Title: Graded fusion categories and homotopy theory

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Abstract:


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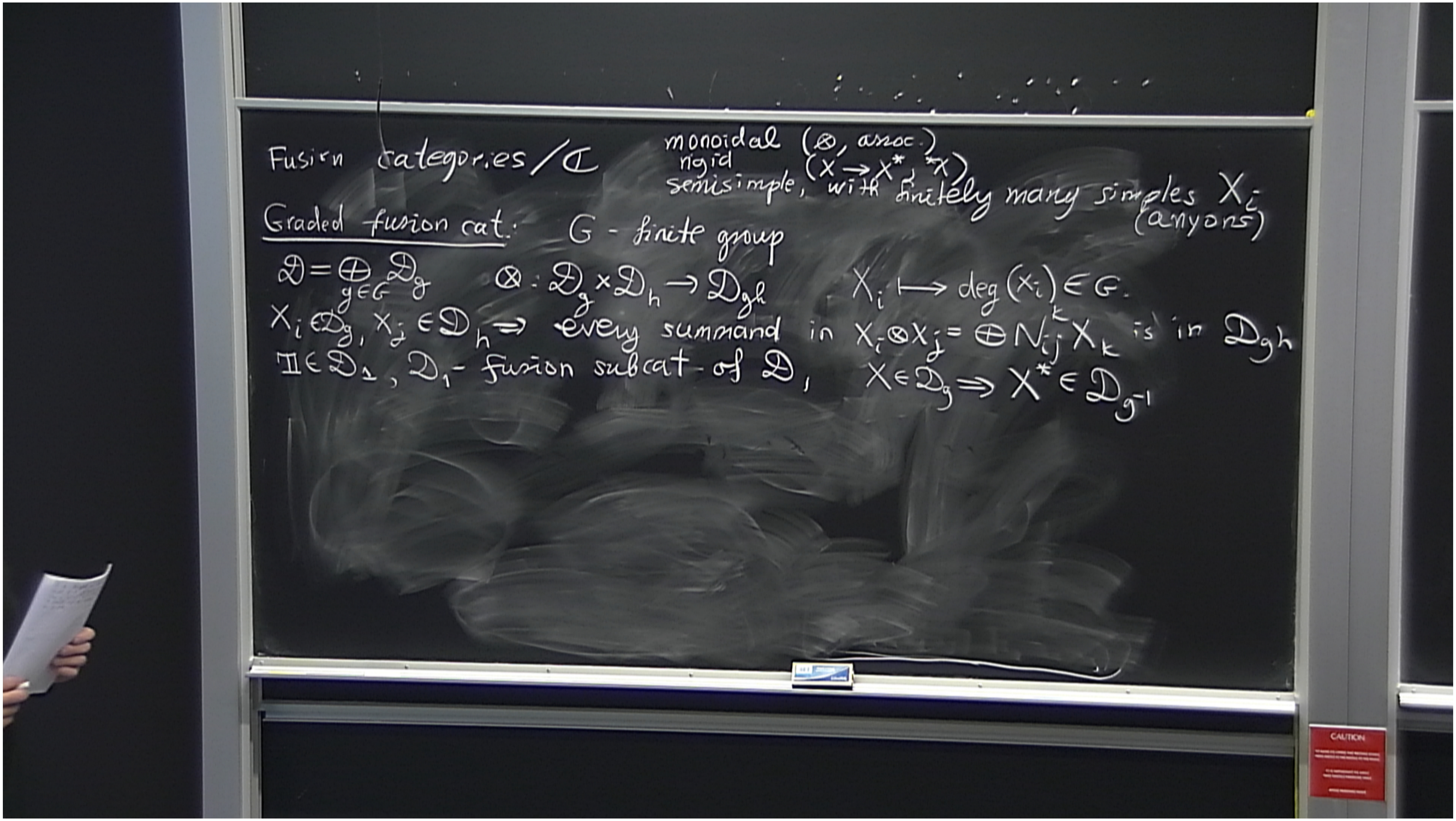
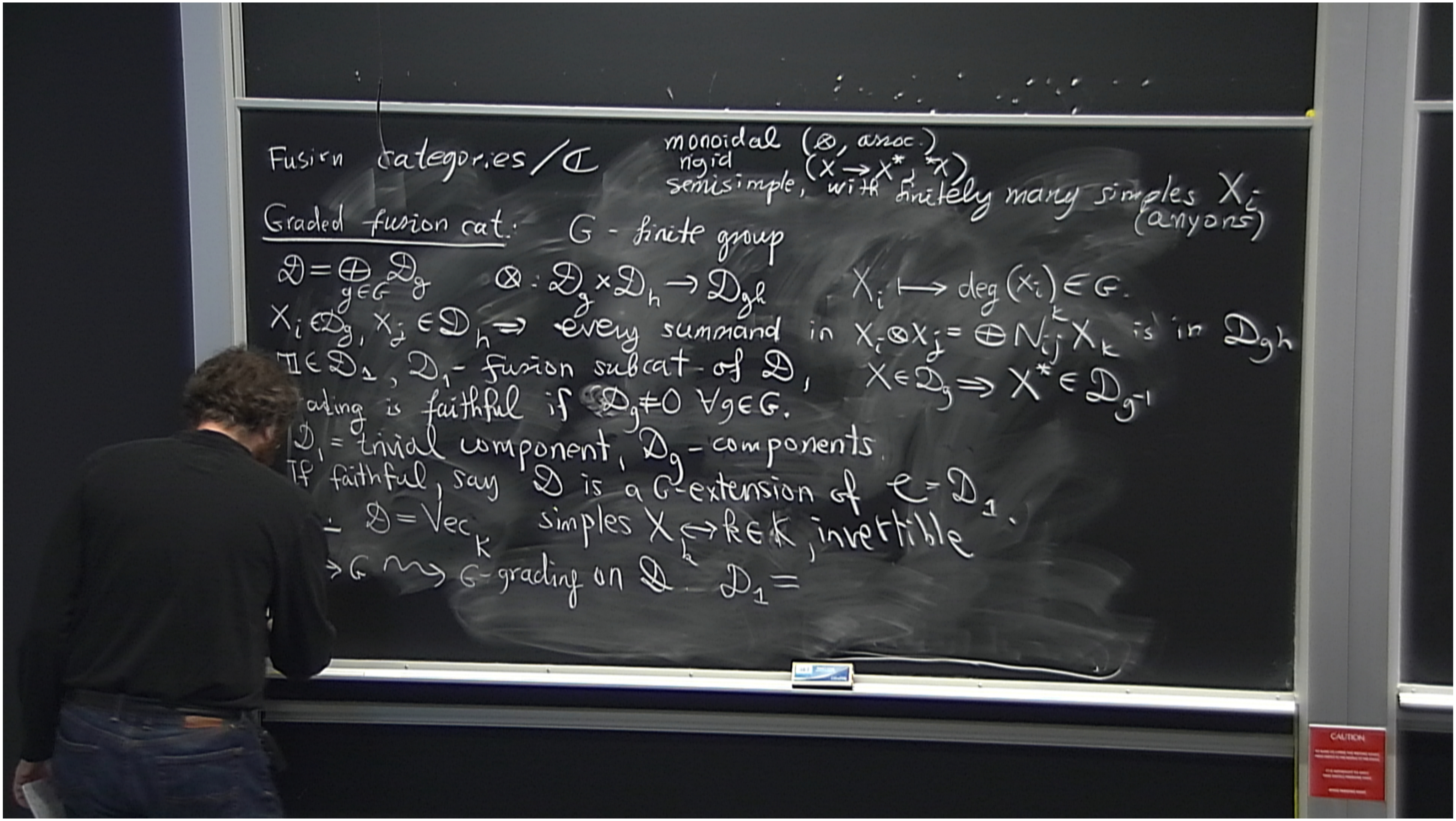
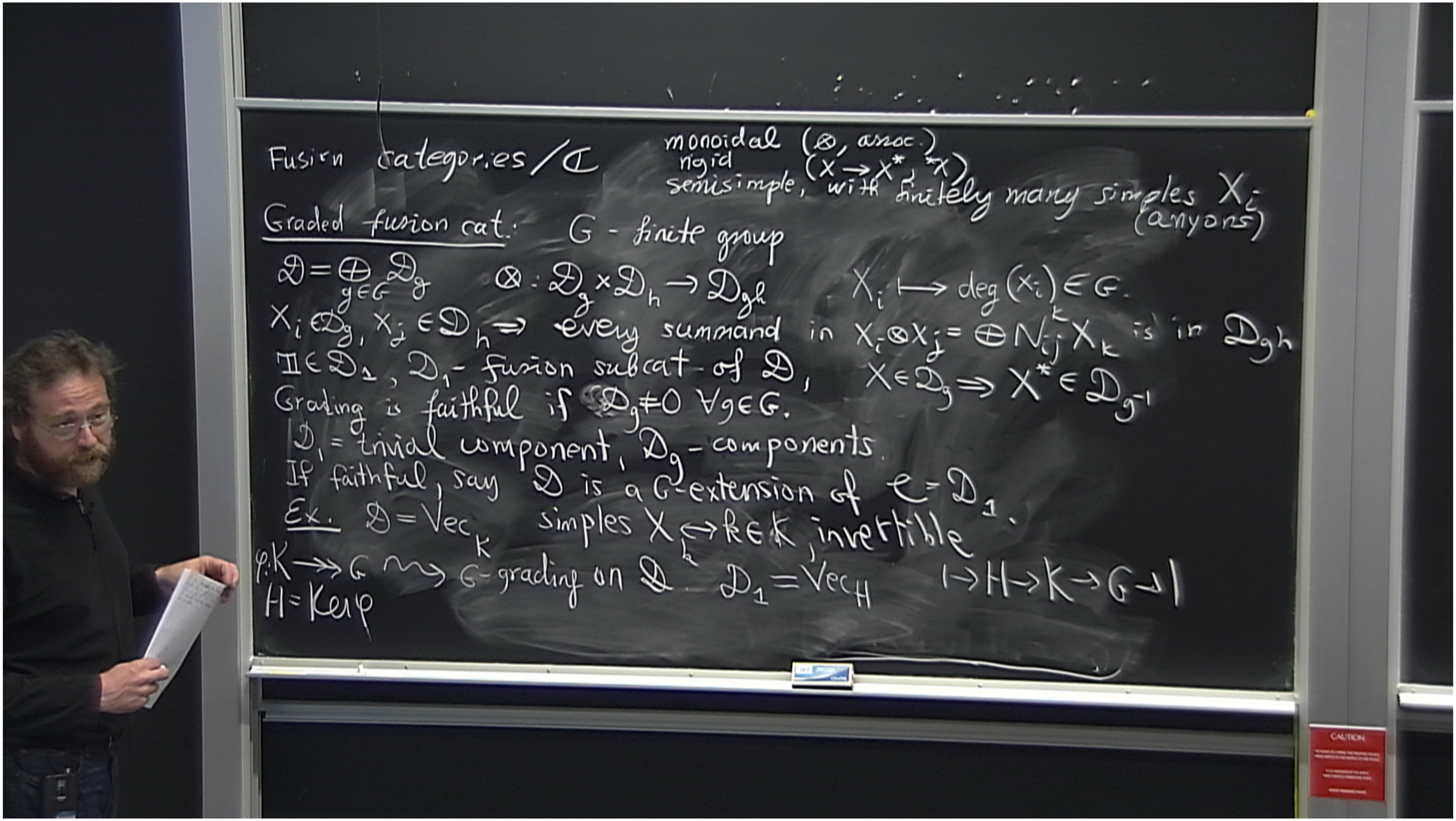
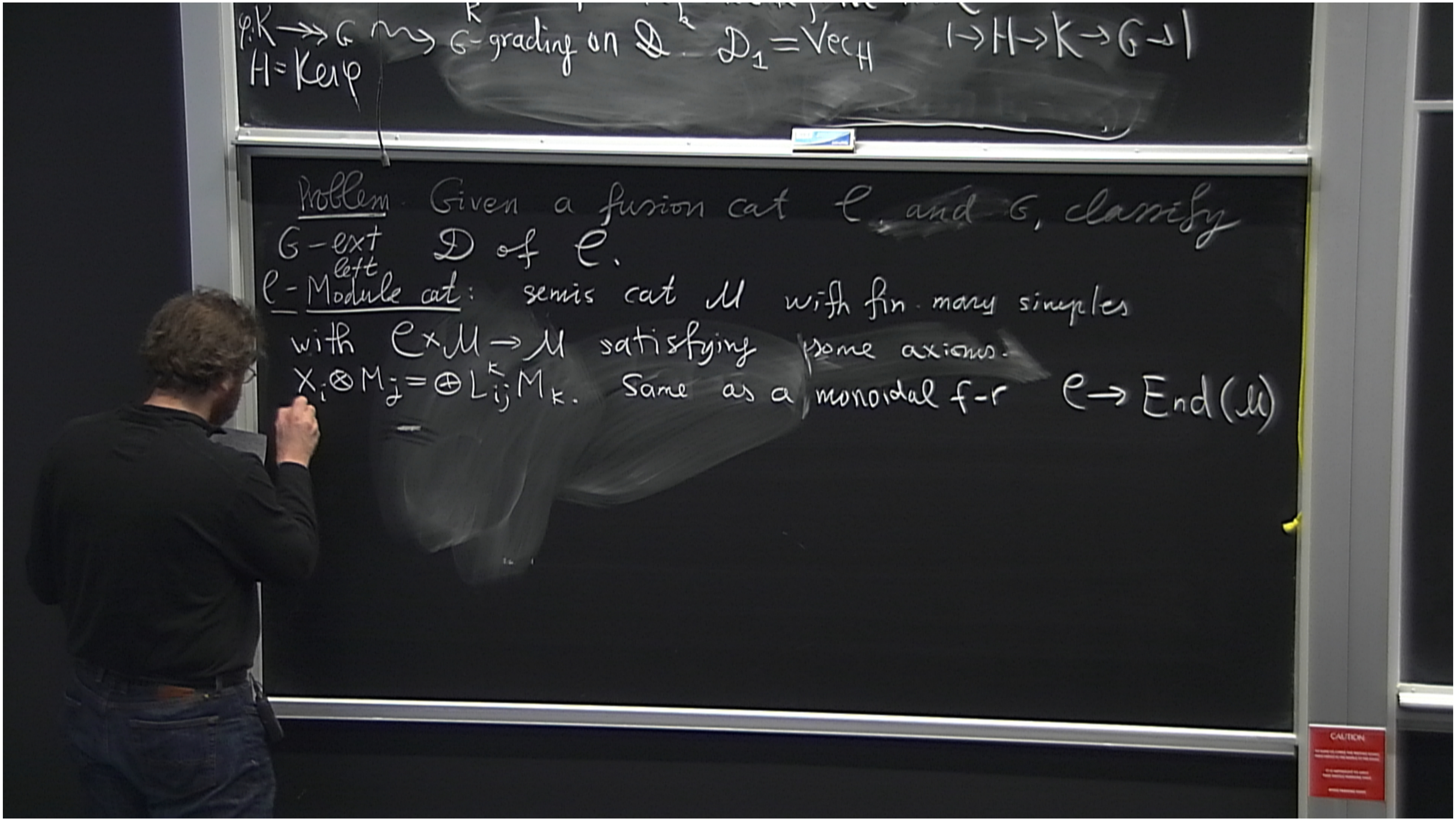
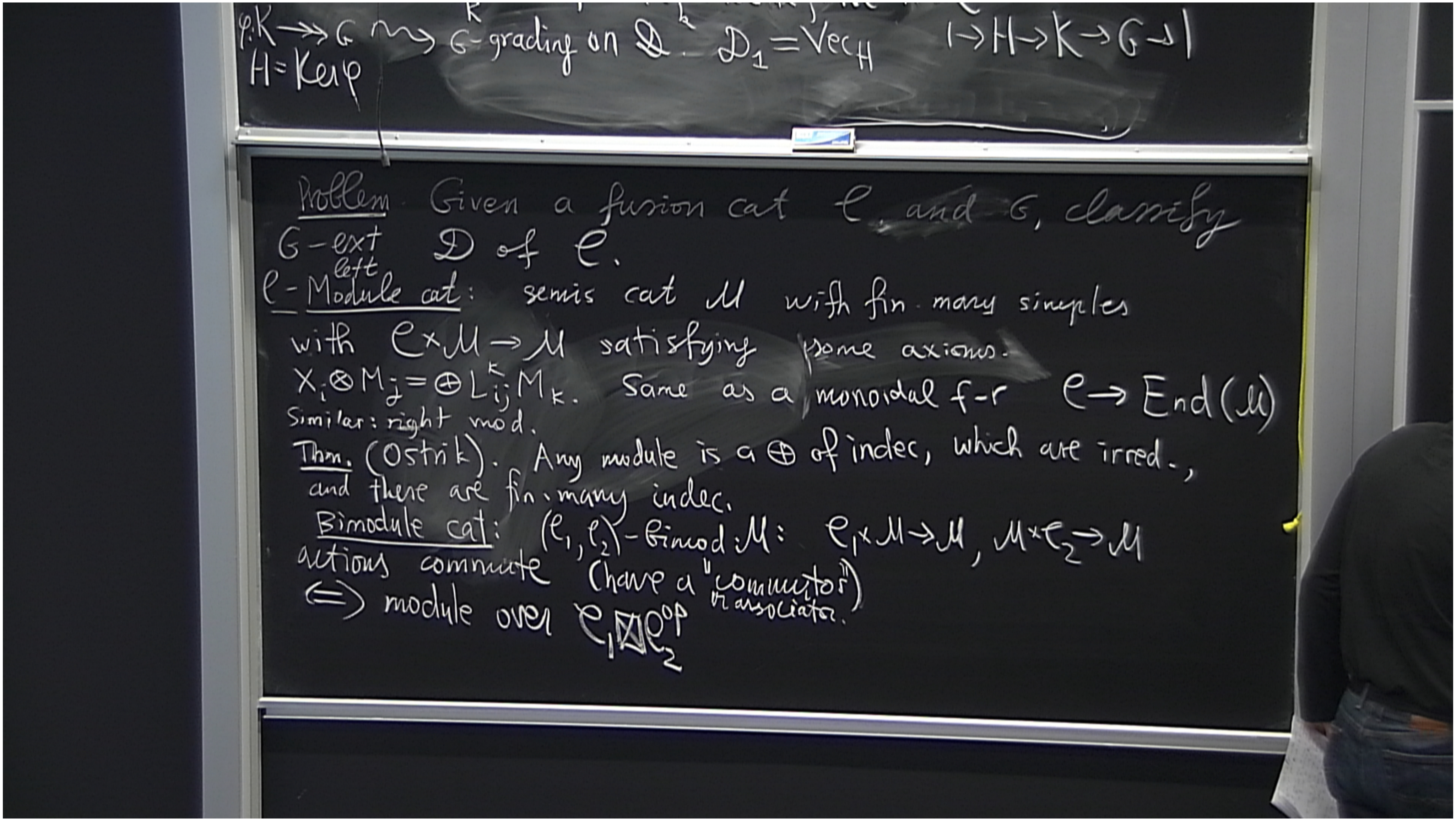

