Title: Topological Quantum Field Theory approach for Bosonic Symmetry-Protected-Topological Phases with Abelian Symmetry in Three Dimensions

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Abstract: Symmetry protected topological(SPT) phase is a generalization of topological insulator(TI). Different from the intrinsic topological phase, e.g., the fractional quantum hall(FQH) phase, SPT phase is only distinguishable from a trivial disordered phase when certain symmetry is preserved. Indeed, SPT phase has a long history in 1D, and it has been shown that the well known Haldane phase of S=1 Heisenberg chain belongs to this class. However, in higher dimensions, most of the previous studies focus on free electron systems. Until very recently, it was realized that SPT phase also exists in interacting boson/spin systems in higher dimensions. In this talk, I will discuss the general mechanism for bosonic SPT phases and propose a corresponding topological quantum field theory(TQFT)descriptions. I will focus on examples in three (spacial) dimensions, including bosonic topological insulators(BTI) .

Hydrodynamic Approach for Bosonic Symmetry-Protected-Topological Phases with Abelian Symmetry in 3+1D

arXiv:1410.2594 (2014)
arXiv:1508.05689 (2015)

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PI. Oct. 2015

Topological phases of quantum matter:

What are topological phases of quantum matter?
Gapped quantum phases without symmetry breaking and long range correlation, but can not be adiabatically connected to a trivial disorder phase without phase transition.

Two basic classes of topological phases:

Intrinsic topological phases (long-range-entanglement)

- adiabatic paths with no symmetry

Symmetry protected topological (SPT) phases

- adiabatic paths with symmetry

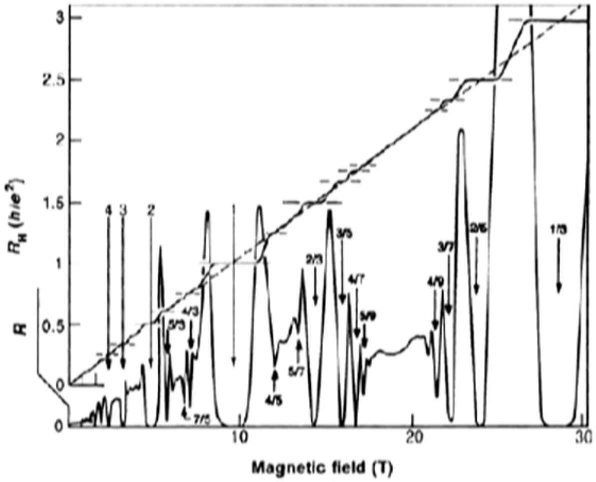
symmetry breaking Hamiltonians

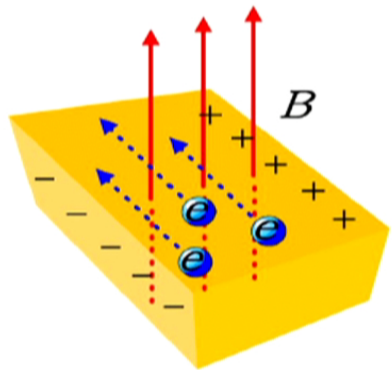
SPT phases $\Longleftrightarrow$ The trivial disorder phase

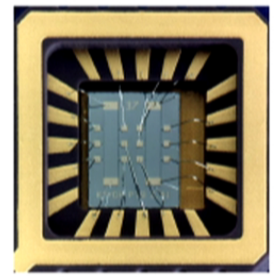
(Z C Gu, X G Wen, 2009)

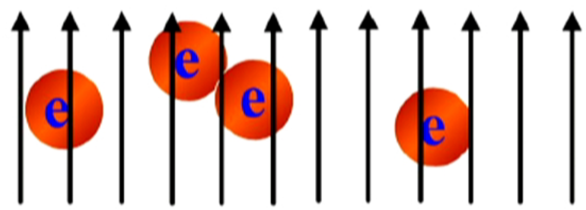
Examples of intrinsic topological phases in interacting systems (no symmetry)

Fractional Quantum Hall Effect(FQHE) D C Tsui, *et al* 1982





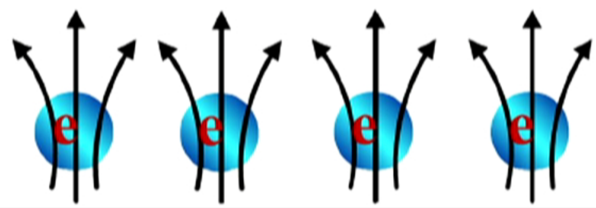




TQFT approach

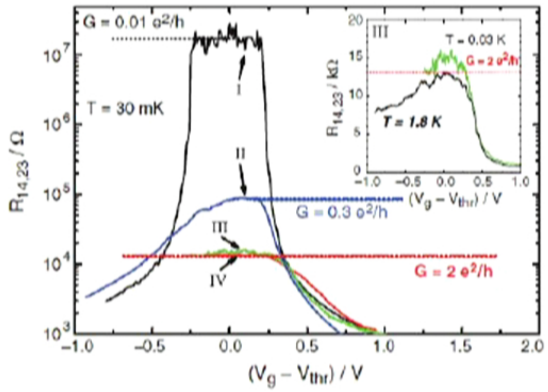
$$\mathcal{L}_{\rm eff} = \frac{2m+1}{4\pi}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda$$

