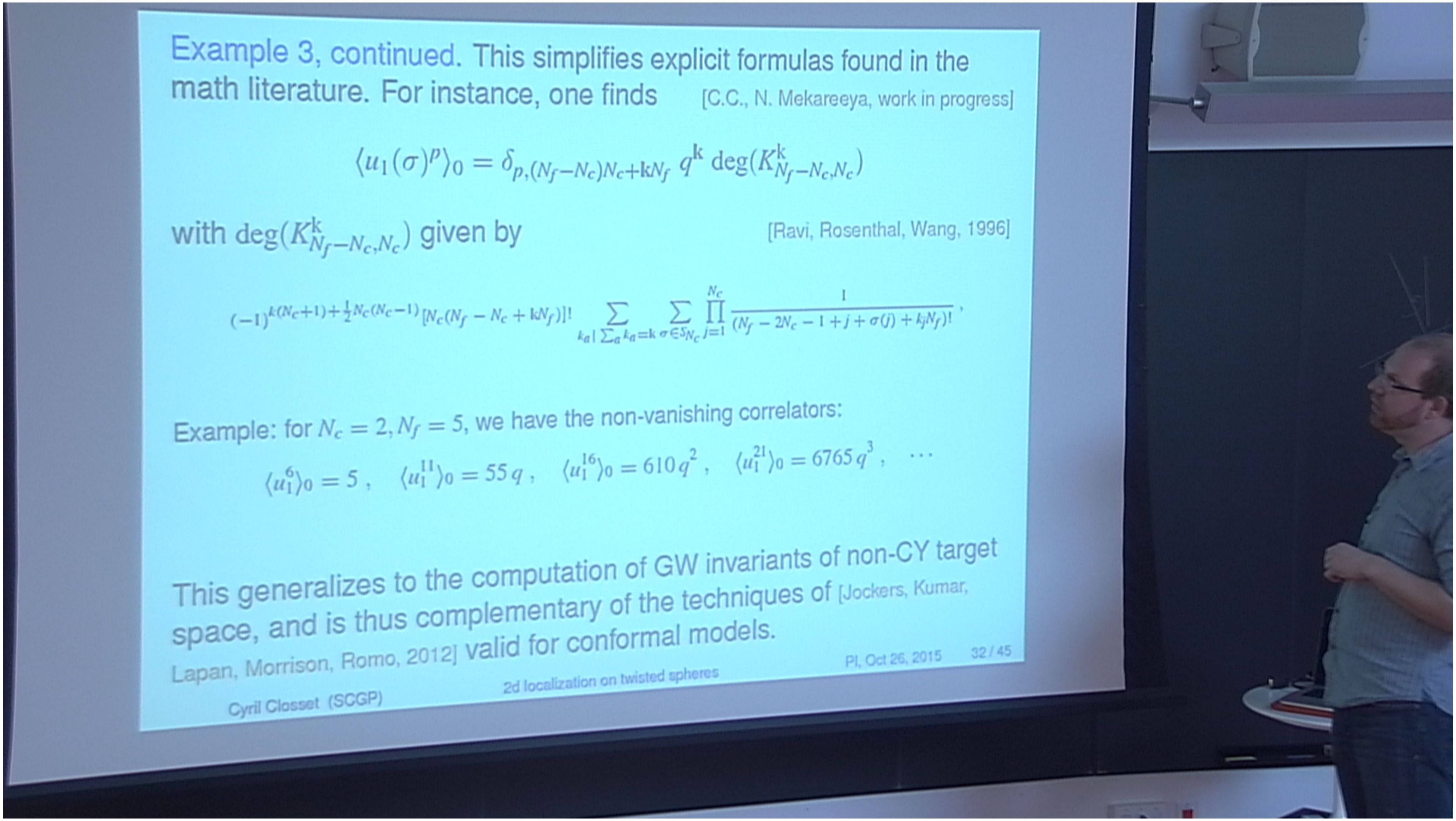
$$\langle \mathcal{O} \rangle_0 = \sum_{\mathrm{k} \in \mathbb{Z}_{\geq 0}} q^{\mathrm{k}} \mathcal{Z}_k(\mathcal{O}) \,,$$

with

$$\mathcal{Z}_{\mathrm{k}} = \frac{1}{N_c!} \sum_{k_a | \sum_a k_a = \mathrm{k}} \frac{(-1)^{2\rho_W(k)}}{(2\pi i)^{N_c}} \oint d^{N_c}\sigma \frac{\prod_{a,b=1}^{N_c} (\sigma_a - \sigma_b)}{\prod_{a=1}^{N_c} \prod_{i=1}^{N_f} (\sigma_a - m_i)^{1+k_a}} \mathcal{O}(\sigma) \,.$$

Here m_i are twisted masses, corresponding to a $SU(N_f)$-equivariant deformation of $Gr(N_c, N_f)$.

For $m_i = 0$, the numbers $\mathcal{Z}_{\mathrm{k}}$ are the $g = 0$ Gromov-Witten invariants.


Example 3, continued. This simplifies explicit formulas found in the math literature. For instance, one finds [C.C., N. Mekareeya, work in progress]
$$\langle u_1(\sigma)^p \rangle_0 = \delta_{p,(N_f-N_c)N_c+\mathrm{k}N_f}\, q^{\mathrm{k}} \deg(K^{\mathrm{k}}_{N_f-N_c,N_c})$$
with $\deg(K^{\mathrm{k}}_{N_f-N_c,N_c})$ given by [Ravi, Rosenthal, Wang, 1996]
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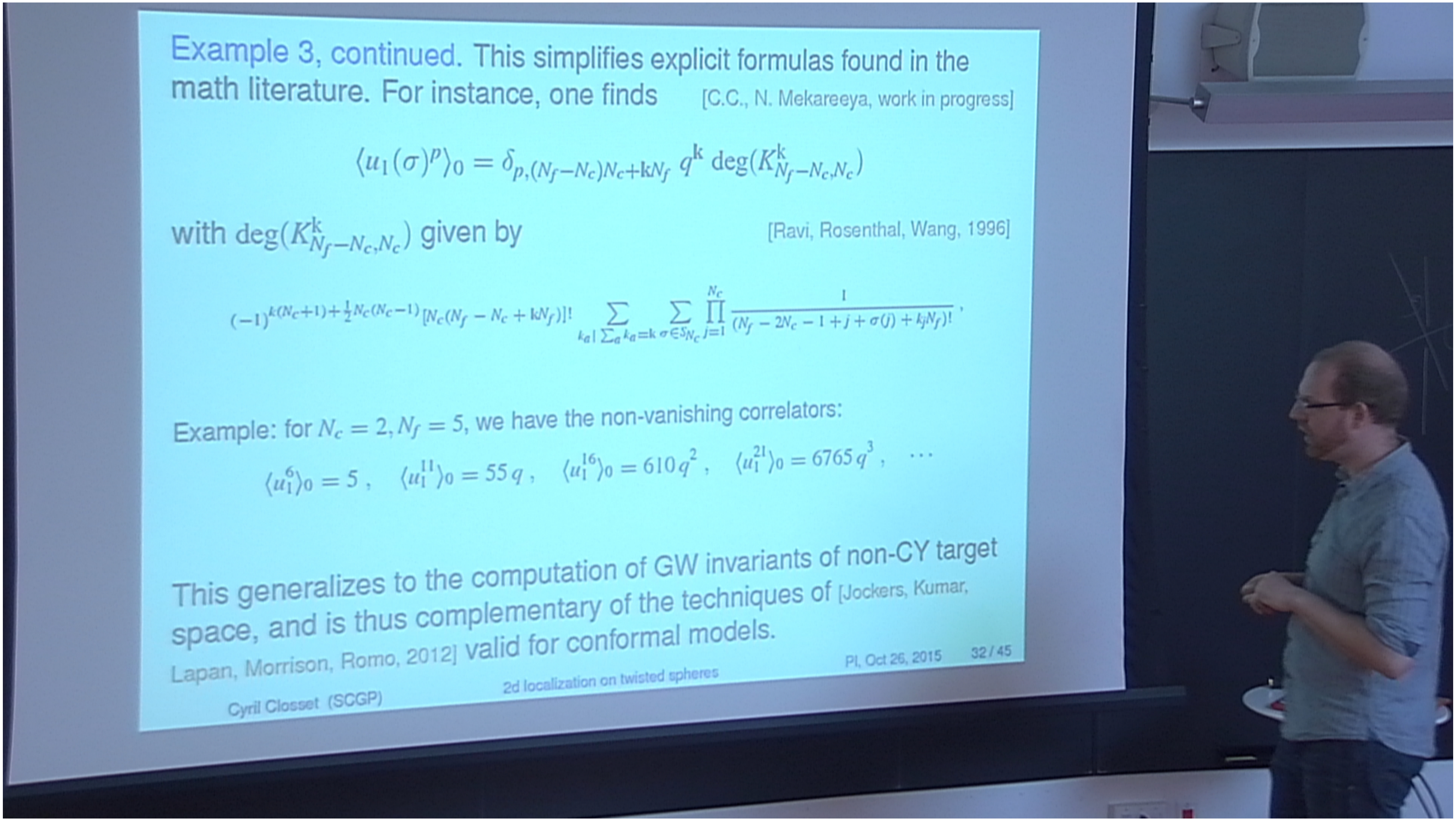
Example: for $N_c=2, N_f=5$, we have the non-vanishing correlators:
$$\langle u_1^6\rangle_0 = 5\,, \quad \langle u_1^{11}\rangle_0 = 55\,q\,, \quad \langle u_1^{16}\rangle_0 = 610\,q^2\,, \quad \langle u_1^{21}\rangle_0 = 6765\,q^3\,, \quad \cdots$$
This generalizes to the computation of GW invariants of non-CY target space, and is thus complementary of the techniques of [Jockers, Kumar, Lapan, Morrison, Romo, 2012] valid for conformal models.
Cyril Closset (SCGP)
2d localization on twisted spheres
PI, Oct 26, 2015
32 / 45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Cyril Closset (SCGP)
2d localization on twisted spheres
PI, Oct 26, 2015
32 / 45