Fusion categories / $\mathbb{C}$
monoidal ($\otimes$, assoc.)
rigid ($X \to X^*, {}^*X$)
semisimple, with finitely many simples X_i (anyons)
Graded fusion cat.: G - finite group
$\mathcal{D} = \bigoplus_{g \in G} \mathcal{D}_g$ $\otimes : \mathcal{D}_g \times \mathcal{D}_h \to \mathcal{D}_{gh}$ $X_i \mapsto \deg(X_i) \in G$.
$X_i \in \mathcal{D}_g, X_j \in \mathcal{D}_h \Rightarrow$ every summand in $X_i \otimes X_j = \bigoplus N_{ij}^k X_k$ is in $\mathcal{D}_{gh}$
$\mathbb{1} \in \mathcal{D}_1$, $\mathcal{D}_1$ - fusion subcat. of $\mathcal{D}$, $X \in \mathcal{D}_g \Rightarrow X^* \in \mathcal{D}_{g^{-1}}$


Fusion categories / ℂ
monoidal (⊗, assoc.)
rigid (X → X*, *X)
semisimple, with finitely many simples X_i
(anyons)
Graded fusion cat.: G - finite group
D = ⊕_{g∈G} D_g   ⊗ : D_g × D_h → D_gh   X_i ↦ deg(X_i) ∈ G.
X_i ∈ D_g, X_j ∈ D_h ⇒ every summand in X_i ⊗ X_j = ⊕ N_ij^k X_k is in D_gh
1 ∈ D_1, D_1 - fusion subcat of D,   X ∈ D_g ⇒ X* ∈ D_g^-1
ading is faithful if D_g ≠ 0 ∀g ∈ G.
D_1 = trivial component, D_g - components.
If faithful, say D is a G-extension of C = D_1.
D = Vec_K   simples X_k ↔ k ∈ K, invertible
→ G ⇝ G-grading on D.   D_1 =


Fusion categories / ℂ
monoidal (⊗, assoc.)
rigid (X → X*, *X)
semisimple, with finitely many simples Xi (anyons)
Graded fusion cat. G - finite group
𝒟 = ⊕g∈G 𝒟g   ⊗ : 𝒟g × 𝒟h → 𝒟gh   Xi ⟼ deg(Xi) ∈ G.
Xi ∈ 𝒟g, Xj ∈ 𝒟h ⇒ every summand in Xi⊗Xj = ⊕ Nijk Xk is in 𝒟gh
𝟙 ∈ 𝒟1, 𝒟1 - fusion subcat of 𝒟,   X ∈ 𝒟g ⇒ X* ∈ 𝒟g-1
Grading is faithful if 𝒟g ≠ 0 ∀g ∈ G.
𝒟1 = trivial component, 𝒟g - components.
If faithful, say 𝒟 is a G-extension of 𝒞 = 𝒟1.
Ex. 𝒟 = VecK   simples Xk ↔ k ∈ K, invertible
φ: K ↠ G ⇝ G-grading on 𝒟   𝒟1 = VecH   1 → H → K → G → 1
H = Ker φ