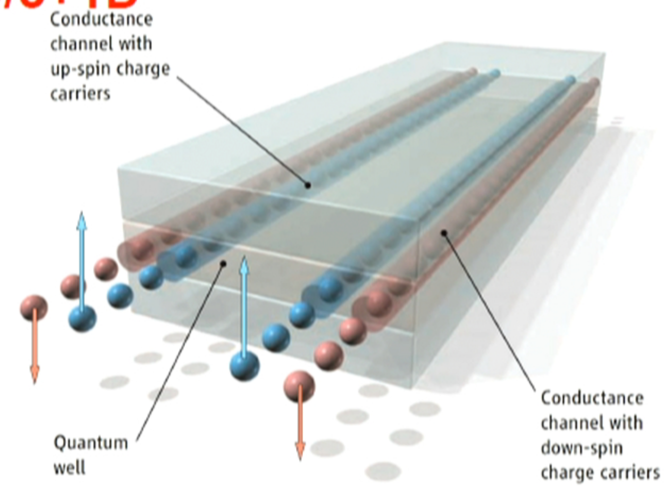
E Witten, 1989
S C Zhang, *et al* 1989
X G Wen, *et al* 1989

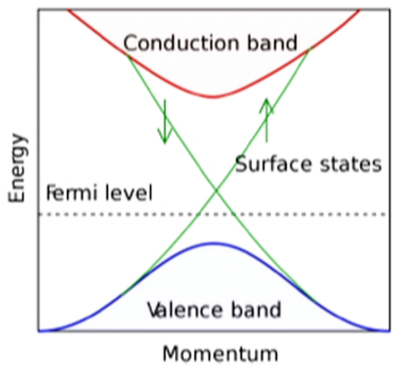


Examples of symmetry protected topological (SPT)phases in free fermion systems

Topological insulator in 2+1D/3+1D

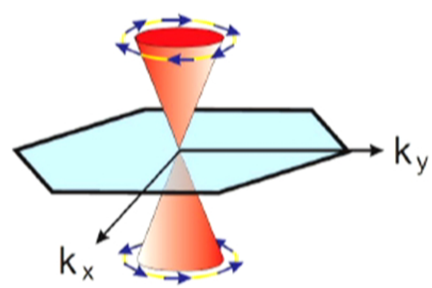






C L Kane, *et al*, 2005
B A Bernevig, et al 2006
W Molenkamp's group 2007
M Zahid Hasan, *et al*, 2008

from Wikipedia

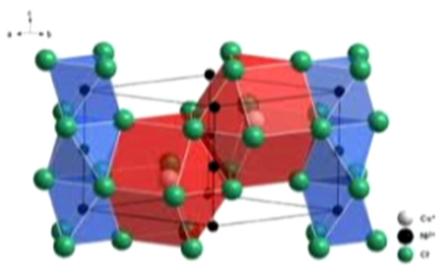


SPT phases in free fermion systems: classified by K theory(Kitaev 2008, A. P. Schnyder, S. Ryu, A. Furusaki, A. W. W. Ludwig 2008)

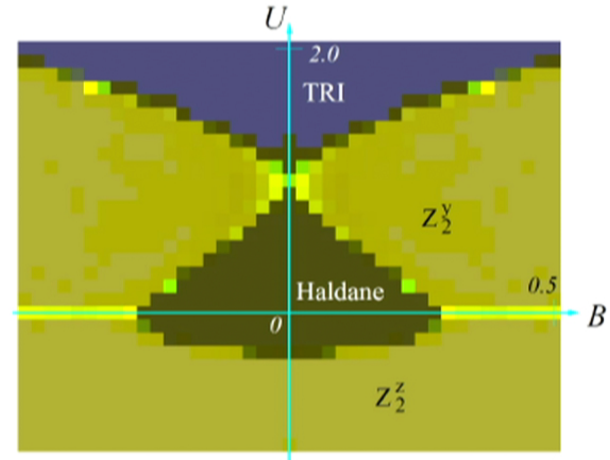
SPT phases in interacting 1+1D systems

Spin-1 Haldane chain realizes 1D topological order

$$H = \sum_i \left(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} + U(S_i^z)^2 + BS_i^x \right)$$



CsNiCl3(U~B~0)



Haldane phase is protected by symmetry! e.g., parity, time reversal Z C Gu and X G Wen, 2009, F Pollmann, *et al*, 2010

Fixed point wavefunction: spin-(1/2,1/2) dimer model

The key observation: edge states form projective representation of the symmetry group! (Xie Chen, Z C Gu, X G Wen, (2011))

SPT phases in interacting boson system in higher dimensions

lattice model from group cohomology theory

(Chen, Gu, Liu, Wen, 2011)

Dijkgraaf-Witten gauge theory

(Dijkgraaf, Witten, 1990)

(Levin Gu, 2012)

high form symmetry

(Gaiotto, Kapustin, Seiberg, Willett, 2014)

bosonic SPT

loop braiding

(Wang Levin, 2014, Jiang, Mesaros, Ran, 2014)

non-perturbative anomaly

(Wen 2013
Kapustin, Thorngren 2014
Wang, Gu, Wen, 2014)

cobordism invertable TQFT

(Kapustin, 2014, Freed 2014)

anomalous surface topological order

(Vishwanath, Senthil, 2013)

Physical mechanism of SPT phases

- Exact solvable lattice models constructed from group cohomology theory are very complicated, and very hard to be realized in any physical lab.

- Just like the Chern-Simons action approach to FQHE, a hydrodynamic approach to SPT phases is very desired.

- **SPT phases are kinds of "nontrivial" disordered phases, therefore they can be achieved from a symmetry breaking phase by "proliferating symmetry defects".**

Advantage of hydrodynamic approach

- At least for Abelian symmetry group, most of the (extended) group cohomology classification results can be easily reproduced in 2+1D and 3+1D.

- Physical mechanism for SPT phases becomes manifested, which might lead to a systematic way towards experimental realization.

