Title: Spin--Statistics and Categorified Galois Groups

Date: Oct 23, 2015 10:30 AM

URL: http://pirsa.org/15100110

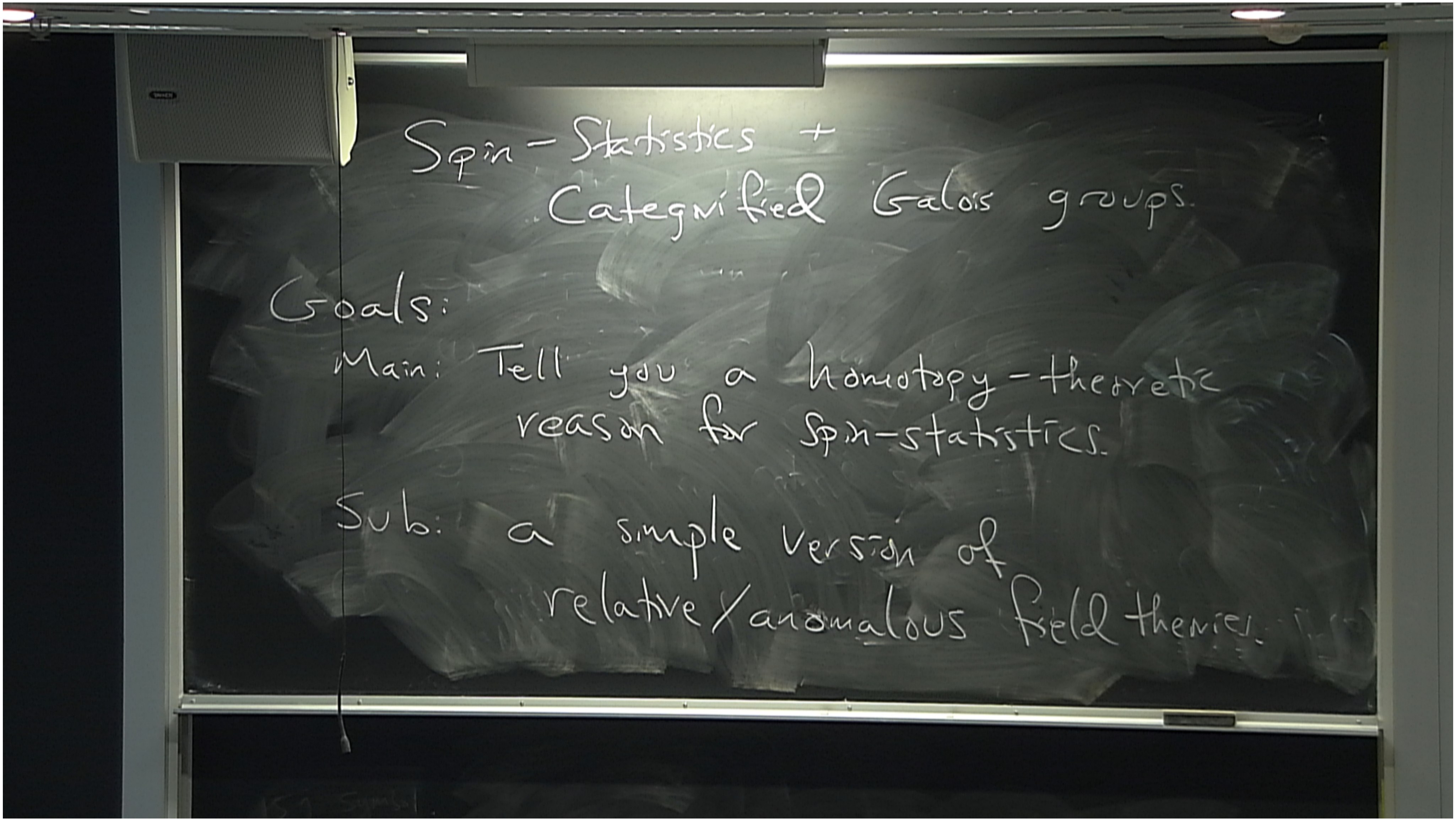
Abstract:

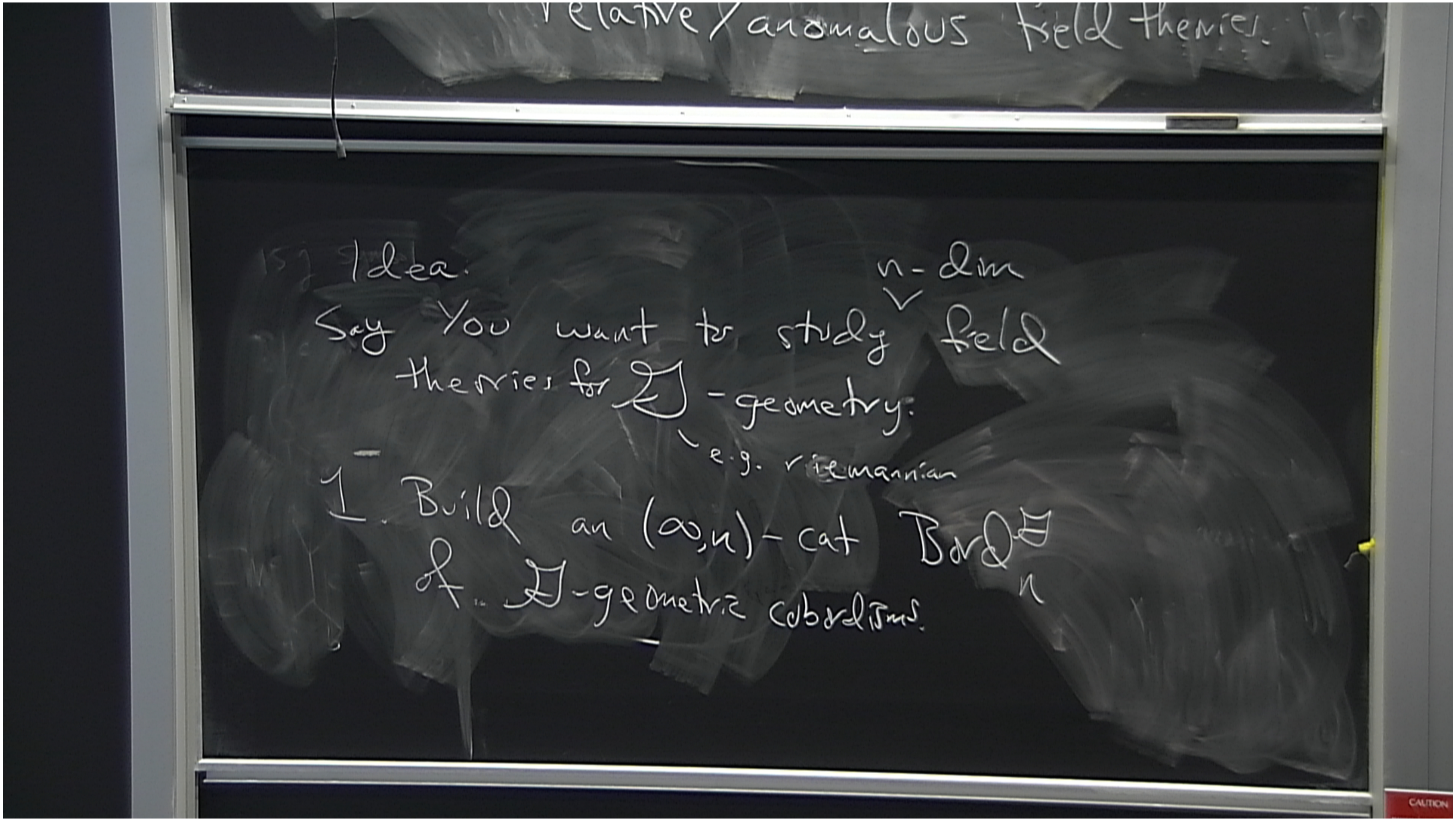
Spin-Statistics +
Categorified Galois groups.

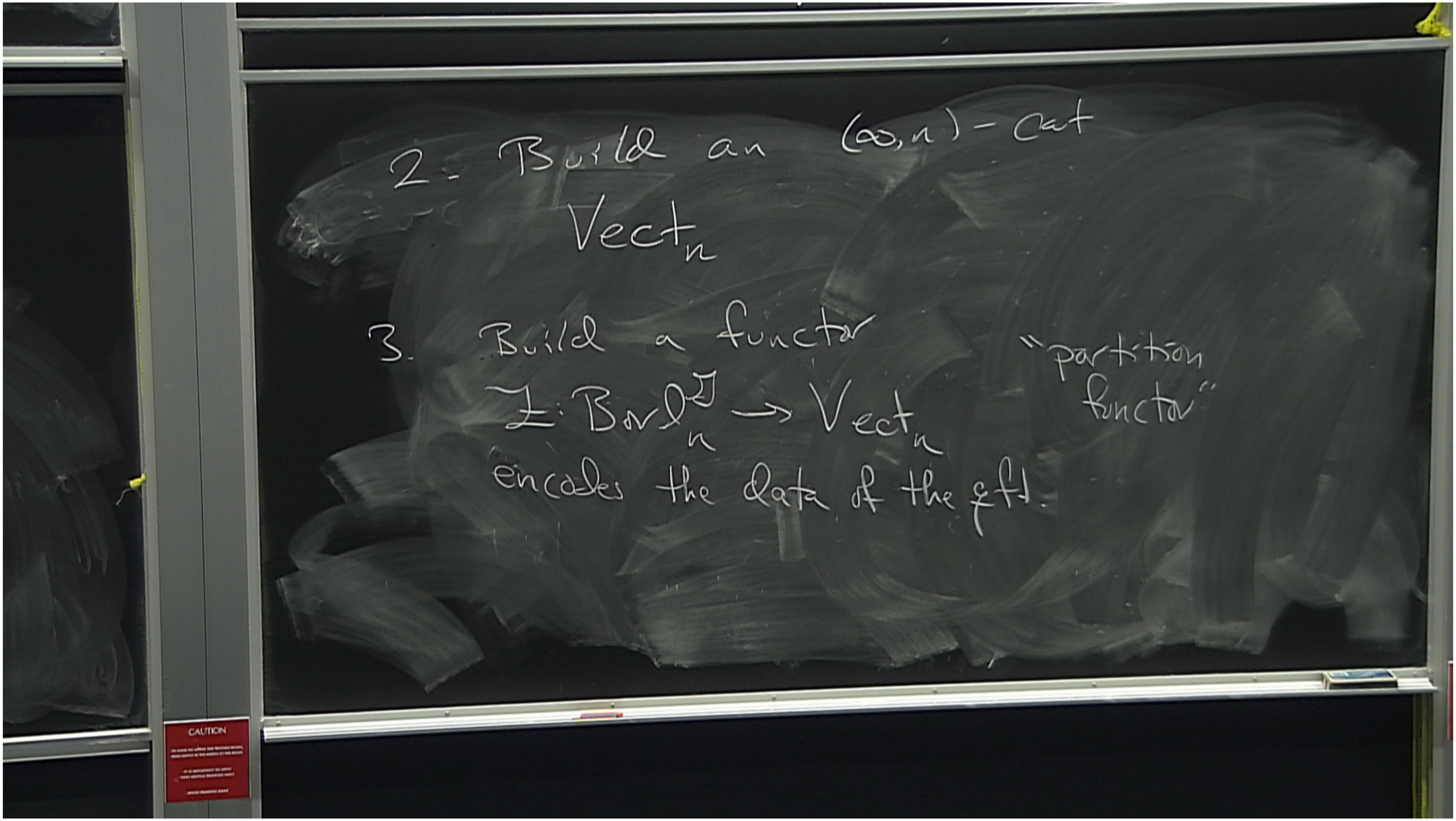
Goals:

Main: Tell you a homotopy-theoretic reason for spin-statistics.

