Title: Bulk/boundary correspondence in topological phases

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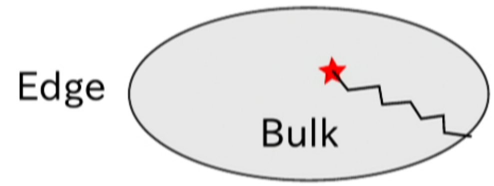
Abstract:

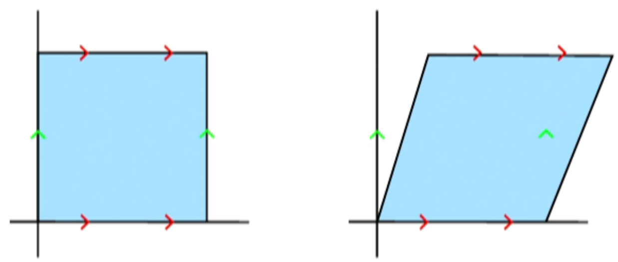
Some key features in (2+1)d

- Bulk quasiparticles <---> twisted boundary conditions at edge

The bulk ground state degeneracy
<---> possible b.c.'s



- Bulk modular S and T matrices of GS wfn on spatial torus
<---> Boundary S and T matrices
of partition functions on spacetime torus



$$|\Psi_i(g)\rangle \to M_{ij}|\Psi_j(g)\rangle$$

- S and T matrices encode quasiparticle spin and statistics.
- Question: Can these features be generalized to (3+1)d ?

Bulk-boundary correspondence in topologically ordered phases in (3+1) dimensions

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- Introduction: Bulk-boundary correspondence
- Model 1: (3+1)D BF theory
- Model 2: (3+1)D BF theory w/ Theta term
- Model 3: Coupled BF theories
- Summary

Results on surface of (3+1)d topological phases

3

[Xiao Chen, Apoorv Tiwari, SR, ArXiv: 1509xxx]

- We have carried out the calculations of S and T matrices in (2+1)d surface of a (3+1)d bulk topological phases.

- Example: Zk gauge theory (BF theory)
  Three-loop braiding statistcs

- Proposed a field theory model realzing 3-loop braiding statistics
  S and T matrices computed from boundary

- Bulk-boundary correspondence was established by extracting S and T matrices from boundary theoy

braiding statistics, (entanglement) entropy, etc.

Results on surface of (3+1)d topological phases

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(3+1)d BF topological theory

- Action: $$S_{bulk} = \int_{\mathcal{M}} \left[\frac{\mathrm{K}}{2\pi} b \wedge da - a \wedge J_{qp} - b \wedge J_{qv} \right]$$

b: two-form field
a: one-form gauge field
J_{qp}: quasi-particle current (3-form, Wilson loop)
J_{qv}: quasi-vortex current (2-form, surface operator)
K: "level" (integer); parameter of the theory.

- Application: BCS superconductors, discrete gauge theories,
  bosonic topological insulators,
  fermionic (topological) insulagors, etc.

(3+1)d BF topological theory

- Action: $$S_{bulk} = \int_{\mathcal{M}} \left[\frac{\mathrm{K}}{2\pi} b \wedge da - a \wedge J_{qp} - b \wedge J_{qv} \right]$$

- Non-trivial particle-string statistics

$$\int \mathcal{D}[a,b] e^{iS_{bulk}} = e^{iS_{eff}} \qquad S_{eff} = \frac{2\pi}{\mathrm{K}} Link(J_{qv}, J_{qp})$$

- K^3 ground state degeneracy on T^3

- Suface theory: $$S_{\partial\mathcal{M}} = \int_{\partial\mathcal{M}} dtdxdy \left[\frac{\mathrm{K}}{2\pi} \epsilon_{ij} \partial_i \zeta_j \partial_t \varphi - V(\varphi, \zeta) \right]$$