- The boundary conformal field theory/anomalous topological order can be derived in a natural way through dimension reduction.

- SPT phases are kinds of "nontrivial" disorder phase, therefore they can be achieved from a symmetry breaking phase by "symmetry defect condensation".

2+1D Mott-insulator from vortex condensation

Hardcore boson: $\mathcal{L}_b = \frac{1}{2}(b^*\partial_\tau b + h.c) + \frac{1}{2m}\partial_x b^* \partial_x b - \mu|b|^2 + g|b|^4$

- superfluid: $b(x) = \sqrt{\rho_0 + \delta\rho}\, e^{i\theta(x)}$ $\mathcal{L}_{\mathrm{XY}} = \frac{\chi}{2}[(\partial_\tau\theta)^2 + v^2(\nabla\theta)^2]$

Mott-insulator from vortex condensation: replace $\partial_\mu\theta$ by $\partial_\mu\theta + \tilde{a}_\mu$

$$\mathcal{L}_{\mathrm{Higgs}} = \frac{1}{2}[(\partial_\mu\theta + \tilde{a}_\mu)^2 + \frac{1}{4\pi^2}\tilde{f}_{\mu\nu}\tilde{f}^{\mu\nu}]$$

- Hubbard-Stratonovich transformation:

$$\frac{1}{2}[(j_\mu)^2 + 2i\theta\partial_\mu j^\mu - 2ij^\mu\tilde{a}_\mu + \frac{1}{4\pi^2}\tilde{f}_{\mu\nu}\tilde{f}^{\mu\nu}]$$

- integrate out smooth fluctuation theta; $j^\mu = \frac{1}{2\pi}\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda$

$$\mathcal{L}_{\mathrm{CS}} = -\frac{i}{2\pi}\varepsilon^{\mu\nu\lambda}\tilde{a}_\mu\partial_\nu a_\lambda + \frac{1}{8\pi^2}[\tilde{f}_{\mu\nu}\tilde{f}^{\mu\nu} + f_{\mu\nu}f^{\mu\nu}]$$

Trivial edge with time reversal and charge conservation symmetry

Boundary Kac-Moody algebra:

$$[\partial_x \phi_I, \partial_y \phi_J] = 2\pi i K^{-1}_{IJ} \partial_x \delta(x-y) \qquad K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \partial_i \phi_I = a_{Ii}$$

Boundary action $\mathcal{L}_{\text{bdry}} = -\frac{i}{4\pi} K_{IJ} \partial_\tau \phi_I \partial_x \phi_J + V_{IJ} \partial_x \phi_I \partial_x \phi_J$

Symmetry transformation for charge/vortex fields

$$U_\theta b_1 U_\theta^{-1} = e^{-i\theta} b_1, \; T b_1 T^{-1} = b_1$$
$$U_\theta b_2 U_\theta^{-1} = b_2, \; T b_2 T^{-1} = b_2^\dagger$$

Boundary symmetry transformation for a trivial 2D Mott-insulator:

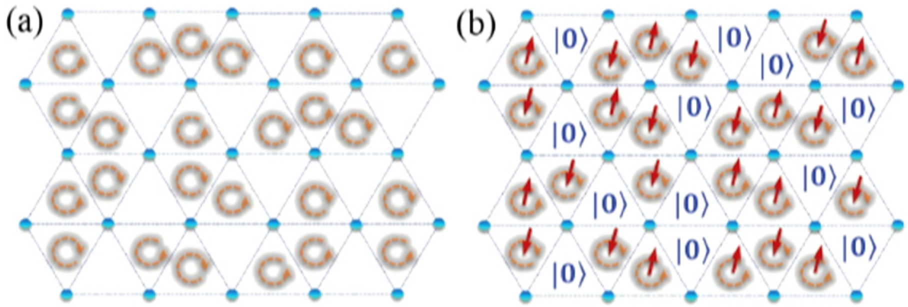
$$\rho_I = \partial_x \phi_I, \quad b_I \sim e^{-i\phi_I} \quad \begin{array}{l} U_\theta \phi_1 U_\theta^{-1} = \phi_1 + \theta, \; T\phi_1 T^{-1} = -\phi_1, \\ U_\theta \phi_2 U_\theta^{-1} = \phi_2, \quad T\phi_2 T^{-1} = \phi_2, \end{array}$$

Relevant operator: $G \cos \phi_2$

2+1D bosonic topological insulator from exotic vortex condensation

Zheng-Xin Liu, Zheng-Cheng Gu, Xiao-Gang Wen, Phys. Rev. Lett. 114, 031601 (2015)

Attach an integer spin to the vortex operator:



$$b'_2 = b_2 S^-$$

$$U_\theta (b'_2) U_\theta^{-1} = b'_2$$

$$T(b'_2)T^{-1} = -(b'_2)^\dagger$$

Anomalous boundary symmetry transformation for a 2D BTI:

$b'_2 = e^{-i\phi'_2}$ $\quad U_\theta \phi'_2 U_\theta^{-1} = \phi'_2, \quad T\phi'_2 T^{-1} = \phi'_2 + \pi$

Relevant operator leads to T-symmetry breaking: $G\cos(2\phi'_2)$

Bulk response for a 2D BTI :

$T|\pi\rangle \propto |-\pi\rangle, \quad T|-\pi\rangle \propto |\pi\rangle$ $\qquad T|\pi\rangle = |-\pi\rangle,$

$b'_2|\pi\rangle = \eta|-\pi\rangle, \quad b'^\dagger_2|-\pi\rangle = \eta|\pi\rangle$ $\qquad T|-\pi\rangle = -|\pi\rangle$