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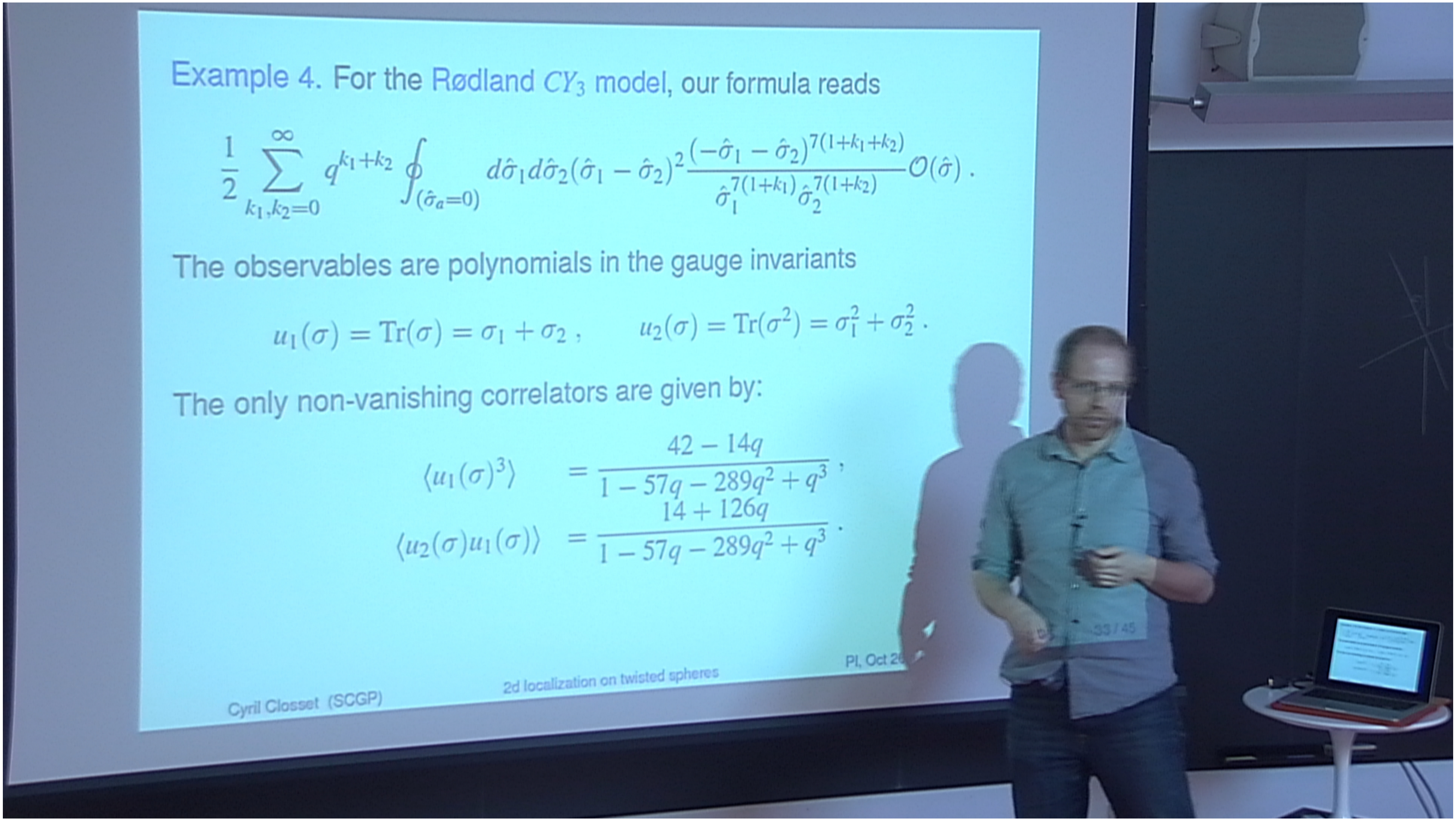
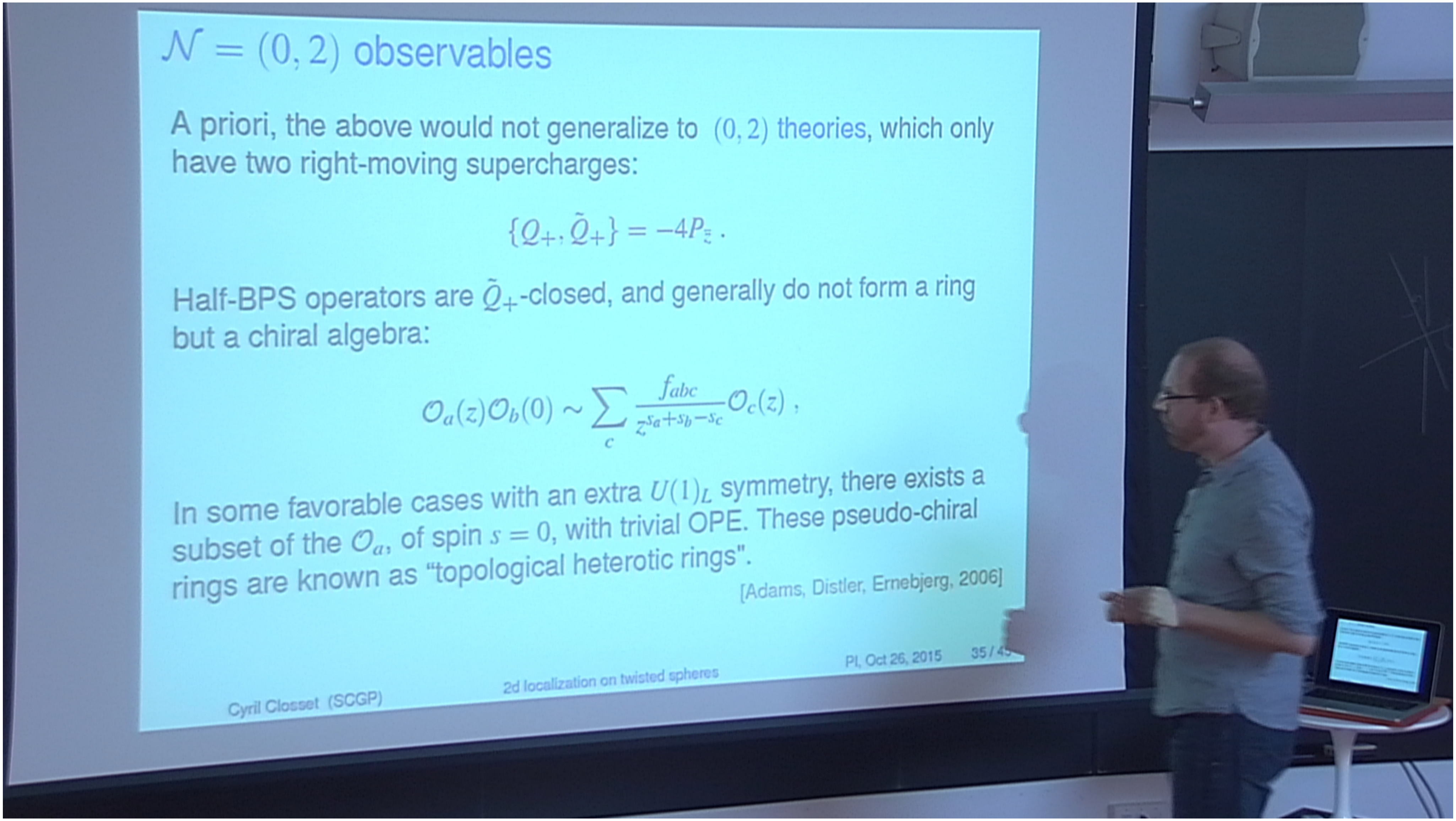
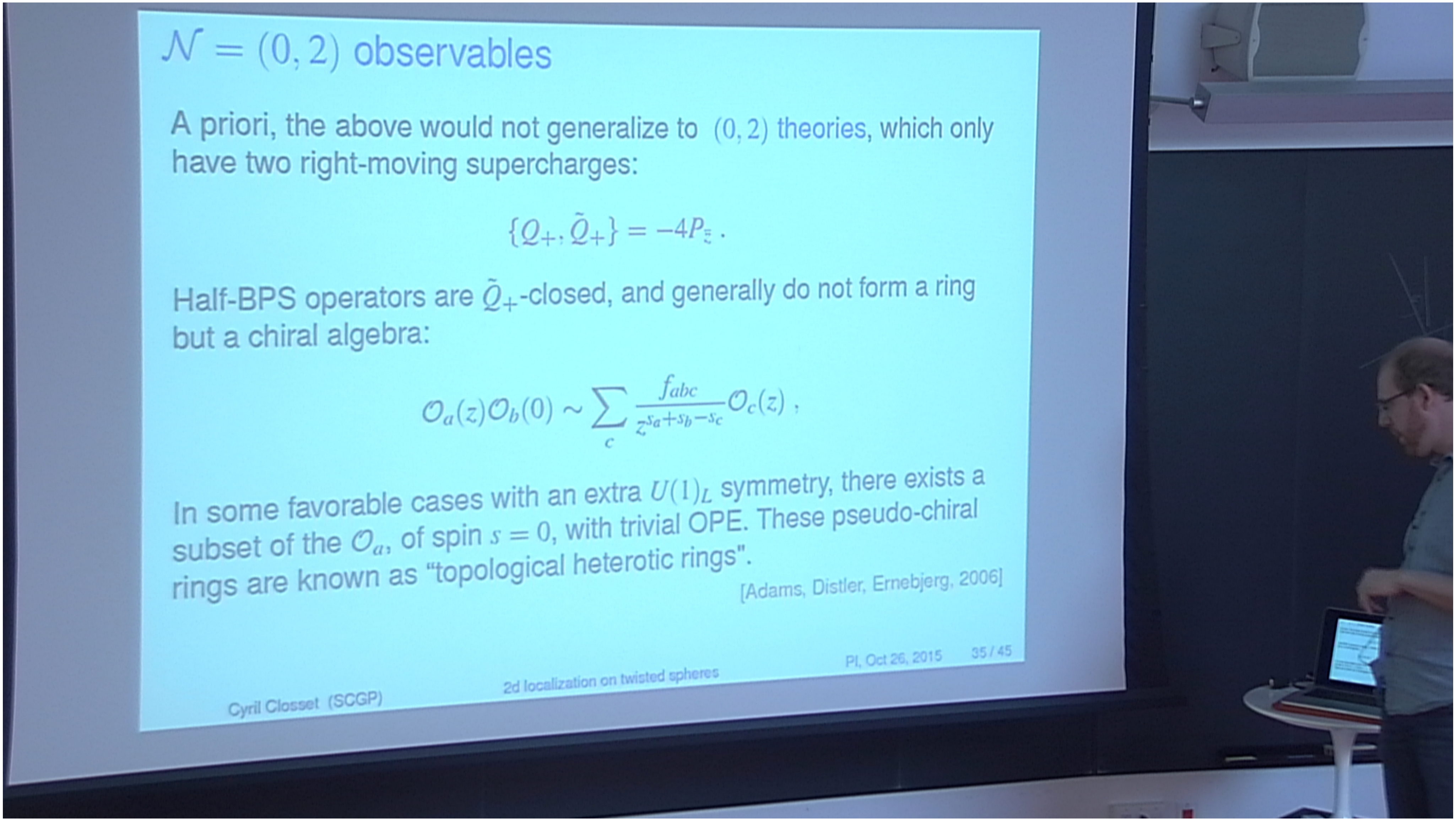