- Goal: Establish bulk-boundary correspondence

"Twist" by Theta term

- From the EOM

$$\frac{\mathrm{K}}{2\pi}\varepsilon^{\mu\nu\lambda\rho}\partial_\lambda a_\rho = j^{\mu\nu}_{qv},$$
$$\frac{\mathrm{K}}{4\pi}\varepsilon^{\mu\nu\lambda\rho}\partial_\nu b_{\lambda\rho} = -\frac{\mathrm{p}}{2\pi\mathrm{K}}\partial_\nu\Theta j^{\mu\nu}_{qv} + j^{\mu}_{qp}$$

- Alternative way:

$$S'_{bulk} = \int d^4x\,\frac{\mathrm{K}}{4\pi}\varepsilon^{\mu\nu\lambda\rho}\partial_\nu b_{\lambda\rho}a_\mu$$
$$- j^{\mu}_{qp}a_\mu - j^{\mu\nu}_{qv}\frac{1}{2}\left[b_{\mu\nu} - \frac{\mathrm{p}}{2\pi\mathrm{K}}(a_\mu\partial_\nu\Theta - a_\nu\partial_\mu\Theta)\right]$$

- Operators in surface theory:

$$\exp im\int_L a = \exp\left[im\varphi(\partial L)\right]$$

$$\exp im\int_{\partial S}\left[\zeta - \frac{\mathrm{p}}{2\pi\mathrm{K}}\varphi\wedge d\Theta\right]$$

Surface theory

- Surface theory $$S_{\partial\mathcal{M}} = \int_{\partial\mathcal{M}} dtdxdy \left[\frac{\mathrm{K}}{2\pi}\epsilon_{ij}\partial_i\zeta_j\partial_t\varphi - V(\varphi,\zeta)\right]$$

- Equivalent to (2+1)D free scalar when K = 1

$$[\varphi(t,\mathsf{r}), \epsilon_{ij}\partial_i\zeta_j(t,\mathsf{r}')] = \frac{2\pi i}{\mathrm{K}}\delta^{(2)}(\mathsf{r}-\mathsf{r}') \qquad \epsilon_{ij}\partial_i\zeta_j \sim \partial_t\varphi$$

<--> $[\phi(t,\mathsf{r}), \partial_t\phi(t,\mathsf{r}')] = i2\pi^2\delta^{(2)}(\mathsf{r}-\mathsf{r}')$

- Modular invariant when K = 1

- Can be viewed as scalar + U(1) gauge theory w/ "self-dual condition"

$$S = \int \left[(d\varphi)^2 + (d\zeta)^2\right] \qquad \partial_\mu\varphi \sim \pm\epsilon_{\mu\nu\lambda}\partial^\nu\zeta^\lambda$$

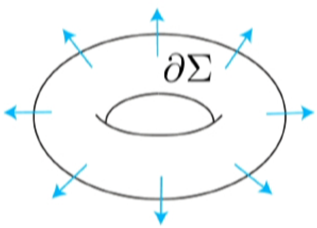
Surface theory

- Surface theory
$$S_{\partial\mathcal{M}} = \int_{\partial\mathcal{M}} dt dx dy \left[\frac{\mathrm{K}}{2\pi}\epsilon_{ij}\partial_i\zeta_j\partial_t\varphi - V(\varphi,\zeta)\right]$$

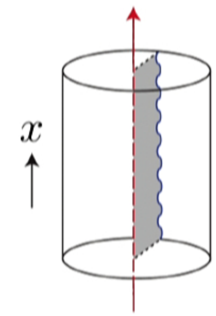
- Twisted b.c.:
$$\mathcal{M} = \Sigma \times S^1$$
$$\int_{\partial\Sigma} dx dy\, \epsilon_{ij}\partial_i\zeta_j = \frac{2\pi M_0}{\mathrm{K}}$$
$$\varphi(t, x+2\pi R_1, y) = \varphi(t,x,y) + 2\pi M_1/\mathrm{K}$$
$$\varphi(t, x, y+2\pi R_2) = \varphi(t,x,y) + 2\pi M_2/\mathrm{K}$$

