Title: integrability and local formulas for spin TQFTs

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Abstract:

Microscopic Integrability and Local Formulas for Spin TQFTs

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Outline

The Toric Code[1]

an easy integrable system

- The Hilbert space is a tensor product of Hilbert spaces $\mathbb{C}^2$ on each edge of a lattice.
- The Hamiltonian is a sum of terms measuring the magnetic flux around each plaquette and terms acting by a gauge transformation on each vertex.

$$H = \sum_P M_P + \sum_i E_i$$

These terms all commute with each other and can be simultaneously diagonalized to diagonalize H. We say that this microscopic Hamiltonian is **integrable**.

[1] **A. Kitaev, C. Laumann, Topological Phases and Quantum Computing, 2009.**

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The Majorana Chain[2]

our motivating example, though deceptively simple

- This 1d lattice system has a vertex Hilbert space equal to the single-particle fermionic Fock space $\mathbb{C}^2$ with creation and annihilation operators $a_j^\dagger, a_j$.
- These define Majorana operators

$$c_{2j-1} = i(a_j - a_j^\dagger)$$

$$c_{2j} = a_j + a_j^\dagger.$$

- The Hamiltonian

$$H = i \sum_j c_{2j} c_{2j+1}$$

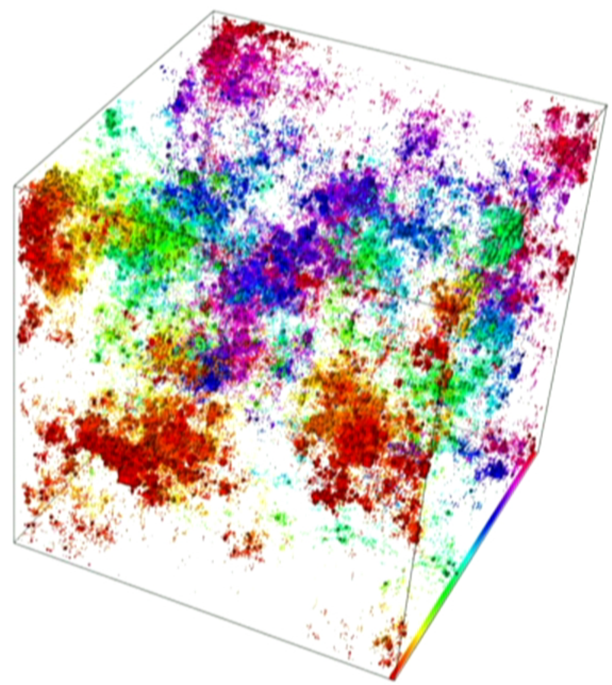
is also integrable. It represents an order 8 phase protected by time reversal.

[2] **L. Fidkowski, A. Kitaev, Topological phases of fermions in one dimension, 2010.**

Question One

for condensed matter theory

- Which phases can be realized by completely integrable systems? Or if you prefer, what are the possible IR limits of integrable systems?



- Experimental Goal:

Question One

which gapped bosonic phases are integrable?

- Integrability implies there is no bulk transport. This probably becomes an obstruction for chiral quantum Hall states, which have edge transport.[3] It also restricts entanglement properties.
- On the other hand, Levin-Wen models can realize all non-chiral bosonic states in 2+1d[4]
- In higher dimensions, I don't know. It seems related to whether the system admits gapped boundary conditions.[5]
- All oriented bosonic phases admitting gapped boundaries can be realized.[6]

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[3] **D.J. Thouless, Wannier functions for magnetic sub-bands, 1984.**
[4] **M. Levin, XG. Wen, 2004.**
[5] **A. Kapustin, N. Saulina, 2010.**
[6] **K. Walker, IPAM 2015.**

Question One

which gapped fermionic phases are integrable?

- Chirality obstruction still exists, so let's focus on SRE/SPT phases.
- $\nu = 1$ Topological superconductor in 1+1d is integrable as we've seen already.
- Thouless' obstruction might rule out $\nu = 1$ free fermion phases in 2+1d and higher, but there is some weirdness.[7]
- Possible obstruction also for the unitary $\mathbb{Z}/2$ phase in 2+1d.[8]
- So this is very different from the bosonic SPT case! These phases all admit gapped boundary conditions. Can we understand this systematically?

[7] **K. Slagle, Z. Bi, YZ. You, C. Xu, 2015.**
[8] **A. Potter, A. Vishwanath, 2015.**

The Majorana Chain
the partition function

- The partition function of the Majorana chain on a (Pin-) surface Σ "of infinite size" has a very nice formula[9]

$$Z(\Sigma, \eta) = \frac{1}{\sqrt{H^1(\Sigma, \mathbb{Z}/2)}} \sum_{a \in H^1(\Sigma, \mathbb{Z}/2)} \exp(i\pi q_\eta(a)/2)$$

- This is the best formula one can ask for, since $Z(\mathbb{RP}^2) = \exp(\pm i\pi/4)$, but for the orientation double cover $Z(S^2) = 1$.
- It looks like a path integral! Why? Stay tuned...