Example 4. For the Rødland CY_3 model, our formula reads
$$\frac{1}{2}\sum_{k_1,k_2=0}^{\infty} q^{k_1+k_2}\oint_{(\hat\sigma_a=0)} d\hat\sigma_1 d\hat\sigma_2 (\hat\sigma_1-\hat\sigma_2)^2 \frac{(-\hat\sigma_1-\hat\sigma_2)^{7(1+k_1+k_2)}}{\hat\sigma_1^{7(1+k_1)}\hat\sigma_2^{7(1+k_2)}}\mathcal{O}(\hat\sigma)\,.$$
The observables are polynomials in the gauge invariants
$$u_1(\sigma)=\mathrm{Tr}(\sigma)=\sigma_1+\sigma_2\,,\qquad u_2(\sigma)=\mathrm{Tr}(\sigma^2)=\sigma_1^2+\sigma_2^2\,.$$
The only non-vanishing correlators are given by:
$$\langle u_1(\sigma)^3\rangle = \frac{42-14q}{1-57q-289q^2+q^3}\,,$$
$$\langle u_2(\sigma)u_1(\sigma)\rangle = \frac{14+126q}{1-57q-289q^2+q^3}\,.$$
Cyril Closset (SCGP)
2d localization on twisted spheres
PI, Oct 2
33 / 45