General bosonic SPT phases with Abelian symmetry in 2+1D

Invertable Abelian Chern-Simons theory $|\det[K]| = 1$

(Michael Levin, Ady Stern, Phys. Rev. B 86, 115131 (2012)
Yuan-Ming Lu, Ashvin Vishwanath Phys. Rev. B 86, 125119 (2012))

$$\mathcal{L}_{\mathrm{CS}} = \frac{\mathrm{i}}{4\pi}\varepsilon^{\mu\nu\lambda}K^0_{IJ}a^I_\mu\partial_\nu a^J_\lambda \qquad {K^0}_{IJ} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathbf{I}_{k\times k}$$

$$\mathcal{L}_{\mathrm{coupling}} = \frac{\mathrm{i}}{2\pi}\varepsilon^{\mu\nu\lambda}q^I_\alpha A^\alpha_\mu\partial_\nu a^I_\lambda; \quad \alpha = 1, 2, \cdots, k$$

- A complete description for $U(1)^k$ bosonic SPT phases
- Naively, one might guess all bosonic $Z_{N_1} \times \cdots \times Z_{N_k}$ SPT phases can be derived from U(1) symmetry breaking

Non-Abelian SPT phase arises even with Abelian symmetry

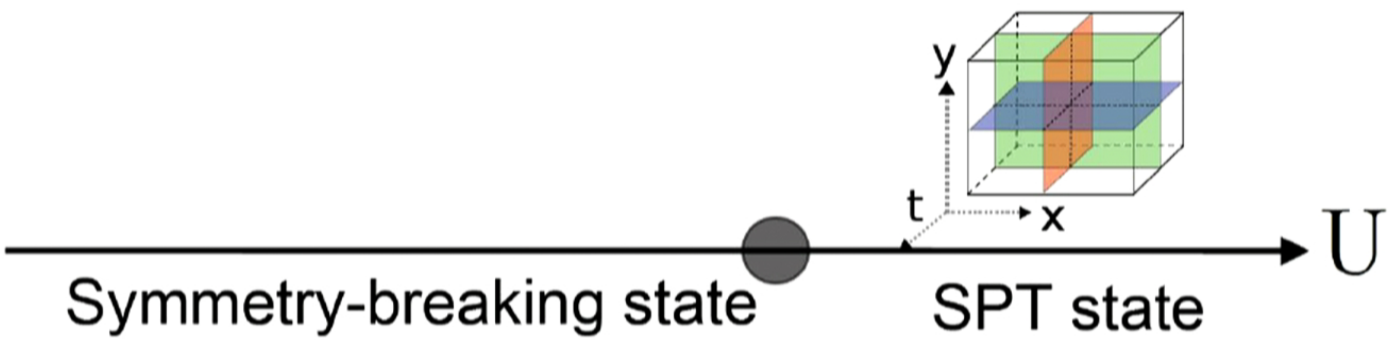
$$H^3[Z_{N_1} \times Z_{N_2} \times Z_{N_3}, U(1)]$$
$$= \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \oplus \mathbb{Z}_{N_3} \oplus \mathbb{Z}_{N_{12}} \oplus \mathbb{Z}_{N_{23}} \oplus \mathbb{Z}_{N_{13}} \oplus \mathbb{Z}_{N_{123}}$$

Beyond Abelian Chern-Simons theory

A GL type action for $Z_{N_1} \times Z_{N_2} \times Z_{N_3}$ bosonic SPT phase

$$\mathcal{L}_{\text{tri-kink}} = \frac{1}{2}(\partial_\mu \theta^I_{\mathrm{s}} + b^I_\mu)^2 + \mathcal{L}^b_{\text{Maxwell}}$$

$$+\frac{\mathrm{i}}{3} C_{IJK} \varepsilon^{\mu\nu\lambda} (\partial_\mu \theta^I_{\mathrm{s}} + b^I_\mu)(\partial_\nu \theta^J_{\mathrm{s}} + b^J_\nu)(\partial_\lambda \theta^K_{\mathrm{s}} + b^K_\lambda)$$



(Gu, Wang, Wen arXiv:1503.01768, 2015)

Quantized coefficient under symmetry protection

- Consider kink configuration on the yt, tx and xy plane

$$\Delta\theta^1 = 2\pi k_1/N_1 \quad \Delta\theta^2 = 2\pi k_2/N_2 \quad \Delta\theta^3 = 2\pi k_3/N_3$$

$$S = \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\;\frac{\mathrm{i}}{3} C_{IJK}\varepsilon^{\mu\nu\lambda}\partial_\mu\theta^I\partial_\nu\theta^J\partial_\lambda\theta^K$$
$$= 16\pi^3\,\mathrm{i}\,C_{123}\frac{k_1k_2k_3}{N_1N_2N_3}.$$