[9] R. Kirby, L. R. Taylor, 1990; A. Kapustin, RT, A. Turzillo, Z. Wang 2014.

Turaev-Viro Formula

a generalization for bosons

- Partition function for a non-chiral bosonic TQFT in 2+1d can also be written as a sort of finite path integral by work of Turaev and Viro. One sums over ways quasiparticles can flow along a triangulation of spacetime.[10]
- Crane-Yetter and Ooguri came up with an analogous formula for a 3+1d invariant, but it gives something invertible like BF theory.[11]
- No such formula is known for chiral TQFT in 2+1d! (Though there is an almost-formula called the Reshitikhin-Turaev formula, which is a state-sum on a surgery diagram for our 3-manifold.)

[10] **A. Turaev, O. Viro, 1990.**
[11] **H. Ooguri, 1992.**

Question Two

for topological field theory

- Which topological field theories have a "state sum" formula for their partition function?

A hope

for everybody

- Question one and question two are equivalent!

The Majorana Chain
the partition function

- Davide and Anton gave a nice explanation of the formula[12]

$$Z(\Sigma, \eta) = \frac{1}{\sqrt{H^1(\Sigma, \mathbb{Z}/2)}} \sum_{a \in H^1(\Sigma, \mathbb{Z}/2)} \exp(i\pi q_\eta(a)/2).$$

  They showed how for oriented (spin) surfaces the signs in the sum are ways of arranging fermionic terms in a tensor product of local Hilbert spaces. (q_η is even in this case).
- I will give a similar derivation that generalizes to a slightly different type of formula.

[12] D. Gaiotto, A. Kapustin, 2015.

The Majorana Chain

local ground states and a special symmetry

- The beauty of an integrable Hamiltonian like the Majorana chain is that there is a notion of ground state on an interval (a ball in general) and these ground states glue to ground states.
- For the Majorana chain, the ground states on an interval are two-fold degenerate. There is an on-site symmetry, similar to fermion parity, which on the ground states acts the same as an operator supported only on the boundary

$$\prod_j ic_{2j-1}c_{2j} = ic_1 c_{2N}.$$

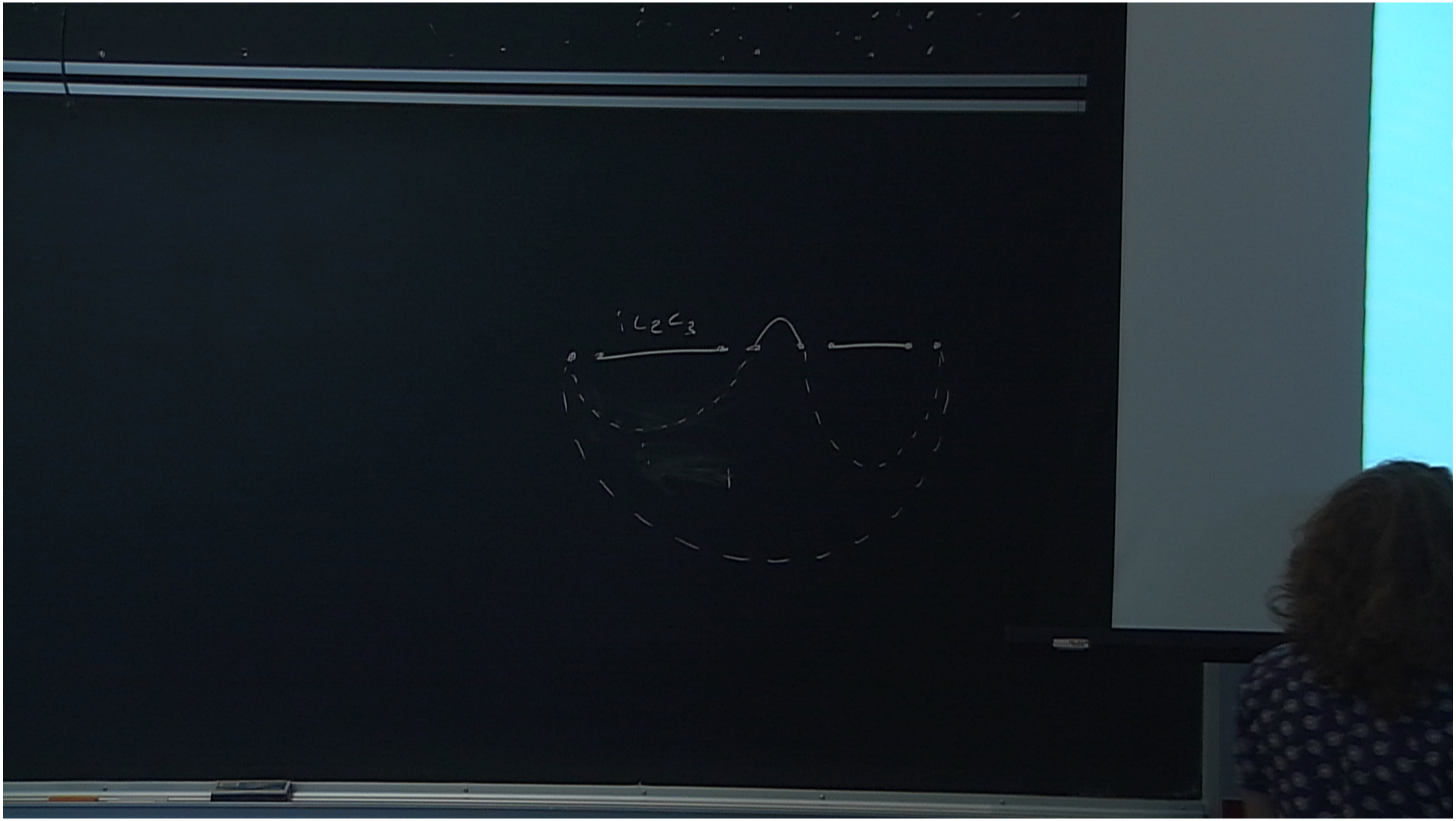
  The two states are distinguished by their charges.
- Let G_{hid} be this hidden symmetry group.