- Twisted b.c. is related to bulk quasi particles/vortices

$$M_\mu = \mathrm{K}N_\mu + n_\mu$$
$$\frac{\mathrm{K}}{4\pi}\int_\Sigma d^3x\, \varepsilon^{0ijk}\partial_i b_{jk} = \int_\Sigma d^3x\, j^0_{qp} = n_0$$
$$L_1 \times \frac{\mathrm{K}}{2\pi}\int dy dz\, \varepsilon^{01ij}\partial_i a_j = n_2 \times L_1$$



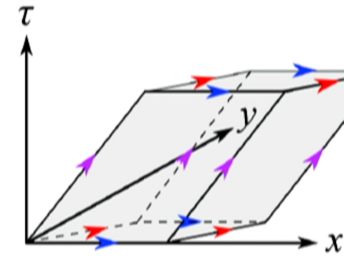
(a)



(b)

Large coord. transformations on T^3

- Surface theory put on T^3 with flat background metric g



5 modular parameters

$$R_1/R_0,\ R_2/R_0,\ \alpha,\ \beta,\ \gamma$$

$$\begin{aligned} ds^2 &= g_{\mu\nu} d\theta^\mu d\theta^\nu \\ &= R_0^2 (d\theta^0)^2 + R_1^2 (d\theta^1 - \alpha d\theta^0)^2 + R_2^2 (d\theta^2 - \beta d\theta^1 - \gamma d\theta^0)^2 \end{aligned}$$

- Symmetry (large diffeo): SL(3,Z) $\quad g_{\mu\nu} \xrightarrow{L} (LgL^T)_{\mu\nu} = L_\mu{}^\rho L_\nu{}^\sigma g_{\rho\sigma}$

- SL(3,Z): generated by two generators $\quad L = U_1^{n_1} U_2^{n_2} U_1^{n_3} \cdots$

$$U_1 = U_1' M, \quad U_1' = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad U_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\alpha \to \alpha - 1, \quad \gamma \to \gamma + \beta$$

Surface theory: results

- Surface theories with twisted b.c.:

$$\int dxdy\,\epsilon_{ij}\partial_i\zeta_j = \frac{2\pi M_0}{\mathrm{K}} \qquad \begin{aligned}\varphi(t,x+2\pi R_1,y) &= \varphi(t,x,y) + 2\pi M_1/\mathrm{K}\\ \varphi(t,x,y+2\pi R_2) &= \varphi(t,x,y) + 2\pi M_2/\mathrm{K}\end{aligned}$$

- Calculated the partition functions on flat T^3 with twisted b.c:

$$Z^{n_0 n_1 n_2} \qquad M_\mu = \mathrm{K}N_\mu + n_\mu$$

- Extracted the modular S and T matrices:

$$\mathcal{S}_{n_i,n_i'} = \frac{1}{\mathrm{K}}\delta_{n_1,n_2'}e^{-\frac{2\pi i}{\mathrm{K}}(n_0' n_2 - n_0 n_1')},$$

$$\mathcal{T}_{n_i,n_i'} = \delta_{n_0,n_0'}\delta_{n_1,n_1'}\delta_{n_2,n_2'}e^{\frac{2\pi i}{\mathrm{K}}n_0 n_1}$$

- The result agrees with the bulk calculations [E.g. Moradi-Wen (14)]
  Established the bulk-boundary correspondence.