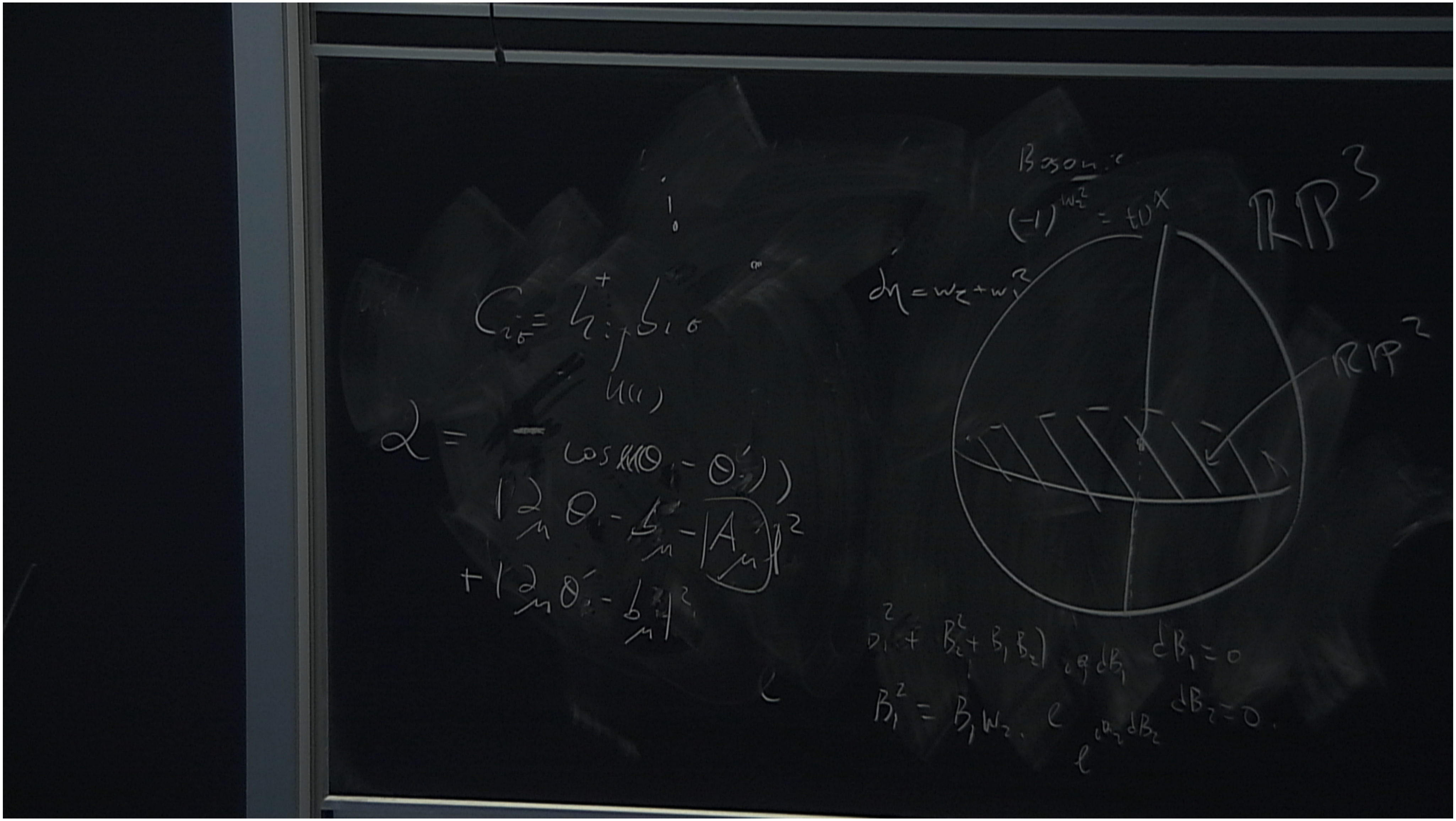
the intersection of theta1 and theta2 kinks carries a

$$Z_{N_3}\text{-charge}\;\; 8\pi^2 C_{123}\frac{k_1k_2}{N_1N_2} \mod N_3$$

$$k_1 = 0 \sim k_1 = N_1$$

$$8\pi^2 C_{123}\frac{k_2}{N_2} = 0 \mod N_3 \qquad\qquad 8\pi^2 C_{123}\frac{k_1}{N_1} = 0 \mod N_3$$

$$\boxed{C_{123} = \frac{p_{\mathrm{III}}}{(2\pi)^2 2!}\frac{N_1N_2N_3}{N_{123}}, \qquad p_{\mathrm{III}} = 0,\cdots,N_{123}-1}$$


Bosonic
$(-1)^{w_2^2} = t0^x$
$\mathbb{RP}^3$
$\mathbb{RP}^2$
$c_{i\sigma} = h_i \, b_{i\sigma}$
$\mathcal{L} = \cos(\theta_i - \theta_j')$
$|\partial_\mu \theta_i - b_\mu - A_\mu|^2$
$+ |\partial_\mu \theta_i' - b_\mu|^2$
$dB_1 = 0$
$B_1^2 = B_1 w_2$
$dB_2 = 0$

Duality transformation and topological quantum field theory

- integrating out all theta leads to(compact a and b):

$$\boxed{\mathcal{L}_{\rm eff} = \frac{\mathrm{i}\varepsilon^{\mu\nu\lambda}}{2\pi} b_\mu^I \partial_\nu a_\lambda^I + \frac{\mathrm{i} C_{IJK}}{3} \varepsilon^{\mu\nu\lambda} b_\mu^I b_\nu^J b_\lambda^K}$$

$$\begin{aligned} b_\mu^I &\to b_\mu^I + \partial_\mu g^I; \\ a_\mu^I &\to a_\mu^I + \partial_\mu f^I - 4\pi C_{IJK} \left(g^J b_\mu^K + \frac{1}{2} g^J \partial_\mu g^K \right) \end{aligned}$$

- Coupling to background flat gauge fields(symmetry twists) gives rise to the correct topological response

$$Z^{\rm twist}_{\text{fixed-point}}(T^3) = e^{\mathrm{i}\, p_{\rm III} \frac{N_1 N_2 N_3}{(2\pi)^2 N_{123}} \int A_1 \wedge A_2 \wedge A_3}, \quad \mathrm{d}A_I = 0$$