General Integrable Systems

gluing ground states

- G_{hid} charges add under gluing.
- Since the charges distinguish the ground states and ground states glue to ground states, we know what ground state we get... up to a phase.
- These phases integrate to a partition function of a bosonic gauge theory

$$Z_{bos}(a, \ldots).$$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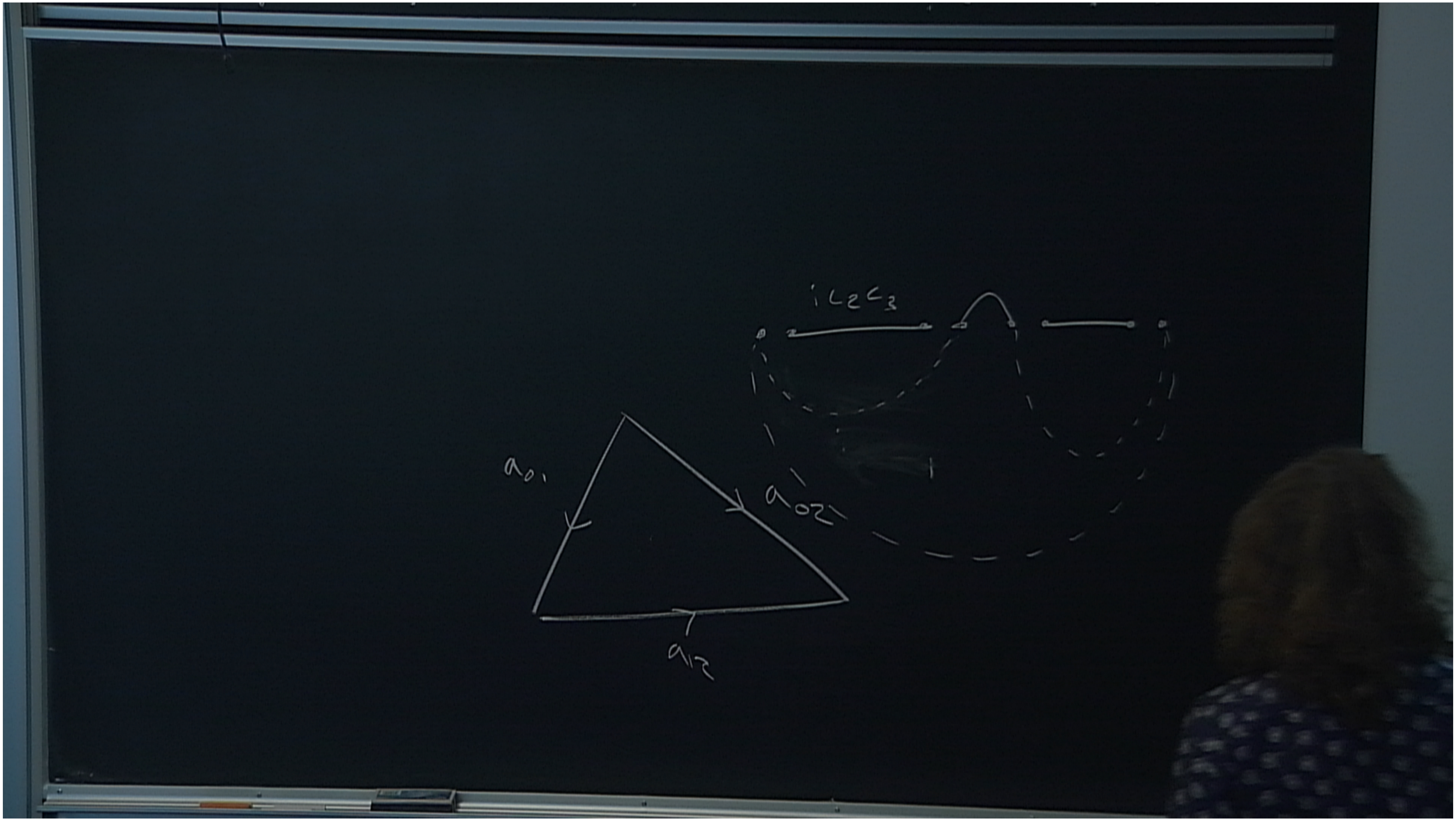