φ: K ↠ G ⤳ G-grading on D. D_1 = Vec_H
H = Ker φ
1 → H → K → G → 1
Problem. Given a fusion cat C and G, classify G-ext D of C.
C-Module cat (left): semis cat M with fin. many simples
with C × M → M satisfying some axioms.
X_i ⊗ M_j = ⊕ L_ij^k M_k. Same as a monoidal f-r C → End(M)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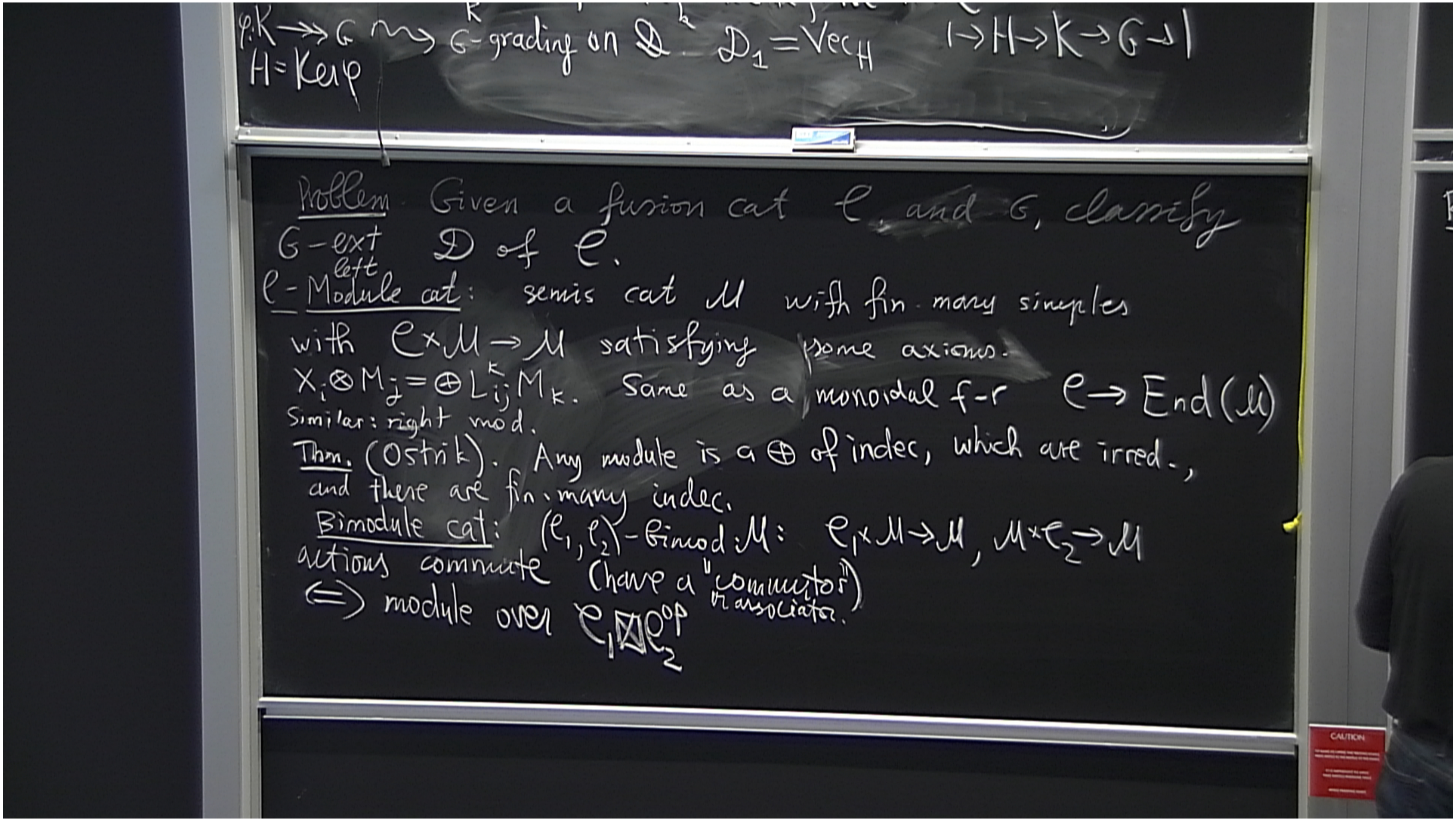
X_i ⊗ M_j = ⊕ L_ij^k M_k. Same as a monoidal f-r C → End(M)
Similar: right mod.
Thm. (Ostrik). Any module is a ⊕ of indec, which are irred.,
and there are fin. many indec.
Bimodule cat: (C_1, C_2)-Bimod M: C_1 × M → M, M × C_2 → M
actions commute (have a "commutor" / associator)
⟺ module over C_1 ⊠ C_2^op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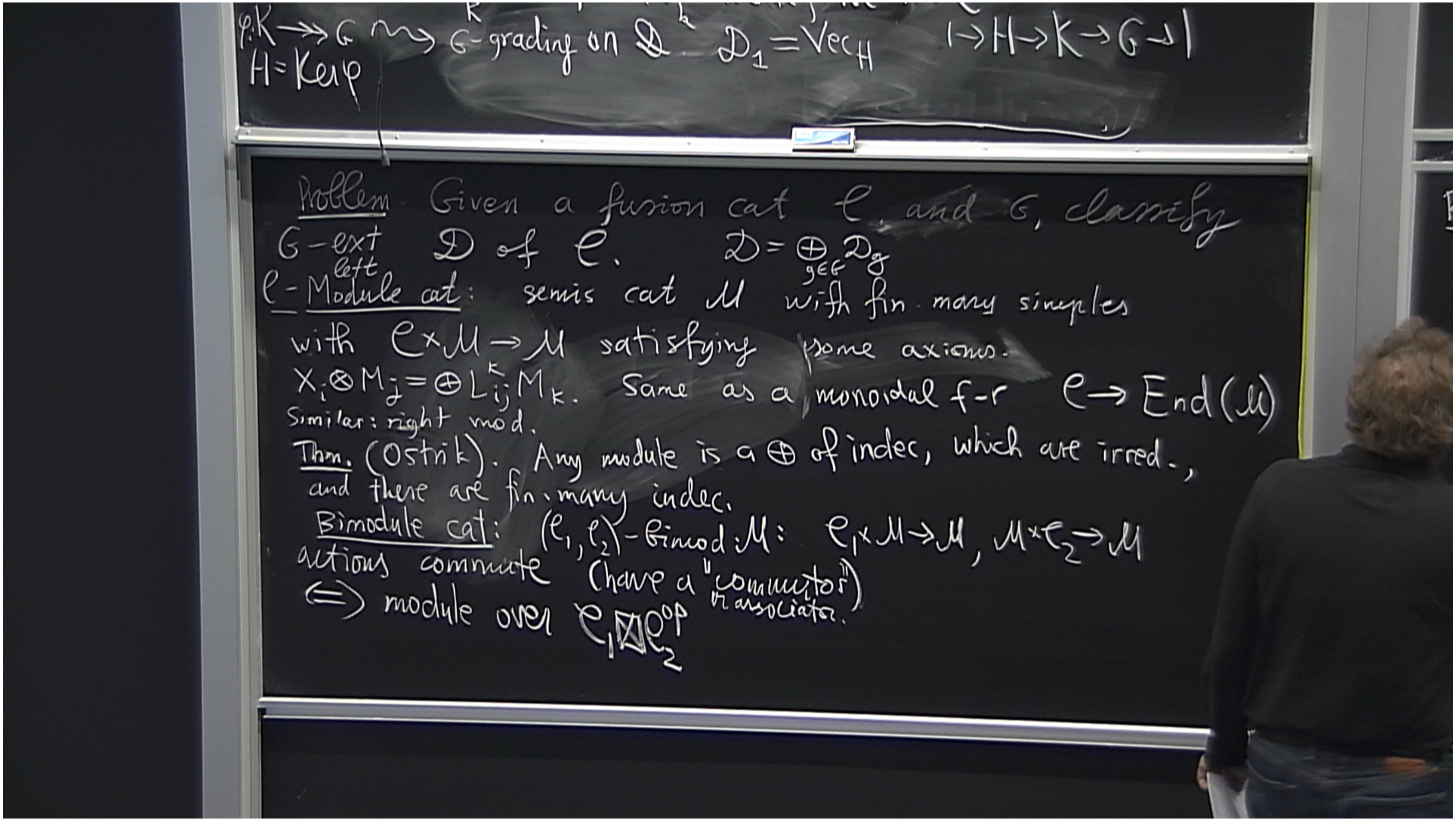
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Problem. Given a fusion cat C and G, classify
G-ext D of C.   D = ⊕_{g∈G} D_g
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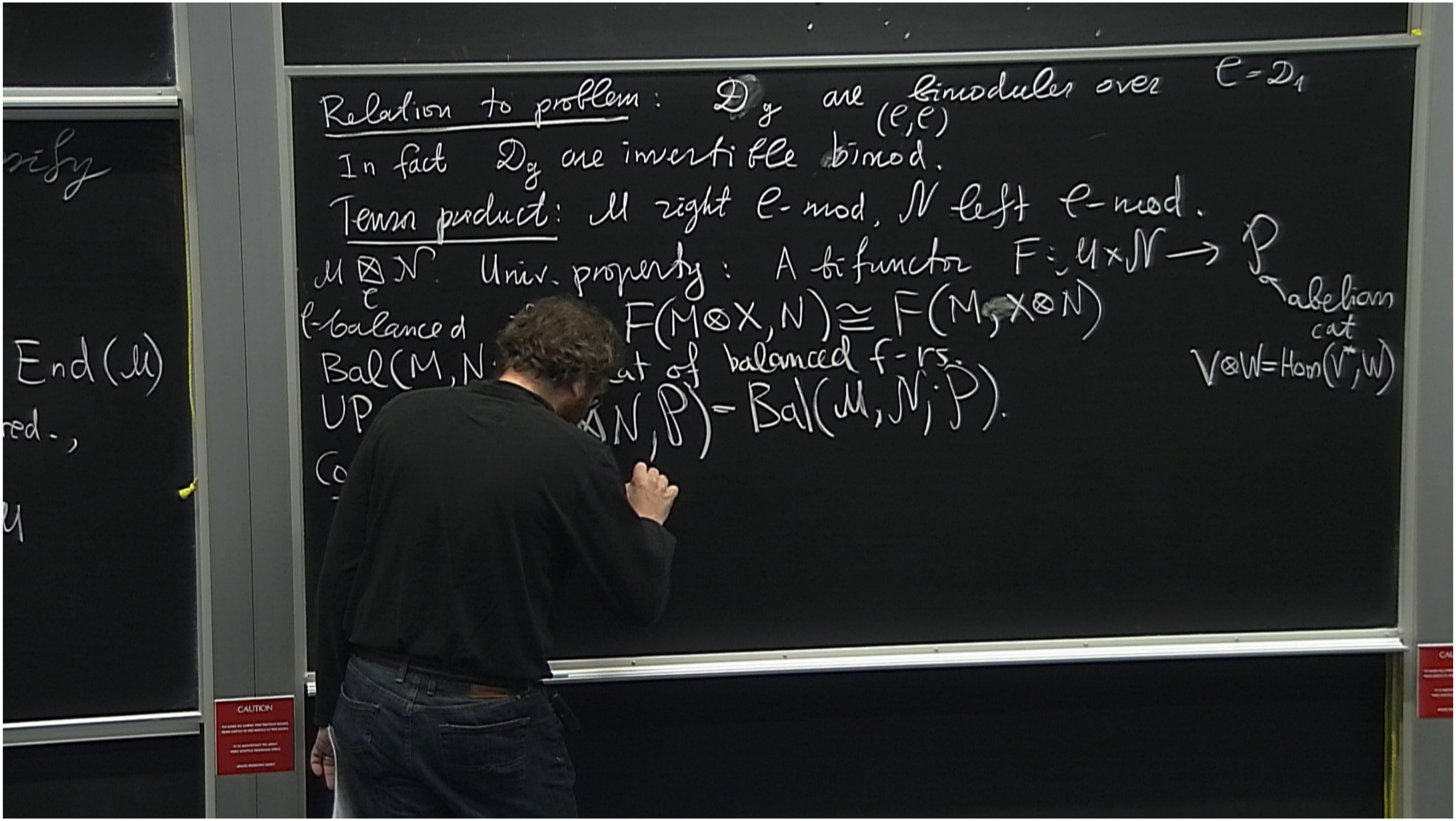
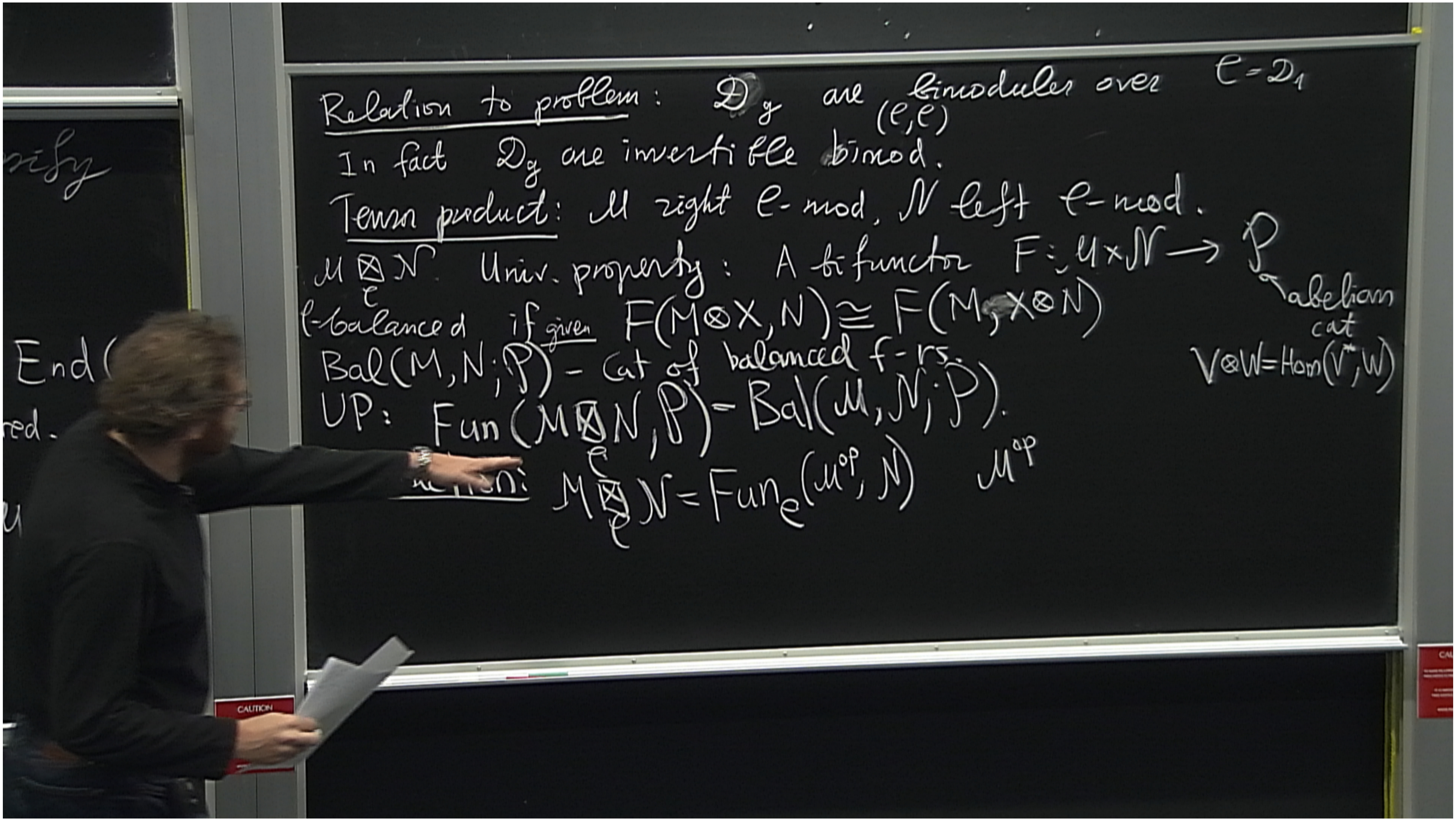

Relation to problem: D_g are bimodules over (C,C) C = D_1
In fact D_g are invertible bimod.
Tensor product: M right C-mod, N left C-mod.
M ⊠_C N. Univ. property: A bifunctor F: M × N → P
abelian cat
C-balanced
F(M⊗X, N) ≅ F(M, X⊗N)
Bal(M, N
cat of balanced f-rs.
UP
N, P) = Bal(M, N; P).
V⊗W = Hom(V*, W)
End(M)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C-balanced if given F(M⊗X, N) ≅ F(M, X⊗N)
Bal(M,N;P) - cat of balanced f-rs.
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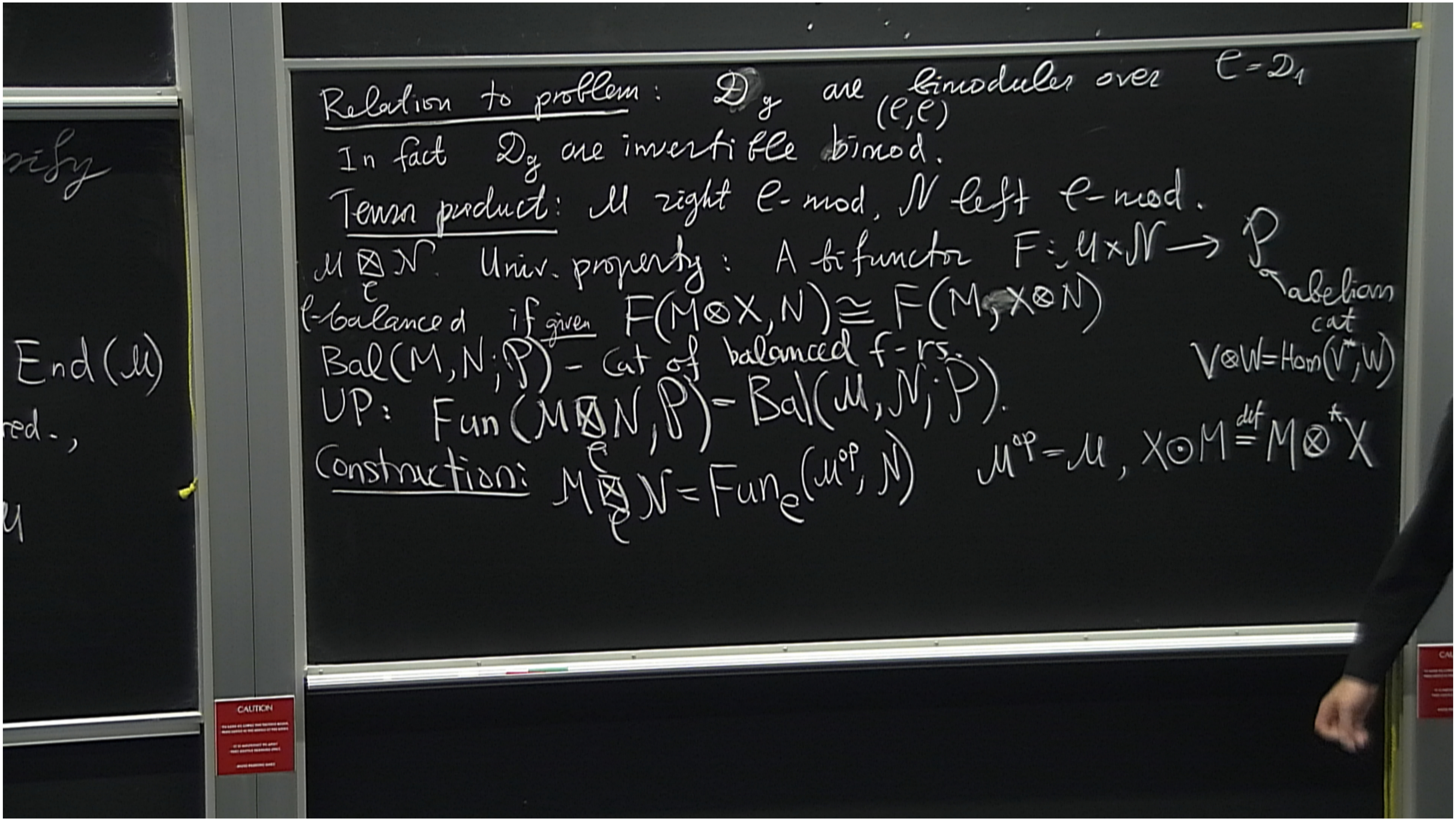
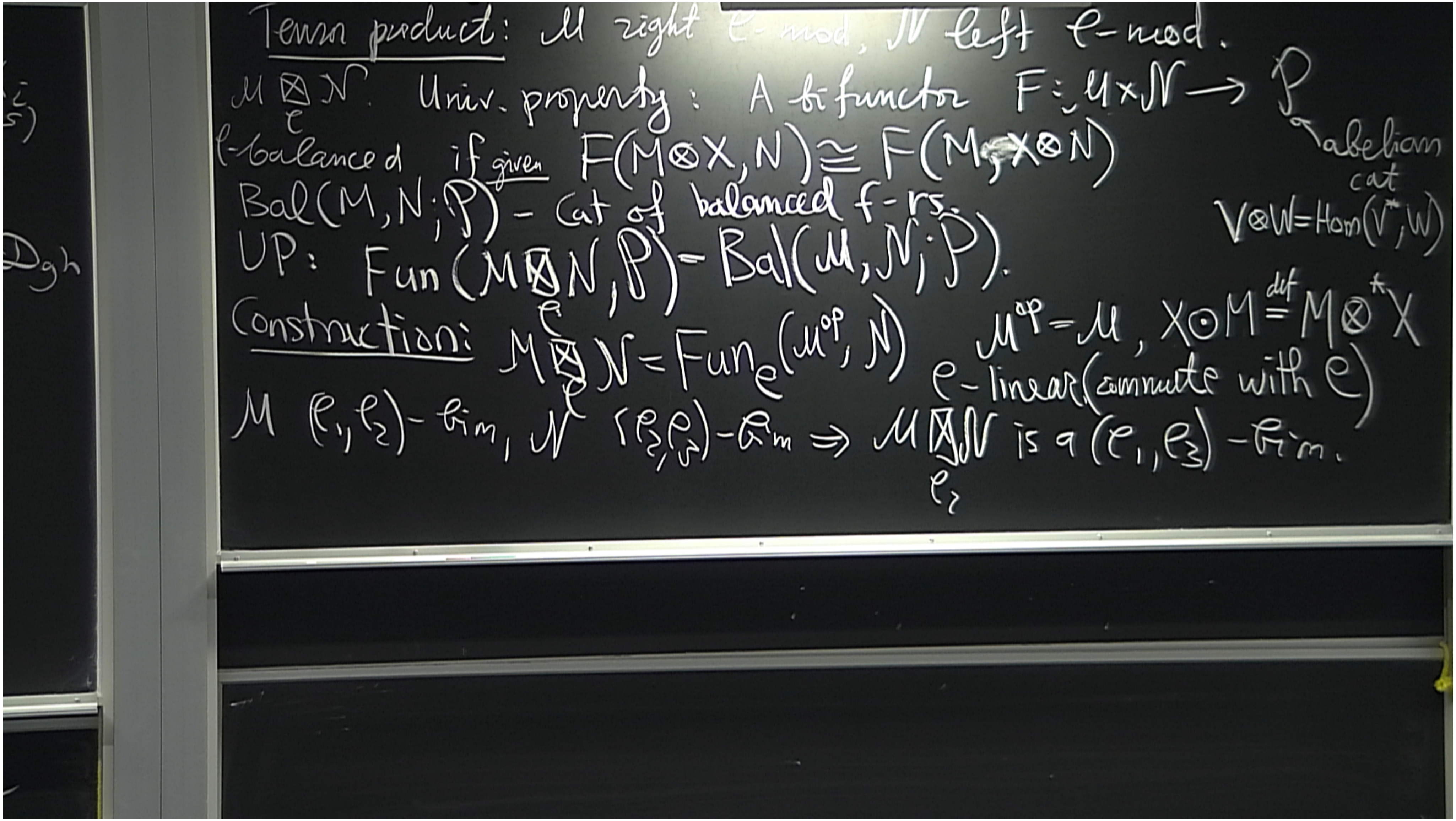
UP: Fun(M ⊠_C N, P) - Bal(M, N; P).
M ⊠_C N = Fun_C(M^op, N)   M^op