$\mathcal{N} = (0,2)$ observables
A priori, the above would not generalize to $(0,2)$ theories, which only have two right-moving supercharges:
$\{Q_+, \tilde{Q}_+\} = -4P_{\bar{z}}$.
Half-BPS operators are $\tilde{Q}_+$-closed, and generally do not form a ring but a chiral algebra:
$\mathcal{O}_a(z)\mathcal{O}_b(0) \sim \sum_c \frac{f_{abc}}{z^{s_a+s_b-s_c}}\mathcal{O}_c(z)$,
In some favorable cases with an extra $U(1)_L$ symmetry, there exists a subset of the $\mathcal{O}_a$, of spin $s = 0$, with trivial OPE. These pseudo-chiral rings are known as "topological heterotic rings".
[Adams, Distler, Ernebjerg, 2006]
Cyril Closset (SCGP)
2d localization on twisted spheres
PI, Oct 26, 2015
35 / 4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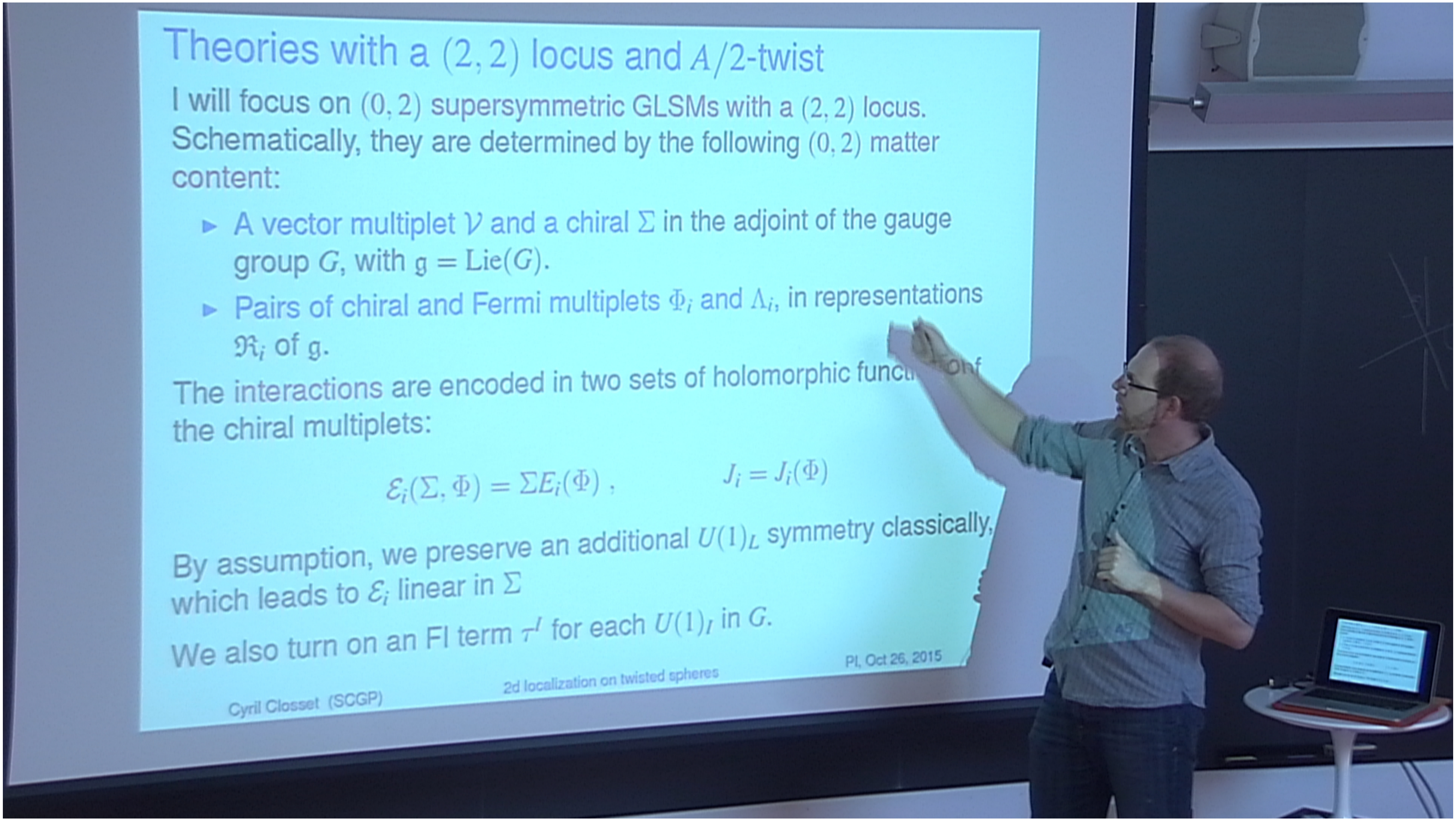
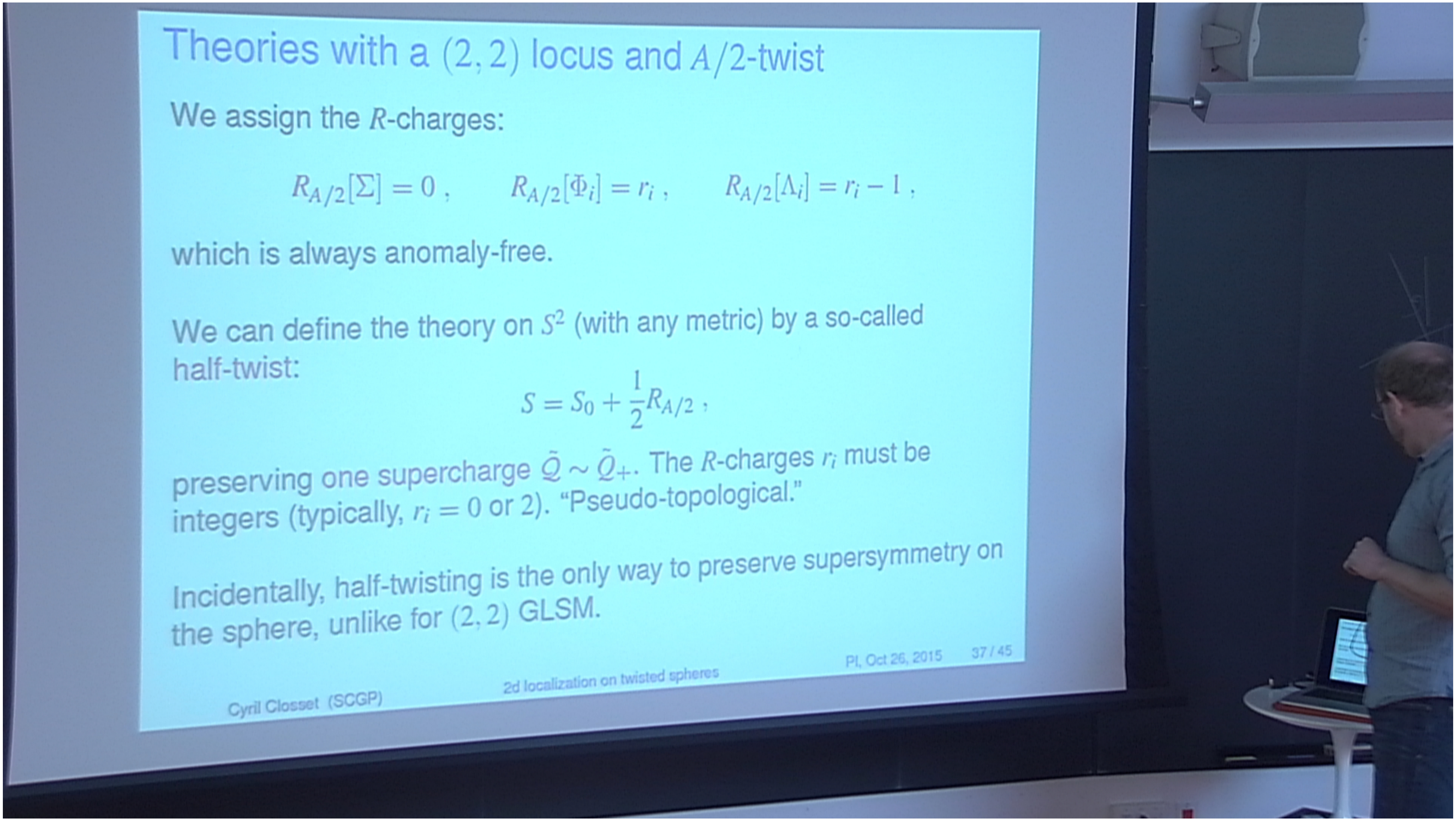
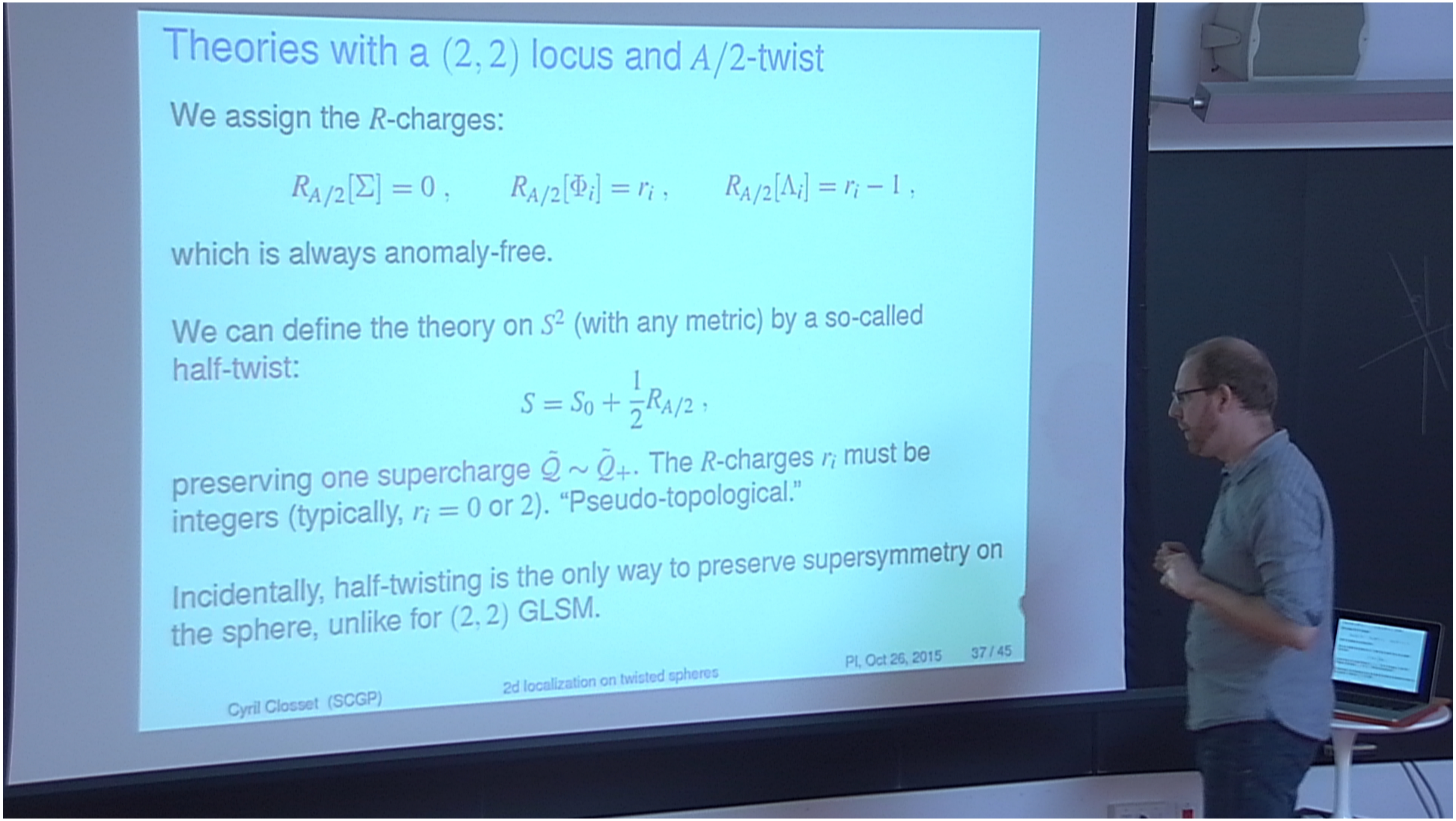
Cyril Closset (SCGP)
2d localization on twisted spheres
PI, Oct 26, 2015
35 / 45