Sub:


Spin-Statistics +
Categorified Galois groups.
Goals:
Main: Tell you a homotopy-theoretic
reason for spin-statistics.
Sub: a simple version of
relative/anomalous field theories.


relative / anomalous field theories.
Idea.
Say you want to study n-dim field theories for $\mathcal{G}$-geometry.
e.g. riemannian
1. Build an (∞,n)-cat $\mathrm{Bord}_n^{\mathcal{G}}$ of $\mathcal{G}$-geometric cobordisms.


2. Build an (∞, n)-cat
Vect_n
3. Build a functor
$Z : \mathrm{Bord}_n^{\mathcal{G}} \to \mathrm{Vect}_n$
"partition functor"
encodes the data of the qft.

relative/anomalous field theories.

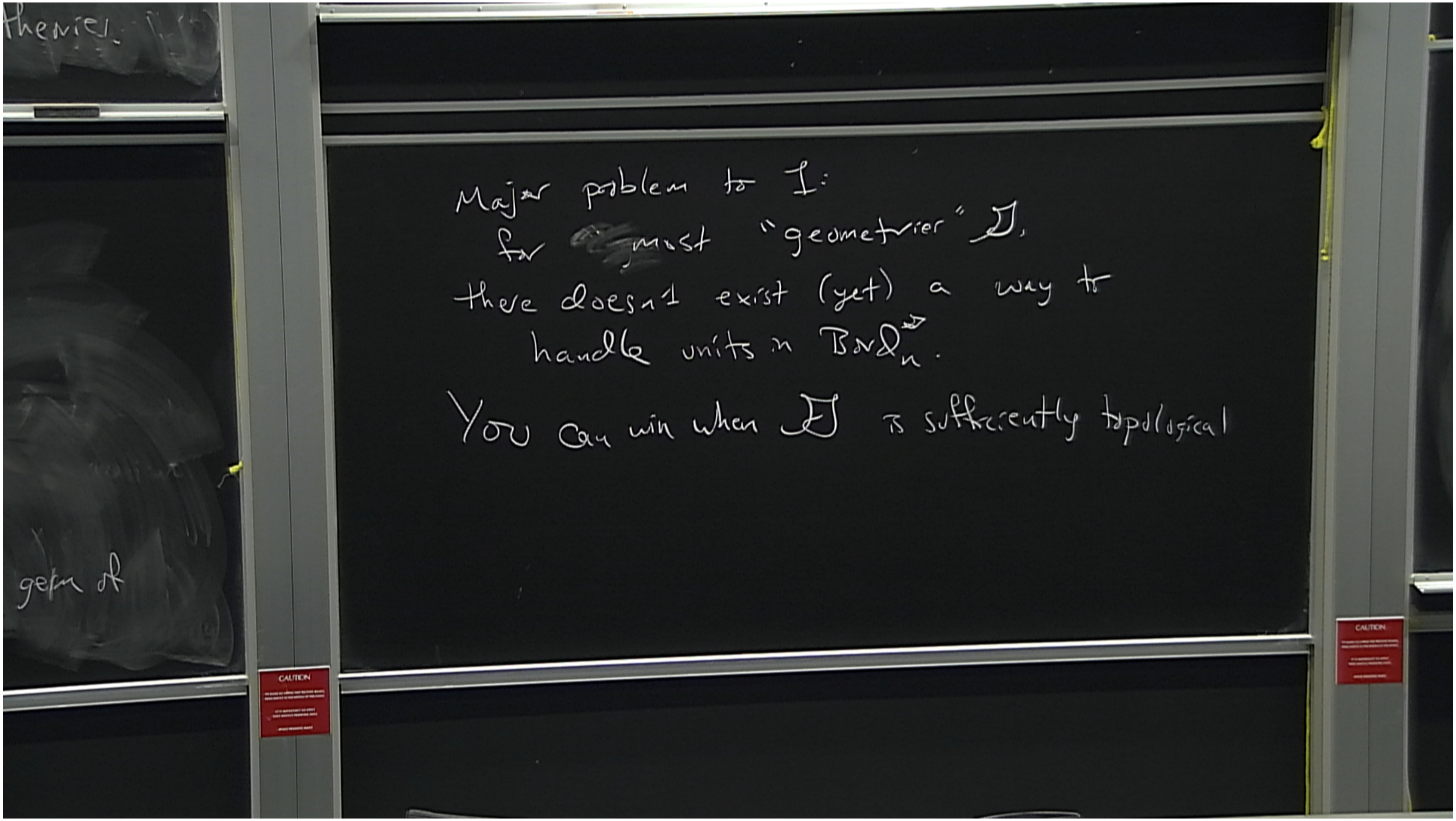
Idea/Wish list

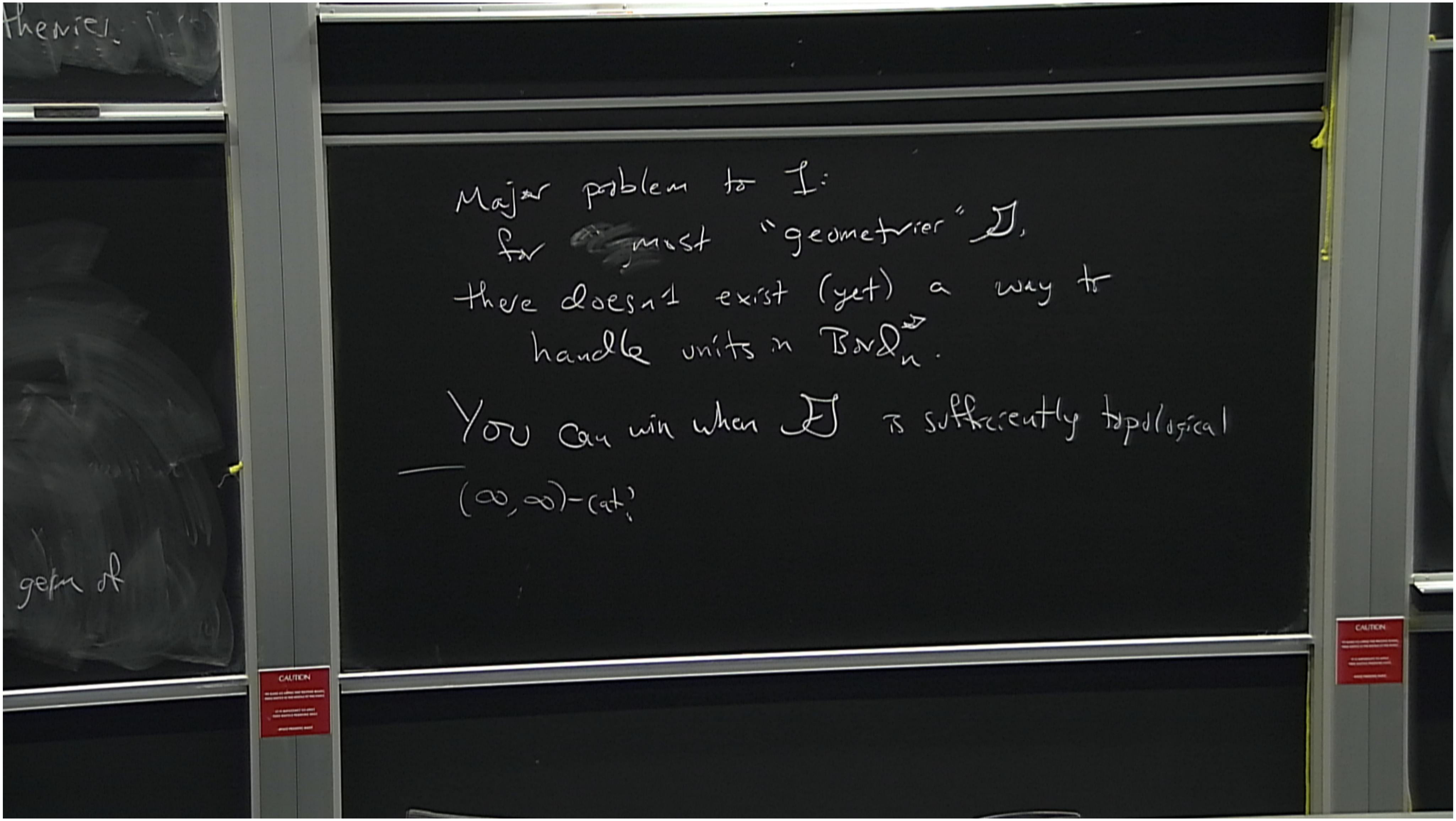
Say you want to study n-dim field theories for $\mathcal{G}$-geometry.