Relation to problem: $\mathcal{D}_g$ are $(\mathcal{C},\mathcal{C})$ bimodules over $\mathcal{C}=\mathcal{D}_1$
In fact $\mathcal{D}_g$ are invertible bimod.
Tensor product: $\mathcal{M}$ right $\mathcal{C}$-mod, $\mathcal{N}$ left $\mathcal{C}$-mod.
$\mathcal{M}\boxtimes_{\mathcal{C}}\mathcal{N}$. Univ. property: A bifunctor $F:\mathcal{M}\times\mathcal{N}\to\mathcal{P}$
abelian cat
$\mathcal{C}$-balanced if given $F(M\otimes X,N)\cong F(M,X\otimes N)$
$Bal(M,N;\mathcal{P})$ - cat of balanced f-rs.
$V\otimes W=Hom(V^*,W)$
UP: $Fun(M\boxtimes_{\mathcal{C}}N,\mathcal{P})=Bal(\mathcal{M},\mathcal{N};\mathcal{P})$.
Construction: $M\boxtimes_{\mathcal{C}}\mathcal{N}=Fun_{\mathcal{C}}(\mathcal{M}^{op},\mathcal{N})$
$\mathcal{M}^{op}=\mathcal{M}$, $X\odot M\overset{def}{=}M\otimes{}^*X$
End($\mathcal{M}$)
CAUTION


Tensor product: M right C-mod, N left C-mod.
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C-balanced if given F(M⊗X, N) ≅ F(M, X⊗N)
Bal(M,N;P) - Cat of balanced f-rs.
V⊗W = Hom(V*, W)
UP: Fun(M ⊠_C N, P) = Bal(M,N;P).
Construction: M ⊠_C N = Fun_C(M^op, N)
M^op = M, X⊙M =def M⊗*X
C-linear (commute with C)
M (C1,C2)-bim, N (C2,C3)-bim ⇒ M ⊠_C2 N is a (C1,C3)-bim.

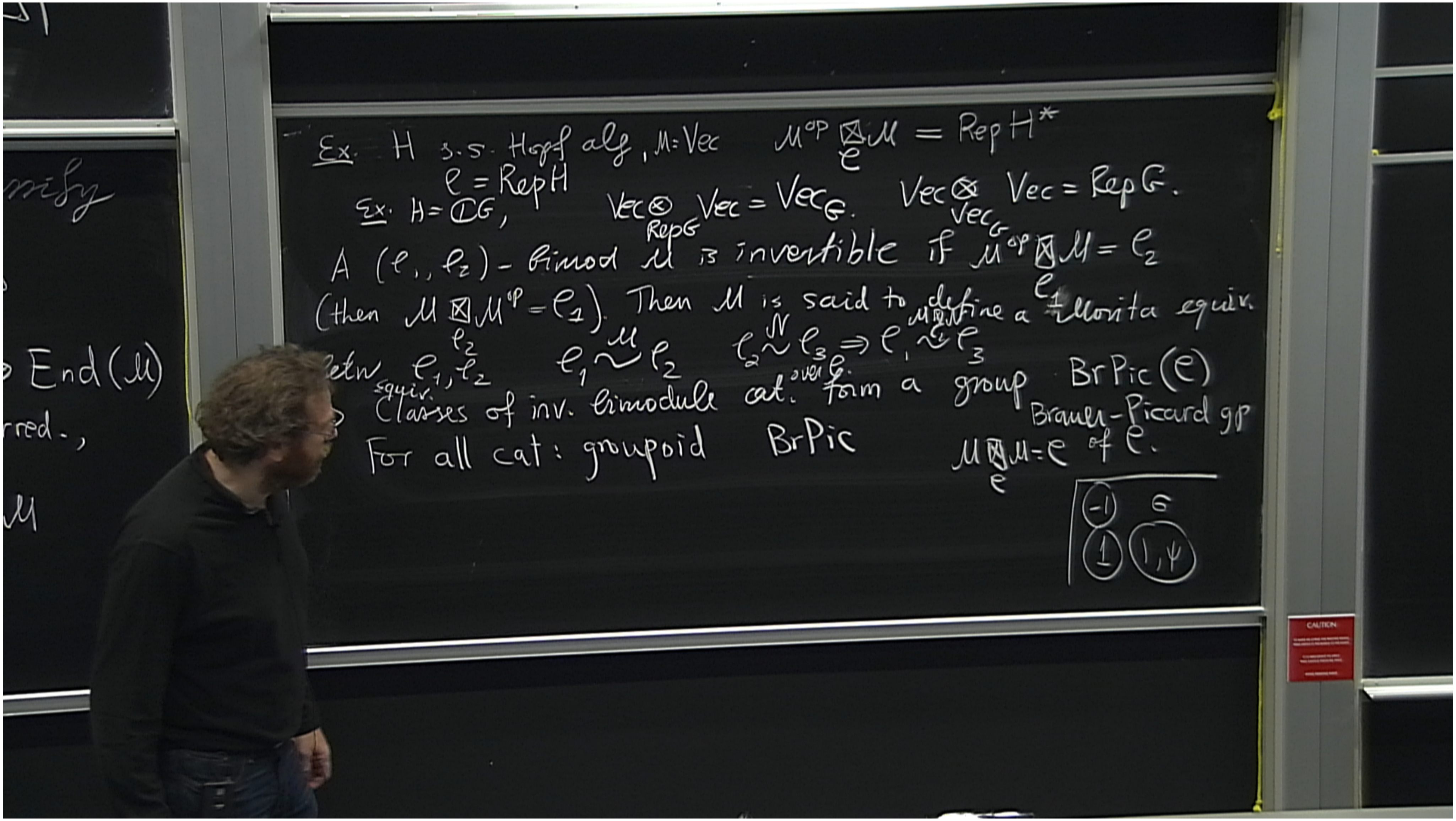
Ex. H s.s. Hopf alg, $\mathcal{M} = \mathrm{Vec}$ $\quad \mathcal{M}^{op} \boxtimes_{\mathcal{C}} \mathcal{M} = \mathrm{Rep}\, H^*$

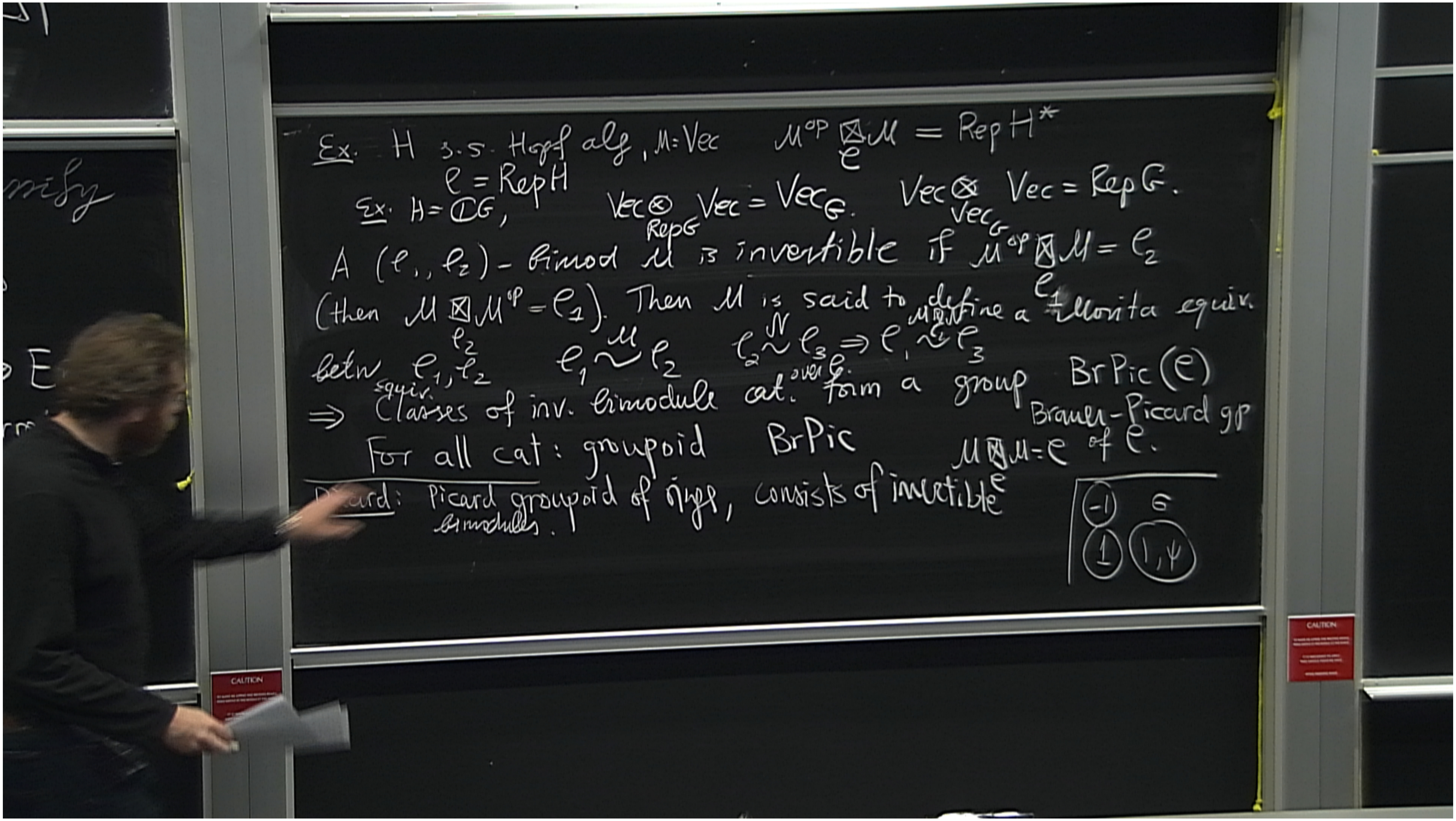
$\mathcal{C} = \mathrm{Rep}\, H$

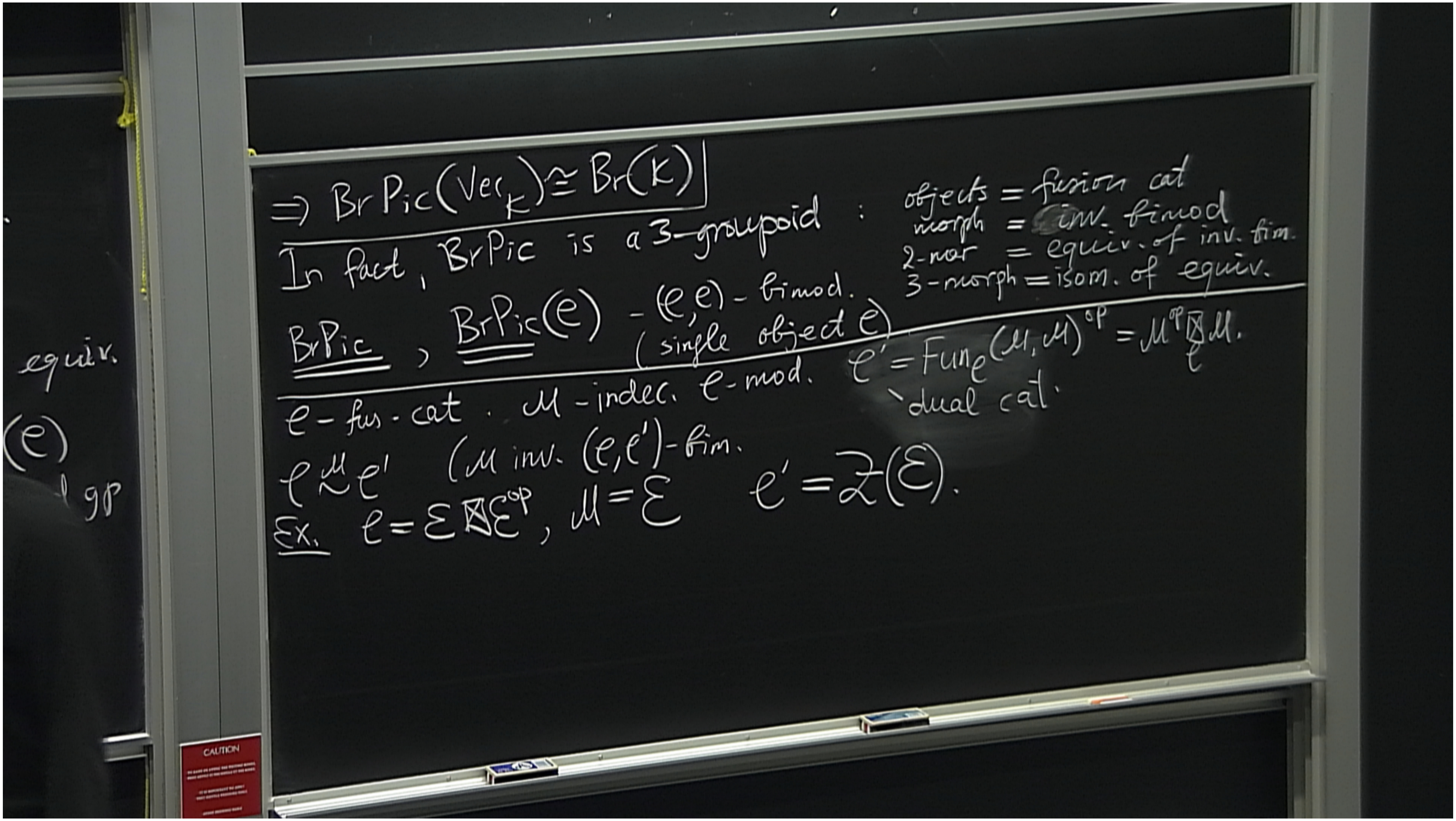
Ex. $H = \mathbb{C}G$, $\quad \mathrm{Vec} \otimes_{\mathrm{Rep}\, G} \mathrm{Vec} = \mathrm{Vec}_G$. $\quad \mathrm{Vec} \otimes_{\mathrm{Vec}_G} \mathrm{Vec} = \mathrm{Rep}\, G$.