Theories with a $(2,2)$ locus and $A/2$-twist
I will focus on $(0,2)$ supersymmetric GLSMs with a $(2,2)$ locus. Schematically, they are determined by the following $(0,2)$ matter content:
A vector multiplet $\mathcal{V}$ and a chiral Σ in the adjoint of the gauge group G, with $\mathfrak{g} = \mathrm{Lie}(G)$.
Pairs of chiral and Fermi multiplets Φ_i and Λ_i, in representations $\mathfrak{R}_i$ of $\mathfrak{g}$.
The interactions are encoded in two sets of holomorphic func
the chiral multiplets:
$\mathcal{E}_i(\Sigma, \Phi) = \Sigma E_i(\Phi)$, $J_i = J_i(\Phi)$
By assumption, we preserve an additional $U(1)_L$ symmetry classically, which leads to $\mathcal{E}_i$ linear in Σ
We also turn on an FI term τ^I for each $U(1)_I$ in G.
Cyril Closset (SCGP)
2d localization on twisted spheres
PI, Oct 26, 2015


Theories with a $(2,2)$ locus and $A/2$-twist
We assign the R-charges:
$R_{A/2}[\Sigma] = 0$, $R_{A/2}[\Phi_i] = r_i$, $R_{A/2}[\Lambda_i] = r_i - 1$,
which is always anomaly-free.
We can define the theory on S^2 (with any metric) by a so-called half-twist:
$S = S_0 + \frac{1}{2} R_{A/2}$,
preserving one supercharge $\tilde{Q} \sim \tilde{Q}_+$. The R-charges r_i must be integers (typically, $r_i = 0$ or 2). "Pseudo-topological."
Incidentally, half-twisting is the only way to preserve supersymmetry on the sphere, unlike for $(2,2)$ GLSM.
Cyril Closset (SCGP)
2d localization on twisted spheres
PI, Oct 26, 2015
37 / 45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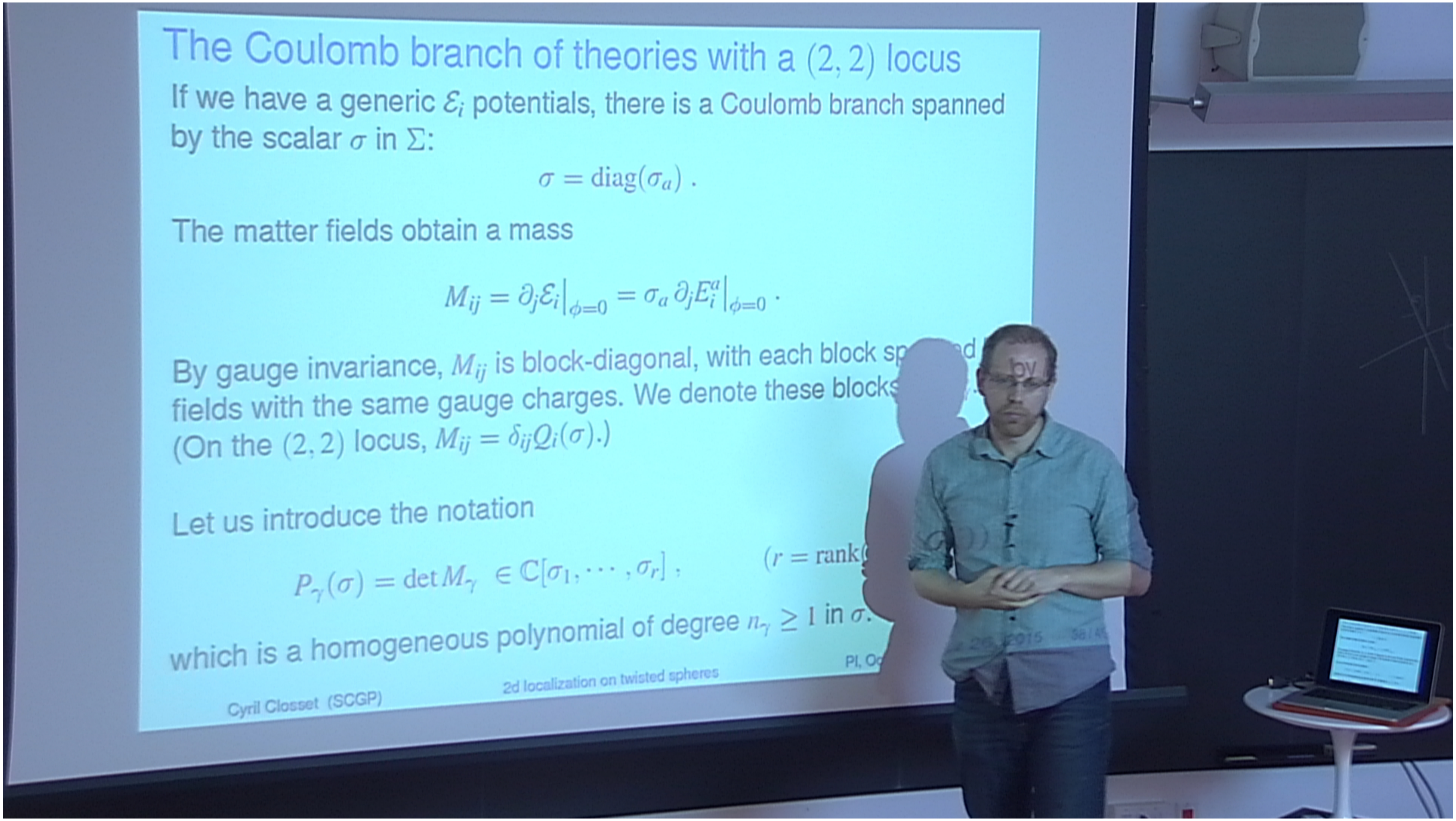
Cyril Closset (SCGP)
2d localization on twisted spheres
PI, Oct 26, 2015
37 / 45