BF topological theory with Theta term

- Action:

$$S_{bulk} = \int_{\mathcal{M}} \Big[\frac{\mathrm{K}}{2\pi} b \wedge da - \frac{\mathrm{p}}{8\pi^2} d\Theta \wedge a \wedge da - a \wedge J_{qp} - b \wedge J_{qv} \Big].$$

- Theta: externally introduced defect:

$$\Theta(x, y, z) = \frac{Q_1 x}{R_1} + \frac{Q_2 y}{R_2}$$

- Theta term induces "twist":

$$M_0 \to M_0 + \frac{Q_1 M_2 - Q_2 M_1}{\mathrm{K}} = M_0 + \frac{Q \times M}{\mathrm{K}},$$

- Surface partition functions are not closed under S and T modular transformations

10

"Twist" by Theta term

- From the EOM

$$\frac{\mathrm{K}}{2\pi}\varepsilon^{\mu\nu\lambda\rho}\partial_\lambda a_\rho = j^{\mu\nu}_{qv},$$
$$\frac{\mathrm{K}}{4\pi}\varepsilon^{\mu\nu\lambda\rho}\partial_\nu b_{\lambda\rho} = -\frac{\mathrm{p}}{2\pi\mathrm{K}}\partial_\nu\Theta j^{\mu\nu}_{qv} + j^{\mu}_{qp}$$

- Alternative way:

$$S'_{bulk} = \int d^4x\, \frac{\mathrm{K}}{4\pi}\varepsilon^{\mu\nu\lambda\rho}\partial_\nu b_{\lambda\rho}a_\mu - j^{\mu}_{qp}a_\mu - j^{\mu\nu}_{qv}\frac{1}{2}\left[b_{\mu\nu} - \frac{\mathrm{p}}{2\pi\mathrm{K}}(a_\mu\partial_\nu\Theta - a_\nu\partial_\mu\Theta)\right]$$

- Operators in surface theory:

$$\exp im\int_L a = \exp\left[im\varphi(\partial L)\right]$$
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$$S'_{bulk} = \int d^4x\, \frac{\mathrm{K}}{4\pi}\varepsilon^{\mu\nu\lambda\rho}\partial_\nu b_{\lambda\rho} a_\mu$$
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12

Coupled BF theories

- Let's now move on to more complicated theory

- Unique topological order with Zk gauge symmetry since

$$H^4[\mathbb{Z}_{\rm K}, U(1)] = 0$$

- K^2 distinct topological orders with Zk x Zk gauge symmetry since

$$H^4[\mathbb{Z}_{\rm K} \times \mathbb{Z}_{\rm K}, U(1)] = \mathbb{Z}_{\rm K} \times \mathbb{Z}_{\rm K}$$

Distinguished by 3-loop braiding statistics [Wang-Levin (14)]

Coupled BF theories

- Motivating cubic theory (I,J=1,2):

$$S_{bulk} = \int_{\mathcal{M}} \Big[\frac{\mathrm{K}}{2\pi} \delta_{IJ} b^I \wedge da^J + \frac{\mathrm{p}_1}{4\pi^2} a^1 \wedge a^2 \wedge da^2 + \frac{\mathrm{p}_2}{4\pi^2} a^2 \wedge a^1 \wedge da^1 - \delta_{IJ} b^I \wedge J^J_{qv} - \delta_{IJ} a^I \wedge J^J_{qp} \Big], \qquad (169)$$

Coupling: p1 and p2:

- Gauge invariance:

$$b^1 \to b'^1 = b^1 + d\zeta^1 - \frac{\mathrm{p}_2}{2\pi\mathrm{K}} \left(a^2 \wedge d\varphi^1 + d\varphi^2 \wedge a^1 \right),$$
$$b^2 \to b'^2 = b^2 + d\zeta^2 - \frac{\mathrm{p}_1}{2\pi\mathrm{K}} \left(a^1 \wedge d\varphi^2 + d\varphi^1 \wedge a^2 \right),$$
$$a^I \to a'^I = a^I + d\varphi^I. \qquad (173)$$

BF topological theory with Theta term

- Action:

$$S_{bulk} = \int_{\mathcal{M}} \left[\frac{\mathrm{K}}{2\pi} b \wedge da - \frac{\mathrm{p}}{8\pi^2} d\Theta \wedge a \wedge da \right.$$
$$\left. - a \wedge J_{qp} - b \wedge J_{qv} \right].$$