Fractionalized "O(3) theta term" on the boundary

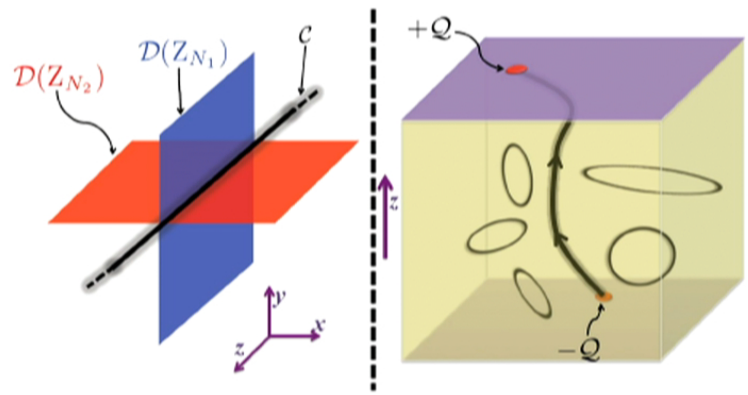
$$\mathcal{L}^1_{\rm edge} = \frac{\mathrm{i}}{3} C_{IJK} \varepsilon^{\mu\nu} \varphi^I \partial_\mu \varphi^J \partial_\nu \varphi^K$$

A 3+1D SPT phase with finite Abelian symmetry group

Group cohomology classification: $\mathcal{H}^4(Z_{N_1} \times Z_{N_2} \cdots, U(1)) =$
$\Pi_{I<J}(\mathbb{Z}_{N_{IJ}})^2 \times \Pi_{I<J<K}(\mathbb{Z}_{N_{IJK}})^2 \times \Pi_{I<J<K<L}\mathbb{Z}_{N_{IJKL}}$

Ginzburg Landau type action for a trivial SPT phase with $G = Z_{N_1} \times Z_{N_2} \times Z_{N_3}$ **symmetry:**

$$S_0 = \sum_I^3 \int d^4x \frac{1}{2}(\partial_\mu \theta^I - a_\mu^I)^2 + \cdots$$



Proliferating nontrivial ZN domain walls by adding topological Berry phase:

$$S_{\mathrm{Top}} = -ip \int d^4x (\partial_\mu \theta^1 - a_\mu^1)(\partial_\nu \theta^2 - a_\nu^2)\partial_\lambda a_\rho^3 \epsilon^{\mu\nu\lambda\rho}$$

Extended BF theory-I

• Start from S0+Stop, apply Hubbard-Stratonovich transformation and integrate out smooth fluctuations theta:

$$S_{\rm TQFT} = \frac{i}{2\pi}\sum_I^3 \int b^I \wedge da^I + ip\int a^1 \wedge a^2 \wedge da^3 + S_M$$

Higher order Maxwell term has a complicated form:

$$\begin{aligned} S_{\rm M} = &\int d^4x \frac{1}{2}(pa_\nu^2 \partial_\lambda a_\rho^3 \epsilon^{\mu\nu\lambda\rho} + \frac{1}{4\pi}\partial_\nu b^1_{\lambda\rho}\epsilon^{\mu\nu\lambda\rho})^2 \\ &+ \int d^4x \frac{1}{2}(-pa_\nu^1 \partial_\lambda a_\rho^3 \epsilon^{\mu\nu\lambda\rho} + \frac{1}{4\pi}\partial_\nu b^2_{\lambda\rho}\epsilon^{\mu\nu\lambda\rho})^2 \\ &+ \int d^4x \frac{1}{2}(\frac{1}{4\pi}\partial_\nu b^3_{\lambda\rho}\epsilon^{\mu\nu\lambda\rho})^2 \end{aligned}$$

Gauge transformation:

$$a^I \to a^I + d\chi^I\,,\; b^I \to b^I + dV^I - 2\pi p\,\epsilon^{IJ3}\chi^J\, da^3$$

Gauge invariant Wilson loop and Wilson surface:

$$\exp\{i \textstyle\int_{\mathcal{M}^1} a^I\}$$

$$\exp\{i \textstyle\int_{\mathcal{M}^2} b^I - i2\pi p \int_{\mathcal{V}^3} \epsilon^{IJ3} a^J \wedge da^3\} \quad \partial\mathcal{V}^3 = \mathcal{M}^2$$

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Level quantization and physical property

Compactness and level quantization:

$$\frac{1}{2\pi}\int_{\mathcal{M}^2} da^I \in \mathbb{Z}\,,\ \frac{1}{2\pi}\int_{\mathcal{M}^3} db^I \in \mathbb{Z};$$

$$\boxed{p = \frac{k}{4\pi^2}\frac{N_1 N_2}{N_{12}}\,, \quad k \in \mathbb{Z}_{N_{123}}}$$

Symmetry condition: $p\int_{\mathcal{M}^3} d\chi^{\bar{I}} \wedge da^3 = N_I \times \mathbb{Z}$

$$\frac{1}{2\pi}\int_{\mathcal{M}^3} db^I \to \frac{1}{2\pi}\int_{\mathcal{M}^3} db^I - p\int_{\mathcal{M}^3} d\chi^{\bar{I}} \wedge da^3 \qquad I = 1, 2\,; \bar{I} = 3 - I$$

Additional symmetry preserving shift condition on b is also used

- On the Z_{N1} symmetry domain wall, we introduce symmetry twists A and $\tilde{A}$

By integrating out dynamic gauge fields a and b, we end up with a Chern-Simons with even integer coefficients, which identifies a Z_{N2}*Z_{N3} 2+1D bosonic SPT phase: (Xie, Lu, Vishwanath 2013)

$$S[A] = \frac{i}{4\pi}\int_{\mathcal{D}(Z_{N_1})} \frac{2N_2 k_1 k}{N_{12}} A \wedge d\tilde{A} \qquad k_1 \in \mathbb{Z}_{N_1}$$