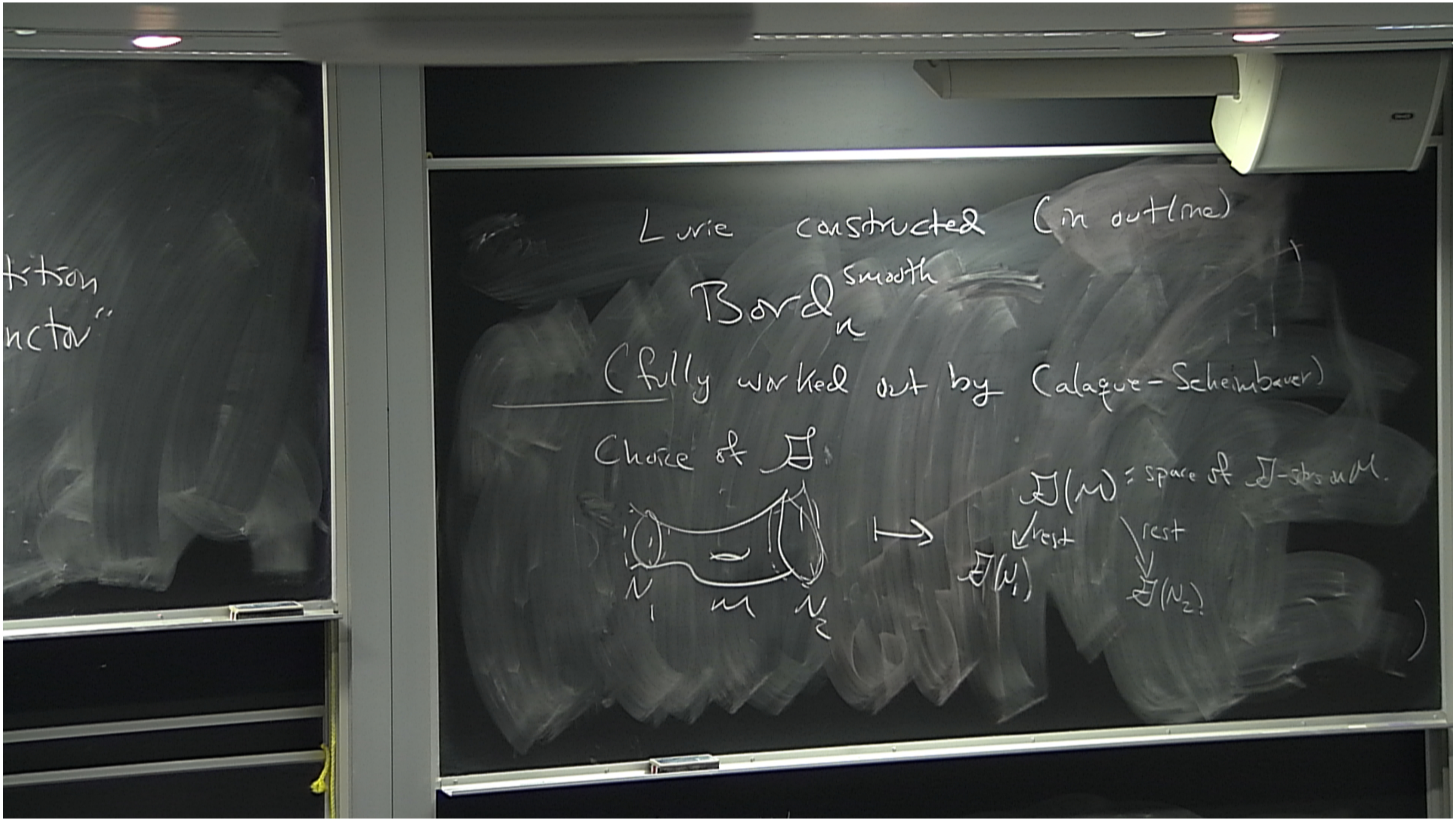
e.g. riemannian

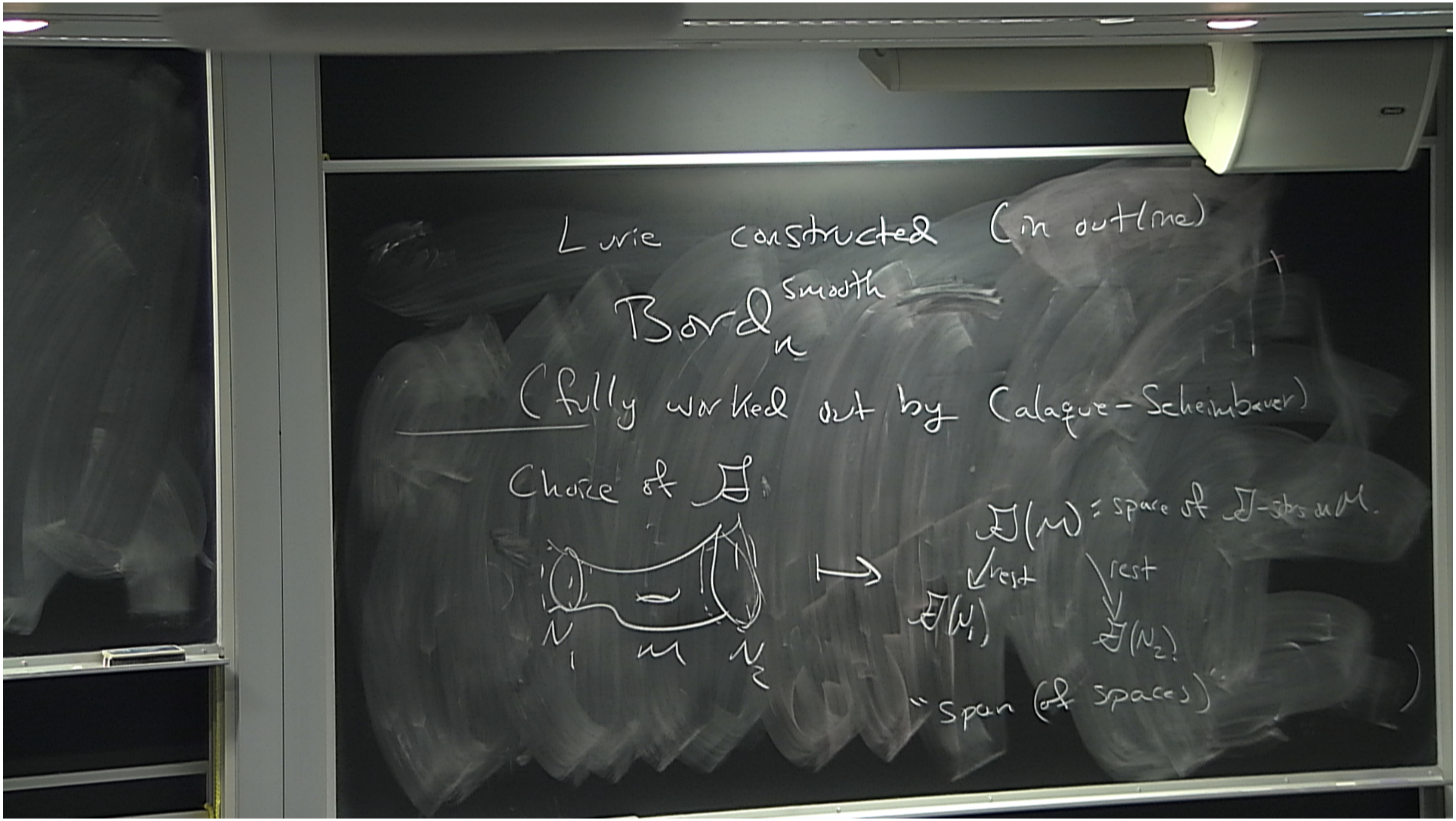
1. Build an (∞,n)-cat $\mathrm{Bord}_n^{\mathcal{G}}$ of $\mathcal{G}$-geometric cobordisms.

i.e. k-morphisms = k-dim cobordisms w/ a germ of n-dim'l $\mathcal{G}$-manifold.

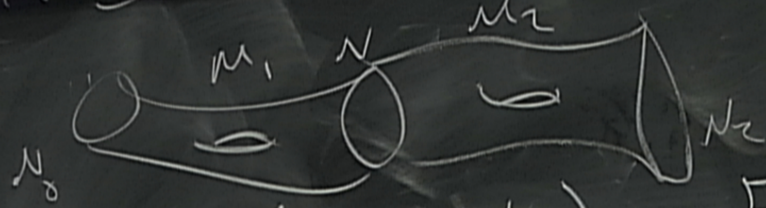

theories.
Major problem to I:
for most "geometries" $\mathfrak{F}$,
there doesn't exist (yet) a way to
handle units in $\mathrm{Bord}_n^{\rightleftarrows}$.
You can win when $\mathfrak{F}$ is sufficiently topological
germ of


Major problem to 1:
for most "geometries" 𝔍,
there doesn't exist (yet) a way to
handle units in $\mathrm{Bord}_n^{\Rightarrow}$.
You can win when 𝔍 is sufficiently topological
(∞,∞)-cat?
theories
germ of


Lurie constructed (in outline)
Bord$_n^{\text{smooth}}$
(fully worked out by Calaque - Scheimbauer)
Choice of $\mathcal{F}$:
$\mathcal{F}(M)$ = space of $\mathcal{F}$-strs on M.
rest
rest
$\mathcal{F}(N_1)$
$\mathcal{F}(N_2)$
N_1
M
N_2


Lurie constructed (in outline)
Bord$_n^{\text{smooth}}$
(fully worked out by Calaque-Scheimbauer)
Choice of $\mathcal{F}$:
N_1 M N_2
$\mapsto$
$\mathcal{F}(M)$ = space of $\mathcal{F}$-strs on M.
rest
rest
$\mathcal{F}(N_1)$
$\mathcal{F}(N_2)$
"Span (of spaces)"

If $\mathcal{F}$ is a "local" geometry.