- Theta: externally introduced defect:

$$\Theta(x, y, z) = \frac{Q_1 x}{R_1} + \frac{Q_2 y}{R_2}$$

- Theta term induces "twist":

$$M_0 \to M_0 + \frac{Q_1 M_2 - Q_2 M_1}{\mathrm{K}}$$
$$= M_0 + \frac{Q \times M}{\mathrm{K}},$$

- Surface partition functions are not closed under S and T modular transformations

Coupled BF theories

- Quadratic theory:

$$S'_{bulk} = \frac{\mathrm{K}}{2\pi}\int \delta_{IJ} b^I \wedge da^J - \int \delta_{IJ} a^J \wedge J^I_{qp}$$
$$- \int \left[b^1 + \frac{\mathrm{p}_2}{2\pi \mathrm{K}} a^1 \wedge a^2\right] \wedge J^1_{qv}$$
$$- \int \left[b^2 + \frac{\mathrm{p}_1}{2\pi \mathrm{K}} a^2 \wedge a^1\right] \wedge J^2_{qv}.$$

- 3-loop braiding statistics

$$S_{eff} = -\frac{2\pi}{\mathrm{K}}\int (d^{-1}J^I_{qv}) \wedge J^I_{qp}$$
$$+ \left(\frac{2\pi}{\mathrm{K}}\right)^3 \mathrm{p}_1 \int (d^{-1}J^1_{qv}) \wedge (d^{-1}J^2_{qv}) \wedge J^2_{qv}$$
$$+ \left(\frac{2\pi}{\mathrm{K}}\right)^3 \mathrm{p}_2 \int (d^{-1}J^2_{qv}) \wedge (d^{-1}J^1_{qv}) \wedge J^1_{qv}$$

- The coupling gives the following twist:

$$M_0 \to M_0 + \frac{Q \times M}{\mathrm{K}}$$
$$Q_0 \to Q_0 + \frac{M \times Q}{\mathrm{K}}$$

Coupled BF theories

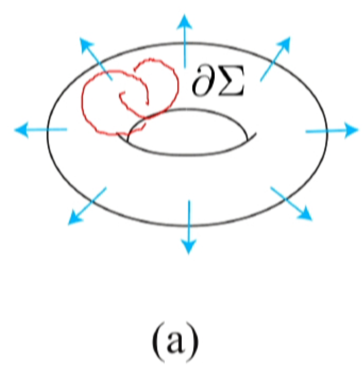
- Surface theories with twisted b.c.:

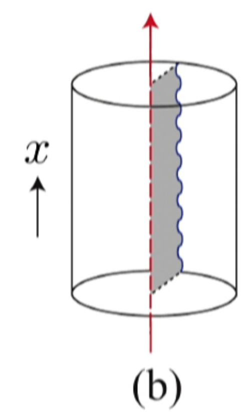
Twisted b.c. is related to bulk quasi particles/vortices

$$\frac{\mathrm{K}}{2\pi}\int_\Sigma db^1 = -\frac{\mathrm{p}_1}{\mathrm{K}^2}\int_\Sigma (d^{-1}J^2_{qv})\wedge J^2_{qv} + \frac{\mathrm{p}_2}{\mathrm{K}^2}\int_\Sigma (d^{-1}J^2_{qv})\wedge J^1_{qv} + \int_\Sigma J^1_{qp}$$

Hopf linking of vortex lines twists b.c.

$$L_1 \times \frac{\mathrm{K}}{2\pi}\int dydz\, \varepsilon^{01ij}\partial_i a_j = n_2 \times L_1$$





Coupled BF theories

- Surface theories with twist

$$M_\mu = \mathrm{K}N_\mu + n_\mu, \quad n_\mu = 0,1,\ldots,\mathrm{K}-1,$$
$$Q_\mu = \mathrm{K}R_\mu + r_\mu, \quad r_\mu = 0,1,\ldots,\mathrm{K}-1,$$

$$M_0 \to M_0 + \frac{Q \times M}{\mathrm{K}}$$
$$Q_0 \to Q_0 + \frac{M \times Q}{\mathrm{K}}$$