The Coulomb branch of theories with a $(2,2)$ locus
If we have a generic $\mathcal{E}_i$ potentials, there is a Coulomb branch spanned by the scalar σ in Σ:
$\sigma = \mathrm{diag}(\sigma_a)$.
The matter fields obtain a mass
$M_{ij} = \partial_j \mathcal{E}_i \big|_{\phi=0} = \sigma_a \, \partial_j E_i^a \big|_{\phi=0}$.
By gauge invariance, M_{ij} is block-diagonal, with each block sp
fields with the same gauge charges. We denote these blocks
(On the $(2,2)$ locus, $M_{ij} = \delta_{ij} Q_i(\sigma)$.)
Let us introduce the notation
$P_\gamma(\sigma) = \det M_\gamma \ \in \mathbb{C}[\sigma_1, \cdots, \sigma_r]$, $(r = \mathrm{rank}($
which is a homogeneous polynomial of degree $n_\gamma \geq 1$ in σ.
Cyril Closset (SCGP)
2d localization on twisted spheres
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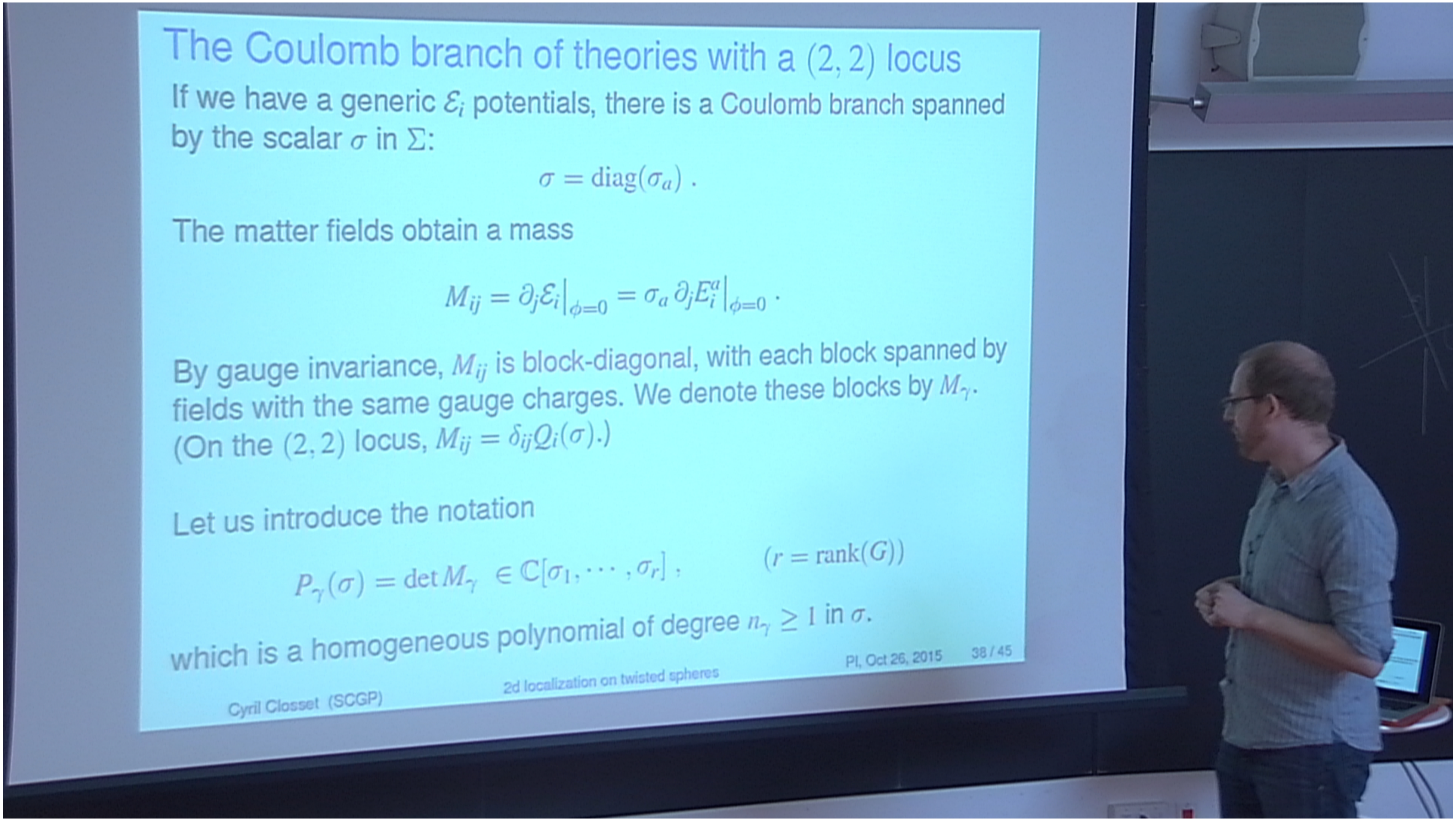
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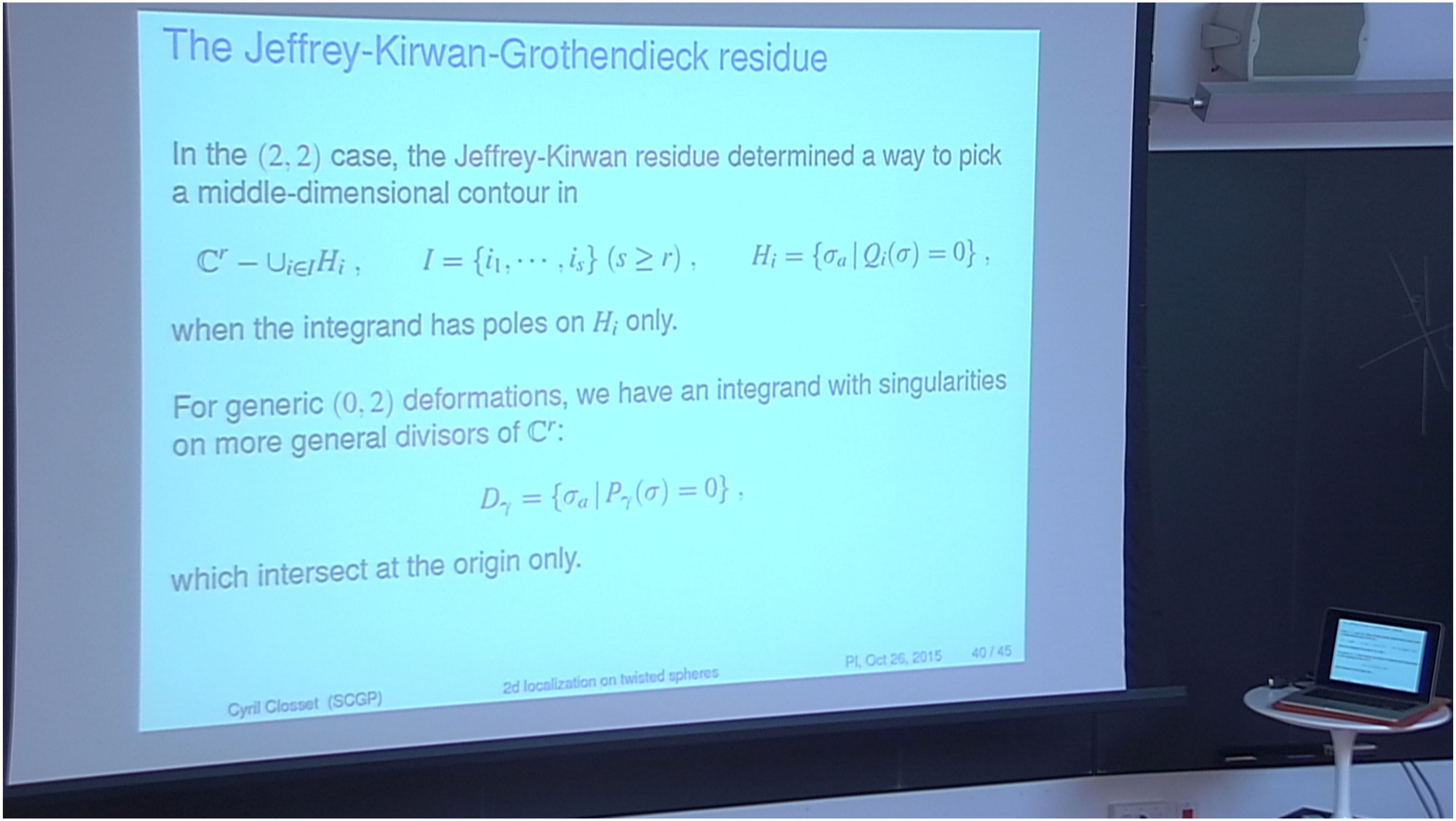
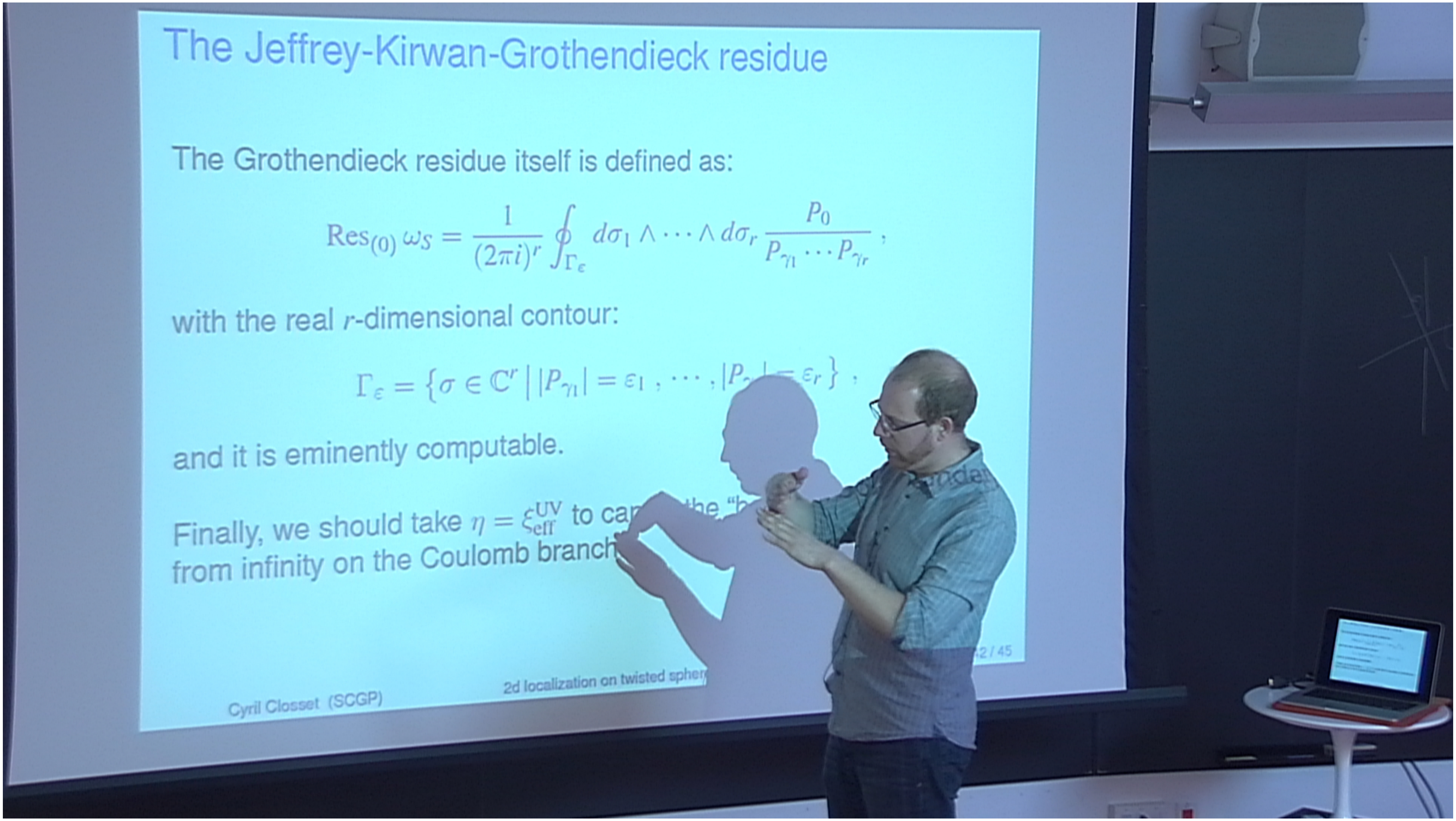
Cyril Closset (SCGP)
2d localization on twisted spheres
PI, Oct 26, 2015
38 / 45