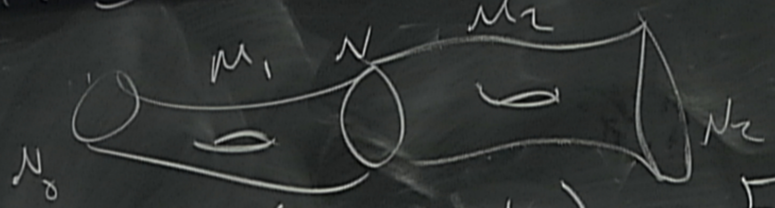


$$\mathcal{F}(M_1 \cup_N M_2) = \mathcal{F}(M_1) \times_{\mathcal{F}(N)} \mathcal{F}(M_2)$$

$$\mathcal{F}(M_1) \to \mathcal{F}(N_0), \qquad \mathcal{F}(M_2) \to \mathcal{F}(N_2)$$

(for this to be a unital functor

If $\mathcal{F}$ is a "local" geometry:

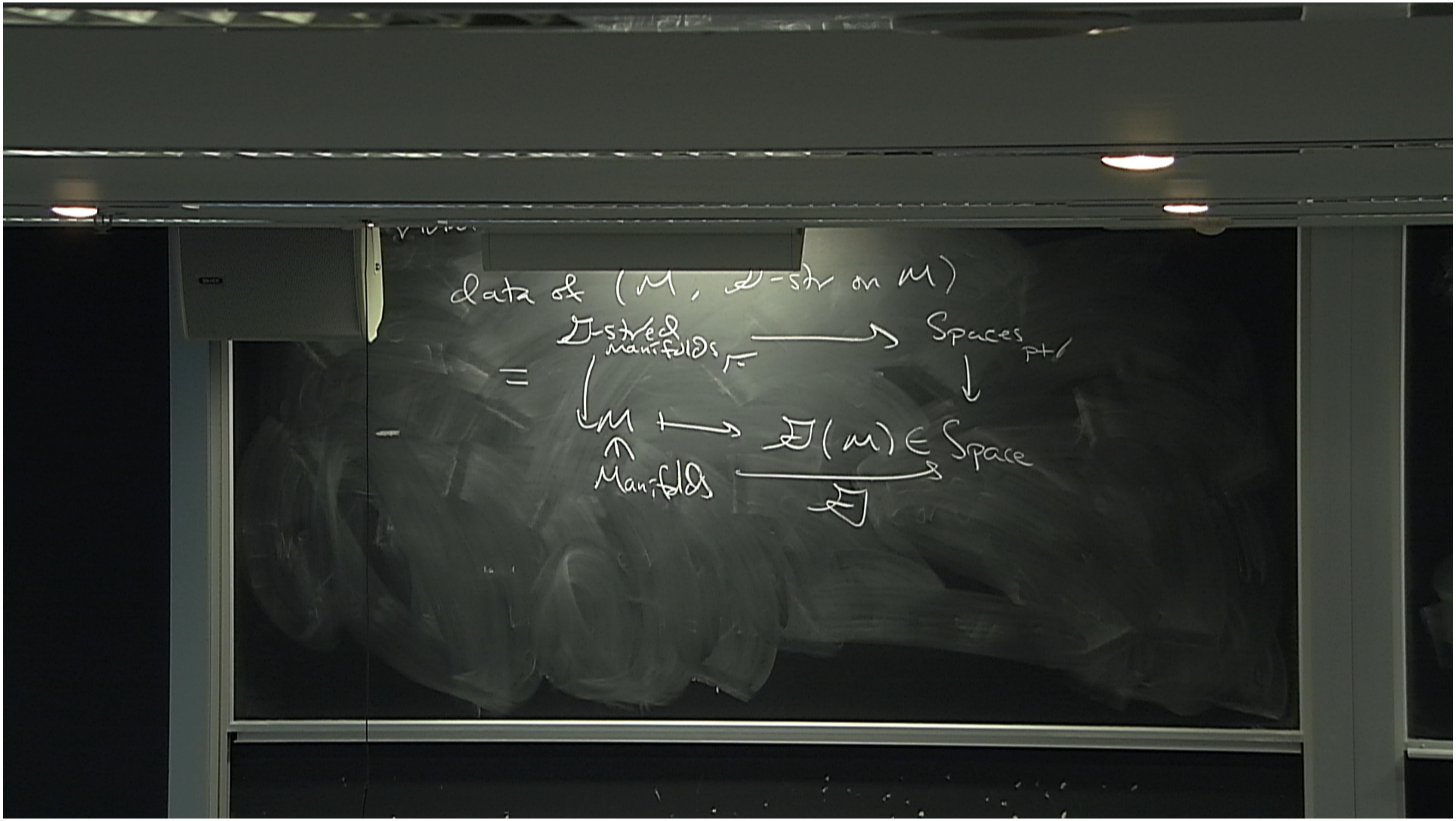


$$\mathcal{F}(M_1 \cup_N M_2) = \mathcal{F}(M_1) \times_{\mathcal{F}(N)} \mathcal{F}(M_2)$$

$$\begin{array}{ccc} \mathcal{F}(M_1) & & \mathcal{F}(M_2) \\ \downarrow & & \downarrow \\ \mathcal{F}(N_0) & & \mathcal{F}(N_2) \end{array}$$

(for this to be a unital functor,

$\mathcal{F}$(def retracts) = embedding


data of (M, 𝔉-str on M)
𝔉-str'ed manifolds
Spaces pt/
M ⟼ 𝔉(M) ∈ Space
Manifolds
𝔉

$$
\begin{tikzcd}
\mathrm{Bord}_n^{\mathfrak{F}} \arrow[r, "\Gamma"] \arrow[d] & \mathrm{Spans}_n(\mathrm{Spaces}_{/pt}) \arrow[d] \\
\mathrm{Bord}_n^{smooth} = \mathrm{Bord}'_n \arrow[r, "\mathfrak{F}"] & \mathrm{Spans}_n(\mathrm{Spaces})
\end{tikzcd}
$$

For nice $\mathfrak{F}$,

$\{$ (∞, n)-cat of n-fold spans

objects = spaces

1^{st}-mor = $\cdot \leftarrow \cdot \rightarrow \cdot$ [Haugseng]

2-mor =

$\mathcal{F}$-structured field thy

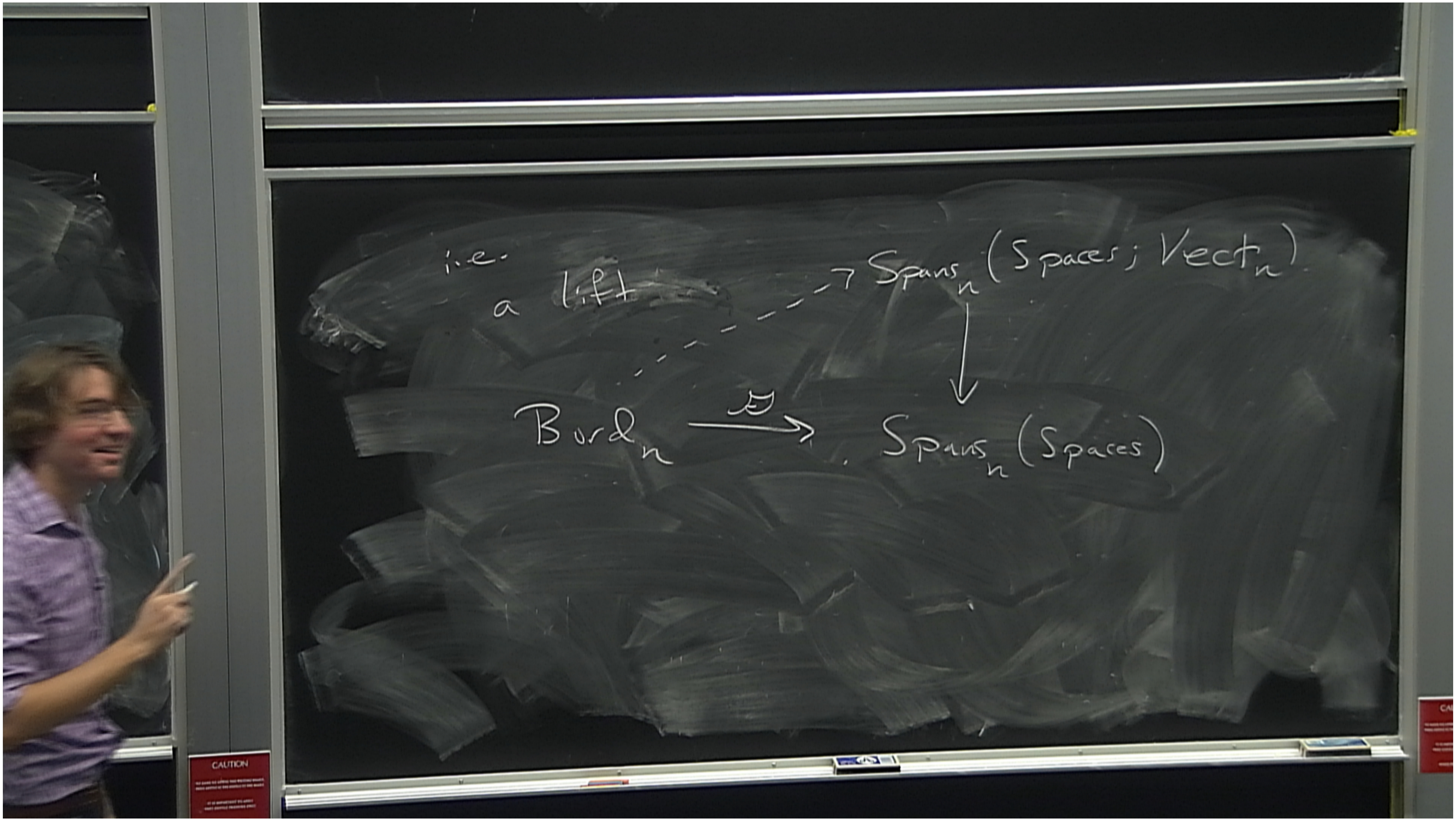
$Z : M, g \mapsto$ something in Vect_n.