- Extracted the modular S and T matrices (when p1=p2=K):

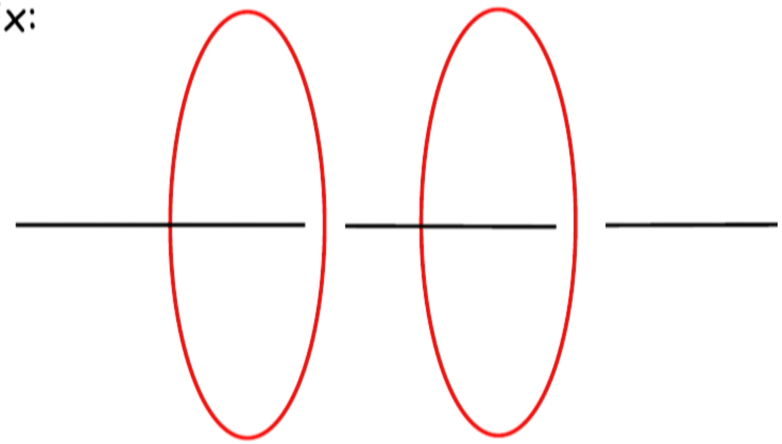
$$\chi^{n_0 n_1 n_2}_{r_0 r_1 r_2}$$

$$\mathcal{S}_{n_\mu, n'_\mu, r_\mu, r'_\mu} = \frac{1}{\mathrm{K}^2}\delta_{n_1,n'_2}\delta_{r_1,r'_2} e^{-\frac{2\pi i}{\mathrm{K}}(\tilde{n}'_0 n_2 - \tilde{n}_0 n'_1 + \tilde{r}'_0 r_2 - \tilde{r}_0 r'_1)}$$
$$\times e^{-\frac{2\pi i}{\mathrm{K}^2}\left[(n_1+r_1)(n_2 r'_1 + n'_1 r_2) - 2n_2 n'_1 r_1 - 2n_1 r_2 r'_1\right]}$$
$$\mathcal{T}_{n_\mu, n'_\mu, r_\mu, r'_\mu} = \delta_{n_\mu, n'_\mu}\delta_{r_\mu, r'_\mu}$$
$$\times e^{\frac{2\pi i}{\mathrm{K}}(\tilde{n}_0 n_1 + \tilde{r}_0 r_1) + \frac{2\pi i}{\mathrm{K}^2}(r_1 n_2 - r_2 n_1)(n_1 - r_1)}.$$

- The result is consistent with the bulk calculations [E.g.Wang-Levin (14) Jiang-Mesaros-Ran (14), Wang-Wen (14)]

- Interpretation of S-matrix:

$$e^{\frac{2\pi i}{K^2} n_1 n_2 r_1'}$$



- Dimensional reduction to (2+1)d S-matrix (K=2):

$$(n_2, r_2) = (0,0) \qquad \mathbf{K} = 2\sigma_x \oplus 2\sigma_x$$

$$(n_2, r_2) = (1,1) \qquad \mathbf{K} = 2\sigma_z \oplus 2\sigma_z$$

$$(n_2, r_2) = (1,0)$$

$$\mathbf{K} = \begin{pmatrix} 0 & 2 & -1 & 0 \\ 2 & 0 & 0 & 0 \\ -1 & 0 & 2 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

Comments:

- K^2 distinct topological orders with Zk x Zk gauge symmetry since

$$H^4[\mathbb{Z}_{\mathrm{K}} \times \mathbb{Z}_{\mathrm{K}}, U(1)] = \mathbb{Z}_{\mathrm{K}} \times \mathbb{Z}_{\mathrm{K}}$$

Choose $p_{1,2}/\mathrm{K} = 0, \ldots, \mathrm{K}-1$

- Can compute (entanglement) entropy from the boundary partition fn

- Is the theory completely healthy ?
  Should the spacetime geometry be formal ?