A $(\mathcal{C}_1, \mathcal{C}_2)$-bimod $\mathcal{M}$ is invertible if $\mathcal{M}^{op} \boxtimes_{\mathcal{C}_1} \mathcal{M} = \mathcal{C}_2$ (then $\mathcal{M} \boxtimes_{\mathcal{C}_2} \mathcal{M}^{op} = \mathcal{C}_1$). Then $\mathcal{M}$ is said to define a Morita equiv. betw $\mathcal{C}_1, \mathcal{C}_2$ $\quad \mathcal{C}_1 \overset{\mathcal{M}}{\sim} \mathcal{C}_2 \quad \mathcal{C}_2 \overset{\mathcal{N}}{\sim} \mathcal{C}_3 \Rightarrow \mathcal{C}_1 \overset{\mathcal{M} \boxtimes \mathcal{N}}{\sim} \mathcal{C}_3$

$\Rightarrow$


Ex. H s.s. Hopf alg, M = Vec M^op ⊠_C M = Rep H*
C = Rep H
Ex. H = ℂG, Vec ⊗_RepG Vec = Vec_G. Vec ⊗_VecG Vec = Rep G.
A (C1, C2) - bimod M is invertible if M^op ⊠_C1 M = C2
(then M ⊠_C2 M^op = C1). Then M is said to define a Morita equiv.
betw C1, C2 C1 ~ C2 C2 ~ C3 ⇒ C1 ~ C3
Classes of inv. bimodule cat. over C form a group BrPic(C)
Brauer-Picard gp of C.
For all cat: groupoid BrPic
M ⊠_C M = C
End(M)
-1 G
1 1,ψ


Ex. H s.s. Hopf alg, M = Vec   M^op ⊠_C M = Rep H*
C = Rep H
Ex. H = ℂG,   Vec ⊗_{Rep G} Vec = Vec_G.   Vec ⊗_{Vec_G} Vec = Rep G.
A (C1, C2) - bimod M is invertible if M^op ⊠_{C1} M = C2
(then M ⊠_{C2} M^op = C1). Then M is said to define a Morita equiv.
betw C1, C2   C1 ~ C2   C2 ~ C3 ⇒ C1 ~ C3
⇒ Classes of inv. bimodule cat. over C form a group BrPic(C)
Brauer-Picard gp of C.
For all cat: groupoid BrPic   M ⊠_C M = C
Picard: Picard groupoid of rings, consists of invertible bimodules.
-1   G
1   1, ψ


⇒ BrPic(Vec_k) ≅ Br(k)
In fact, BrPic is a 3-groupoid :
objects = fusion cat
morph = inv. bimod
2-mor = equiv. of inv. bim.
3-morph = isom. of equiv.
BrPic , BrPic(C) - (C,C) - bimod.
(single object C)
C - fus - cat . M - indec. C - mod.
C' = Fun_C(M, M)^op = M^op ⊠_C M.
dual cat.
C ≃^M C' (M inv. (C,C') - bim.
Ex. C = E ⊠ E^op , M = E
C' = Z(E).
equiv.
(C)
gp
CAUTION

$\Rightarrow \mathrm{BrPic}(\mathrm{Vec}_k) \cong \mathrm{Br}(k)$

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$\underline{\underline{\mathrm{BrPic}}}$, $\underline{\underline{\mathrm{BrPic}}}(\mathcal{C})$ - $(\mathcal{C},\mathcal{C})$ - bimod.
(single object $\mathcal{C}$)

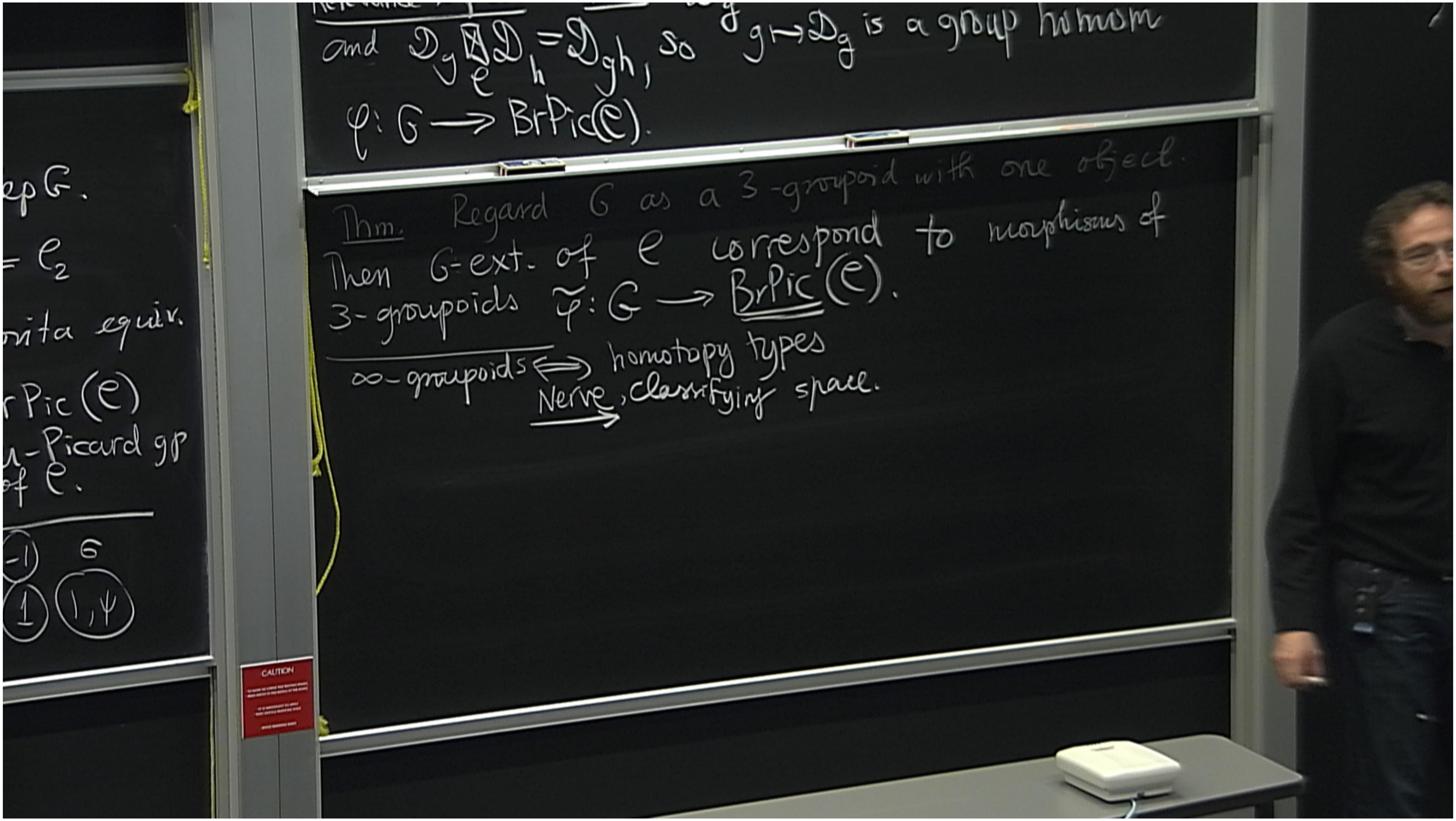
$\mathcal{C}$ - fus. cat . $\mathcal{M}$ - indec. $\mathcal{C}$-mod. $\mathcal{C}' = \mathrm{Fun}_{\mathcal{C}}(\mathcal{M},\mathcal{M})^{op} = (\mathcal{M}^{op} \boxtimes_{\mathcal{C}} \mathcal{M}.)^{op}$
dual cat.

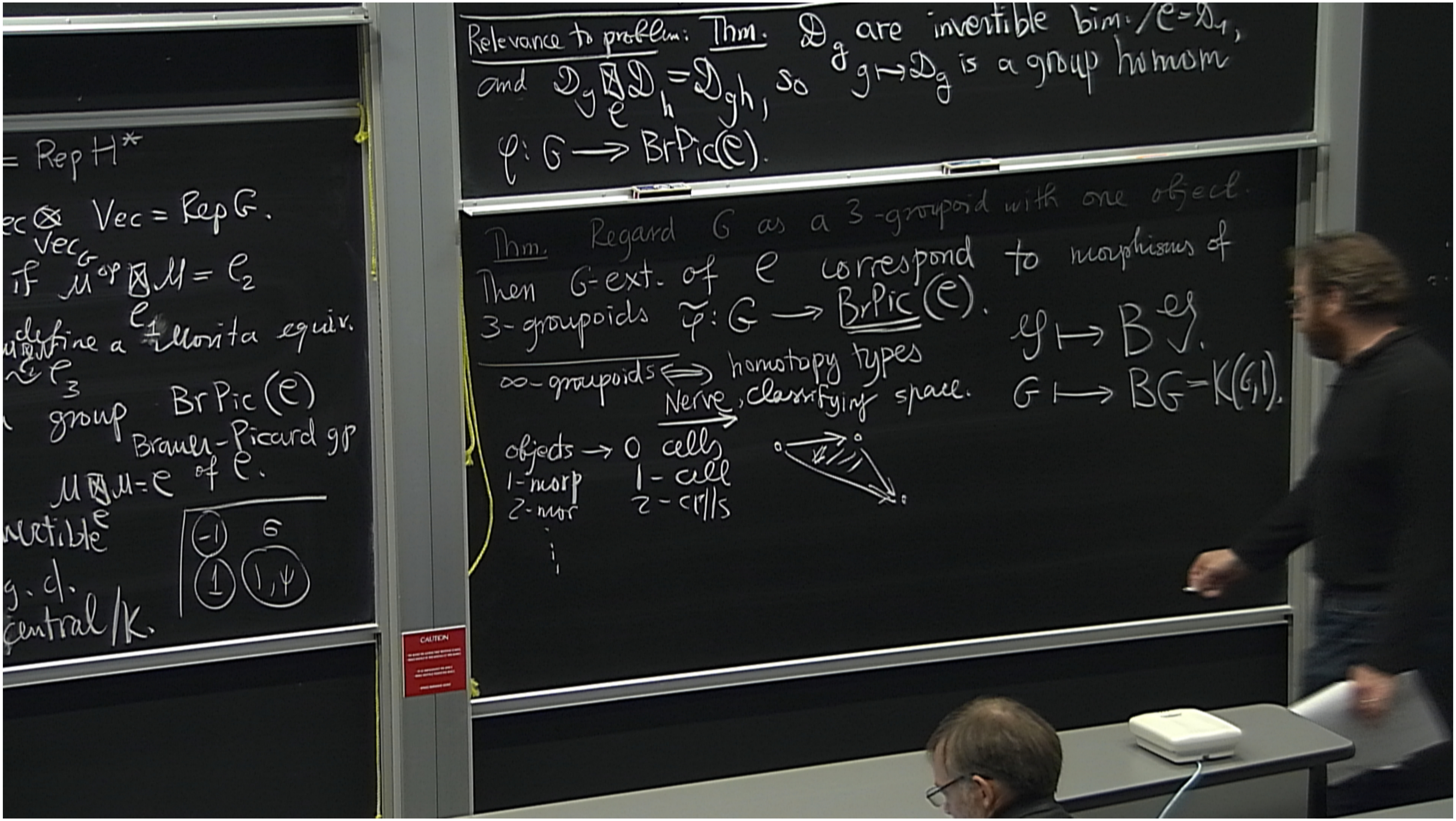
$\mathcal{C} \overset{\mathcal{M}}{\simeq} \mathcal{C}'$ ($\mathcal{M}$ inv. $(\mathcal{C},\mathcal{C}')$-bim.

Ex. $\mathcal{C} = \mathcal{E} \boxtimes \mathcal{E}^{op}$, $\mathcal{M} = \mathcal{E}$ $\quad \mathcal{C}' = \mathcal{Z}(\mathcal{E})$.