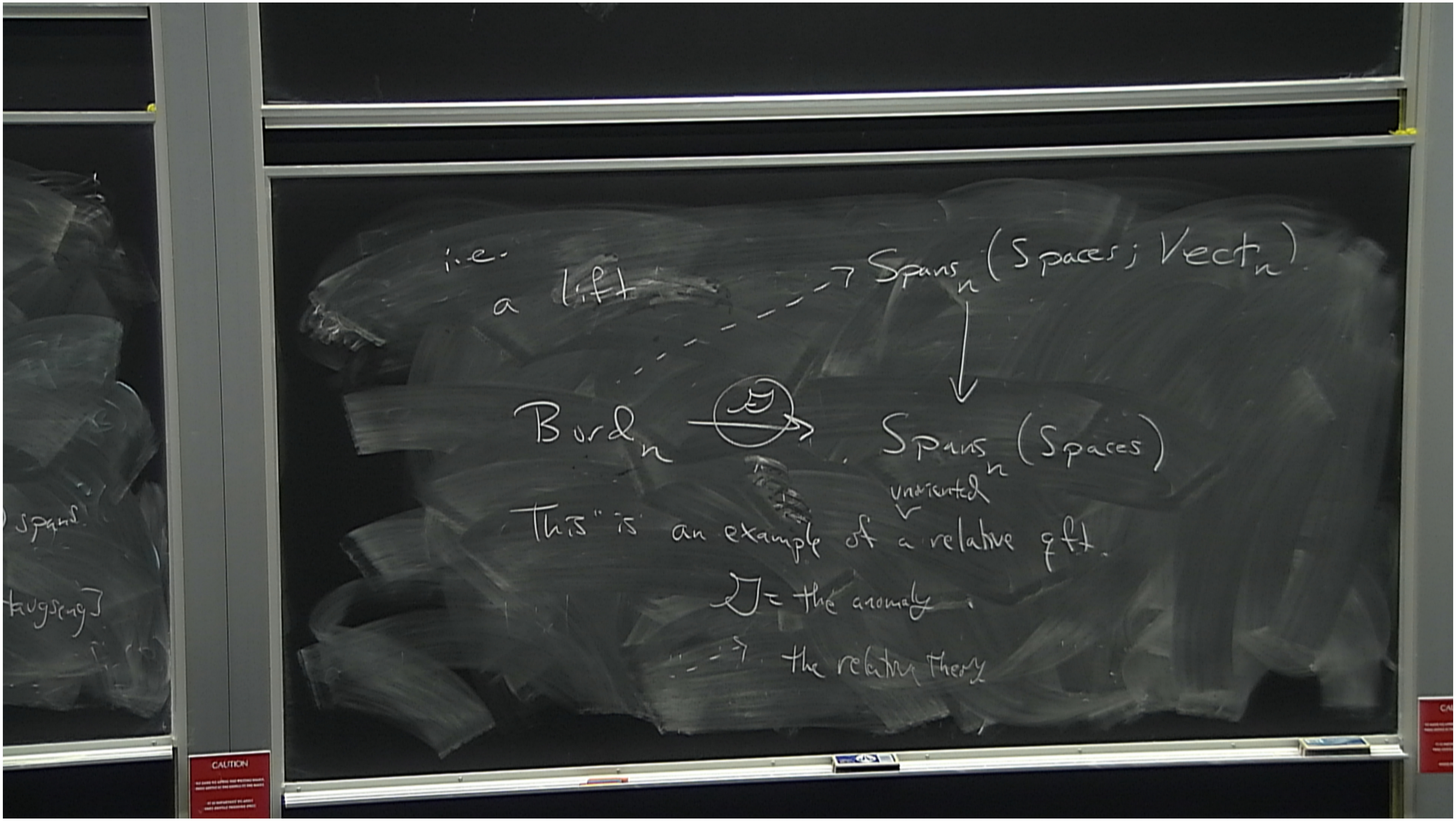
$\uparrow$

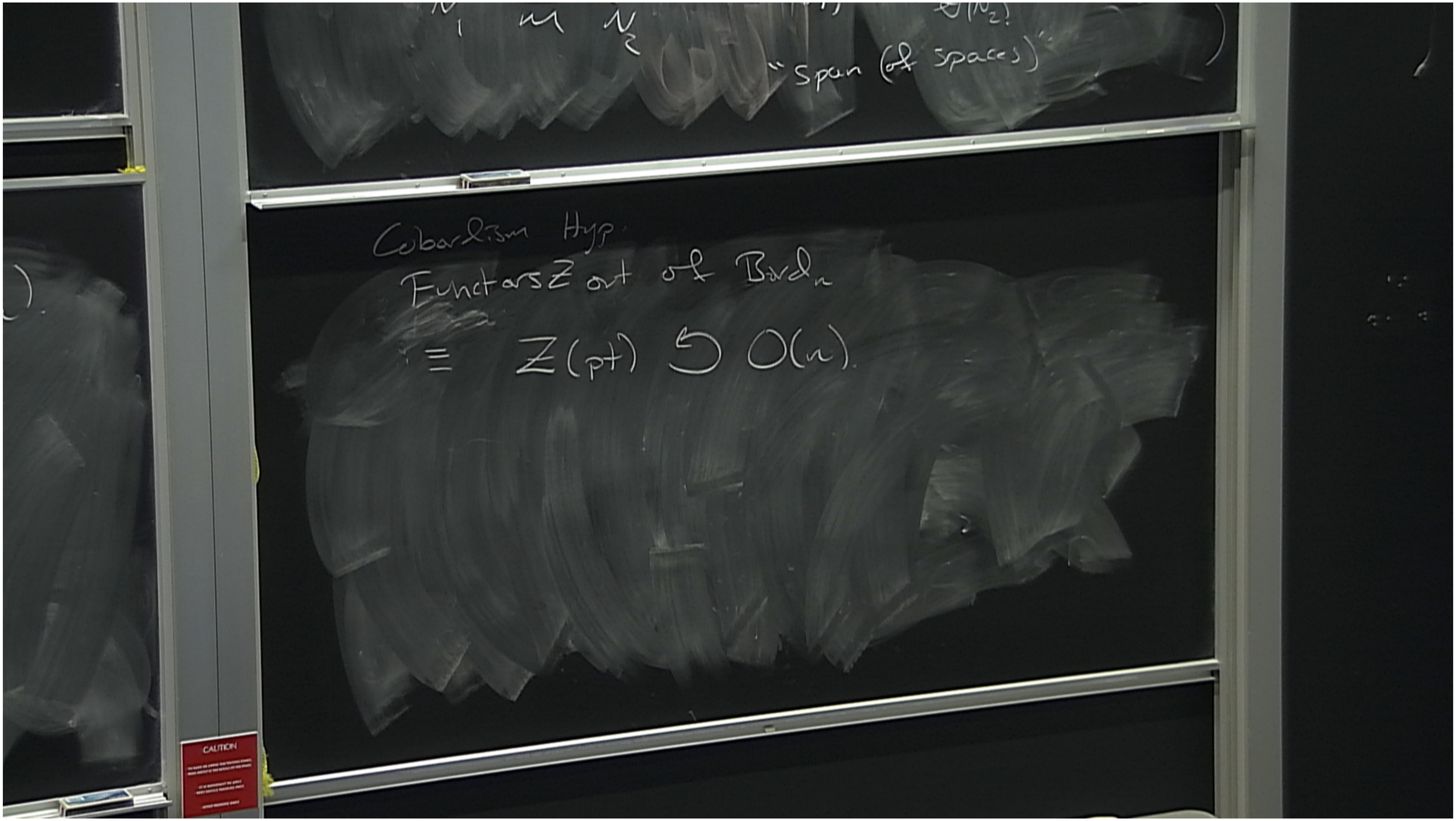
$\mathrm{Bord}_n^{\mathcal{F}}$

is the same

$M \longmapsto$ bundle over $\mathcal{F}(M)$ of things in Vect_n.


i.e. a lift
Spans_n (Spaces; Vect_n)
Bord_n
Z
Spans_n (Spaces)


i.e.
a lift --> Spans_n (Spaces; Vect_n)
Bord_n --𝔄--> Spans_n (Spaces)
unoriented
This "is" an example of a relative qft.
𝔄 = the anomaly
--> the relative theory
spans
haugseng]


"Span (of spaces)"
Cobordism Hyp.
Functors Z out of Bord_n
≡ Z(pt) ↻ O(n).

N_1 m N_2

"Span (of Spaces)"

Cobordism Hyp.

Functors $Z : \mathrm{Bord}_n \to \mathcal{C}$

$\simeq$ $Z(pt) \in \mathcal{C}^{O(n)}$

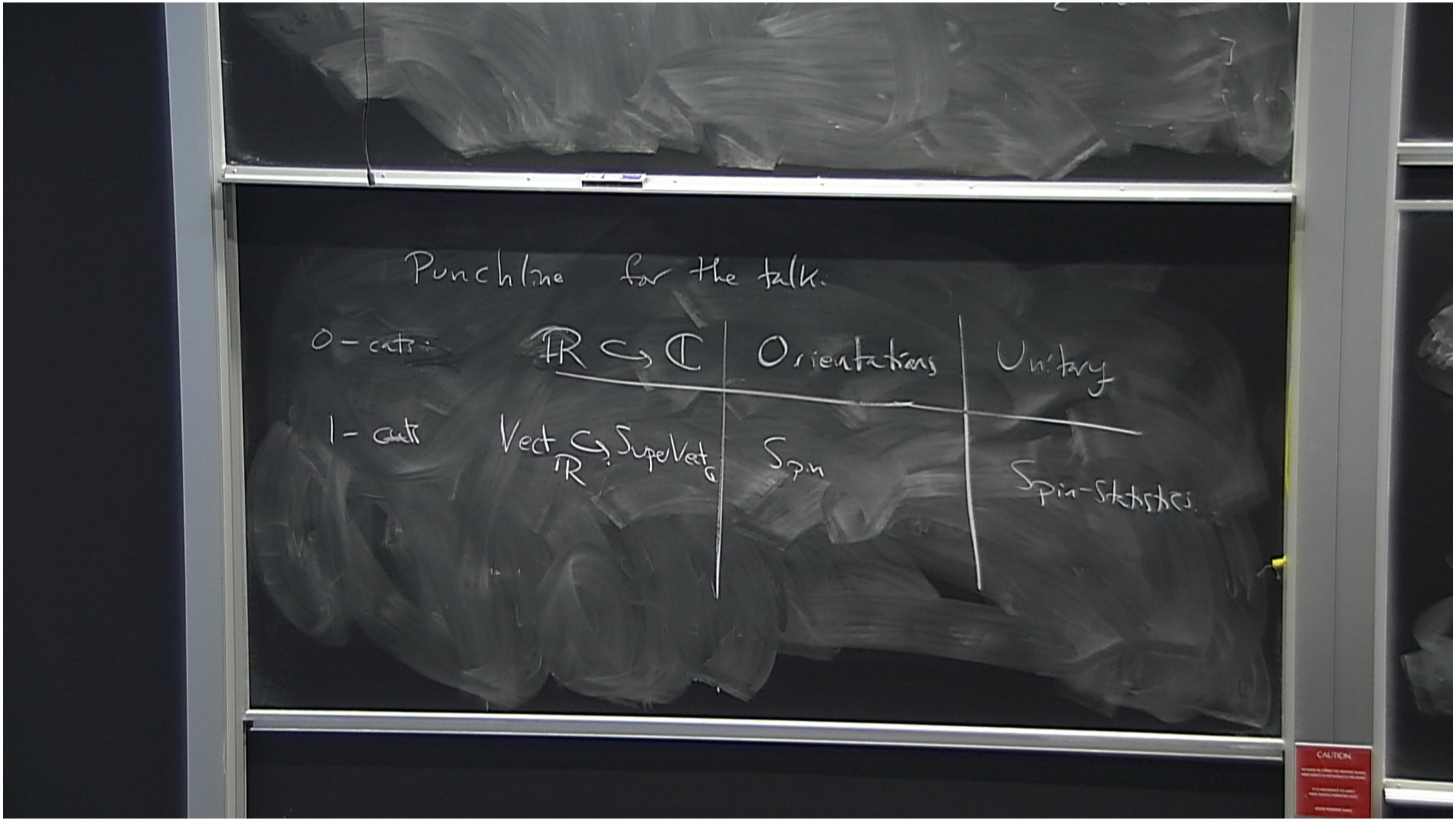
fixed pt for the action $\forall g \in O(n)$!