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A dual formulation of 3+1D superfluid

- 3+1D superfluid:

$$\mathcal{L} = \frac{\rho}{2}(\partial_\mu \theta)^2$$

- Hubbard-Stratonovich transformation and dual representation:

$$\mathcal{L} = \frac{1}{2\rho}(\mathcal{J}_\mu)^2 + i\mathcal{J}^\mu(\partial_\mu \theta^{\mathrm{s}} + \partial_\mu \theta^{\mathrm{v}})$$

- Integrate out the smooth fluctuation of theta:

$$\mathcal{J}^\mu \overset{\text{def.}}{=\!=} \frac{1}{4\pi}\epsilon^{\mu\nu\lambda\rho}\partial_\nu b_{\lambda\rho} \qquad b_{\mu\nu} \to b_{\mu\nu} + \partial_{[\mu}\xi_{\nu]}$$

- Vortex-line current and field strength of b:

$$\Sigma^{\mu\nu} \overset{\text{def.}}{=\!=} \frac{1}{2\pi}\epsilon^{\mu\nu\lambda\rho}\partial_\lambda\partial_\rho\theta^{\mathrm{v}} \qquad h_{\mu\nu\lambda} \overset{\text{def.}}{=\!=} \partial_\mu b_{\nu\lambda} + \partial_\nu b_{\lambda\mu} + \partial_\lambda b_{\mu\nu}$$

$$\mathcal{L} = \frac{1}{48\pi^2\rho}h^{\mu\nu\lambda}h_{\mu\nu\lambda} + \frac{i}{2}b_{\mu\nu}\Sigma^{\mu\nu}$$

The Goldstone mode of the 3+1D superfluid is described by the Maxwell term of the two-form gauge field b.

The linking-Berry phase term in a vortex line condensate

Add a linking-Berry phase term:

$$\mathcal{L} = \frac{1}{2}\phi_0^2\left(\partial_{[\mu}\Theta^s_{\nu]} - b_{\mu\nu}\right)^2 + \mathcal{L}_h$$
$$- i\frac{\Lambda}{16\pi}\epsilon^{\mu\nu\lambda\rho}\left(\partial_{[\mu}\Theta^s_{\nu]} - b_{\mu\nu}\right)\left(\partial_{[\lambda}\Theta^s_{\rho]} - b_{\lambda\rho}\right)$$

- Hubbard-Stratonovich transformation:

$$\mathcal{L} = i\frac{1}{2}\Xi^{\mu\nu}(\partial_{[\mu}\Theta^s_{\nu]} - b^{\mu\nu}) + \frac{1}{8\phi_0^2}\Xi_{\mu\nu}\Xi^{\mu\nu} + i\frac{\Lambda}{8\pi}\epsilon^{\mu\nu\lambda\rho}\partial_{[\mu}\Theta^s_{\nu]}b_{\lambda\rho}$$
$$- i\frac{\Lambda}{16\pi}\epsilon^{\mu\nu\lambda\rho}b_{\mu\nu}b_{\lambda\rho} + \mathcal{L}_h$$

modified constraint: $\boxed{\partial_\nu\left(\Xi^{\mu\nu} + \frac{\Lambda}{4\pi}\epsilon^{\mu\nu\lambda\rho}b_{\lambda\rho}\right) = 0}$

$$= i\Theta^s_\mu\partial_\nu\left(\Xi^{\mu\nu} + \frac{\Lambda}{4\pi}\epsilon^{\mu\nu\lambda\rho}b_{\lambda\rho}\right) - i\frac{1}{2}\Xi_{\mu\nu}b^{\mu\nu}$$
$$- i\frac{\Lambda}{16\pi}\epsilon^{\mu\nu\lambda\rho}b_{\mu\nu}b_{\lambda\rho} + \frac{1}{8\phi_0^2}\Xi_{\mu\nu}\Xi^{\mu\nu} + \mathcal{L}_h,$$

solve the constraint: $\boxed{\Xi^{\mu\nu} \overset{\text{def.}}{=\!=} -\frac{1}{2\pi}\epsilon^{\mu\nu\lambda\rho}\partial_\lambda a_\rho - \frac{\Lambda}{4\pi}\epsilon^{\mu\nu\lambda\rho}b_{\lambda\rho}}$

Extended BF theory II

$$\mathcal{L}_{\rm top} = i\frac{1}{4\pi}\epsilon^{\mu\nu\lambda\rho}b_{\mu\nu}\partial_\lambda a_\rho + i\frac{\Lambda}{16\pi}\epsilon^{\mu\nu\lambda\rho}b_{\mu\nu}b_{\lambda\rho}$$

(G. T. Horowitz, Commun. Math. Phys. 125, 417,1989)

Gauge transformation:

$$b_{\mu\nu} \longrightarrow b_{\mu\nu} + \partial_{[\mu}\xi_{\nu]}\ ,\ a_\mu \longrightarrow a_\mu + \partial_\mu\eta - \Lambda\xi_\mu$$

Multiple components generalization:

$$\mathcal{L}_{\rm top} = i\frac{K^{IJ}}{4\pi}\epsilon^{\mu\nu\lambda\rho}b^I_{\mu\nu}\partial_\lambda a^J_\rho + i\frac{\Lambda^{IJ}}{16\pi}\epsilon^{\mu\nu\lambda\rho}b^I_{\mu\nu}b^J_{\lambda\rho}$$

Unique ground state degeneracy and canonical form:

$$|\det K| = 1 \qquad\qquad K = \mathrm{diag}(1, 1, \cdots, 1)_{N\times N} = \mathbb{I}$$

Gauge transformation:

$$b^I_{\mu\nu} \longrightarrow b^I_{\mu\nu} + \partial_{[\mu}\xi^I_{\nu]}\ ,\ a^I_\mu \longrightarrow a^I_\mu + \partial_\mu\eta^I - (K^{-1}\Lambda)^{II'}\xi^{I'}_\mu$$

Both a and b are compact.