a_{01}
a_{02}
a_{12}

Bosonic SPTs

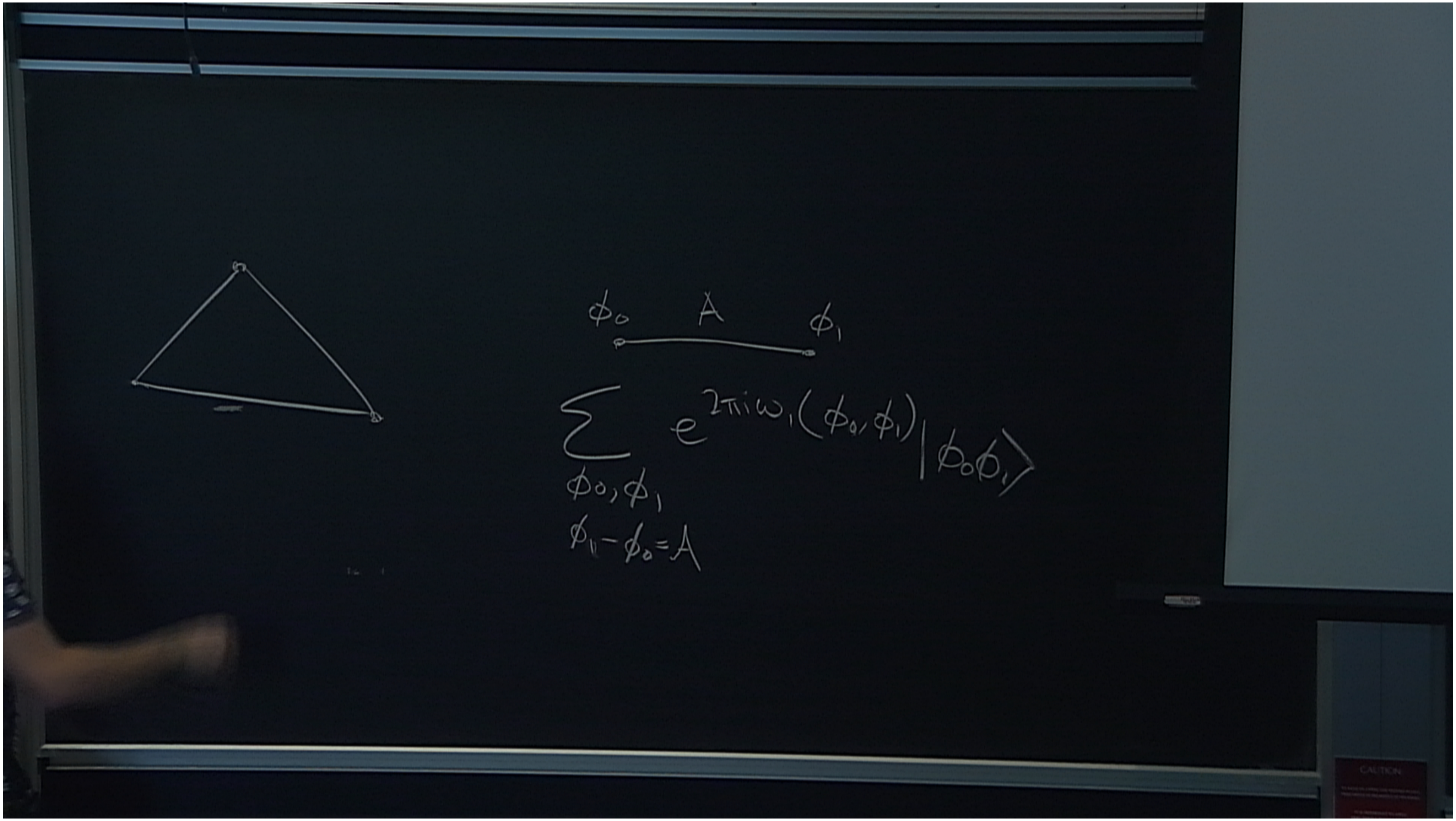
a case where $G_{hid} = 1$

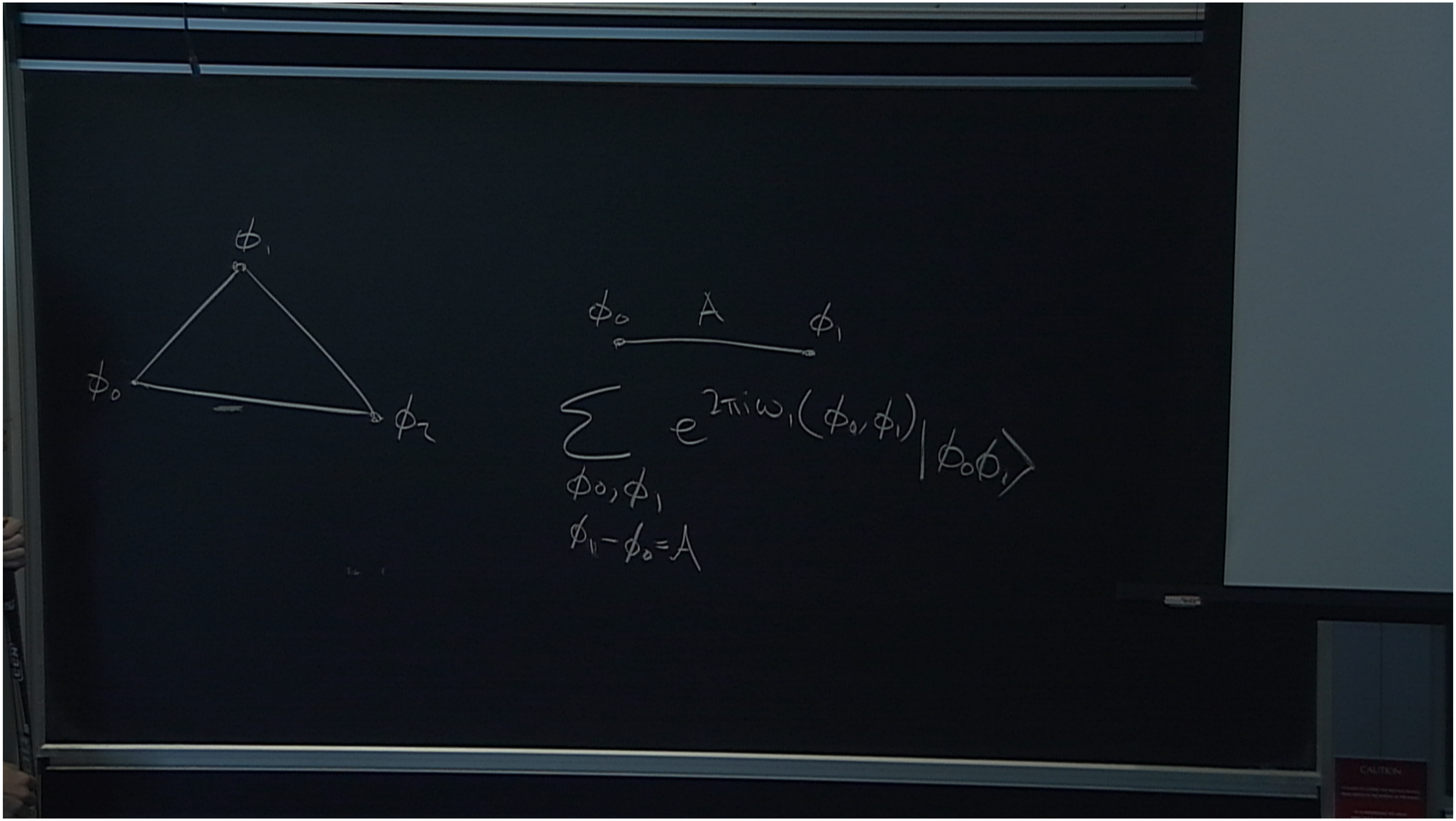
- For example, there is a unique local ground state for a bosonic SPT phase. In the group cohomology picture, it looks like[13]

$$|\omega\rangle = \sum_{\phi} \exp\left(2\pi i \int_B \omega_1(\phi)\right)|\phi\rangle$$

- The gluing phase is $d\omega_1(\phi) = \omega(d\phi)$, so $Z_{bos}(A) = e^{2\pi i \omega(A)}$.