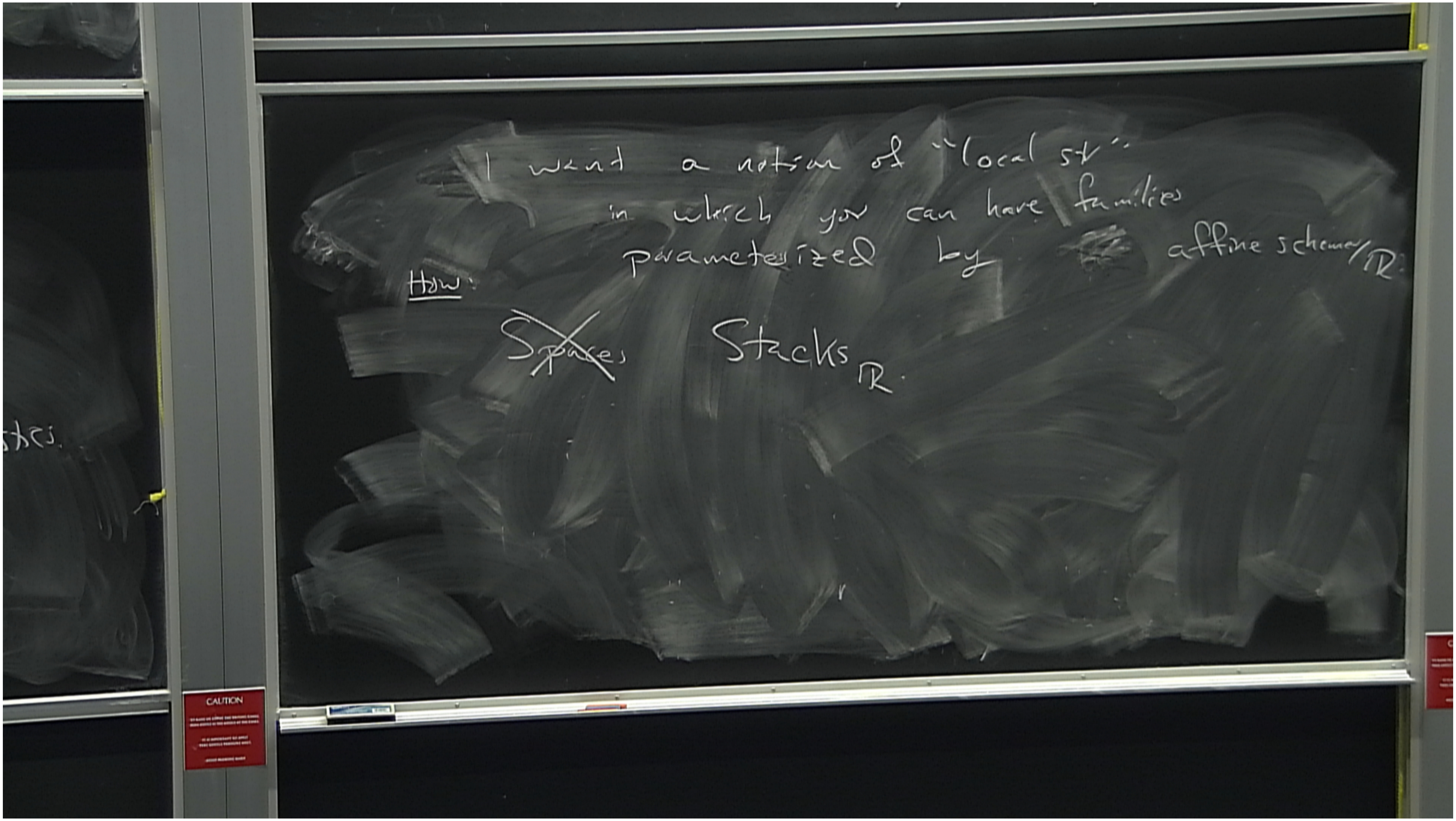
$g \cdot Z(pt) \underset{\varphi(g)}{\simeq} Z(pt)$

implicit $O(n)$ acts on every (f.d.) (∞, n)-cat

For $\mathrm{Bord}_n \longrightarrow \mathrm{Spans}_n(\mathrm{Spaces})$

turns out the $O(n)$-action on $\circlearrowright$ is trivial


Punchline for the talk.
0-cats: ℝ ↪ ℂ | Orientations | Unitary
1-cats Vect_ℝ ↪ SuperVect_ℂ | Spin | Spin-Statistics


I want a notion of "local str".
in which you can have families
parameterized by affine schemes/$\mathbb{R}$:
How:
~~Spaces~~ $\text{Stacks}_{\mathbb{R}}$.

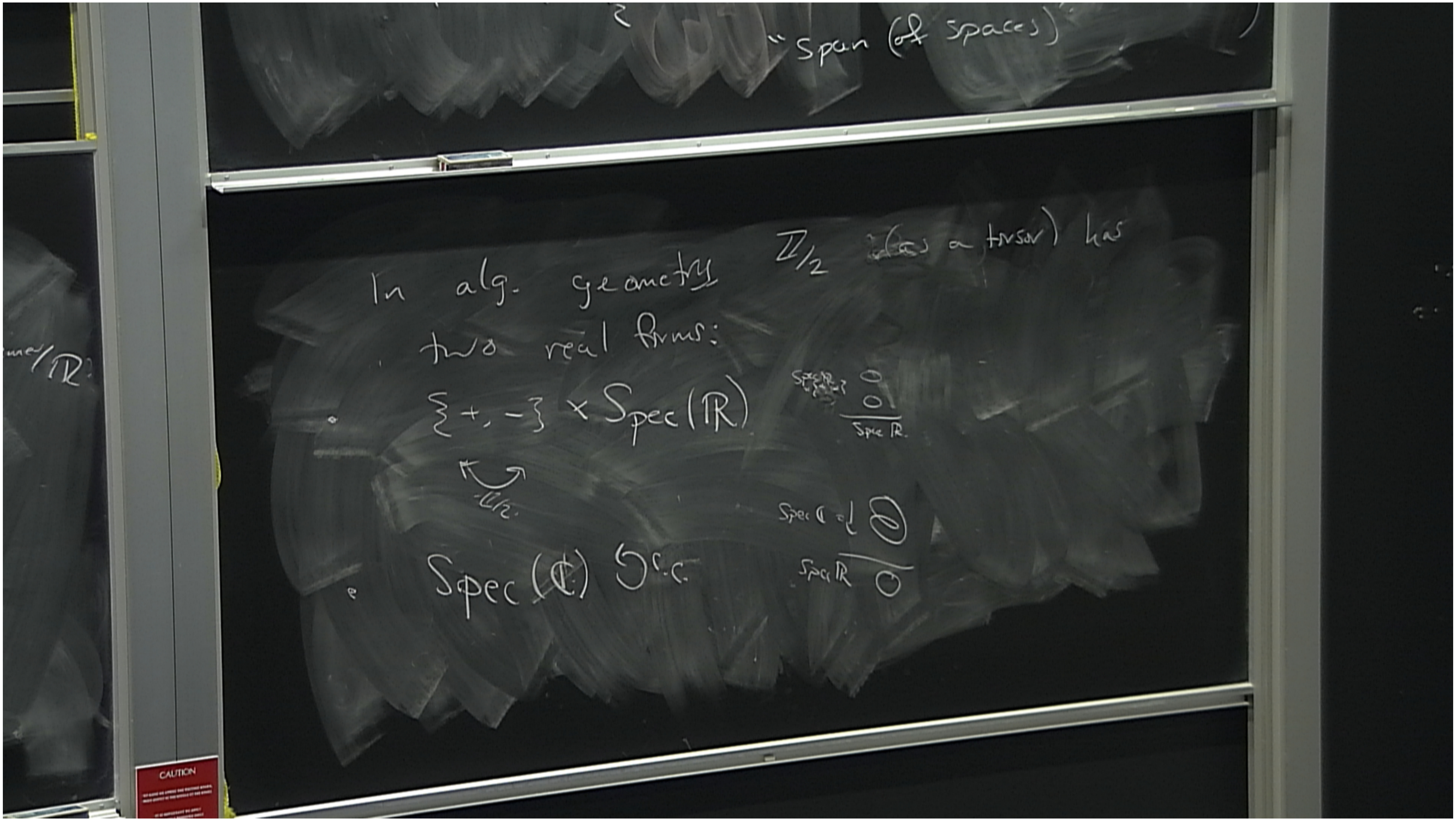
I want a notion of "local str."
in which you can have families
parameterized by affine schemes/$\mathbb{R}$:

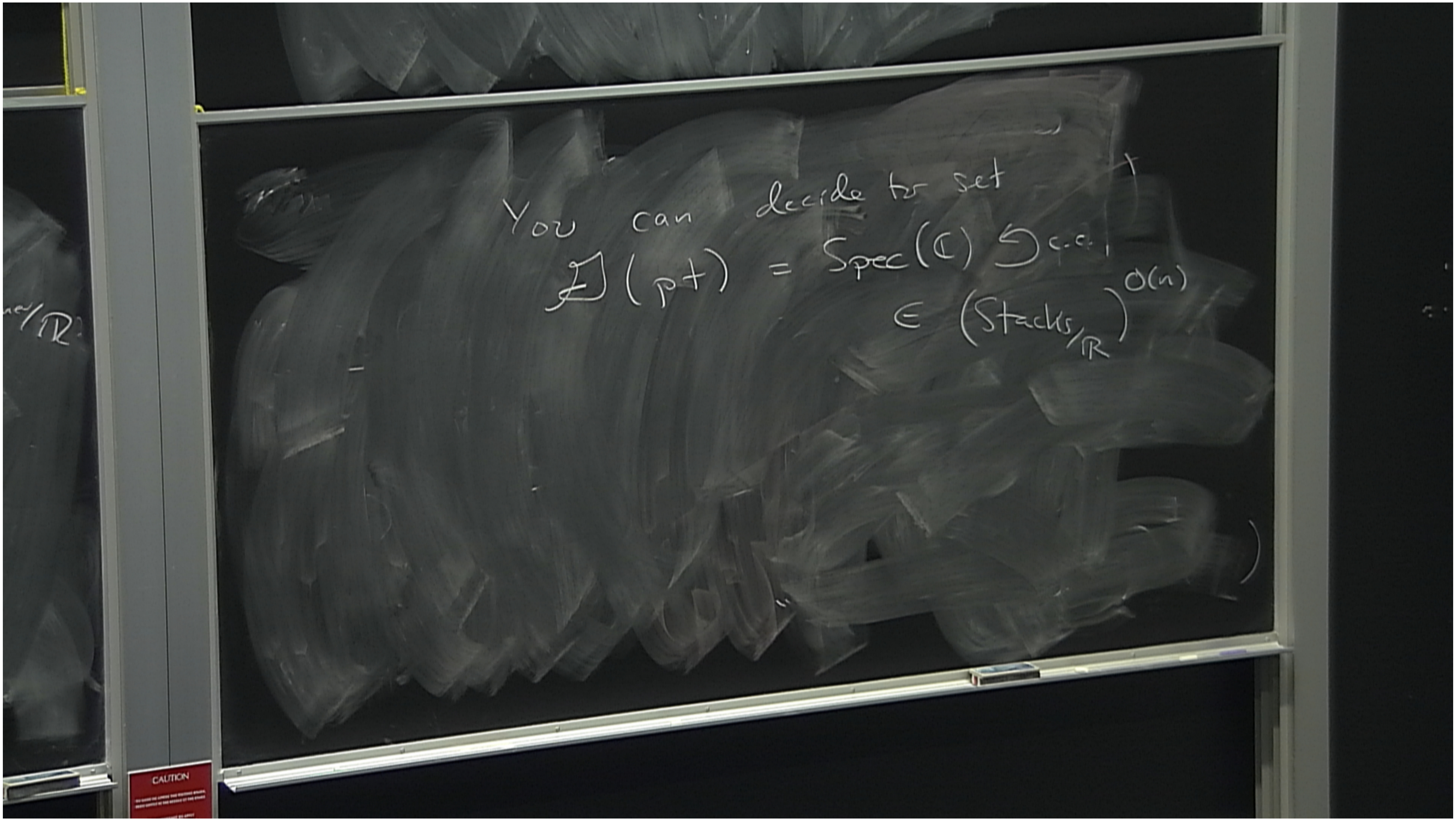
How:

~~Spaces~~ $\text{Stacks}_{\mathbb{R}}$.

you end up:

e.g. $\mathrm{Borel}_n^{\mathcal{G}}$ has $\mathbb{R}$-algebraic structure.


"Span (of spaces)"
In alg. geometry Z/2 (as a torsor) has
two real forms:
{+, −} × Spec(R)
Z/2
Spec(C) c.c.
Spec R
Spec C
Spec R


You can decide to set
$\mathcal{B}(pt) = \mathrm{Spec}(\mathbb{C}) \circlearrowleft c.c.$
$\in (\mathrm{Stacks}_{/\mathbb{R}})^{O(n)}$
CAUTION

You can decide to set

$\mathcal{Z}(M)$ $\qquad \mathcal{Z}(pt) = \mathrm{Spec}(\mathbb{C}) \circlearrowleft$ c.c.

$= \mathrm{Orientations}(M) \times \mathrm{Spec}\,\mathbb{C} \,/\, \mathbb{Z}/2$ $\qquad \in (\mathrm{Stacks}_{/\mathbb{R}})^{O(n)}$

What are $\mathcal{Z}$-structured f.t.'s in $\mathrm{Vect}_{\mathbb{R}}$

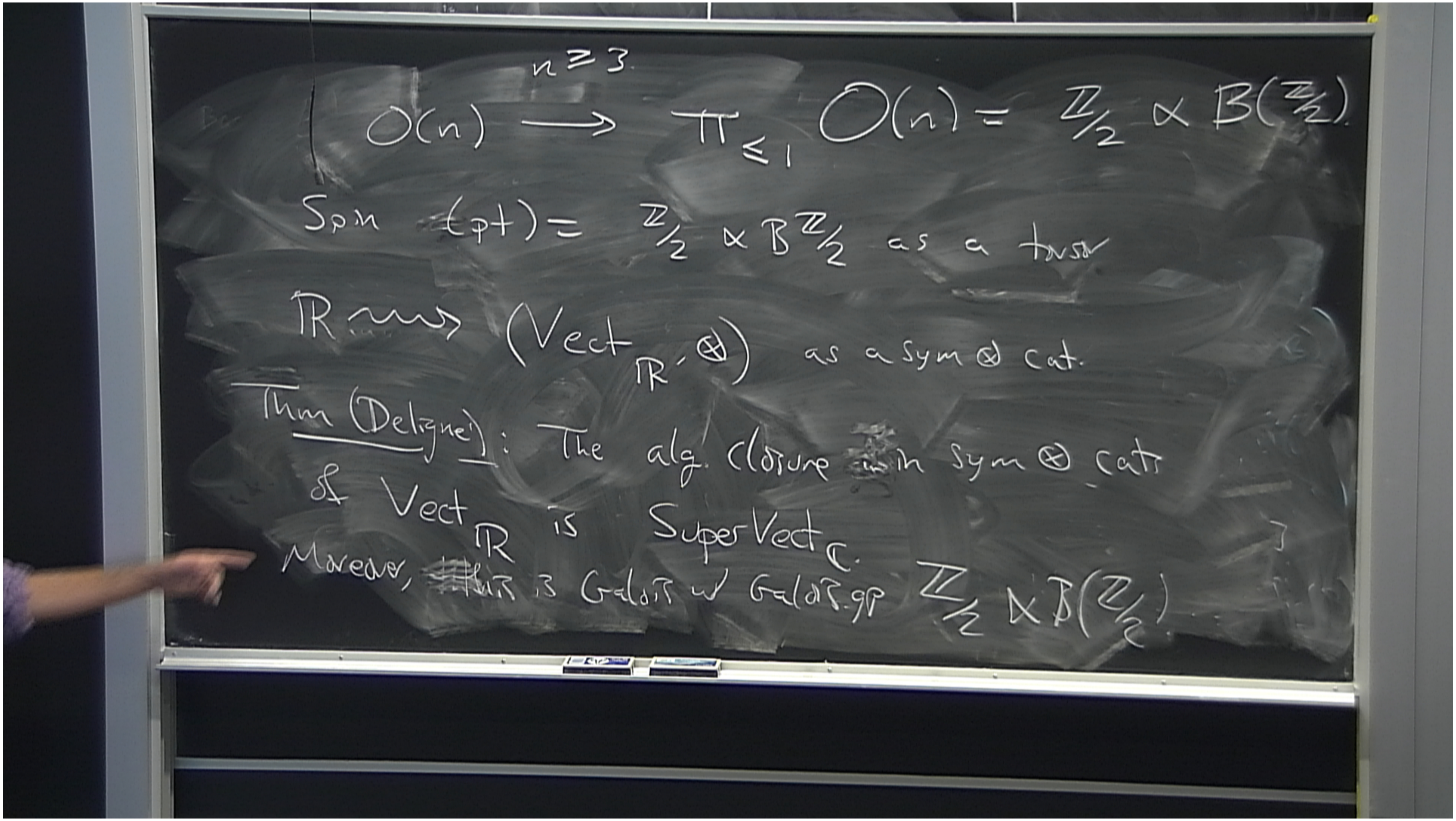
$n=1$: $\qquad Z(pt) = V$ $\qquad$ i.e. Complex v.s.

$\qquad\qquad \downarrow$

$\qquad \mathcal{Z}(pt) = \mathrm{Spec}(\mathbb{C})$ $\qquad$ with a hermitian inner product

$(\lambda, \bar{\lambda}) \longleftarrow\!\shortmid \lambda$

$\mathcal{Z}(\supset) = \mathrm{Spec}(\mathbb{C}) \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C}) \longleftarrow \mathrm{Spec}(\mathbb{C})$


$n \geq 3$
$O(n) \longrightarrow \pi_{\leq 1} O(n) = \mathbb{Z}/2 \ltimes B(\mathbb{Z}/2)$
Spin $(pt) = \mathbb{Z}/2 \ltimes B\mathbb{Z}/2$ as a torsor
$\mathbb{R} \rightsquigarrow (\mathrm{Vect}_{\mathbb{R}}, \otimes)$ as a sym $\otimes$ cat.
Thm (Deligne): The alg. closure in sym $\otimes$ cats
of $\mathrm{Vect}_{\mathbb{R}}$ is $\mathrm{SuperVect}_{\mathbb{C}}$.
Moreover, this is Galois w/ Galois gp $\mathbb{Z}/2 \ltimes B(\mathbb{Z}/2)$

$$n \geq 3$$

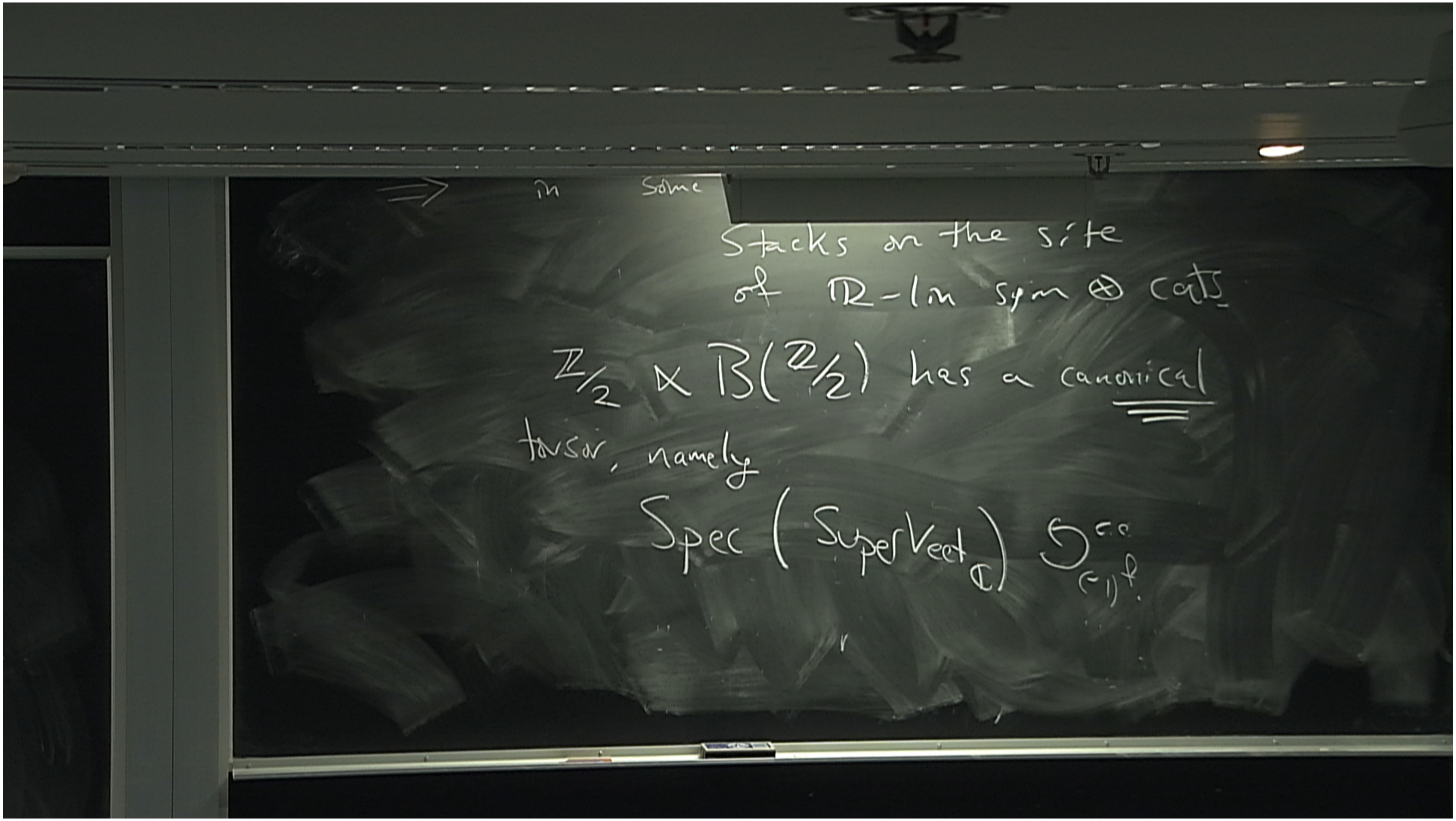
$$O(n) \longrightarrow \pi_{\leq 1} O(n) = \mathbb{Z}/2 \propto B(\mathbb{Z}/2)$$