Relevance to problem: Thm. $\mathcal{D}_g$ are invertible bim. $/\mathcal{C} = \mathcal{D}_1$,
and $\mathcal{D}_g \boxtimes_{\mathcal{C}} \mathcal{D}_h = \mathcal{D}_{gh}$, so $g \mapsto \mathcal{D}_g$ is a group homom

$\varphi : G \longrightarrow \mathrm{BrPic}(\mathcal{C})$.


and $\mathcal{D}_g \boxtimes_{\mathcal{C}} \mathcal{D}_h = \mathcal{D}_{gh}$, so $g \mapsto \mathcal{D}_g$ is a group homom
$\varphi : G \to BrPic(\mathcal{C})$.
Thm. Regard G as a 3-groupoid with one object.
Then G-ext. of $\mathcal{C}$ correspond to morphisms of
3-groupoids $\tilde{\varphi} : G \to BrPic(\mathcal{C})$.
∞-groupoids ⟺ homotopy types
Nerve, classifying space.
rep G.
= $\mathcal{C}_2$
rita equiv.
r Pic ($\mathcal{C}$)
-Picard gp
of $\mathcal{C}$.
CAUTION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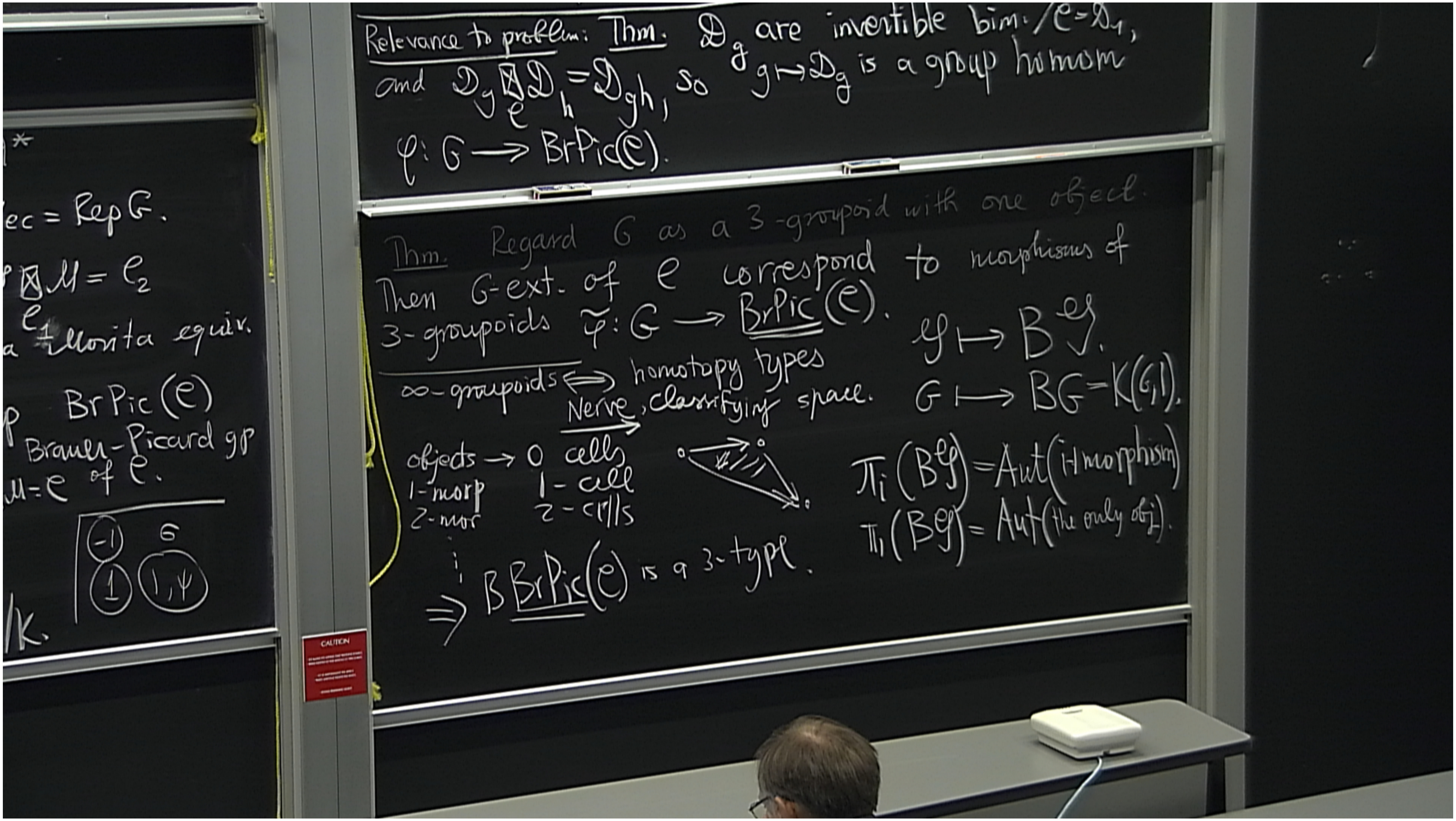
3-groupoids $\tilde{\varphi}: G \to BrPic(\mathcal{C})$. $\mathcal{G} \mapsto B\mathcal{G}$.
∞-groupoids ⟺ homotopy types
Nerve, classifying space. $G \mapsto BG = K(G,1)$.
objects → 0 cells
1-morp 1-cell
2-mor 2-cells
= Rep H*
Vec = Rep G.
Morita equiv.
group Br Pic ($\mathcal{C}$)
Brauer-Picard gp of $\mathcal{C}$.
central/K.
CAUTION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φ: G → BrPic(C).
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Then G-ext. of C correspond to morphisms of
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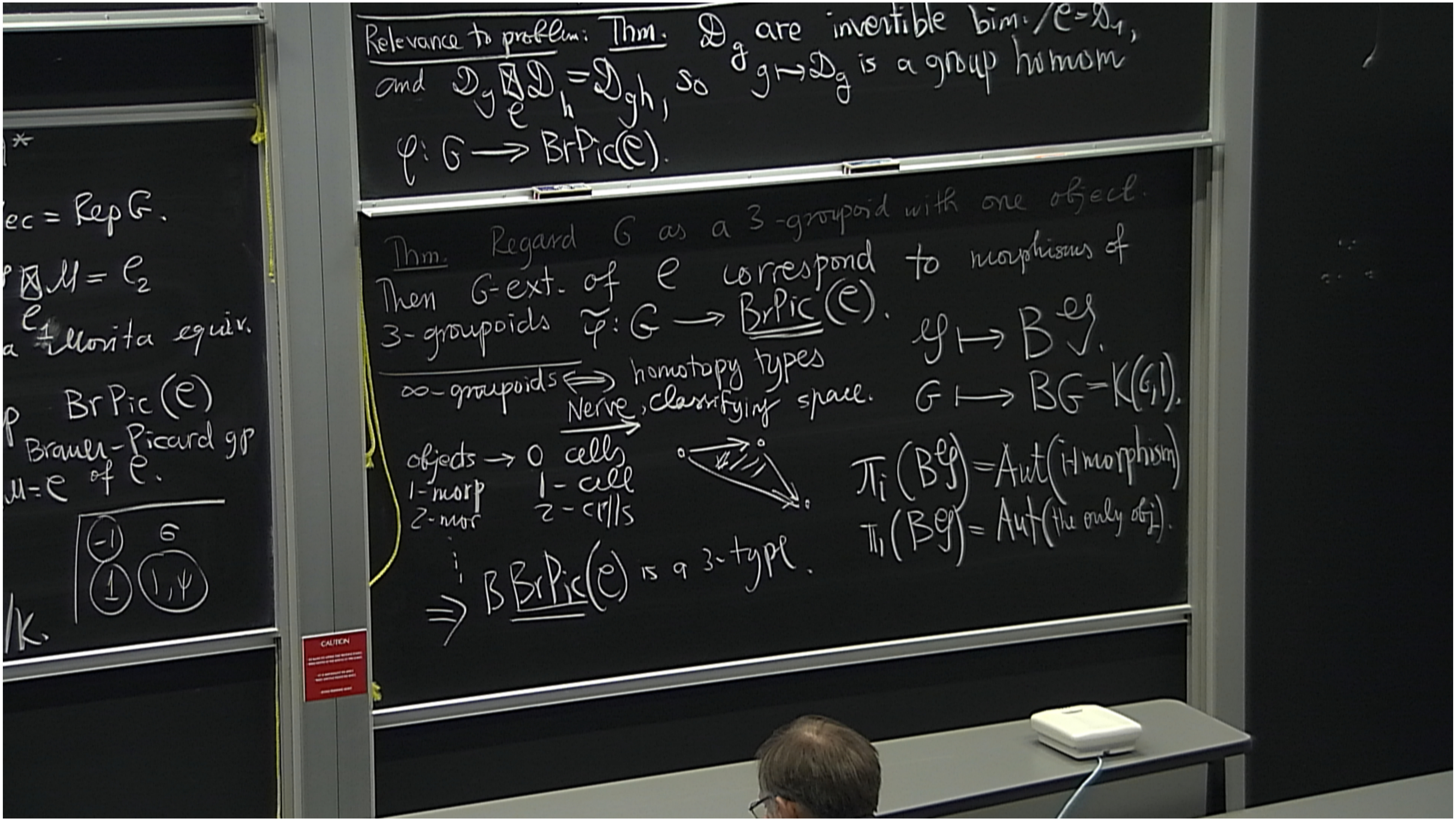
𝒢 ↦ B𝒢.
∞-groupoids ⟺ homotopy types
Nerve, classifying space.
G ↦ BG = K(G,1).
objects → 0 cells
1-morp 1-cell
2-mor 2-cells
π_1(B𝒢) = Aut(1-morphism)
π_1(B𝒢) = Aut(the only obj).
⇒ B BrPic(C) is a 3-type.
Vec = Rep G.
Morita equiv.
Br Pic(C)
Brauer-Picard gp of C.
CAUTION