[13] **X. Chen, ZC. Gu, ZX. Liu, XG. Wen, 2011.**


ϕ_0 A ϕ_1
$\sum_{\phi_0, \phi_1} e^{2\pi i \omega_1(\phi_0, \phi_1)} |\phi_0 \phi_1\rangle$
$\phi_1 - \phi_0 = A$


ϕ_1
ϕ_0
ϕ_2
ϕ_0 A ϕ_1
$\sum_{\phi_0, \phi_1} e^{2\pi i \omega_1(\phi_0, \phi_1)} |\phi_0 \phi_1\rangle$
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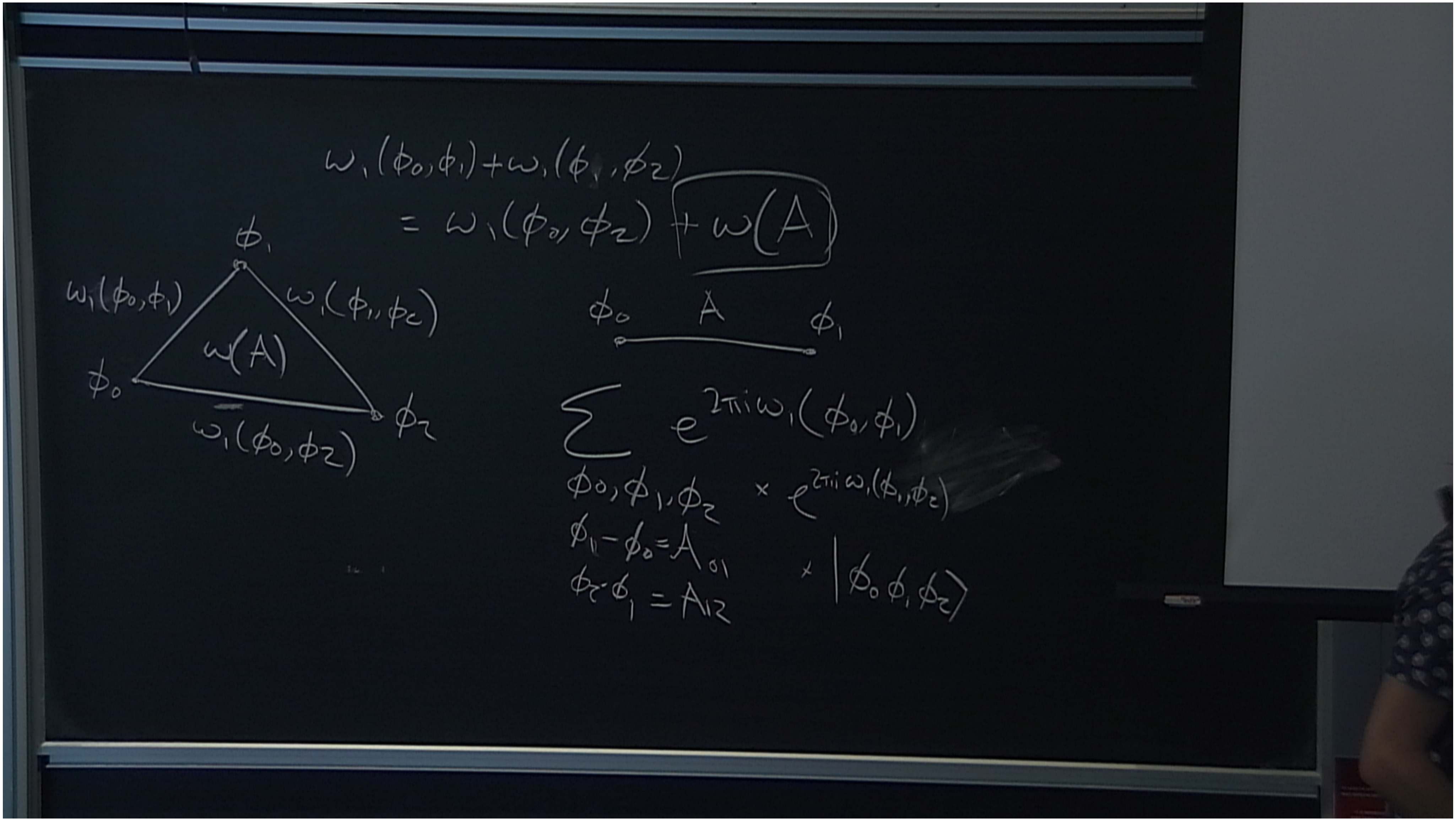
ω₁(φ₀,φ₁) + ω₁(φ₁,φ₂)
= ω₁(φ₀,φ₂) + ω(A)
φ₁
ω₁(φ₀,φ₁)
ω₁(φ₁,φ₂)
ω(A)
φ₀
φ₂
ω₁(φ₀,φ₂)
φ₀ A φ₁
Σ e^{2πi ω₁(φ₀,φ₁)}
φ₀,φ₁,φ₂
× e^{2πi ω₁(φ₁,φ₂)}
φ₁ − φ₀ = A₀₁
φ₂ − φ₁ = A₁₂
× |φ₀ φ₁ φ₂⟩