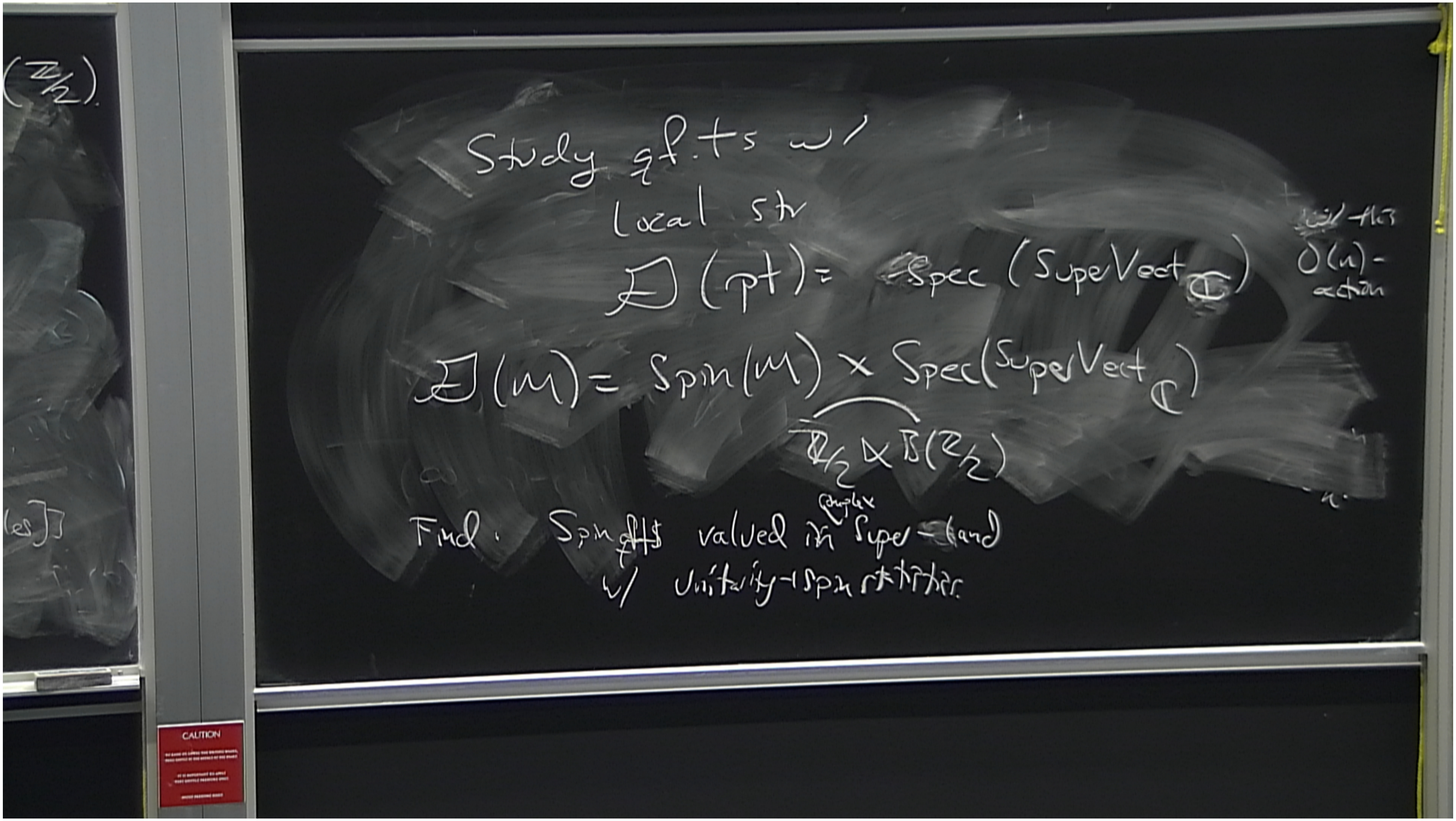
$\mathrm{Spin}\,(pt) = \mathbb{Z}/2 \propto B\mathbb{Z}/2$ as a torsor

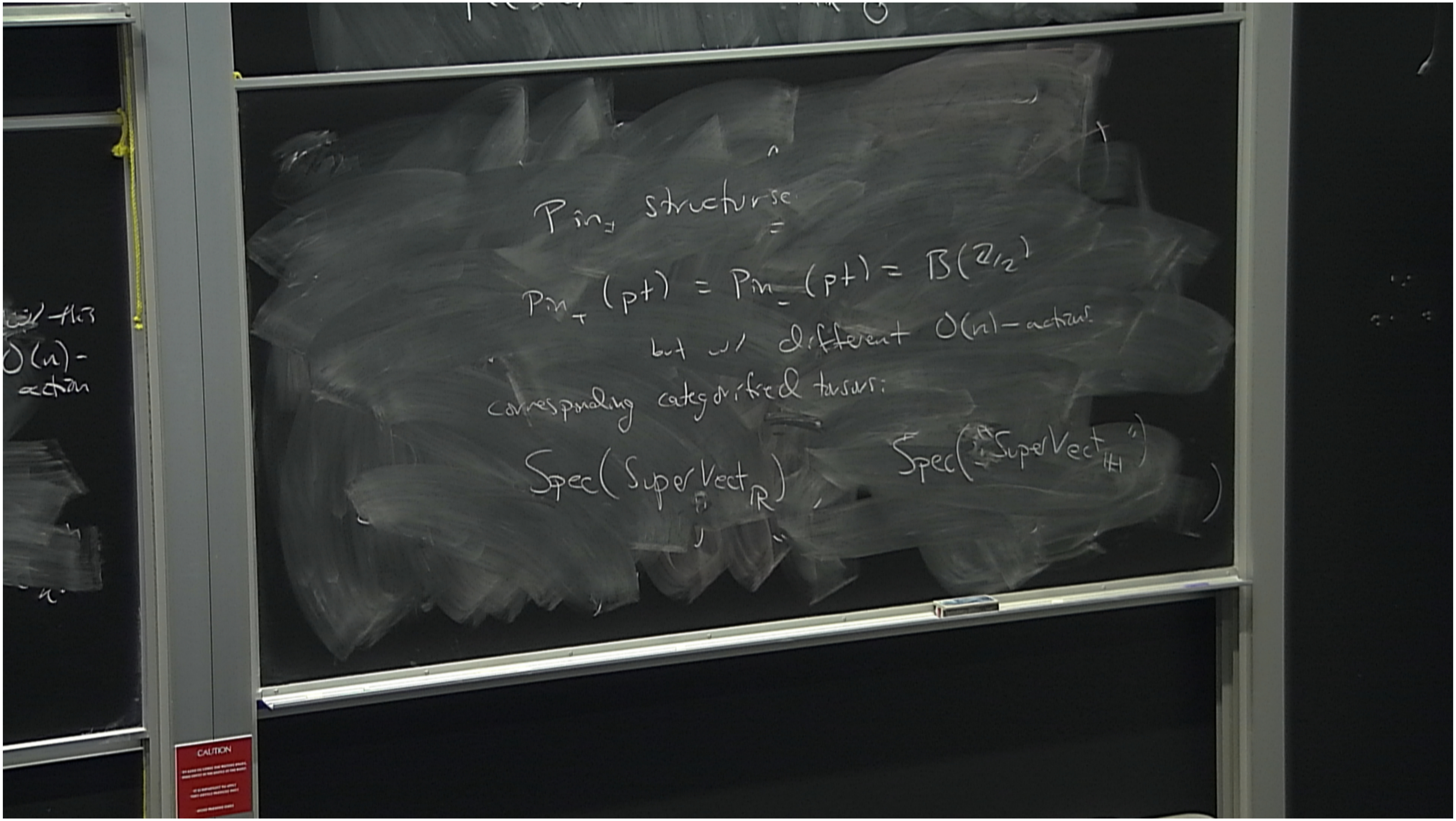
$\mathbb{R} \rightsquigarrow (\mathrm{Vect}_{\mathbb{R}}, \otimes)$ as a sym $\otimes$ cat.

Thm (Deligne): The alg. closure in sym $\otimes$ cats of $\mathrm{Vect}_{\mathbb{R}}$ is $\mathrm{SuperVect}_{\mathbb{C}}$. [Catégories Tensorielles]

Moreover, this is Galois w/ Galois gp $\mathbb{Z}/2 \propto B(\mathbb{Z}/2)$.


⇒ in some
Stacks on the site
of ℝ-lin sym ⊗ cats
ℤ/2 ⋉ B(ℤ/2) has a canonical
torsor, namely
Spec(SuperVect_ℂ) ⟲ (-1)^f


Study q.f.t.s w/
local str
F(pt) = Spec(SuperVect C)
F(M) = Spin(M) × Spec(SuperVect C)
Z/2 ⋉ B(Z/2)
Find: Spin qfts valued in complex Super-(and)
w/ Unitarity + spin statistics.
O(n)-
action


Pin± structures
Pin+(pt) = Pin−(pt) = B(Z/2)
but w/ different O(n)-actions
corresponding categorified torsors:
Spec(SuperVect_R)
Spec("SuperVect_H")
O(n)-
action