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Thm. Regard G as a 3-groupoid with one object.
Then G-ext. of C correspond to morphisms of
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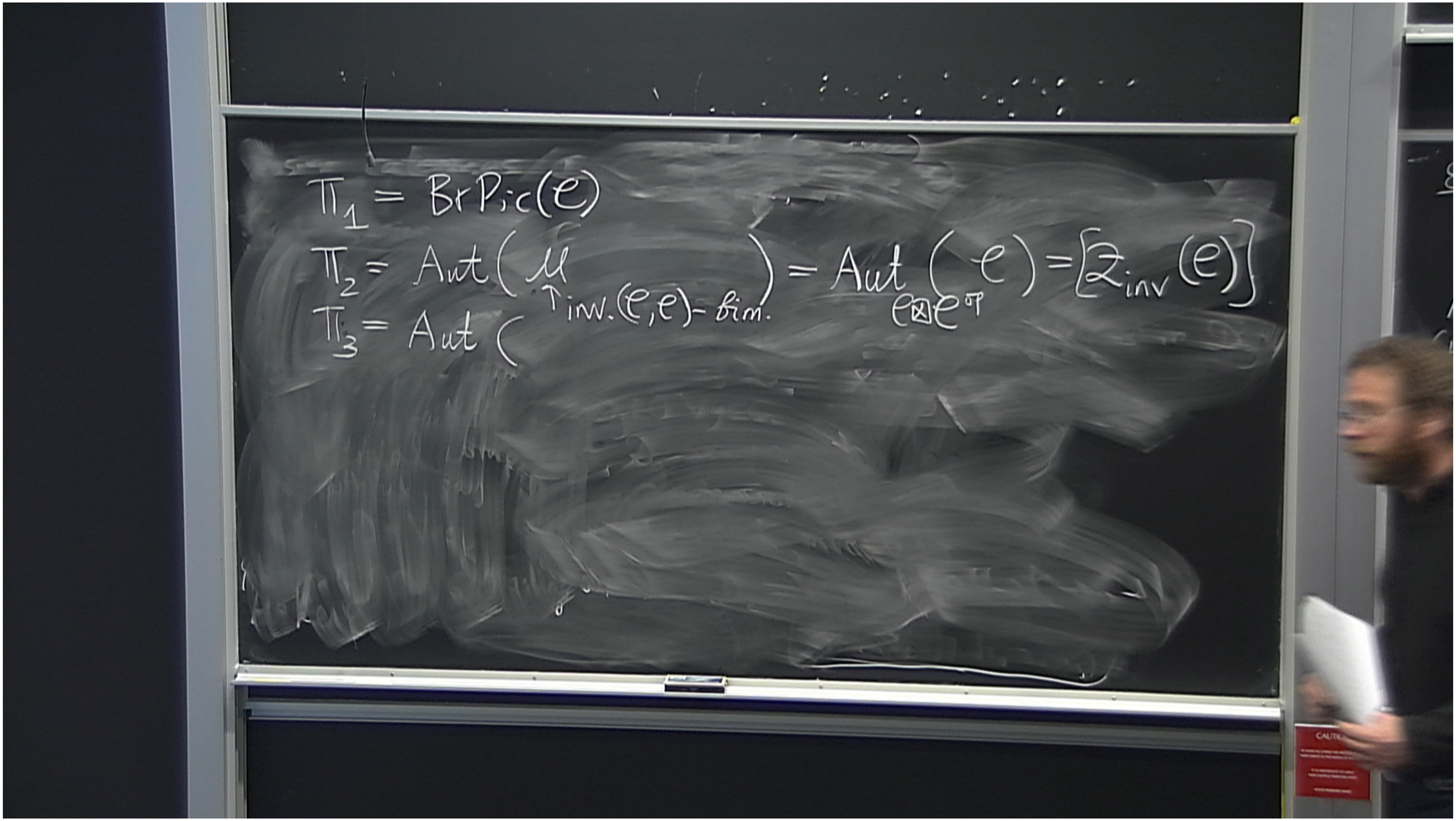
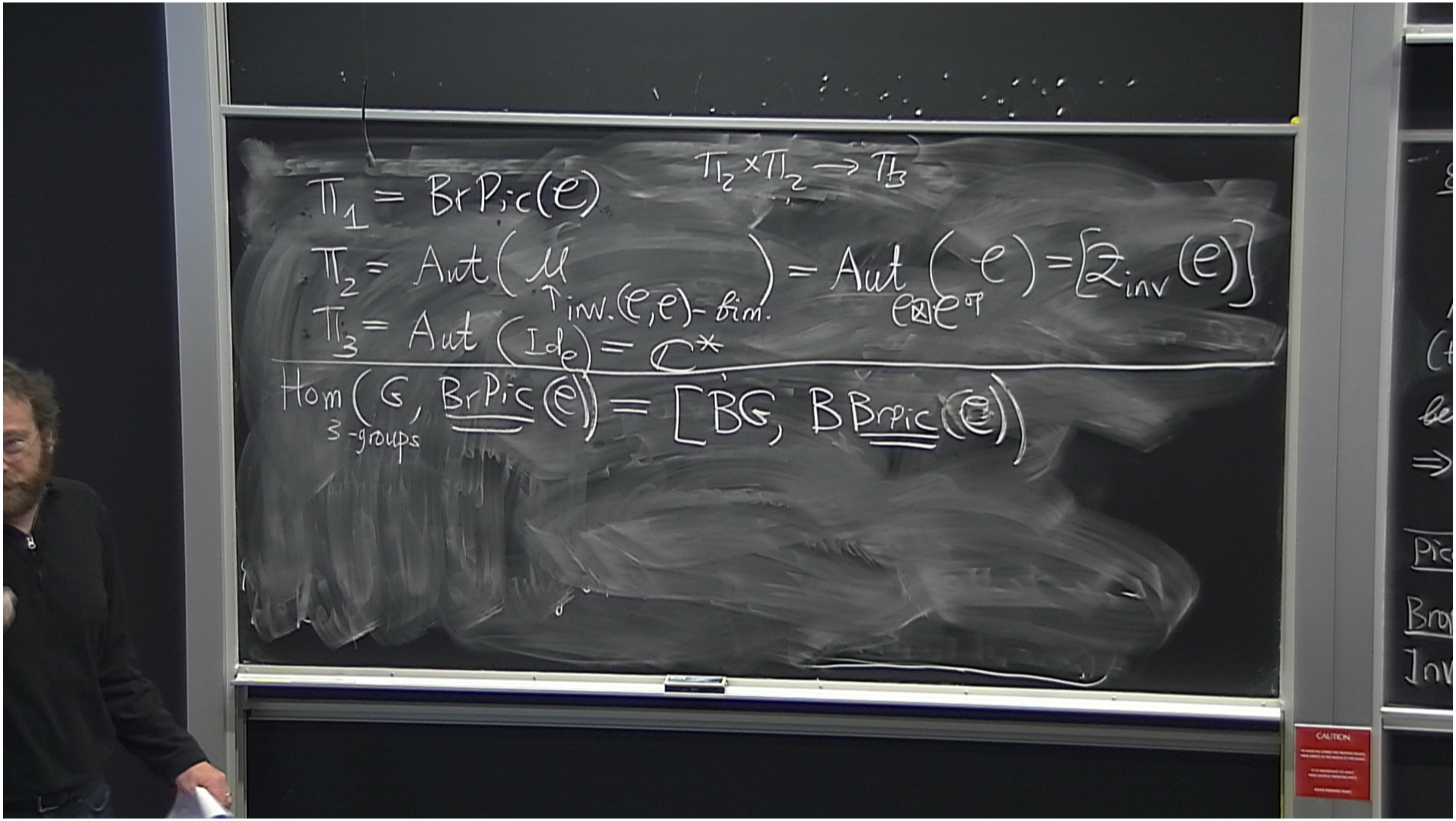
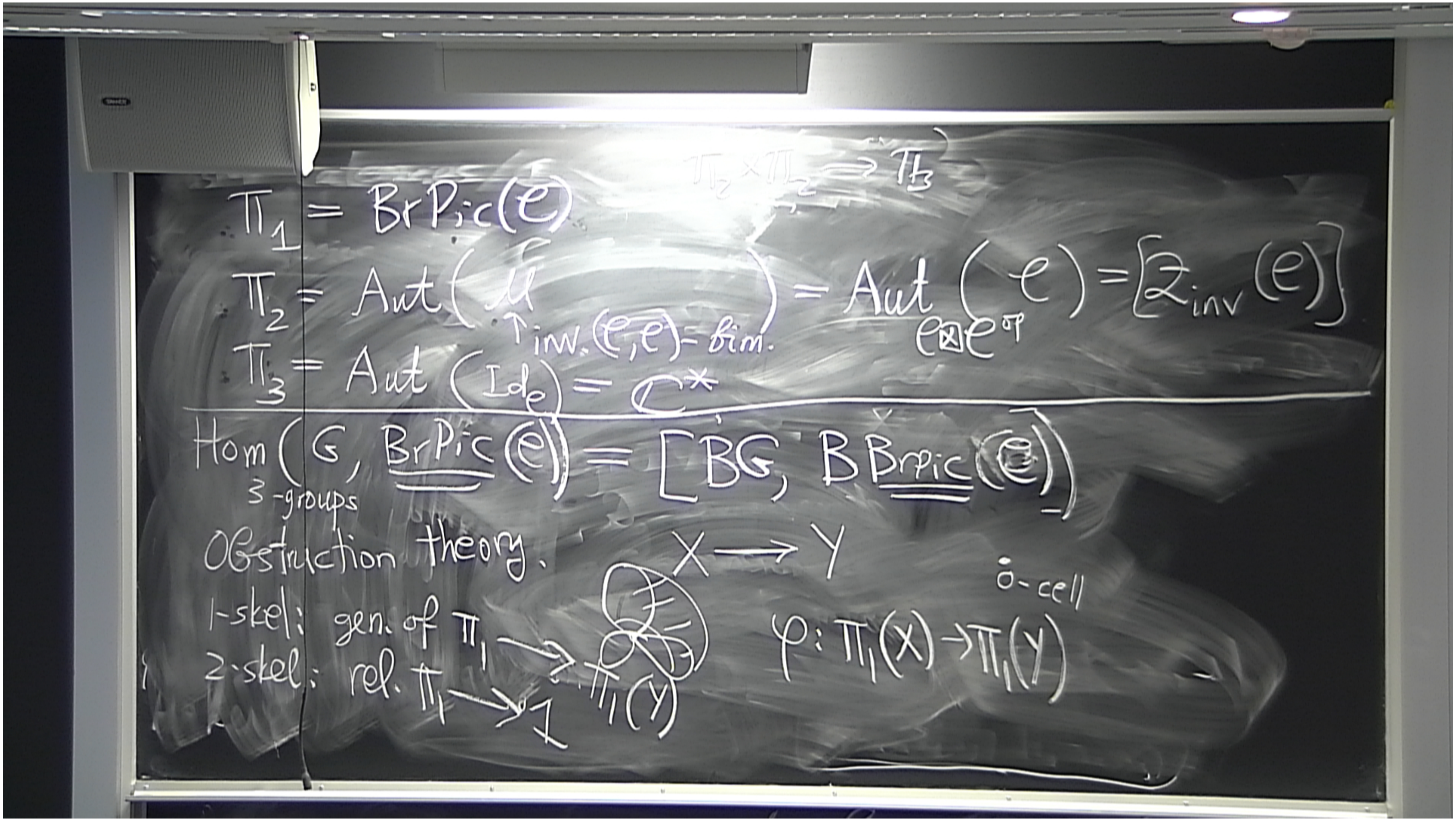
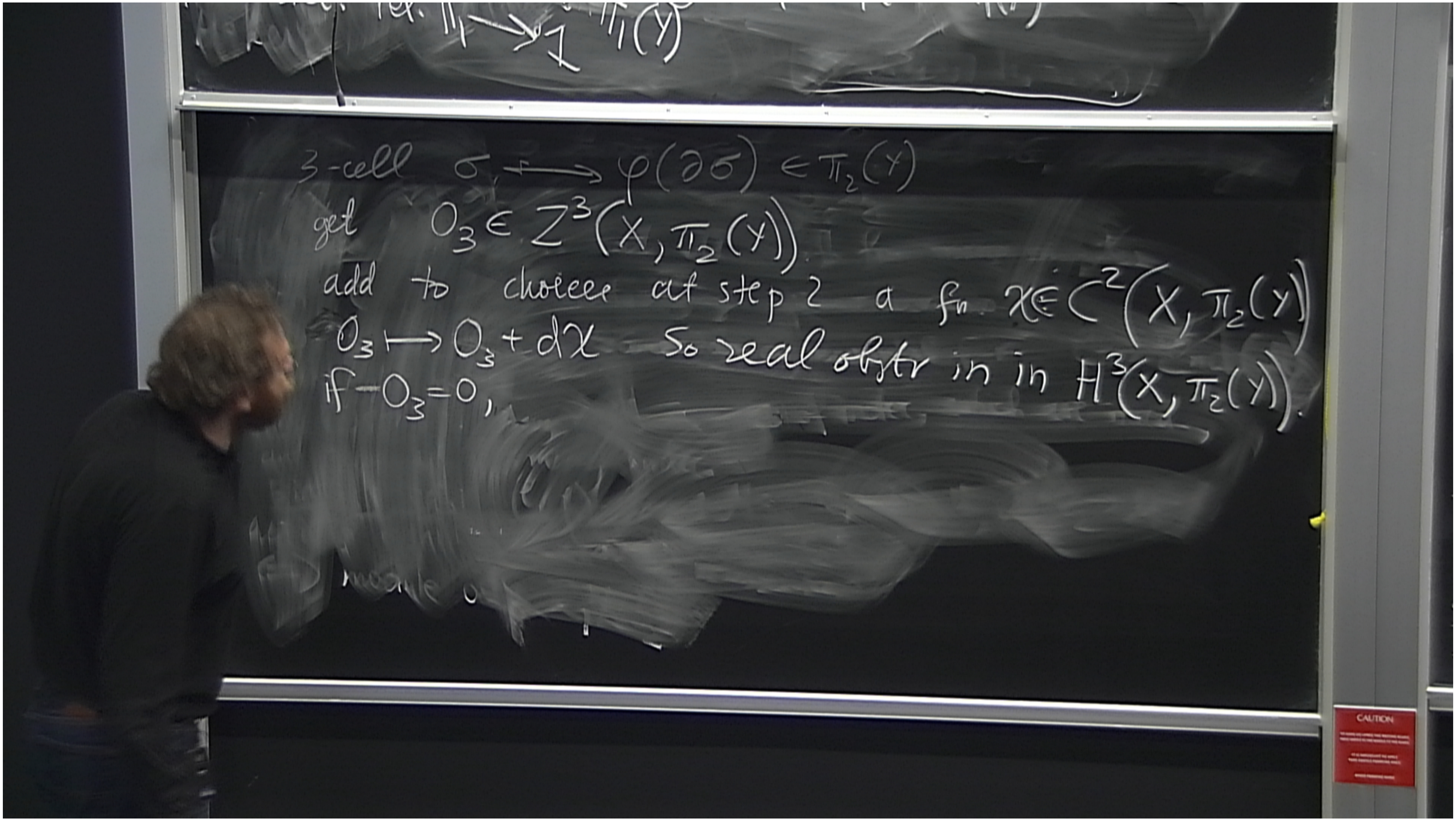
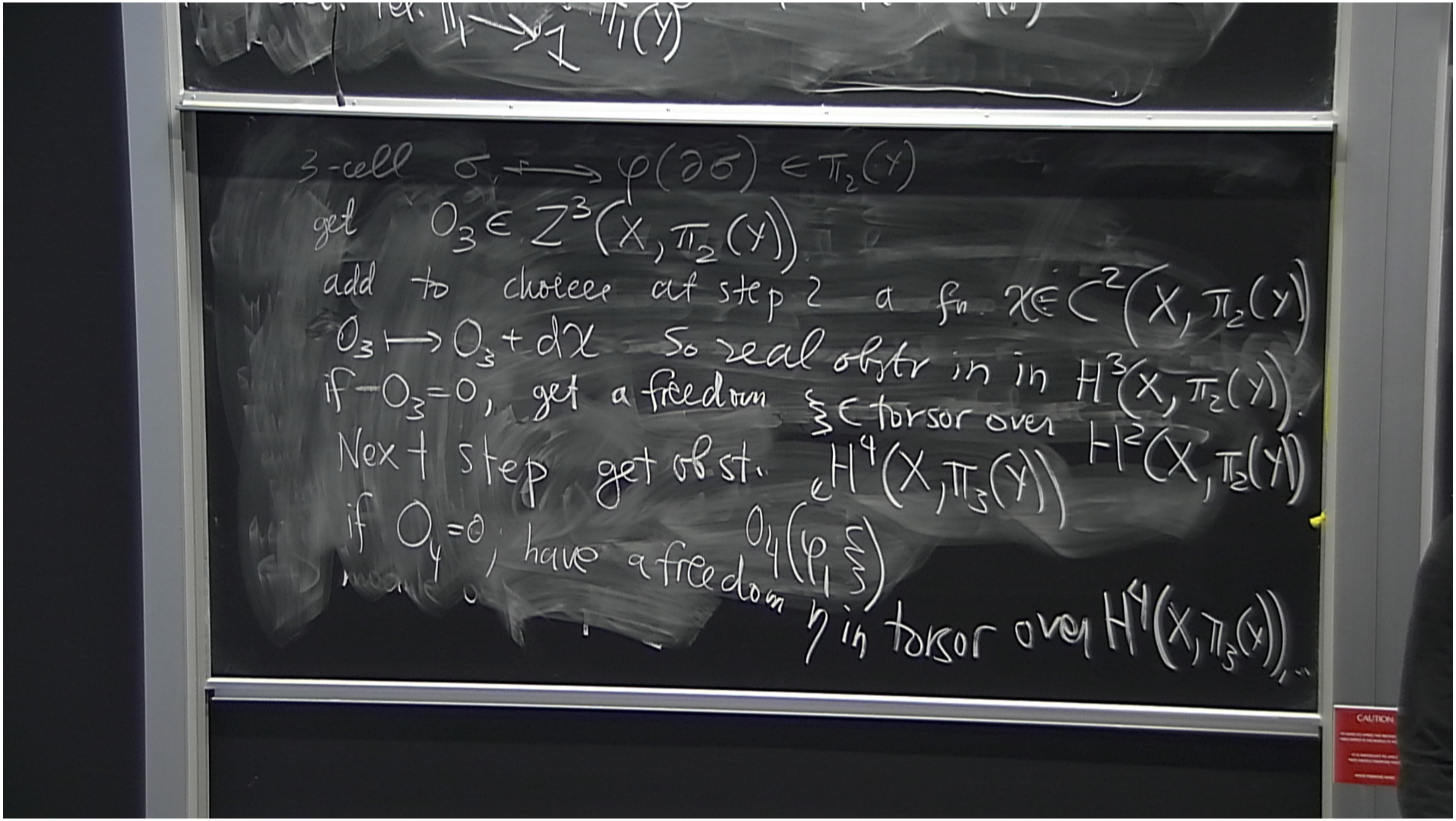

$\pi_1 = \mathrm{BrPic}(\mathcal{C})$
$\pi_2 = \mathrm{Aut}(\mathcal{M}_{\text{inv.}(\mathcal{C},\mathcal{C})\text{-bim.}}) = \mathrm{Aut}_{\mathcal{C}\boxtimes\mathcal{C}^{op}}(\mathcal{C}) = [Z_{inv}(\mathcal{C})]$
$\pi_3 = \mathrm{Aut}($