Fermions, However

have an anomaly

- Different ground states on the ball have fermion parity some function of their charge and the background fields $\beta(a, ...)$.
- Since tensor products must be ordered when $\beta(a, ...) \neq 0$, this theory has an anomaly that was computed by Davide and Anton in the orientable case but more generally depends on how the fermions transform under time-reversal symmetry:

$$d\omega(a) = w_2\beta(a, ...) \text{ or } (w_2 + w_1^2)\beta(a, ...).$$

- This anomaly is always corrected by a choice of sign that depends only the (s)pin$\pm$ structure and β

$$Z_{fer}(\eta, \beta(a, ...)).$$

The General Formula

at last

- So finally, for any integrable system, we get a formula for the partition function like

$$Z(\eta, \ldots) = \frac{1}{\#} \sum_{a \in H^1(X, G_{int})} Z_{fer}(\eta, \beta(a, \ldots)) Z_{bos}(a, \ldots).$$

- When $G_{hid} = 1$, possible $Z_{bos}(\ldots)$ are classified by the E-cohomology of B(...)[14].
- Equivalently, the $G_{hid} = 1$ phases are the ones classified by group supercohomology.[15]

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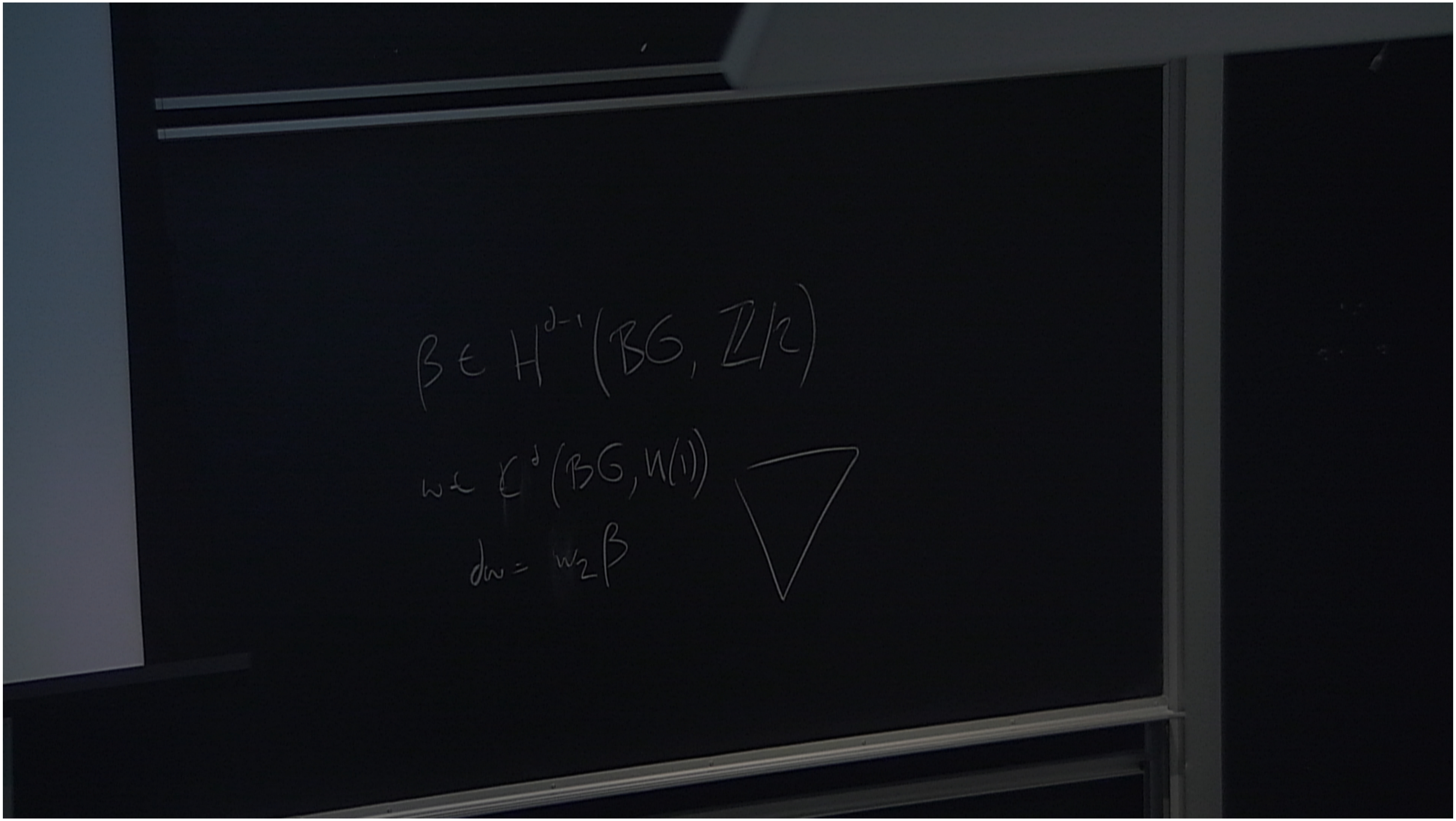
[14] D. Freed, Pions and Generalized Cohomology, 2007.