Time reversal symmetry and quantized cosmological constant term

Flux quantization under time reversal symmetry:

$$\int_{\mathbb{T}_{0x}} b_{0x} d\tau dx = \pi \times \mathcal{N}_{0x}, \int_{\mathbb{T}_{yz}} b_{yz} dy dz = \pi \times \mathcal{N}_{yz}$$

$$\boxed{\Lambda \to \Lambda + 4}$$

B^B term is time reversal symmetry odd:

$$\frac{\Lambda^{IJ}}{16\pi} \epsilon^{\mu\nu\lambda\rho} b^I_{\mu\nu} b^J_{\lambda\rho} \Longrightarrow -\frac{\Lambda^{IJ}}{16\pi} \epsilon^{\mu\nu\lambda\rho} b^I_{\mu\nu} b^J_{\lambda\rho}$$

A quantized B^B term:

$$\Lambda^{II} = 0, \pm 2 \, , \Lambda^{IJ} = 0, \pm 1 \ (\text{for } I \neq J)$$

Trivial and nontrivial SPT phases

Two special kinds of trivial SPT phases: $|\det\Lambda| = 1$

$$\Lambda_{t1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Lambda_{t2} = \begin{pmatrix} 2&1&0&0&0&0&0&0 \\ 1&2&1&0&0&0&0&0 \\ 0&1&2&1&0&0&0&1 \\ 0&0&1&2&1&0&0&0 \\ 0&0&0&1&2&1&0&0 \\ 0&0&0&0&1&2&1&0 \\ 0&0&0&0&0&1&2&0 \\ 0&0&1&0&0&0&0&2 \end{pmatrix}$$

$|\det\Lambda| > 1$

Inconsistent theories that realize different topological orders under time reversal symmetry:

$$\Lambda = 2 \quad \Rightarrow \quad \Lambda = -2$$

Trivial SPT phases with time reversal symmetric boundary topological order that can be realized in purely 2D:

$$W^T \Lambda W = -\Lambda, \ \exists W \in \mathbb{GL}(N, \mathbb{Z}) \qquad \Lambda = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$$

A nontrivial SPT phase with half-E8 boundary topological order:

$$\Lambda_{so8} = \begin{pmatrix} 2&1&1&1 \\ 1&2&0&0 \\ 1&0&2&0 \\ 1&0&0&2 \end{pmatrix} \qquad W^T(\Lambda_{so8} \sum^4 \oplus \Lambda_{t1})W = (-\Lambda_{so8}) \oplus \Lambda_{t2}$$

$$\exists W \in \mathbb{GL}(12, \mathbb{Z}).$$

Conclusions and future works

TQFT descriptions for (irreducible) bosonic SPT phases with Abelian group symmetry in 3+1D.

Symmetry	Topological Quantum Field Theory
$\mathrm{Z}_{N_1} \times \mathrm{Z}_{N_2}$	$\frac{i}{2\pi}\int\sum_I^2 b^I \wedge da^I + ip_1 \int a^1 \wedge a^2 \wedge da^2$ $(\mathbb{Z}_{N_{12}})$; $\frac{i}{2\pi}\int\sum_I^2 b^I \wedge da^I + ip_2 \int a^2 \wedge a^1 \wedge da^1$ $(\mathbb{Z}_{N_{12}})$
$\prod_I^3 \mathrm{Z}_{N_I}$	$\frac{i}{2\pi}\int\sum_I^3 b^I \wedge da^I + ip_1 \int a^1 \wedge a^2 \wedge da^3$ $(\mathbb{Z}_{N_{123}})$; $\frac{i}{2\pi}\int\sum_I^3 b^I \wedge da^I + ip_2 \int a^2 \wedge a^3 \wedge da^1$ $(\mathbb{Z}_{N_{123}})$
$\prod_I^4 \mathrm{Z}_{N_I}$	$\frac{i}{2\pi}\int\sum_I^4 b^I \wedge da^I + ip \int a^1 \wedge a^2 \wedge a^3 \wedge a^4$ $(\mathbb{Z}_{N_{1234}})$
Z_2^T	$\frac{iK^{IJ}}{2\pi}\int b^I \wedge da^J$ $(\mathbb{Z}_2)$; $\frac{iK^{IJ}}{2\pi}\int b^I \wedge da^J + \frac{i\Lambda^{IJ}}{4\pi}\int b^I \wedge b^J$ $(\mathbb{Z}_2)$
$\mathrm{U}(1) \rtimes \mathrm{Z}_2^T$	$\frac{iK^{IJ}}{2\pi}\int b^I \wedge da^J$ $(\mathbb{Z}_2)$
$\mathrm{U}(1) \times \mathrm{Z}_2^T$	$\frac{iK^{IJ}}{2\pi}\int b^I \wedge da^J$ $(\mathbb{Z}_2)$; $\frac{iK^{IJ}}{2\pi}\int b^I \wedge da^J$ $(\mathbb{Z}_2)$
$\mathrm{Z}_{N_1} \times \mathrm{Z}_{N_2} \times \mathrm{U}(1)$	$\frac{i}{2\pi}\int\sum_I^3 b^I \wedge da^I + ip \int a^1 \wedge a^2 \wedge da^3$ $(\mathbb{Z}_{N_{12}})$

- Boundary conformal field theory(CFT).
- Spin TQFT for Fermionic SPT phases.
- Phase transitions among different SPT phases