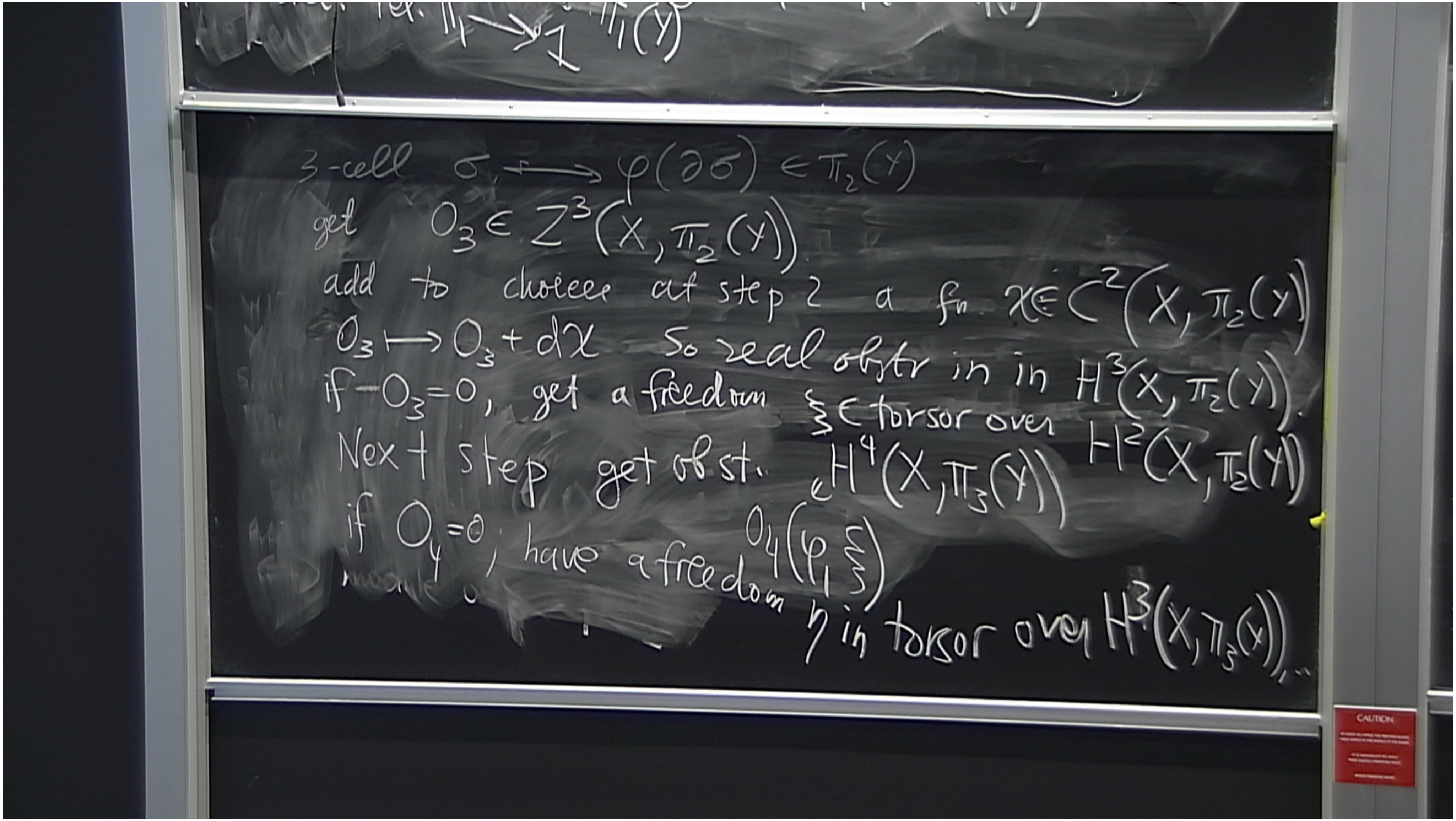

π2 × π2 → π3
π1 = BrPic(C)
π2 = Aut(M) = Aut(C) = [Zinv(C)]
inv.(C,C)-bim.
C⊠C^op
π3 = Aut(IdC) = ℂ*
Hom(G, BrPic(C)) = [BG, BBrPic(C)]
3-groups

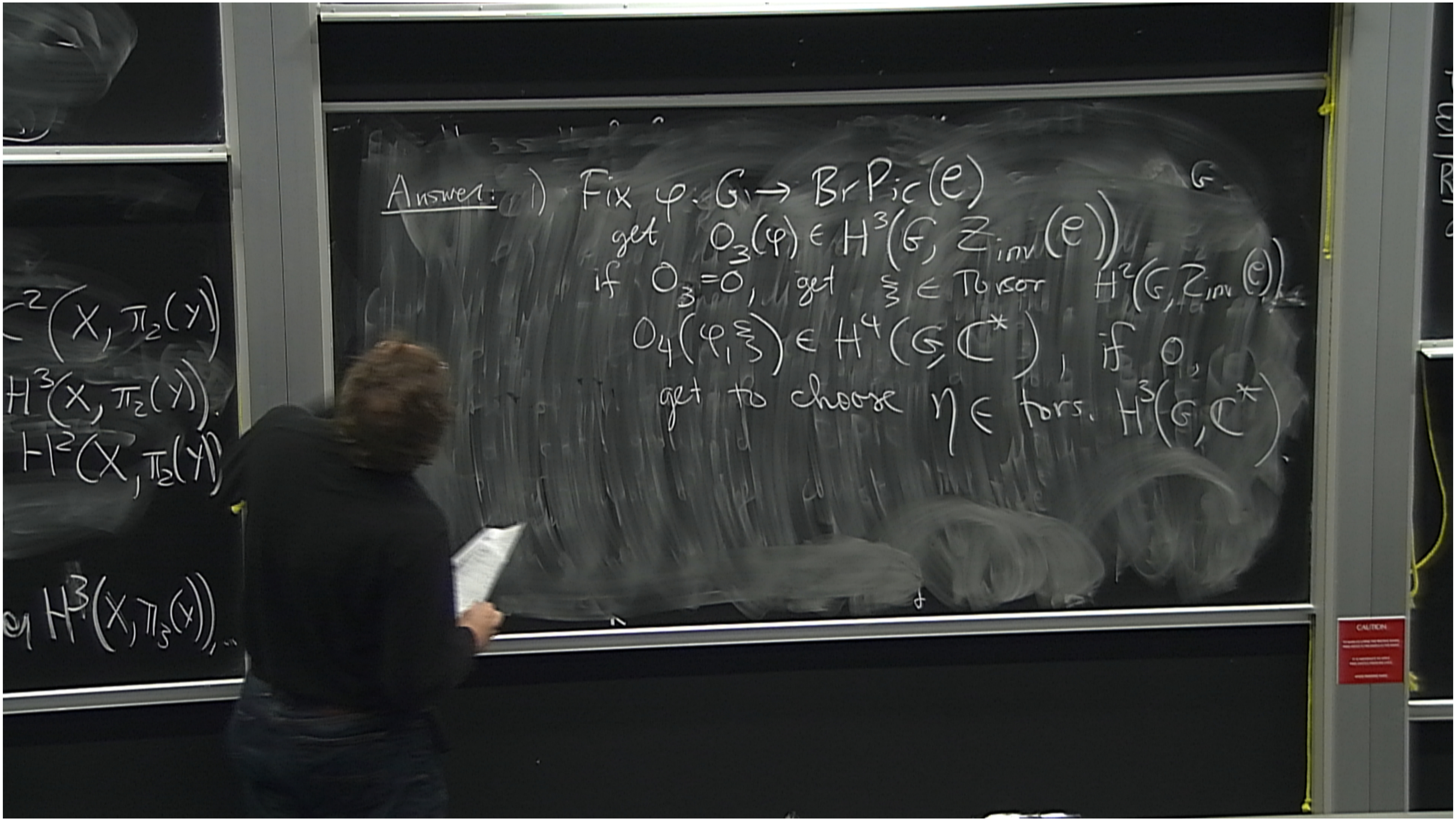

$\pi_2 \times \pi_2 \to \pi_3$
$\pi_1 = BrPic(\mathcal{C})$
$\pi_2 = Aut(\mathcal{M}) = Aut_{\mathcal{C} \boxtimes \mathcal{C}^{op}}(\mathcal{C}) = [Z_{inv}(\mathcal{C})]$
inv. $(\mathcal{C},\mathcal{C})$-bim.
$\pi_3 = Aut(Id_{\mathcal{C}}) = \mathbb{C}^*$
$Hom_{3\text{-groups}}(G, \underline{\underline{BrPic}}(\mathcal{C})) = [BG, B\underline{\underline{BrPic}}(\mathcal{C})]$
Obstruction theory. $X \to Y$
0-cell
1-skel: gen. of $\pi_1 \to \pi_1(Y)$
2-skel: rel. $\pi_1 \to 1$
$\varphi: \pi_1(X) \to \pi_1(Y)$

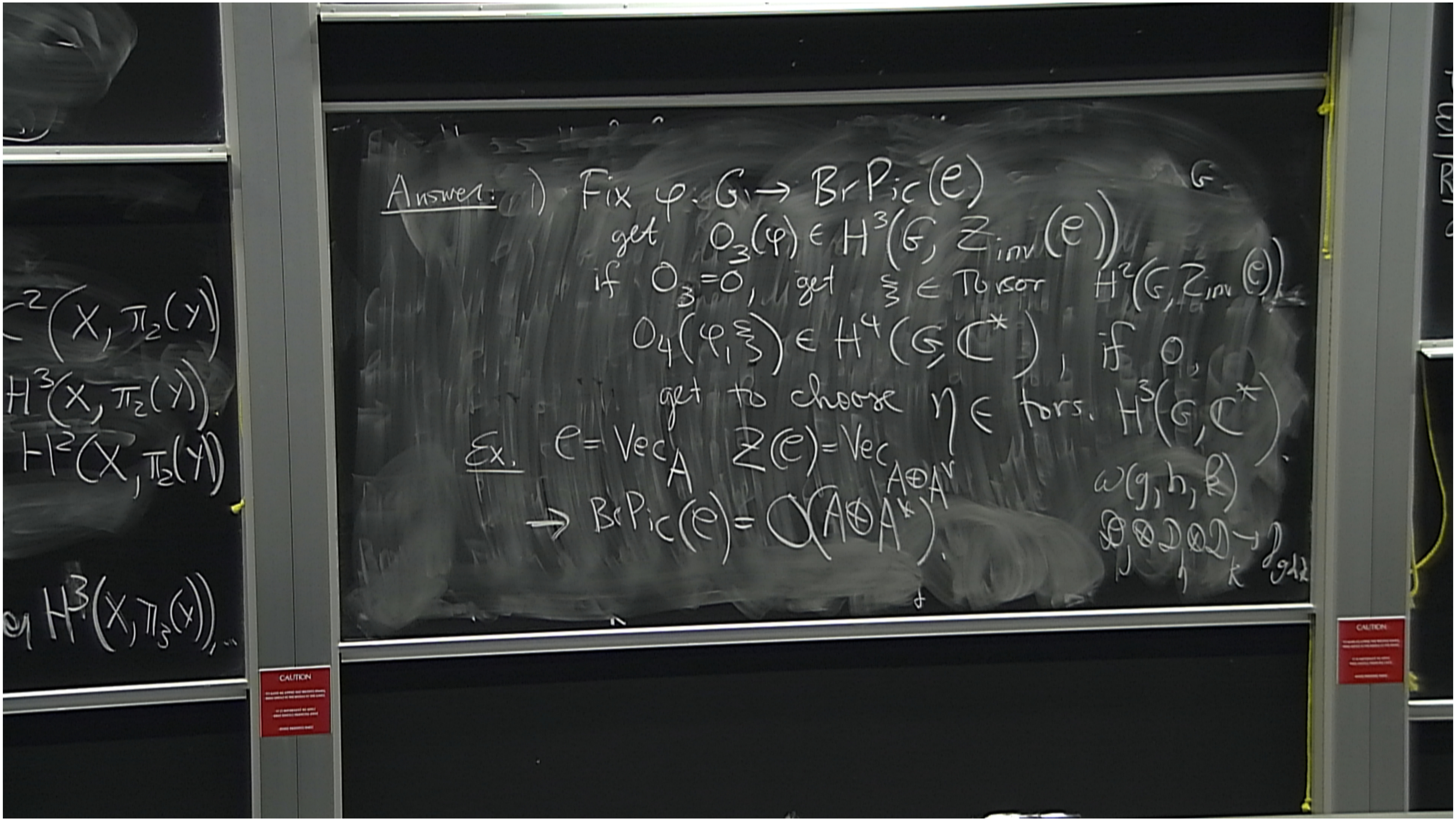

3-cell σ ↦ φ(∂σ) ∈ π2(Y)
get O3 ∈ Z3(X, π2(Y)).
add to choice at step 2 a fn. χ ∈ C2(X, π2(Y))
O3 ↦ O3 + dχ So real obstr in in H3(X, π2(Y)).
if O3 = 0,



$\pi_1 \to 1$ $\pi_1(Y)$
3-cell $\sigma \mapsto \varphi(\partial\sigma) \in \pi_2(Y)$
get $O_3 \in Z^3(X, \pi_2(Y))$
add to choices at step 2 a fn $\chi \in C^2(X, \pi_2(Y))$
$O_3 \mapsto O_3 + d\chi$ So real obstr in in $H^3(X, \pi_2(Y))$.
if $O_3 = 0$, get a freedom $\xi \in$ torsor over $H^2(X, \pi_2(Y))$
Next step get obst. $\in H^4(X, \pi_3(Y))$
$O_4(\varphi, \xi)$
if $O_4 = 0$, have a freedom η in torsor over $H^4(X, \pi_3(Y))$, ...


π1 → 1 π1(Y)
3-cell σ ↦ φ(∂σ) ∈ π2(Y)
get O3 ∈ Z3(X, π2(Y))
add to choices at step 2 a fn χ ∈ C2(X, π2(Y))
O3 ↦ O3 + dχ So real obstr in in H3(X, π2(Y)).
if O3 = 0, get a freedom ξ ∈ torsor over H2(X, π2(Y))
Next step get obst. ∈ H4(X, π3(Y))
O4(φ, ξ)
if O4 = 0, have a freedom η in torsor over H3(X, π3(Y)), ...


Answer: 1) Fix φ: G → BrPic(C)
get O_3(φ) ∈ H^3(G, Z_inv(C))
if O_3 = 0, get ξ ∈ Torsor H^2(G, Z_inv(C))
O_4(φ, ξ) ∈ H^4(G, C^*), if O_4 = 0,
get to choose η ∈ tors. H^3(G, C^*).
H^3(X, π_2(Y))
H^2(X, π_2(Y))
H^3(X, π_3(Y)), ...


Answer: 1) Fix $\varphi: G \to BrPic(\mathcal{C})$
get $O_3(\varphi) \in H^3(G, Z_{inv}(\mathcal{C}))$
if $O_3 = 0$, get $\xi \in$ Torsor $H^2(G, Z_{inv}(\mathcal{C}))$
$O_4(\varphi, \xi) \in H^4(G, \mathbb{C}^*)$, if $O_4 = 0$
get to choose $\eta \in$ tors. $H^3(G, \mathbb{C}^*)$.
Ex. $\mathcal{C} = Vec_A$ $Z(\mathcal{C}) = Vec_{A \oplus A^\vee}$
$\to BrPic(\mathcal{C}) = O(A \oplus A^*)$.
$\omega(g, h, k)$
$\mathcal{D}_g \otimes \mathcal{D}_h \otimes \mathcal{D}_k \to \mathcal{D}_{ghk}$
$H^2(X, \pi_2(Y))$
$H^3(X, \pi_2(Y))$
$H^2(X, \pi_2(Y))$
$H^3(X, \pi_3(Y))$, ...