[15] ZC. Gu, XG. Wen, 2012.


β ∈ H^{d-1}(BG, Z/2)
ω ∈ C^d(BG, U(1))
dω = w_2 β

The Majorana Chain

state sum

- $\beta(a) = a$.
- Things are deceptively simple in the oriented case since $w_2 = 0$, so it looks like $Z_{bos}(a) = 1$.
- Actually one finds

$$Z_{bos}(a) = \exp\left(\frac{i\pi}{2}\int w_1 a\right).$$

- These also combine into a quadratic refinement

$$Z_{fer}(\eta, a) Z_{bos}(a) = e^{i\pi q_\eta(a)/2}.$$

No State Sum for $\nu = 1$ TSC in 3d

and hence no integrability

- We can use the general formula to sometimes rule out the existence of a state sum model.
- According to the cobordism classification, as well as some explicit computations[16], the partition function of the $\nu = 1$ topological superconductor on K3 is -1.
- However, K3 is simply connected, so the state sum is a single term, and there is no way to get -1 from a purely bosonic partition function.
- Thus, the $\nu = 1$ TSC cannot be protected by Anderson localization.
- Note also this TQFT detects smooth structure.
- $\nu = 2$?

[16] **E. Witten, 2015.**

A 2+1D phase

from decorating domain walls

- There is a $\mathbb{Z}/8$ of SPT phases in 2+1d protected by a unitary $\mathbb{Z}/2$ symmetry. According to the cobordism classification, the partition function of such an SPT phase can be computed on any 3-manifold X with a $\mathbb{Z}/2$ gauge field A and a spin structure is a cobordism invariant of such data.[17]
- To compute for a configuration of A, one looks at the domain walls, embedded surfaces in X Poincaré dual to A. These inherit a Pin-structure, and this $\mathbb{Z}/8$ invariant is the partition function of a Majorana chain sweeping out this surface in spacetime.

[17] **A. Kapustin, RT, A. Turzillo, Z. Wang, 2014.**

A 2+1d phase

a first guess at a state sum

- The simplest guess is to take $G_{hid} = 1$ and $\beta(A) = dA/2$. Then we can use

$$Z_{bos}(A) = \exp\left(\frac{2\pi i}{4}\int A\frac{dA}{2}\right).$$

- We check

$$d\frac{1}{4}A\frac{dA}{2} = \frac{1}{2}\beta^2 = \frac{1}{2}w_2\beta.$$

- Only an order 4 phase...

A 2+1d phase

a conjectural state sum

- The next simplest guess is to use $G_{hid} = \mathbb{Z}/2$, $\beta(a, A) = dA/2 + Aa$

$$\sum_{a \in H^1(X,\mathbb{Z}/2)} z_{fer}(\beta) \exp\Big(\frac{2\pi i}{4} \int A(da/2 + dA/2)\Big).$$

- Has correct anomaly and partition function on $\mathbb{RP}^3$, but is it invertible?

Summary

mostly questions

- A gapped integrable system gives a TQFT with a state sum model.
- As a corollary, not all fermionic SPT phases are integrable.
- Challenge 1: rule out or construct state sum models for $\nu = 1$ in 2+1D, $\nu = 2$ in 3+1D.
- Challenge 2: understand pin state sums more concretely, in particular the phase in 1d with $T^2 = -1$ which has a state sum most naturally defined on the orientation double cover of spacetime.
- Challenge 3: when does gauging an SPT give an SPT again?