The Jeffrey-Kirwan-Grothendieck residue
In the $(2,2)$ case, the Jeffrey-Kirwan residue determined a way to pick a middle-dimensional contour in
$\mathbb{C}^r - \cup_{i \in I} H_i$, $I = \{i_1, \cdots, i_s\}$ $(s \geq r)$, $H_i = \{\sigma_a \,|\, Q_i(\sigma) = 0\}$,
when the integrand has poles on H_i only.
For generic $(0,2)$ deformations, we have an integrand with singularities on more general divisors of $\mathbb{C}^r$:
$D_\gamma = \{\sigma_a \,|\, P_\gamma(\sigma) = 0\}$,
which intersect at the origin only.
Cyril Closset (SCGP)
2d localization on twisted spheres
PI, Oct 26, 2015
40 / 45


The Jeffrey-Kirwan-Grothendieck residue
The Grothendieck residue itself is defined as:
$$\mathrm{Res}_{(0)}\,\omega_S = \frac{1}{(2\pi i)^r}\oint_{\Gamma_\varepsilon} d\sigma_1 \wedge \cdots \wedge d\sigma_r \, \frac{P_0}{P_{\gamma_1}\cdots P_{\gamma_r}} ,$$
with the real r-dimensional contour:
$\Gamma_\varepsilon = \{\sigma \in \mathbb{C}^r \,|\, |P_{\gamma_1}| = \varepsilon_1 , \cdots ,$
and it is eminently computable.
Finally, we should take $\eta = \xi_{\text{eff}}^{\text{UV}}$ to ca
from infinity on the Coulomb branch
Cyril Closset (SCGP)
2d localization on twisted spher
/ 45

$\mathbb{C}P^1 \times \mathbb{C}P^1$, continued.

We have two sets $\gamma = 1, 2$:

$$\det M_1 = \det(A\sigma_1 + B\sigma_2)\ , \qquad \det M_2 = \det(C\sigma_1 + D\sigma_2)\ .$$

The Coulomb branch residue formula gives

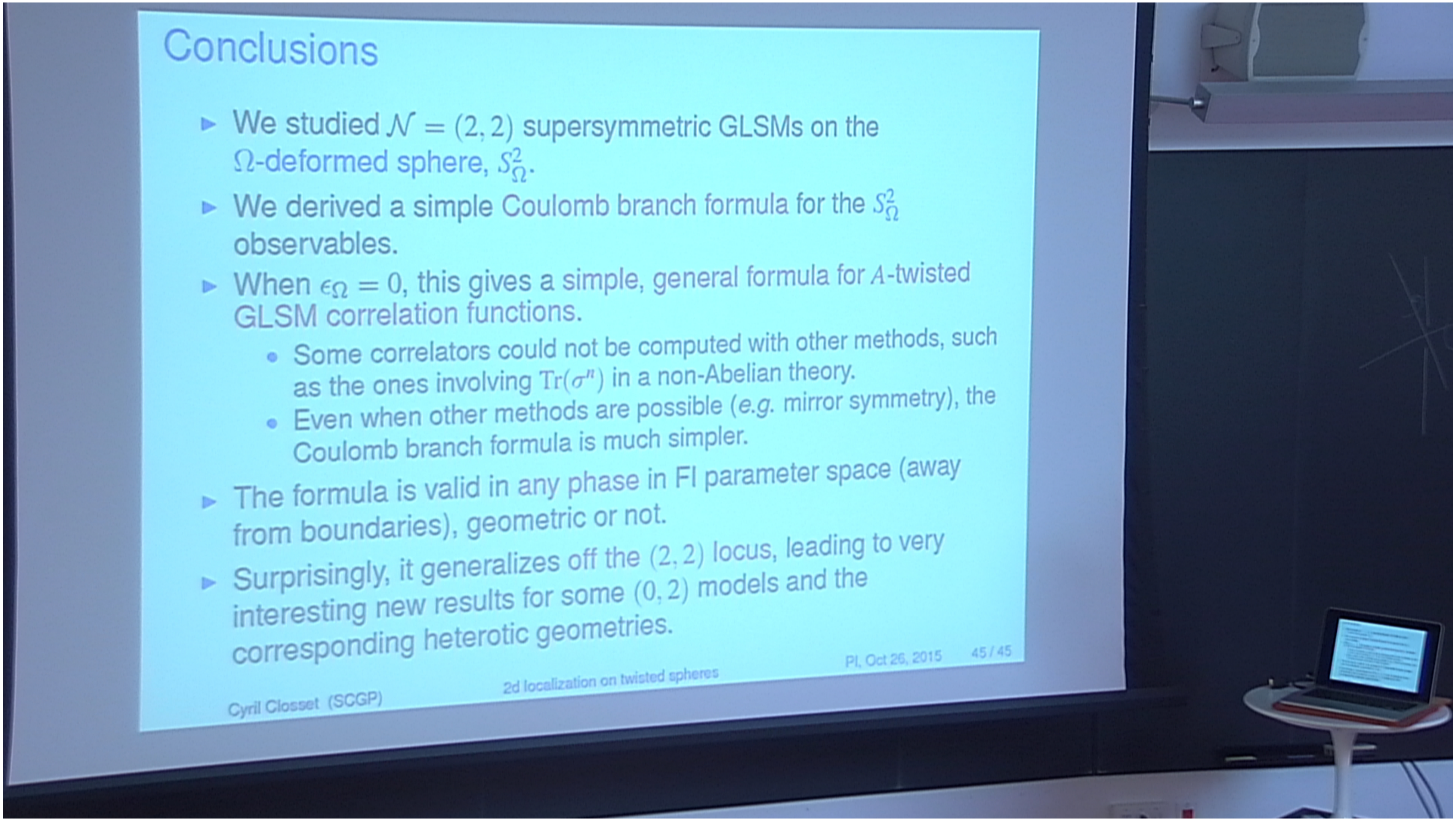
$$\langle \sigma_1^{p_1} \sigma_2^{p_2} \rangle = \sum_{k_1, k_2 \in \mathbb{Z}} q_1^{k_1} q_2^{k_2} \oint_{\mathrm{JKG}} d\sigma_1 d\sigma_2 \, \frac{\sigma_1^{p_1} \sigma_2^{p_2}}{(\det M_1)^{1+k_1} (\det M_2)^{1+k_2}}$$

This can be checked against independent mathematical computations of sheaf cohomology groups.

This result also implies the "quantum sheaf cohomology relations":

$$\det M_1 = q_1\ , \qquad \det M_2 = q_2\ ,$$

in the $A/2$-ring. This can also be derived from a standard argument on the Coulomb branch. [McOrist, Melnikov, 2008]


Conclusions
We studied $\mathcal{N} = (2,2)$ supersymmetric GLSMs on the Ω-deformed sphere, S^2_Ω.
We derived a simple Coulomb branch formula for the S^2_Ω observables.
When $\epsilon_\Omega = 0$, this gives a simple, general formula for A-twisted GLSM correlation functions.
Some correlators could not be computed with other methods, such as the ones involving $\mathrm{Tr}(\sigma^n)$ in a non-Abelian theory.
Even when other methods are possible (*e.g.* mirror symmetry), the Coulomb branch formula is much simpler.
The formula is valid in any phase in FI parameter space (away from boundaries), geometric or not.
Surprisingly, it generalizes off the $(2,2)$ locus, leading to very interesting new results for some $(0,2)$ models and the corresponding heterotic geometries.
Cyril Closset (SCGP)
2d localization on twisted spheres
PI, Oct 26, 2015
45 / 45